Winter 1-2018

Inverse Function: Pre-Service Teachers' Techniques and Meanings

Teo Paoletti
Montclair State University, paolettit@mail.montclair.edu

Irma E. Stevens
University of Georgia

Natalie L.F. Hobson
Sonoma State University

Kevin C. Moore
University of Georgia

Kevin R. LaForest
University of Georgia

Published Citation

Paoletti, T., Stevens, I.E., Hobson, N.L.F. et al. Educ Stud Math (2018) 97: 93. https://doi.org/10.1007/s10649-017-9787-y

Inverse function: Pre-service teachers' techniques and meanings

Teo Paoletti[1], Irma E. Stevens[2], Natalie L. F. Hobson[3], Kevin C. Moore[2], and Kevin R. LaForest[2]

[1] *Mathematical Sciences Department, Montclair State University, 1 Normal Ave. Montclair, NJ 07043*

Phone: 856.912.3435

paolettit@mail.montclair.edu

[2] *Department of Mathematics and Science Education, University of Georgia, 105 Aderhold Hall, Athens, GA 30602*

[3] *Department of Mathematics and Statistics, Sonoma State University, 1801 E. Cotati Ave., Rohnert Park, CA 94928*

Acknowledgements: This material is based upon work supported by the NSF under Grant No. DRL-1350342. Any opinions, findings, conclusions, or recommendations expressed are those of the authors. We thank Neil Hatfield for directing us to a notation for inverse function and Nick Wasserman for his feedback on an earlier draft.

Abstract

Researchers have argued teachers and students are not developing connected meanings for function inverse, thus calling for a closer examination of teachers' and students' inverse function meanings. Responding to this call, we characterize 25 pre-service teachers' inverse function meanings as inferred from our analysis of clinical interviews. After summarizing relevant research, we describe the methodology and theoretical framework we used to interpret the pre-service teachers' activities. We then present data highlighting the techniques the pre-service teachers used when responding to tasks that involved analytical and graphical representations of functions and inverse functions in both decontextualized and contextualized situations and discuss our inferences of their meanings based on their activities. We conclude with implications for the teaching and learning of inverse function and areas for future research.

Keywords: Function; Inverse Function; Pre-service Secondary Teachers; Meanings

Inverse function: Pre-service teachers' techniques and meanings

1. Introduction

Inverse function is important in secondary and post-secondary mathematics. Both United States (U.S.) and international standards include inverse function as a topic for students to study (Bergeron, 2015), setting the expectation that mathematics teachers afford their students opportunities to construct connected inverse function meanings. Hence, pre-service teachers should construct and operationalize inverse function meanings that enable them to foster such opportunities for their future students repeatedly in order to prepare their students to develop many ideas in post-secondary mathematics including families of functions and calculus topics.

Although numerous researchers have investigated students' function meanings (e.g., Breidenback, Dubinsky, Hawks, & Nichols, 1992; Leinhardt, Zaslavsky, & Stein, 1990; Paz & Leron, 2009; Thompson & Carlson, 2017), fewer researchers have focused on students' inverse function meanings. Researchers who have investigated students' inverse function meanings have argued post-secondary students, pre-service teachers, and in-service teachers do not construct productive inverse function meanings (Brown & Reynolds, 2007; Engelke, Oehrtman, & Carlson, 2005; Even, 1992; Lucus, 2005; Vidakovic, 1996). We add to the body of research on individuals' inverse function meanings by providing insights into the pre-service teachers' (hereafter 'students'') inverse function meanings we inferred from their activities during clinical interviews.

2. Research on Students' Inverse Function Meanings

Vidakovic (1996) presented a genetic decomposition (i.e., a description of how students might learn a concept, including possible constructions of their schemas) in which she hypothesized students need to maintain a process-object conception of function (Breidenback et al., 1992) and schema rooted in the operation of composition. In characterizing the university calculus students' meanings, Vidakovic described that their most common technique for determining the inverse function when given an analytic rule was the action of switching the independent and dependent variables of the original function and solving for the dependent variable as defined for the original function (hereafter '*switching-and-solving*').[1] Based on her analysis of students' activities, Vidakovic adjusted her genetic decomposition to entail the schemas: "Inverse function as reverse of the function process," "Inverse function as coordination of two function processes to get the identity," and "Inverse function as an action of switching *x* and *y* and solving it for *y*" (p. 311). She conjectured students could coordinate these schemas with their function and composition of function schemas to address situations that call for inverse functions. However, the extent to which students construct or coordinate such schemas remains an open question (cf. Even, 1992).

More generally, researchers have shown that students and teachers often hold compartmentalized inverse function meanings (Brown & Reynolds, 2007; Engelke et al., 2005; Lucus, 2005; Vidakovic, 1996). These researchers have argued students are restricted to carrying out particular techniques in analytic (equation rule) or graphical representations that (from the researchers' perspectives) may not be related to one another. For instance, Brown and Reynolds (2007) noted that the seven university pre-calculus students in their study infrequently relied on their general descriptions of inverse function when determining an inverse function rule given a specific function rule. That is, when asked, "What does f^{-1} mean?" five students referenced switching the domain and range (or *x* and *y*), but when determining f^{-1} of a specific analytically defined function, the students' responses had "little to no connections" (p. 191) with their previously descriptions involving domain and range.

---

[1] Here we use "variable" to refer to a notational letter or symbol.

Engelke et al. (2005) reported 1,031 pre-calculus students' responses to four questions the researchers conceived as related to inverse function on the Precalculus Concept Assessment. They reported 35% of the students, when given a graphical representation of f, successfully determined x such that $f(x) = 3$ (i.e., determining $f^{-1}(3)$ given a graph of f); we return to this question when discussing inverse function notation in Section 6.1. No more than 16% of students responded correctly to any of the three questions that included inverse function notation, and only 1% of the students answered all three questions correctly. Additionally, in their report on 601 Calculus 1 students' responses to the Calculus Concept Readiness instrument, Carlson, Madison, and West (2015) noted 53% of students chose the option $f^{-1}(t) = 1/(100^t)$ when asked, "Given that f is defined by $f(t) = 100^t$, which of the following is a formula for f^{-1}?". Collectively, Carlson, Engelke, and colleagues' results suggest few pre-calculus and calculus students maintained inverse function meanings that supported them in consistently addressing tasks related to inverse function correctly.

Gaps in the literature on students' inverse function meanings remain. Brown & Reynold's (2007) inferences were limited because all seven students in their sample provided similar responses to tasks. Engelke et al. (2005) and Carlson et al. (2015) had large samples, but their analyses were limited to reporting students' responses to multiple-choice items. Based on these collective findings and limitations, we were interested in better understanding the techniques students use to determine inverse functions and the extent to which students' techniques indicate they maintain connected systems of meanings for 'function inverse' across several tasks. Relatedly, we were interested in providing a more detailed characterization of students' meanings than researchers have previously provided.

3. Theoretical Perspective

We approach knowledge as actively built up by an individual in ways idiosyncratic to that individual (von Glasersfeld, 1995); researchers can only make inferences about students' meanings based on interpretations of students' words and actions. When making such inferences, models that Steffe and Thompson (2000) referred to as the *mathematics of students*, we rely on definitions Thompson, Carlson, Byerley, and Hatfield (2014) attributed to Thompson and Harel. Specifically, Thompson and Harel defined *understanding* to be a *cognitive state* of equilibrium resulting from assimilation. Whereas understanding refers to a cognitive state, *meaning* refers to the scheme(s) associated with an understanding including the scheme's *space of implications*, which can consist of (a collection of) actions, objects, or schemes an individual brings to mind when an understanding is achieved (Thompson et al., 2014). The students in our study had multiple instructional experiences prior to our interactions in which they established inverse function meanings. Our interest was to gain insights into these meanings.

To do so, we characterized the techniques the students used when addressing our prompts. We use the term *technique* to refer to a student's words and observable activity as she addressed a single task. Relating Mason and Spence's (1999) construct of *knowing-to*—"active knowledge which is present in the moment when it is required" (p. 135)— to Piaget's genetic epistemology, Thompson (2016) stated:

> Knowing-to, as described by Mason and Spence, can be characterized more expansively by appealing to Piaget's notion that a scheme is a meaning—an organization of actions, images, and other meanings. Thus, one knows-to act in a particular way in a particular context because the actions implied by one's understanding of a context are in the scheme to which you assimilated the context. In this regard, I hasten to add that Piaget had an

> expansive meaning of action, as "all movement, all thought, all emotion that responds to a need" (Piaget, (1968), p. 6). (p. 436).

By characterizing and analyzing the students' techniques, we provide descriptions of what the students "knew-to" do when addressing each prompt.

We did not make inferences about a student's inverse function meanings based solely on her technique in one task or representational system. Instead, we characterized and compared a student's techniques within and across representations and tasks to make inferences about her meanings. By designing tasks (see *Task Design*) that prompted students to determine, use, and interpret inverse functions across a variety of representations and settings, we intended to develop a collection of viable explanations for their techniques (i.e., their "know-to" acts) and use these explanations to make inferences about the students' meanings. For instance, and consistent with Thompson and Harel's description of knowing (Thompson et al., 2014), we hoped to characterize the extent to which students' activities across tasks, representations, and settings indicated their techniques were interrelated parts of a meaning's space of implications, thus enabling them to assimilate all prompts to the same meaning. In contrast, for students who engaged in disconnected techniques, we consider the student's space of implications resulting from an act of assimilation to consist (almost) solely of the technique with little to no operative connections to other techniques used in other representational systems or contexts. In such a case, the technique *is* essentially the student's meaning for inverse functions in that representation or context, as opposed to the technique being a part of a larger, connected meaning entailing numerous techniques.

4. Methods

4.1 Subjects and setting

We conducted interviews with 25 students (18 female, 7 male) enrolled in a large university in the southeastern U.S. The students were, according to coursework, in their third year of university studies and were between 18 and 22 years of age, which is traditional in U.S. universities. We chose the students on a volunteer basis from a convenience sample of students accessible to the research team in terms of program, location, and scheduling. The students were beginning their first pair of courses (content and pedagogy) in a secondary mathematics teacher preparation program. Consistent with most U.S. teacher preparation programs, the students' program-specific courses occurred during their last two years of university studies. We purposefully chose to work with students *prior* to the onset of their program coursework in order to examine their inverse function meanings before these courses potentially influenced their meanings. Each student had completed a full calculus sequence and at least two additional mathematics courses (e.g., linear algebra, differential equations) with a minimum grade of a C in each course. The students' extensive mathematical experiences allowed us to examine if such students maintained more connected meanings than the pre-calculus and calculus students reported elsewhere (Brown & Reynolds, 2007; Carlson et al., 2015; Engelke et al., 2005).

With each student, we conducted one 60-90 minute task-based interview (Goldin, 2000) designed to investigate students' knowledge and reasoning, thus aligning with Clement's (2000) descriptions of clinical interviews. Each interview consisted of the same series of tasks with the same initial prompts, thus aligning with structured task-based interviews. The interviews were semi-structured in that we had leeway to ask follow-up questions based on the student's responses and activity (Merriam & Tisdell, 2005). At least one researcher was present at each interview. If two researchers were present, one served as lead interviewer, and the other took observational notes based on the student's activities and was free to ask follow-up questions. We emphasized to each student that we were not concerned with correct answers and we asked the students to 'think aloud' (Clement, 2000; Goldin, 2000) as they worked through tasks. We attempted to gain insights into the student's thinking by using the follow-up suggestions offered by Hunting (1997, p. 153) (e.g., asking open-ended questions). We used minimal heuristics to probe student thinking because we did not intend to promote shifts in student thinking (Goldin, 2000; Hunting, 1997).

4.2 Task design

The interview tasks included decontextualized and contextualized tasks in analytic and graphical representations. These variations enabled us to explore the techniques students enacted when addressing each task type and to analyze the students' techniques across task types. As an example, consider the decontextualized graphed functions in *Figure 1*. From our perspective, these tasks involved both linear (a and d) and non-linear (b and implicitly c) curves as well as coordinate systems with square (a and b), rectangular (d), and unlabeled (c) axes. By using different curves, we intended to examine if students' perceived curves impacted their techniques. By providing different axes partitions, we intended to delineate between students who engaged in particular actions regardless of axes scaling versus those who engaged in actions attentive to axes scaling.

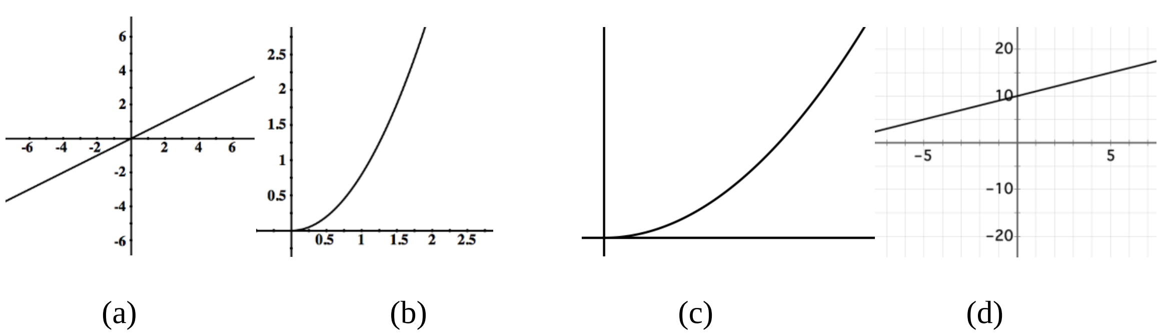


Figure 1. Four decontextualized graphical tasks.

Other tasks consisted of decontextualized functions represented analytically. These tasks included (a) identifying if several pairs of functions are inverse functions (e.g., determining if $f(x) = x + 1$ and $g(x) = x - 1$ define inverse functions), (b) evaluating $f^{-1}(6)$ *after* using the given rule $f(x) = x^3 - x$ to determine $f(2) = 6$ (e.g., $f^{-1}(6) = 2$), and (c) determining x such that $f(x) = 1$ given the analytic rule defining f^{-1} (*Figure 2*). Through these tasks, we intended to explore the extent to which the students' inverse function meanings were tied to carrying out particular techniques. For instance, we designed the task in *Figure 2* so that determining a rule for $f(x)$ would be unmanageable. A student who understands inverse functions in relation to function composition might understand that $f^{-1}(f(x)) = x$ and leverage this to determine x by evaluating $f^{-1}(1)$. In contrast, a student whose inverse function meanings are constrained to switching-and-solving (see *Figure 3* for two examples of this technique) might be unable to determine the rule for f, and thus not obtain a value for x.

Suppose that $f(x)$ is a one-to-one function whose inverse is $f^{-1}(x) = (x + 1)^3 - 5x^2 + 2$. Find a value of x so that $f(x) = 1$.

Figure 2. An example analytical inverse function task.[2]

$f(x) = x + 1$ $\quad\quad$ $f(x) = x^2$

$x = y + 1$ $\quad\quad$ $x = y^2$

$x - 1 = y$ $\quad\quad$ $\sqrt{x} = y$

Figure 3. A student's switching-and-solving technique for determining the inverse function of $f(x) = x + 1$ and $f(x) = x^2$.

[2] When creating tasks, we attempted to use notation consistent with what we conjectured the students were introduced through schooling. No students questioned or objected to the notation in any prompt. For instance, students could have questioned using $f(x)$ (i.e., a function's output value) to name a function (i.e., f).

We also posed graphical and analytical tasks asking students to determine and interpret the inverse function of a function that defined a contextual relationship. Specifically, we provided students an analytic rule and graph each defining the relationship between temperature measured in degrees Fahrenheit and degrees Celsius, and asked them to determine the inverse function (*Figure 7*). Most researchers examining students' inverse function meanings have investigated these meanings without reference to contextualized situations. Phillip's (2015) study of (a university mathematics major) Britney's understanding of domain is one exception. When determining the inverse function of a given function that related the number of patrons entering a theme park to the park's income, Britney claimed the inverse function did not exist because, "Common sense-wise it's not invertible." We conjecture Britney understood a function that inputs the park's income and outputs the number of patrons entering the park was not viable because she conceived that the park's income was dependent upon the number of patrons and not vice versa. Britney's case illustrates an added complexity when interpreting a contextual meaning of an inverse function; if a student conceives of a situation in which one quantity is necessarily dependent on the other, then the inverse function may not be a viable representation of the situation.

Our intention in using contextualized functions was to examine if students used a technique similar to one used with decontextualized functions and how students interpreted their determined inverse function with regards to the context. For the Fahrenheit/Celsius task, a student who understands that a function and its inverse function represent the same relationship between quantities may reflect over the line determined by equality of the axes variable values (i.e., the line $y = x$) when graphically determining an inverse function while holding in mind that this technique requires switching the quantitative referent of each axis to maintain the relationship between temperature measures. In contrast, a student whose meanings foreground the technique of reflecting over the line $y = x$ without attending to quantitative referents could interpret their constructed graph as representing a new, non-normative, relationship between temperature measures.

Table 1 provides a summary of the aforementioned task types. We did not ask students to define "inverse function" at the onset of the interview. We hypothesized that a definition prompt (e.g., "Define inverse function.") at the onset of the interview might have led to students producing some associated technique they would attempt to apply on subsequent problems. We also intended to determine if the students *spontaneously* identified some meaning for inverse function that connected their techniques both within and across representations and contexts.

Table 1
The number of graphical and analytical tasks by decontextualized and contextualized tasks.

	Graphical	Analytical
Decontextualized	4	3
Contextualized	1	1

4.3 Analysis

We videotaped each interview and digitized all written work. We selectively transcribed the videos, capturing words and actions. To analyze the data, we used open and axial techniques (Clement, 2000; Strauss & Corbin, 1998) in combination with conceptual analysis techniques (Thompson, 2008) in order to construct explanatory models of the students' thinking (i.e., *mathematics of students* (Steffe & Thompson, 2000)). Each researcher analyzed a subset of students, describing each student's technique to solve each task. We discussed our observations, looking for common techniques on specific tasks or types of tasks. As patterns developed, we created and revised codes to characterize the techniques we observed. When a researcher was unsure how to code a particular instance, we watched this instance collectively to reach an agreement regarding the code. This process sometimes led to the refinement of a code or creation of a new code. Through this iterative process, we developed a final set of codes to represent students' techniques, described in the next section. After we created final codes, each researcher re-coded his or her students' activities to ensure the codes satisfactorily described each student's techniques.

By comparing and contrasting the students' techniques across the various task types, we hoped to gain insights into students' inverse function meanings. We discerned how *connected* a student's inverse function meanings were based on how consistent her techniques were (from our perspective) while addressing various tasks. For instance, if a student determined an inverse function by using the reciprocal of an analytically defined function (e.g., $f^{-1}(x) = 1/f(x)$) for all decontextualized analytical tasks, then we classified that student as having a *consistent technique* when working with analytically defined functions. However, if that same student reflected over the horizontal axis when prompted to graph the inverse of a linear function represented graphically (e.g., *Figure 1*a), but reflected over the line $y = x$ when graphing the inverse function of a non-linear function represented graphically (e.g., *Figure 1*b), we classified the student as having an *inconsistent technique* for graphically determining inverse functions. This analysis supported our identifying the extent to which a student's techniques addressing various representations and contexts were part of a connected meaning versus a student's techniques implying he or she drew on meanings that were restricted to executing techniques constrained to some feature of the task, thus not appearing to connect to techniques used to address other representations or contexts (i.e., disconnected meanings).

5. Results

We first describe our coding of the students' activities addressing decontextualized tasks. We then present our coding of the students' activities addressing contextualized tasks and compare the students' activities in the decontextualized and contextualized tasks.

5.1 Students' activities addressing decontextualized tasks

In total, 19 of the 25 students (see Table 2) exhibited consistent techniques to determine the inverse function of a graphically displayed decontextualized function. Eleven students consistently switched coordinate values (i.e., the point (a, b) defined by the original curve was mapped to (b, a) on the curve determining the inverse function) or reflected over the line they perceived as $y = x$. Caroline is one student who consistently exhibited this technique (*Figure 4*). When she could not identify coordinate values to switch, Caroline reflected the curve over the line she perceived as $y = x$ (*Figure 4*c), and expressed that this reflection produced the same outcome as switching coordinate values.

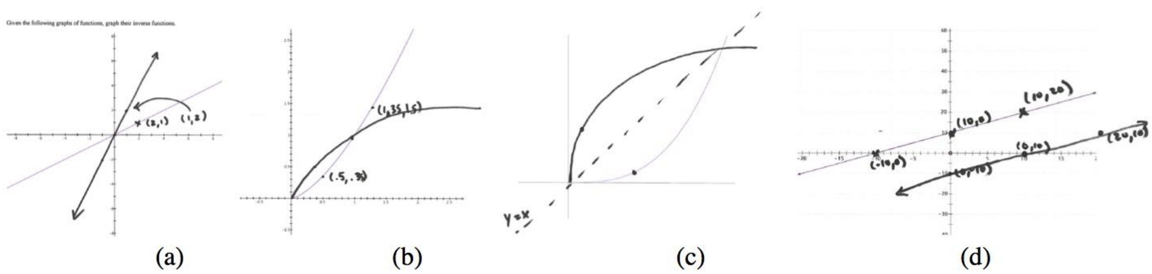
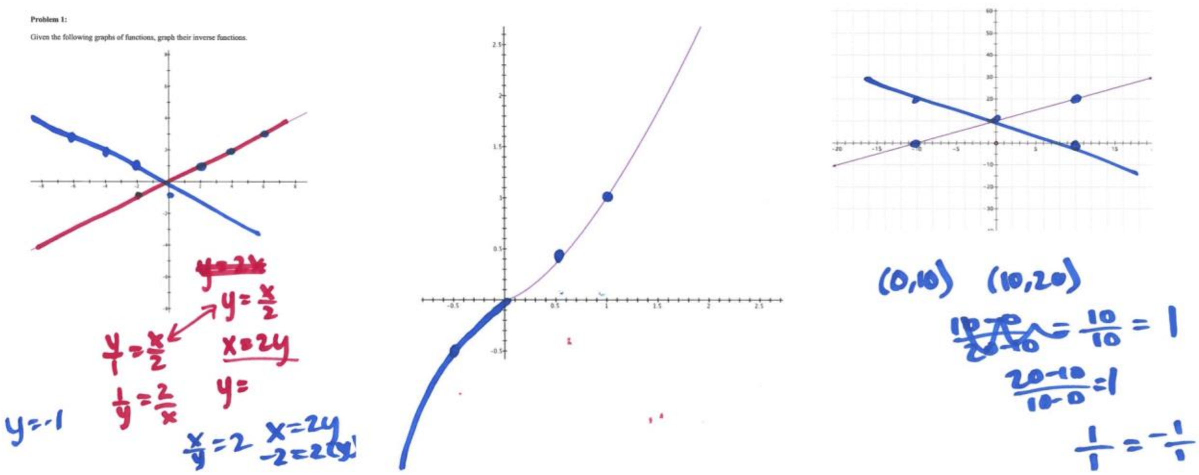


Figure 4. Caroline's solutions to the four decontextualized graphical tasks.

Three students consistently reflected over the horizontal or vertical axis. Three other students determined (exact or approximate) analytic rules in the form "$y =$" associated with each graph, and then determined the inverse function by switching-and-solving and graphing the resulting analytic rule. One student mapped all points (x, y) to the points (x, $1/y$), and the final student determined her inverse function graphs by negating the "slope" for the linear curves and changing an increasing graph to a decreasing graph for the non-linear curves. We summarize these techniques in Table 2.

Alyssa is a student who did not use a consistent technique across the decontextualized graphical tasks. Alyssa first attempted to determine the "slope" of the line representing the original function and to calculate "one over" this slope to determine what she anticipated to be the slope of the line representing the inverse function. After having difficulty, she negated *x*-values of the initial curve (i.e., the point (a, b) became the point ($-a$, b) in *Figure 5*a). For *Figure 5*b (and for the graph in *Figure 1*c), Alyssa negated both coordinate values (i.e., the point (a, b) in the initial curve became the point ($-a$, $-b$)). For *Figure 5*c, Alyssa used another technique; she calculated the slope of the line representing the original function and negated this value to identify what she called the slope of the inverse function. She then graphed a line with this slope and the same *y*-intercept as the original function's graph. Although Alyssa's final graphs in *Figure 5*a and 5c are reflections over the *y*-axis, her techniques for determining the inverse function in each case differed. Due to the variety of her techniques, we characterized Alyssa as one of the six students who did not exhibit a consistent technique across decontextualized graphical tasks.[3]

[3] It can be argued all of her techniques consistently involve switching something or performing a transformation. However, we did not infer Alyssa conceived each technique in terms of some underlying invariance.



(a) (b) (c)

Figure 5. Alyssa's solutions to the decontextualized graphing tasks.

Across the decontextualized analytical tasks, 24 students exhibited consistent techniques. Fifteen students engaged in switching-and-solving[4]. Eight students determined the inverse function to be the reciprocal of the original function (e.g., stated $f(x) = x^2$ and $g(x) = x^{-2}$ were inverse functions). One student determined the inverse function by solving for x using the original function rule (*Table 2*). The one student who did not exhibit a consistent technique across these tasks used function composition to determine if two functions were inverse functions, but did not have a technique for determining an analytic representation of the inverse function of a given function.

Table 2
Categorizations of students' techniques in decontextualized tasks (# denotes the number of students who exhibited the technique)

Graphical Technique	#	Analytical Technique	#
Switched coordinate values or reflected over the line $y = x$	11	Switched x and y (with or without solving for the switched y variable)	15
Reflected a curve over an axis or a line other than $y =$	3	Used the reciprocal (e.g., $f^{-1}(x) =$	8

[4] Students used $f(x)$ and y interchangeably. See Thompson (2013) for more on muddled uses of function notation.

x		1/*f* (*x*))	
Determined an analytic function then solved for *x* or switched *x* and *y* and solved *y*	3	Solved for *x* using the original function rule	1
Created a new curve by transforming each point (*x*, *y*) to (*x*, 1/*y*)	1		
Negated slope, and/or reflected concavity	1		

We highlight that a majority of the students used a technique for determining inverse functions that involved 'switching' (i.e., switching-and-solving for functions represented analytically and switching coordinate values for functions represented graphically). In total, eight students consistently used a 'switching' technique for both decontextualized graphical and decontextualized analytical tasks. An additional 10 students used a switching technique for one but not the other representation (Table 3).

Table 3
Number of students consistently using 'switching' or other (possibly inconsistent) techniques in graphical and analytic representations

	Consistently used 'switching' technique for graphical representations	Other (possibly inconsistent) technique(s) for graphical representations
Consistently used 'switching' technique for analytic representations	8	7
Other (possibly inconsistent) technique(s) for analytic representations	3	7

We found students' responses to the task in *Figure 2* especially interesting. Two students immediately recognized they could input $x = 1$ to the inverse rule $f^{-1}(x)$ to determine x such that $f(x) = 1$. Both of these students also used function composition to determine if two given functions were inverse functions. Fourteen students attempted to determine the rule for $f(x)$ by switching-and-solving, obtaining $x = (y + 1)^3 - 5y^2 + 2$ then intending to solve for y and substitute $x = 1$ in their new formula. When they were unable to obtain a rule of the form "$y =$…", 10 of these students were unsure how to proceed. The other four students eventually substituted $y = 1$ in $x = (y + 1)^3 - 5y^2 + 2$ to determine the value for x.

5.2 Students' activities addressing contextualized tasks and a comparison across contextualized and decontextualized tasks

Our goals for the Celsius/Fahrenheit tasks included establishing whether a student used the same technique across contextualized and decontextualized tasks and examining how the students interpreted their determined inverse function with respect to the context. *Figure 6* illustrates the number of students who (i) maintained a consistent technique across decontextualized (graphical or analytical) tasks, (ii) maintained this same technique in the contextualized task, and (iii) interpreted their inverse function as continuing to represent the relationship between degrees Celsius and degrees Fahrenheit defined by the *given* function.

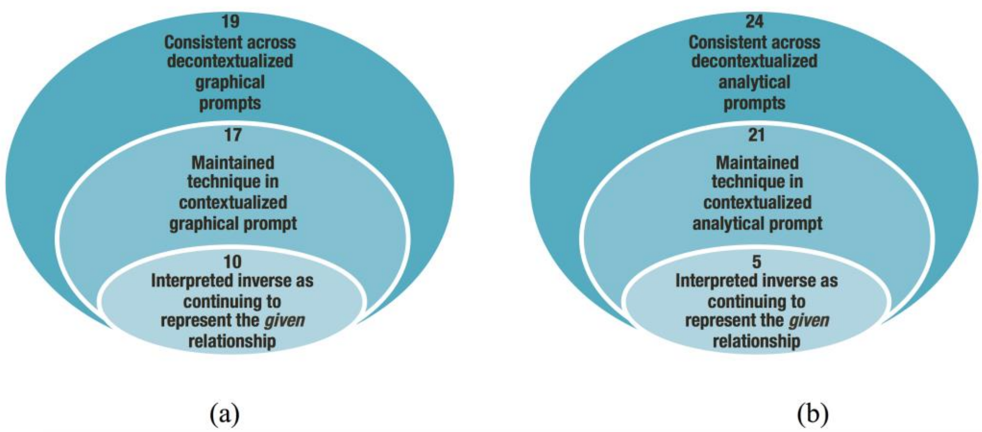


Figure 6. Students' (a) graphical and (b) analytical consistency across decontextualized and contextualized tasks.

We present Kate's activity as one example (*Figure 7*). Kate used the same 'switching' techniques for both *contextualized* tasks that she had used on the *decontextualized* tasks. However, in context, Kate was perturbed as to how to interpret her constructed inverse functions. In the graphical task, after describing that the point (10, 50) on the given line represented 10 degrees Celsius corresponded to 50 degrees Fahrenheit, the interviewer asked Kate to explain what the point (50, 10) on her line representing the inverse function meant in relation to degree measures. Kate responded, "That's a good question. If I start at, if I have 50 degrees Celsius, then I have 10 degrees Fahrenheit. That doesn't make sense." After a long pause, Kate pointed to the line determining her inverse function (*Figure 7*) and said, "I can't wrap my mind around what that means."

Similarly, Kate was unsure how to interpret the analytically represented inverse function she obtained by switching-and-solving (i.e., $(5/9)F + 32 = C^{-1}(F)$ in *Figure 7*). She conjectured if the given function had an input of degrees Fahrenheit, then the inverse function would have an input of degrees Celsius. However, she quickly noted, "I'm thinking you can [find degrees Fahrenheit from degrees Celsius] if you don't take the inverse"; Kate raised the possibility she could use the given rule to determine degrees Fahrenheit if given a specific Celsius value. As she subsequently attempted to interpret her inverse analytic rule, she added, "…if it was zero degrees Fahrenheit it'd be thirty two degrees Celsius but that's not, correct. [*long pause*]. I don't know, I can't wrap my head around it right now." Kate did not consider switching the quantitative referents of the variables; she did not interpret C and F in her inverse function rule as representing degrees Fahrenheit and degrees Celsius, respectively. Although Kate exhibited a consistent technique for both graphical and analytical tasks, she remained perturbed when interpreting a contextualized meaning for the inverse functions she obtained using these techniques.

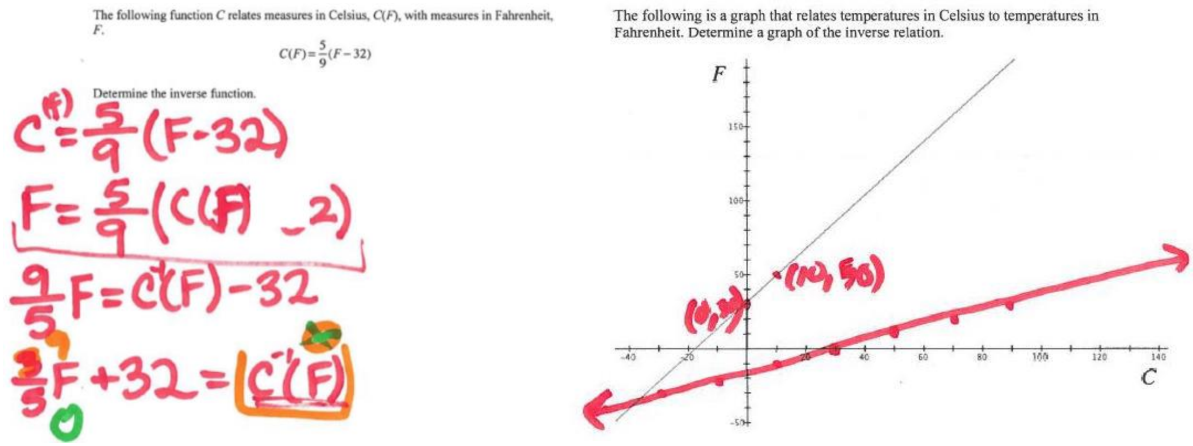


Figure 7. Kate's solutions to the Celsius/Fahrenheit tasks.

Including Kate, 17 of the 19 students who exhibited a consistent technique for decontextualized graphical tasks used the same technique when given a contextualized graph (*Figure 6*a). Ten of these 17 students concluded that the quantitative referent of each axis must switch (i.e., the Fahrenheit axis became the Celsius axis) to interpret their created graph as representing what they considered to be a correct relationship between temperature measures. Six (including Kate) of the remaining seven students maintained the quantitative referents of each axis and had difficulty interpreting their created graph.[5] As another example, one of these students claimed, "I don't know what the point of having the inverse function of a graph like this [is]… it's not gonna give you any useful information, I don't feel like." This student conceived that her new graph still represented a relationship between degrees Celsius and degrees Fahrenheit, though not the normative relationship, and it, therefore, did not provide "any useful information."

Students also experienced difficulties interpreting their determined analytical inverse function. Twenty-one of the 24 students used the same consistent technique for both

[5] The other student turned her focus to determining and working with analytically represented functions and did not return to the context.

contextualized and decontextualized analytic tasks (*Figure 6*b). Only five of these students interpreted their determined inverse function as maintaining the normative relationship in terms of the temperature context. The other 16 students concluded that their inverse function was not meant to define the same contextual relationship as the original function or remained unsure what their inverse function defined with respect to the context. This result is not surprising, however, considering most of these students switched-and-solved which, once applied, no longer maintained the relationship between the temperature measures *unless* the student simultaneously coordinated different variable referents depending on the analytic representation under consideration (i.e., for the given function, *F* represents a number of degrees Fahrenheit and *C* represents a number of degrees Celsius, and for the inverse function, *F* represents a number of degrees Celsius and *C* represents a number of degrees Fahrenheit).

6. Discussion and Implications

In this section, we discuss our inferences regarding the students' meanings. Recall, from our interpretation of the consistency of a student's techniques across tasks, we inferred if the student maintained a connected system of meanings. From these inferences, we also deduced the extent to which a student's meanings were restricted to carrying out an activity they "knew-to do" in a particular representation or context (i.e., if the student maintained disconnected meanings).

Nine of the 25 students exhibited techniques when working with *decontextualized* graphical and analytical tasks that result in determining compatible inverse functions across these two representations; eight students used a 'switching' technique (Table 3) and one student used reciprocals in both representations. We infer these students were more likely to hold connected inverse function meanings across decontextualized tasks. The other students engaged in inconsistent techniques (from our perspective) and made no explicit (prompted or unprompted) connections between their techniques within or across tasks. Based on each of these latter student's activities, we inferred that her or his meanings were restricted to engaging in a specific technique for that specific task or subset of tasks. In essence, carrying out the technique in order to obtain a new equation or graph was potentially the entirety of each student's meanings for inverse function for that representation.

Five of the 25 students maintained meanings such that they understood that a function and its inverse function represented the same relationship between quantities' values in a contextualized situation. The other students maintained inverse function meanings that did not entail anticipating how their determined inverse functions related to a given context in graphical or analytical representations. Finally, we note only one student engaged in activities that we inferred to imply the student maintaining a connected meaning she drew on addressing each task. Specifically, she used 'switching' techniques for decontextualized and contextualized tasks while anticipating that, in a context, the function and its inverse represent the same relationship between quantities, thus requiring switching the quantitative referents of her used variables. We return to this finding at the end of this section.

6.1 Inverse function notation

Compatible with previous findings (Brown & Reynolds, 2007; Carlson et al., 2015; Engelke et al., 2005), several of the students in our study responded to questions about inverse function by producing (what we perceived to be) a function representing the inverse of the given function under the operation of multiplication (i.e., the reciprocal) rather than the inverse function under the operation of function composition. Considering that the conventional notation of these two operations look similar (e.g., $f^{-1}(x)$ for inverse element under composition and

$(f(x))^{-1}$ for inverse element under multiplication), it is easy to understand the confusion students experience when interpreting which operation they are meant to use in a given context or representation. The lack of inverse function notation in the first question described by Engelke et al. (2005) (i.e., determine x such that $f(x) = 3$ when given a graphical representation of f) may help explain why significantly more students correctly answered this question (35%) when compared to any of the three questions that included inverse function notation (no more than 16%).

Although some may argue that this notational similarity can help students understand an invariance across inverse elements under different operations (e.g., Zazkis & Kontorovich, 2016), we take the prevalence across this and other studies (Carlson et al. 2015; Engelke et al. 2005) of students interpreting inverse function notation as representing the reciprocal of the given function rule or values to indicate students do not hold normative understandings of this nuanced notational distinction. Hence, there is an apparent need for a notation in school mathematics that differentiates inverse elements under composition and inverse elements under multiplication. One possible notation for inverse function that could be adopted is arrow notation ($f^{\leftarrow}$). Blyth (2006) used this notation to identify the inverse image map of an order-preserving mapping. Such notation may help students relate the inverse of a function, f, as the reverse mapping of the original function while also differentiating it from the notation for inverse under the operation of multiplication.

6.2 Textbook approaches to inverse function

We did a cursory, non-rigorous, review of U.S. pre-calculus textbooks we had used for instruction or were readily available to examine how they presented inverse functions in relation to our findings (i.e., Dugopolski 2009; Larson, Hostetler, & Edwards 2001; Stewart, Redlin, &

Watson 2012; Sullivan & Sullivan III 2007; Swokowski & Cole 2012). We note the authors of each of these texts used boxes emphasizing techniques for determining inverse functions represented analytically and graphically (i.e., reflecting over the line $y = x$ and switching coordinate values in graphical situations and switching-and-solving for analytic rules). Only one of the authors provided a contextual example involving, albeit discrete, quantities' values (Dugopolski, 2009). No authors discussed the techniques for determining analytic and graphical representations of inverse functions in ways that specify how to interpret a function and its inverse with respect to the relevant context. Hence, we are unsurprised that most students were unable to construct what they considered a viable contextual meaning for their determined inverse function.

We infer that our students' switching techniques often served as didactic obstacles to their developing connected inverse function meanings. Extending the work of Brousseau (1997), Harel and Sowder (2005) described didactic obstacles as barriers "one encounters in developing a way of thinking... [as a] result of narrow or faulty instruction" (p. 34). There are many parallels between the textbook presentations and the students' techniques and meanings characterized here. Although most students had techniques for determining inverse functions, they often did not anticipate how these techniques applied in a contextualized situation. Our findings indicate researchers and curriculum designers in the U.S. (and internationally) should consider the institutional role given to these techniques and if learning these techniques supports students developing productive inverse function meanings.

6.3 Connected inverse function meanings

Similar to Carlson's (1998) discussion of the complexities students encounter as they attempt to make sense of function, we describe complexities students encounter as they attempt to make sense of inverse function. We, as educators, expect students to (a) learn how to determine if a function has an inverse function, (b) know when to interpret the superscript '–1' as representing an inverse function or exponent, (c) be able to use the definition of inverse function to determine if two functions are inverses of each other via function composition, (d) construct inverse functions of given functions in a variety of representations, (e) *either* understand that techniques for determining inverse functions differ for contextualized and decontextualized functions *or* pay explicit attention to the quantitative referent of variables and axes, and (f) recall all of these ideas when learning about inverses of specific function families (i.e., trigonometric functions) after the initial treatment of inverse function. To develop powerful inverse function meanings, we contend we must give students ample opportunities to wrestle with these expectations in productive ways.

7. Limitations and Future Research

Harel and Sowder (2005) noted teaching practices are often difficult to eradicate even when researchers agree they serve as didactic obstacles. Mathematics educators in the U.S. and internationally should continue to consider ways in which curriculum and instruction addressing inverse function may emphasize techniques that serve as didactic obstacles. Further, future researchers might explore ways to support students in constructing inverse function meanings that are productive for both contextualized and decontextualized situations. For instance, researchers might use teaching experiments to examine ways to afford students opportunities to make accommodations to their inverse function meanings in order to develop connected inverse function meanings. Researchers taking a non-standard approach to the teaching and learning of inverse function (Moore, 2014; Moore, Silverman, Paoletti, & LaForest, 2014) have had preliminary successes in such efforts.

We collected our data using only one individual interview with 25 students from one university, which limited our study. Because our students came from one U.S. university, we do not make any claims regarding the generalizability of our findings to students at other institutions, nationally and internationally. Future researchers might examine the connectedness of students' inverse function meanings, especially in countries where the switching-and-solving technique is not explicit in curricula. Such investigations can provide more insight into how switching techniques can serve as didactic obstacles. Researchers may also be interested to explore how students' inverse function techniques differ between function classes (e.g., trigonometric, exponential, and logarithmic functions) or how students might develop different function classes in relation to their understanding of inverse functions. Finally, our analysis of textbooks' treatment of inverse function was done in retrospect and we only analyzed textbooks immediately available to us. Researchers may be interested in further exploring textbook approaches to inverse function, including the extent these texts afford students opportunities to develop connected inverse function meanings.

8. Contributions and Concluding Remarks

We have extended previous research in several ways. First, we provided a more fine-grained analysis of students' techniques than previous research, identifying several techniques the students "knew-to" (Mason & Spence, 1999) engage in when addressing inverse-related prompts both within and across a variety of task types. Second, we examined a population—pre-service teachers post-calculus—that has received little attention. Compatible with the findings of previous researchers (e.g., Carlson et al., 2015; Engelke et al., 2005; Vidakovic, 1996), we found that many of the pre-service teachers used techniques to determine, use, or interpret inverse functions that were reliant on a particular representation. Third, we addressed students' interpretations of a contextual meaning of an inverse function for a given contextualized function, identifying that a majority of students had difficulty interpreting the contextual meaning for their determined inverse function. Fourth, we outlined a theoretical lens that supported our making inferences about students' inverse function meanings based on their techniques addressing a variety of tasks. By examining the consistency of the students' enacted techniques, we inferred the extent to which students maintained connected or disconnected inverse function meanings, with most students maintaining disconnected inverse function meanings. By relating our findings to previous research, we drew important implications for the teaching and learning of inverse function. We hope this work serves as a catalyst for future research examining teaching and curricula in school mathematics that promote students developing connected inverse function meanings.

References

Bergeron, L. (2015). *IB mathematics comparability study: Curriculum & assessment comparison*.

Blyth, T. S. (2006). *Lattices and ordered algebraic structures*. Springer Science & Business Media.

Breidenback, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics, 23*(3), 147–185.

Brousseau, G. (1997). *Theory of didactical situations in mathematics* (N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield, Eds. & Trans.). Dordrecht, Netherlands: Kluwer Academic.

Brown, C., & Reynolds, B. (2007). *Delineating four conceptions of function: A case of composition and inverse*. In Lamberg, T., & Wiest, L. R. (Eds.). *Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 190–193). Lake Tahoe, NV.

Carlson, M. P. (1998). A cross-sectional investigation of the development of the function concept. In A. H. Shoenfeld, J. J. Kaput, & E. Dubinsky (Eds.), *Research in collegiate mathematics education, III. Issues in mathematics education* (Vol. 7, pp. 114–162). Providence, RI: American Mathematical Society.

Carlson, M. P., Madison, B., & West, R. D. (2015). A study of students' readiness to learn calculus. *International Journal of Research in Undergraduate Mathematics Education 1*(2), 209–233.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 547–589). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Dugopolski, M. (2009). *Fundamentals of precalculus*. Boston, MA: Pearson Education, Inc.

Engelke, N., Oehrtman, M., & Carlson, M. P. (2005). Composition of functions: Precalculus students' understandings. In Lloyd, G. M., Wilson, M., Wilkins, J. L. M., & Behm, S. L. (Eds.), *Proceedings of the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 570–577). Roanoke, VA.

Even, R. (1992). The inverse function: Prospective teachers use of 'undoing'. *International Journal of Mathematical Education in Science and Technology, 23*(4), 557–562.

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517–545). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Harel, G., & Sowder, L. (2005). Advanced mathematical-thinking at any age: Its nature and its development. *Mathematical Thinking and Learning, 7*(1), 27–50.

Hunting, R. P. (1997). Clinical interview methods in mathematics education research and practice. *Journal of Mathematical Behavior, 16*(2), 20.

Larson, R., Hostetler, R. P., & Edwards, B. H. (2001). *Precalculus with limits: A graphing approach* (3 ed.). Boston, MA: Houghton Mifflin Company.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research, 60*(1), 1–64.

Lucus, C. A. (2005). *Composition of functions and inverse function of a function: Main ideas as perceived by teachers and preservice teachers.* (Doctor of Philosophy Dissertation), Simon Fraser University, BC, Canada.

Merriam, S. B., & Tisdell, E. J. (2005). *Qualitative research: A guide to design and implementation*. San Francisco, CA: John Wiley & Sons.

Mason, J., & Spence, M. (1999). Beyond mere knowledge of mathematics: The importance of knowing-to act in the moment. *Educational Studies in Mathematics 38*, 135–161.

Moore, K. C. (2014). Signals, symbols, and representational activity. In L. P. Steffe, K. C. Moore, L. L. Hatfield, & S. Belbase (Eds.), *Epistemic algebraic students: Emerging models of students' algebraic knowing* (Vol. 4, pp. 211–235). Laramie, WY: University of Wyoming.

Moore, K. C., Silverman, J., Paoletti, T., & LaForest, K. R. (2014). Breaking conventions to support quantitative reasoning. *Mathematics Teacher Educator, 2*(2), 141–157.

Paz, T., & Leron, U. (2009). The slippery road from actions on objects to functions and variables. *Journal for Research in Mathematics Education, 40*(1), 21.

Phillips, N. (2015). Domain, co-domain and causation: A study of Britney's conception of function. In T. Fukawa-Connelly, N. Infante, K. Keene, and M. Zandieh (Eds.), *Proceedings of the 18th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 893–895), Pittsburgh, PA.

Piaget, J. (1968). *Six psycological studies* (A. Tenzer, Trans.). New York: Random House.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. Kelly & R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 267–306). Mahwah, NJ: Erlbaum.

Stewart, J., Redlin, L., & Watson, S. (2012). *Precalculus: Mathematics for calculus, sixth edition* (6th ed.). Belmont, CA: Brooks/Cole Cegage Learning.

Strauss, A. L., & Corbin, J. M. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (2nd ed.). Thousand Oaks: Sage Publications.

Sullivan, M., & Sullivan III, M. (2007). *Precalculus: Concepts through functions, a unit circle approach to trigonometry*. Upper Saddle River, NJ: Pearson Education.

Swokowski, E. W., & Cole, J. A. (2012). *Precalculus: Functions and graphs* (12 ed.): Cengage Learning.

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sépulveda (Eds.), *Proceedings of the Annual Meeting of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 31–49). Morélia, Mexico: PME.

Thompson, P. W. (2013). "Why use f(x) when all we really mean is y?". *OnCore, The Online Journal of the AAMT*.

Thompson, P. W. (2016). Researching mathematical meanings for teaching. In L. D. English & D. Kirshner (Eds.), *Handbook of international research in mathematics education* (pp. 435–461). New York, NY: Taylor & Francis.

Thompson, P. W., Carlson, M. P., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: A hypothesis about foundational reasoning abilities in algebra. In L. P. Steffe, K. C. Moore, L. L. Hatfield, & S. Belbase (Eds.), *Epistemic algebraic students: Emerging models of students' algebraic knowing* (Vol. 4, pp. 1–24). Laramie, WY: University of Wyoming.

Thompson, P. & Carlson, M. P. (2017) Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed). *The compendium for research in mathematics education*. NCTM: Washington, DC.

Vidakovic, D. (1996). Learning the concept of inverse function. *Journal of Computers in Mathematics and Science Teaching*, *15*(3), 295–318.

Vidakovic, D. (1997). Learning the concept of inverse function in a group versus individual environment. In E. Dubinsky, D. Mathews, & B. E. Reynolds (Eds.), *Readings in cooperative learning for undergraduate mathematics.* (Vol. 44, pp. 175–196). Washington, DC: The Mathematical Association of America.

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Washington, D.C.: Falmer Press.

Zazkis, R., & Kontorovich, I. (2016). A curious case of superscript (− 1): Prospective secondary mathematics teachers explain. *Journal of Mathematical Behavior, 43*, 98–110.