Connecting Advanced and Secondary Mathematics

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Abstract

There is an ongoing debate among scholars in understanding what mathematical knowledge secondary teachers should have in order to provide effective instruction. We explore connections between advanced and secondary mathematics as an entry point into this debate. In many cases, advanced mathematics is considered relevant for secondary teachers simply because the content is inherently related. In this paper, we instead argue that there are connections between advanced mathematics and secondary mathematics that directly influence teaching. These are not discussions of the mathematical connections, per se, but rather discussions of specific ways in which knowing mathematical connections might influence secondary teachers' pedagogical choices in the classroom. With a focus on abstract algebra, we exemplify three categories of connections between advanced and secondary mathematics that influence teaching practice. Through exploring these examples, we aim to identify and situate more advanced mathematics knowledge within a practice-based lens of teacher knowledge.

Introduction

Over the past few decades, research has increasingly focused on the specialized mathematics knowledge that teachers need to know to teach mathematics. Recommendations of the *Mathematics Education of Teachers* reports (Conference Board of the Mathematical Sciences [CBMS], 2001, 2012) provide specific content recommendations for the preparation and ongoing professional development of teachers. In addition, much work has been done identifying connections between advanced and secondary mathematics (e.g., Connecting Middle School and College Mathematics [(CM)2] (Papick, n.d.); Mathematical Understanding for Secondary Teaching: A Framework and Classroom-Based Situations (Heid, Wilson, & Blume, 2016)). Yet many questions remain regarding connections to advanced mathematics, including, which connections are important, and how knowledge of such connections could influence practice.

Often the argument is made that because mathematical connections exist between the worlds of secondary and advanced mathematics, teachers should know them, on the premise that this knowledge will impact their instruction as secondary teachers and in turn positively affect student learning. Yet there is very little evidence in the literature that supports this claim - neither based on teachers' own self-reports (e.g., Zazkis & Leikin, 2010) nor on students' performance (e.g., Darling-Hammond, 2000; Monk, 1994).

With a view toward bridging this gap, our work has an explicit focus on the implications for teaching that stem from understanding connections between secondary and advanced mathematics. In this article, rather than a "trickle-down" argument that more advanced mathematics knowledge should impact instruction because the content is related, we discuss and exemplify three categories for how connections between advanced and secondary mathematics influence teaching practices. These are not discussions of the *mathematical* connections, per se, but rather discussions of specific ways in which knowing mathematical connections might influence secondary teachers' *pedagogical* choices in the classroom. Our goals are to connect more explicitly the ways in which understanding connections between advanced and secondary mathematics might actually influence instruction, as a means to identify and situate more advanced mathematics knowledge within a practice-based lens of teacher knowledge.

Background

In November 2015, the authors led a working group at the North American Chapter of the Psychology of Mathematics Education annual meeting. The working group focused specifically on the knowledge of abstract algebra a secondary teacher might need that goes beyond high school mathematics, how abstract algebra might enable a teacher to unpack a secondary mathematics topic, and how an understanding of abstract algebra might influence instructional choices in the secondary classroom.

Our work naturally fit into three components, which we focused on during three meetings over the course of the conference. For the first component, which was about a working definition of connections between abstract algebra and secondary mathematics, we considered connections to be about both: i) mathematical content and ii) ways of thinking about and engaging with that content. With respect to mathematical content, we used two related, but slightly different frameworks – horizon content knowledge (HCK) (Ball, Thames, & Phelps, 2008) and key developmental understandings (KDUs) (Simon, 2006).

When we refer to HCK, we are not just referencing any mathematical content beyond the content of the school curriculum, but specifically content that can inform and be related to the practice of teaching. So our framing of HCK is forward-looking in that we want to consider teachers' practice in relation to knowing content that is 'down the road,' which could be abstract algebra or even pre-calculus, depending on what one teaches. KDUs are critical mathematical understandings that are necessary for mathematical development to take place. For instance, Simon (2006) suggests that a KDU might be students shifting from understandings of fractions simply as an arrangement of congruent parts of a whole to understanding fractions also as quantities. According to Simon, KDUs involve a conceptual advance, and students are not likely to gain such knowledge through explanation or demonstration. Silverman and Thompson (2008) add that for teachers, KDUs need to be transformed for how the KDU can empower student learning as well as actions teachers can take to support student learning and why. Specifically, advanced mathematics can support teachers to re-conceptualize, re-structure, and re-understand their knowledge of secondary mathematics (e.g., Wasserman, 2016). In this way, advanced mathematics can support teachers to develop "knowledge that supports conceptual teaching of a particular mathematical topic" (Silverman & Thompson, 2008, p. 508). In order for an advanced topic such as abstract algebra to be useful for teachers at all, we contend that it has to serve as a KDU for the secondary content; however, just because a topic in abstract algebra might apply to or connect with secondary content does not necessarily mean that it will change teachers' instruction. One of ours goal is to understand what connections to abstract algebra will impact instruction.

When we refer to ways of thinking and engaging in mathematics, we consider both mathematical habits of mind (Cuoco, Goldenberg, & Mark, 1996) and the standards for mathematical practice (e.g., Council of Chief State School Officers [CCSSO], 2010). These include looking for patterns, giving precise descriptions, utilizing visualizations, making conjectures, attending to precision, and connecting representations. Such habits and practices cut across content areas and levels of mathematical study. For example, Baldinger (2014) found that pre-service teachers engaging in mathematical practices in an abstract algebra course could improve their ability to enact these practices (particularly proof and attention to precision) with respect to high school content. We want to think about how engagement in these habits and practices around advanced mathematics are similar to and different from engaging in those same habits and practices around secondary content, particularly as they relate to good instructional approaches. Therefore, in our view, connections might go beyond knowledge *of* mathematics and encompass engagement *in* mathematics.

The discussion above serves as a jumping off point for the second component of our work; that is to consider the possible influence of connections on classroom instruction. Mathematical content connections can include solving equations, polynomials and the zero-product property, divisibility rules, factorability, and conjugacy classes. In related work, Wasserman (2016) categorized four content connections, organized around school mathematics topics, as opposed to abstract algebra topics. We can also identify connections by looking at engagement in mathematical practices. Such connections include proof, questioning, examples and tasks, and attending to structure. Many general ideas concerning connections between abstract algebra and secondary mathematics exist. For example, knowledge of abstract algebra can help secondary teachers communicate mathematics to their students by connecting approaches, content, and principles, thus helping students see relevance in mathematics (e.g., When is something fully factored or non-factorable?). Further, knowledge of abstract algebra can help secondary teachers explain seemingly obvious properties (e.g., commutativity) and may dissuade them from talking strictly in ‘procedural language’. Beyond content knowledge of abstract algebra, engaging in problem solving related to abstract algebra may have direct connection to teaching. In particular, providing teachers with challenging abstract algebra problems may not only cause teachers to give greater consideration to a concept, but may also help teachers learn perseverance. Learning perseverance may impact pedagogy by making teachers more aware that students need time to work through problems on their own. Moreover, engaging in abstract algebra problem solving may help teachers, who are comfortable with basic concepts, recall how mathematical habits of mind are important for mathematical development, thus reminding them to encourage their own students to do things like look for patterns, attend to precision, and connect representations.

The third component of our work stems from these mathematical ideas, and considers how knowing such connections might influence specific aspects of instruction. It is in this regard that our work specifically moves beyond others, in not just identifying mathematical connections but ways in which those might influence pedagogical choices made by the teacher. Based on our own experiences and work with pre- and in-service teachers, we conjecture some of the different aspects of instruction that may change based on such connections. For example, teachers might change the language they use, what examples or mathematics tasks they select, or how they support students to develop mathematical habits of mind. The working group pushed for specific examples of instructional change, rather than general categories, as a means to flesh out and depict these instructional changes in some concrete, and specific, way. Therefore, during our working group, we focused on particular instructional changes that could occur, using (and completing) the following statements as a guide:

1. Without abstract algebra, one might teach [high school topic] in [this specific way]

2. One learns [this specific abstract algebra topic]
3. Now one might teach [high school topic] in [this new way]

Below we present three examples that stemmed from this work, addressing each of these three statements in turn.

Examples of Connections

Attending to Language. In mathematics teaching, language is an important part of how mathematical ideas get communicated. Particularly in connection to abstract algebra, words such as factor, term, and cancel may get used imprecisely in the secondary mathematics classroom when teaching students about, say, simplifying polynomials. A teacher may not clearly distinguish between what a factor is and what a term is within a polynomial; similarly, it may not be clear what canceling refers to in the context of manipulating an algebraic expression - or why or when one is or is not allowed to cancel parts of an algebraic expression. Moreover, canceling is not a specific mathematical operation, but rather a word commonly used to describe what is happening in several varying contexts. We include that word here because of its ubiquity and also to show how the use of that language might shift through an understanding of topics in abstract algebra.

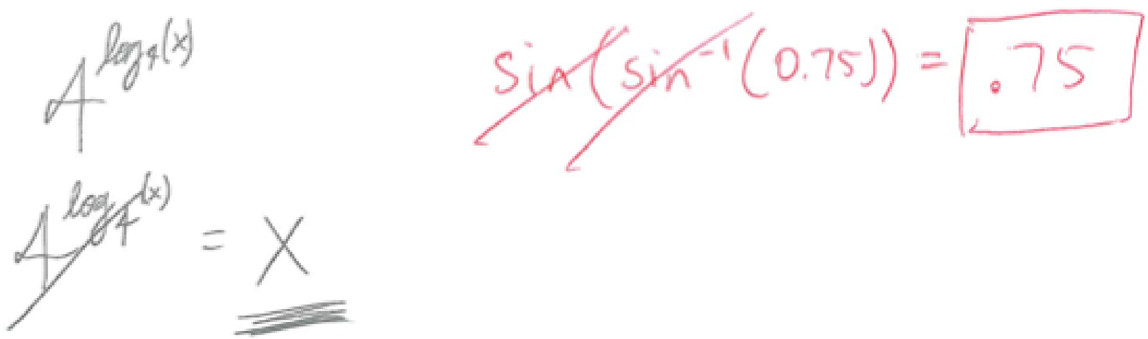
In abstract algebra, as teachers think deeply about algebraic structures, they have the opportunity to develop an explicit definition of an inverse in relation to a set, operation, and identity element. By working in familiar algebraic groups, for example, teachers might observe more formally that in the integers, -2 is the inverse of 2 under addition, and 0 is the identity element, whereas when considering the multiplicative group of positive rational numbers the inverse of 2 is ½, and 1 is the identity element. However, in other algebraic groups, for example addition modulo 12, 10 is the inverse of 2 under addition, and 0 is the identity element. The point, however, is a deeper understanding that the various instantiations of the word inverse in secondary school mathematics are examples of a more general notion of inverse, as well as recognition of the importance of the set and operation under consideration for identifying inverse and identity elements.

We argue that such explorations can influence the ways in which teachers use language in the classroom, such as the words factor, term, and cancel. Through their exploration of algebraic structures, teachers will understand that the distinction between the words "factor" and "term" has to do with the operation under consideration. For example, the expression $x + 5$ has two terms under addition, but just one factor under multiplication. Understanding this distinction also lends clarity to the word cancel. When the operation is addition, we can talk about two terms canceling (that is, summing to the additive identity). Under multiplication, in contrast, we can talk about two factors canceling (that is, multiplying to the multiplicative identity). And once summed to 0 or multiplied to 1, the identity properties come into play, that for any element, $a+0=a$ and $a*1=a$, which allows one to effectively remove these two elements from the algebraic expression. More generally, the teacher can help students see how the word cancel can be used with respect to other operations (e.g., composition of functions). To understand why we can cancel the 4 and the $\log_4$ in the expression $4^{\log_4(x)}$ or why we can cancel the sin and $\sin^{-1}$ in the expression $\sin(\sin^{-1}(x)$, and what results from this cancellation, requires a nuanced understanding of composition of two functions and the identity function, $i(x)=x$. (See Figure 1).

Beyond being more precise in their own use of language, teachers with this sort of understanding might also push students, when they use the word "cancel" (or "factor" or "term") to be more explicit about what they mean, rather than to take the terminology as shared knowledge.

Figure 1

Examples of cancellation within composition of two functions (http://blogs.ams.org/matheducation/2015/12/10/connections-between-abstract-algebra-and-high-school-algebra-a-few-connections-worth-exploring/#more-1068)



We argue that to support teachers to make a meaningful change to their language use, it is important that they develop a rich understanding of the underlying algebraic structure and see how it connects to various topics, such as simplifying polynomials, within school mathematics. Developing this understanding may not require an entire course in abstract algebra, but it does require thoughtful consideration of a big idea within abstract algebra.

Use of Examples and Tasks. In mathematics, many properties - of operations, numbers, functions, etc. - are often treated as intuitively obvious, as just an interesting side note, or as something to be memorized. Consider, for example, the commutative property of multiplication. Teachers using an area model for multiplication might quickly mention the commutative property by telling students to just “flip the rectangle.” Other teachers might treat the commutative property as something to be memorized while learning multiplication facts. Overall, the examples or tasks chosen for use with students may not emphasize the commutative property of multiplication or support students development of a deep understanding. At the elementary level, even teachers who incorporate potentially productive models such as area for multiplication may not choose examples for students that explicitly problematize commutativity; at the secondary level, teachers might assume commutativity is something students already understand, and these teachers might also not think to include examples or tasks specifically focused on this topic.

In abstract algebra, particularly group theory, teachers have the opportunity to explore the properties of non-commutative groups. Outside work with matrix multiplication, it is unlikely that teachers have had extensive prior experience with non-commutative groups. In particular, working with non-commutative groups helps emphasize that commutativity is non-trivial and not a property of the operation alone. That is to say, if one only considers subtraction and/or division for examples of non-commutativity, one might miss the point that commutativity is a characteristic of particular group structures. Because both subtraction and division are also not associative, they do not exemplify a group structure, and thus do not help illuminate the connection between this property, the operation, and the set. Working, for example, with S_3 (the symmetric group on three elements), matrix groups, or the group of invertible functions under composition provides opportunities to explore non-commutative groups under different operations. Teachers are able to expand their example space to include both commutative and non-commutative groups.

We argue that experiences with non-commutative groups such as those described above may shift the nature of the tasks or examples teachers select when exploring the commutative property of multiplication with their students. For example, rather than asking students to just observe that 7x4 is the same as 4x7, a teacher might instead demonstrate that 7x4 means seven groups of 4, or 4+4+4+4+4+4+4, while 4x7 means four groups of 7, or 7+7+7+7. Then students would be asked to show or justify why these two results are the same, since it is not obvious that they should be. We see, in fact, that when you add up some of the 4s, you are not guaranteed to get a common value in the summing of 7s until the last term. Indeed, one valuable way to understand their equivalence in this form is by "taking 1" from each of the seven 4s, which forms one 7; taking 1 again from each of the seven 3s (now), forms a second 7; etc. (see Figure 2). This visualization of groups could even be rearranged so that the four groups of 7 (four groups containing seven different colors) are rows and the seven groups of four (seven groups, one of each color) are columns, aligning closely with the area model for multiplication (see Figure 3).

Figure 2
Seeing 7 groups of 4 inside of 4 groups of 7

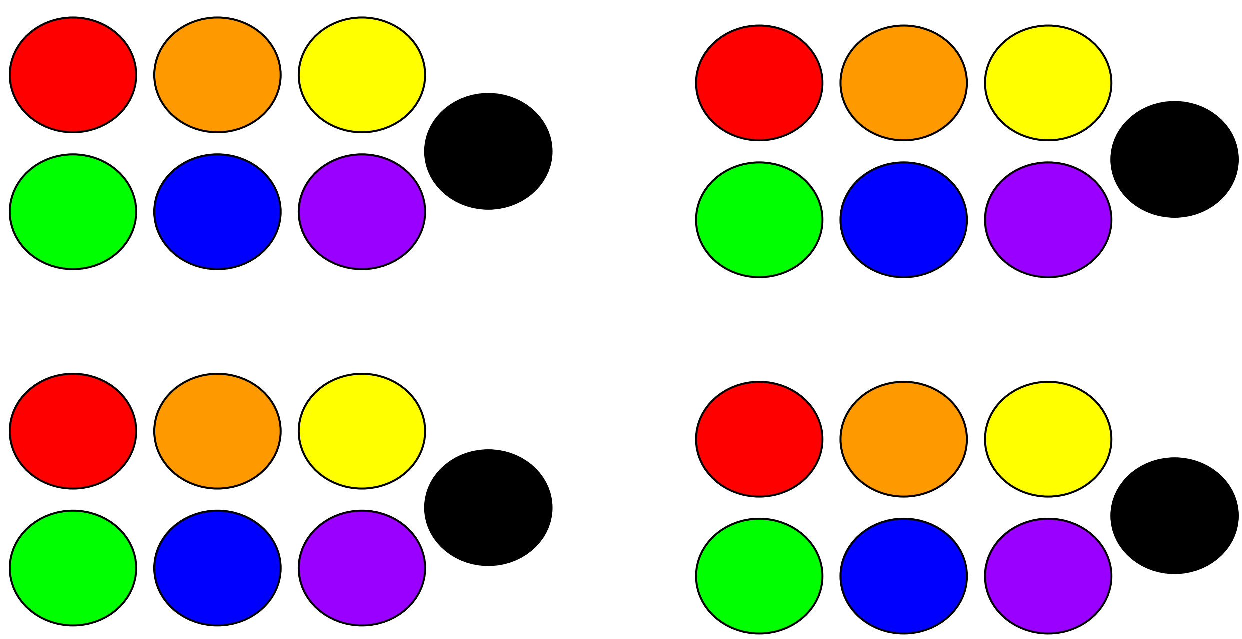


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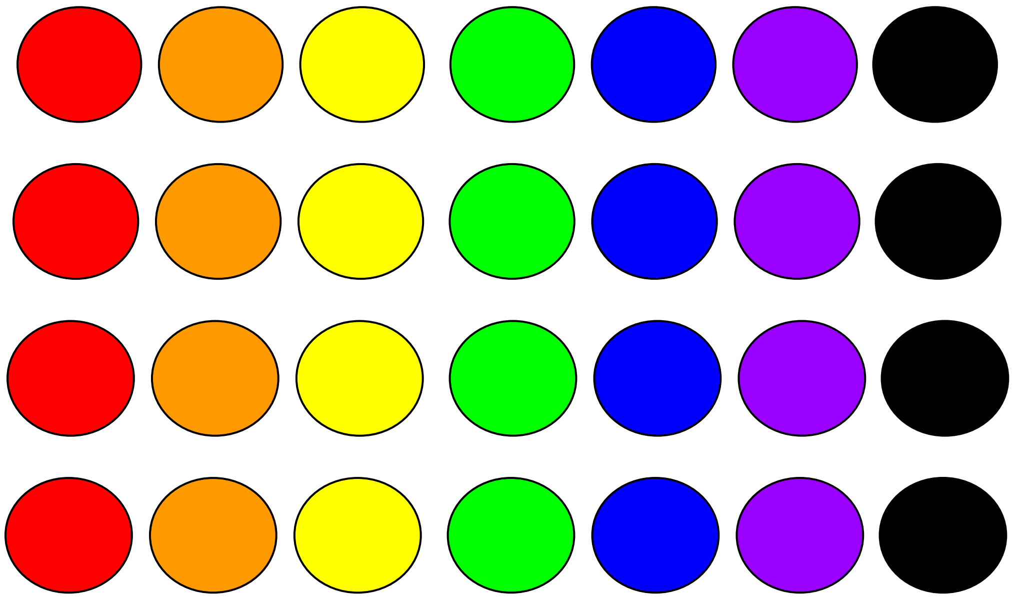
Figure 3

Aligning groups with the area model, where rows represent 4 groups of 7 and columns represent 7 groups of 4



A teacher might then connect this to an area model for multiplication – thereby demonstrating in a variety of contexts why the operation of multiplication will be commutative, at least on the set of natural numbers. Rather than giving a trivial explanation of commutativity as "flipping the rectangle," a teacher with a deeper understanding of the connection between commutative and non-commutative groups and the commutative property of multiplication might give students the opportunity to explore and make sense of the area model in a non-trivial way.

Focusing on Habits of Mind. In mathematics teaching, helping students develop mathematical habits of mind (Cuoco et al., 1996) is an important part of their ability to learn and do mathematics. It is important that teachers and students are "bringing mathematical meaning to problems and statements through definition, systematization, abstraction, or logical connection making" (p. 376). For both advanced mathematics and secondary mathematics, habits of mind, such as those found in the Common Core standards of mathematical practice (2010; e.g., generalizing from examples and looking for and making use of structure), can help students solidify the mathematics they have learned and enable them to understand new mathematics. These practices help connect new mathematics to their existing structures. When teachers themselves work at becoming proficient at particular habits, they become better at helping their students learn mathematics. For example, if teachers know how to construct viable arguments and critique the reasoning of others, then they can press students to do so as well, thus enabling students to improve their mathematical thinking and abilities.

In abstract algebra, many common algebraic notions from K-12 mathematics are revisited in a more general setting. For example, properties of exponents are a topic that students first explore in the context of integers in the 8th grade Common Core Standards (e.g., CCSS.Math.Content.8.EE.A.1). These are revisited in high school in the more general setting of polynomials and rational numbers (e.g., CCSS.Math.Content.HSN.RN.A.1). In college, the meaning and properties of exponents are addressed again in other particular contexts such as matrices. Abstract algebra unifies these contexts through the development of rings and the associated properties of exponents in that setting. In order to make sense of the properties of exponents in this more general context, teachers look for patterns and begin to make sense of the underlying algebraic structure. Through this process they are engaging in critical mathematical practices and developing mathematical habits of mind.

We argue that experience with how exponents work in arbitrary rings may influence how teachers support students to also develop these key mathematical habits of mind during instruction focused on exponents. For example, rather than showing the rules of exponents and telling students to apply or use them in specific ways, a teacher might now provide students with opportunities to explore the structure and behavior of exponents to develop the rules themselves. A teacher might present tasks that necessitate the application and understanding of the underlying structure of exponents such as "What is the units digit of 7^{95}?" By presenting this example, a teacher promotes students' development of mathematical habits of mind as they look for and make use of structure and search for and justify patterns. A teacher with a deeper understanding of the connection between how exponents work in rational numbers and how they work in arbitrary rings might also be inclined to help the students understand symbolization in meaningful ways, thus allowing students to make conjectures, and construct arguments for why exponential rules work the way they do.

Conclusion

Mathematicians and mathematics educators understand the vertical connections that occur throughout mathematics disciplines. We see these connections as integral to our subject and believe knowledge of connections strengthens an individual's depth of understanding of particular mathematics content and ability to engage in mathematical practices. In our working group, we chose to focus on the knowledge of abstract algebra a secondary teacher might need that goes beyond high school mathematics, but even more specifically on how abstract algebra might enable a teacher, for example, to unpack a secondary mathematics topic or, more generally, might influence various instructional choices in the secondary classroom.

As a field, we believe this work could, and should, go further. We need specific examples of connections within content areas to think about where these connections live within the teaching world. In the working group we focused on number systems, but we could think about abstract algebra across the curriculum to get even more out of the connections and their importance in mathematics teacher preparation. Additionally, just knowing about and understanding connections is not enough. The work in this area should consider the pedagogical impact of any connection between advanced and secondary mathematics. That is, one goal would be to identify portions of abstract algebra that are particularly salient for undergraduate learning as too often undergraduates see no purpose, and many mathematics faculty agree. But, if we have identified meaningful connections in our advanced mathematics courses, which permeate future pedagogical practice, then the argument for requiring abstract algebra for prospective teachers has more validity.

We are continuing and expanding the research regarding connections between advanced and secondary mathematics through multiple projects. The first project is to examine algebra tasks and curriculum in both colleges and secondary schools. The goals are to find interesting tasks that are currently available and to assess the connections important to those tasks. Additionally,

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this project aims to look at curriculum currently used in advanced and secondary courses to provide a broader spectrum to interrogate connections and to possibly use in professional development for practicing teachers.

The second project is to design and implement professional development (PD) with secondary teachers around connections between abstract algebra and secondary mathematics. The leading questions include: What would PD look like that was focused on not just the mathematical connections between abstract algebra and secondary mathematics, but mathematical connections in relation to teachers' pedagogical practices? What are the goals of such professional development? What are the core values? What are the research questions aligned with this PD? How do we answer these questions, or measure what we are interested in? To further this work, we are in the process of creating explicit learning goals, deciding on topics to be covered and collecting a set of scenarios of secondary mathematics teaching that attend to the desired topics and connections.

For many teachers, the divide between high school algebra and abstract algebra is significant. However, the connections between advanced and secondary mathematics serve as key bridges to crossing this divide. Our working group seeks to further research that is focused on the knowledge of advanced mathematics (e.g., abstract algebra) a secondary teacher might need that goes beyond high school mathematics, how advanced mathematics might support a teacher's efforts to unpack a secondary mathematics topic, and how an understanding of advanced mathematics might impact pedagogy in the secondary classroom. By continuing this work, we hope to illustrate if, why, and how knowledge of advanced mathematics is important for secondary instruction, and thus help motivate current and prospective teachers in their journey towards mathematical understanding.

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