11-15-2008

An Optimal Principle in Fluid-Structure Interaction

Bong Jae Chung
Montclair State University, chungb@mail.montclair.edu

Ashuwin Vaidya
Montclair State University, vaidyaa@mail.montclair.edu

Available online at www.sciencedirect.com
ScienceDirect
Physica D 237 (2008) 2945–2951
PHYSICA D
www.elsevier.com/locate/physd

An optimal principle in fluid–structure interaction

Bong Jae Chung, Ashwin Vaidya*

Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599, USA

Received 29 May 2007; received in revised form 15 April 2008; accepted 21 April 2008
Available online 4 May 2008

Communicated by D. Lohse

Abstract

We study the steady terminal orientation of a fore-aft symmetric body as it settles in a viscous fluid. An optimal principle for the settling behavior is discussed based upon entropy production in the system, both in the Stokes limit and the case of near equilibrium states when inertial effects emerge. We show that in the Stokes limit, the entropy production in the system is zero allowing any possible terminal orientation while in the presence of inertia, the particle assumes a horizontal position which coincides with the state of maximum entropy production. Our results are seen to agree well with experimental observations.

PACS: 47.61.Jd; 47.63.mf; 47.90.+a

Keywords: Entropy production; Fluid–structure interaction; Orientation

1. Introduction

This paper is concerned with examining the thermodynamic principles behind fluid structure interactions. More specifically, we are concerned with the orientation behavior of a sedimenting body. It is well established that homogeneous bodies of revolution around an axis, a, with fore-aft symmetry, when dropped in a quiescent liquid, will orient themselves in certain ways with respect to the direction of gravity. The orientation is seen to depend upon the shape of the body and also upon the nature of the fluid in which they are immersed. In a highly viscous fluid, in creeping flow regimes, the body is seen to keep its initial orientation as it falls [18]. In a Newtonian fluid when the velocity of the body begins to experience inertial forces, the body falls with a eventually becoming perpendicular to the direction of gravity. If the same body falls in a viscoelastic fluid, such as a polymer, where the inertial and elastic effects compete, then, a will eventually become parallel to the direction of gravity. In fact, the orientation behavior becomes very complex in viscoelastic fluids since, at critical concentrations of the polymer, it can also allow for some intermediate angles, referred to in the literature as *tilt angles* [4,3,16]. Theoretical explanations of these observations have been provided in a variety of fluid models, Newtonian and non-Newtonian by considering that in the terminal state, the net torque imposed by the body on the fluid, due to viscosity (constant and shear dependent), inertia and viscoelasticity must be in equilibrium. Hence, in its steady state, the terminal angle can be obtained from the vanishing of the net torque [8–10,28]. The previous, mechanical approach successfully explains the orientation phenomena in various cases. However, it still remains to be seen if the preferred states are, after all based upon a higher optimal principle (see for example [2,5,20,26]).

It is very often found that problems concerning pattern formation are intricately related to optimal principles and conservation laws such as the principle of minimum potential energy, principle of least action, Fermat's principle of least time etc. It has always seemed to us therefore, that some such quantity must be optimized in the orientation problem, since the problem that we are studying is one of pattern selection. We note that such a principle does after all exist and is related to entropy production in the system, since fluid systems are essentially dissipative. Our goal in this paper is two-fold: (i) The first is to establish that the problem of terminal orientation

* Corresponding author. Tel.: +1 919 962 9622; fax: +1 919 962 2568.
E-mail address: avaidya@email.unc.edu (A. Vaidya).

0167-2789/$ - see front matter
doi:10.1016/j.physd.2008.04.017

of a symmetric body in a fluid is governed by an optimal principle which is related to the entropy production of the system, i.e. an optimal principle exists and (ii) secondly, we want to show that the nature of the extremum is also an important issue and is related to the choice of the extremizing variable.

2. Thermodynamics of irreversible processes

In this section, we review some essential points concerning dissipative structures in fluid mechanics.[1] The application of non-equilibrium thermodynamics to fluid mechanics has been a subject of some importance in the past [1,11,13,30,31] but seems to have been sidelined in the recent literature. The greatest relevance of this subject to fluid motion, of course, lies in the regime of turbulence. However, it is observed that even in the case of slow flows, thermodynamics plays an essential role, since fluid motion is inherently dissipative in nature. Motivated by the work of Zeigler [31], there have been several studies, in particular, concerning the application of thermodynamic principles to the constitutive modeling of complex fluids (non-Newtonian fluids). Specifically, the requirement that viscous energy dissipation be non -negative is very useful in obtaining restriction on the material parameters of the constitutive models [7].

The second law of thermodynamics leads us to the local entropy equation which describes a system out of equilibrium, namely

$$\frac{\partial(\rho_f s)}{\partial t} + \operatorname{div} \mathbf{j}_s = \sigma_s \tag{1}$$

where ρ_f represents the fluid density, s represents the entropy density, $\mathbf{j}_s$ is the entropy flux density and σ_s is the local entropy production. It has been established [21] that the equation for entropy production can be given by the product of thermodynamic forces[2] (denoted X) and fluxes (denoted Y)

$$\sigma_s = \sum_i X_i Y_i + \sum_j \mathbf{X}_j \mathbf{Y}_j + \sum_{kl} \mathbf{X}_{kl} \mathbf{Y}_{kl} \tag{2}$$

which may be represented as scalars, vectors or second order tensors. Onsager suggested that for looking at near equilibrium phenomena, we may represent the fluxes as a linear function of the forces, $Y_i = \sum_{i,j} L_{ij} X_j$ where L_{ij} represent phenomenological constants which satisfy the well known Onsager reciprocity relations [21,22]. In the case of motion of an incompressible fluid, σ_s takes the specific form

$$\sigma_s = \frac{1}{T}\mathbf{T} : \mathbf{D} + \mathbf{j}_q \cdot \nabla \left(\frac{1}{T}\right) \tag{3}$$

where T is the temperature, $\mathbf{j}_q$ is the heat flux, $\mathbf{T}$ is the Cauchy stress tensor corresponding to a Newtonian fluid and $\mathbf{D}$ is the symmetric part of the velocity gradient

$$\mathbf{D} = \frac{1}{2}(\nabla \mathbf{u} + \nabla^{\mathrm{T}} \mathbf{u}).$$

The first term on the right hand side represents the viscous dissipation term while the second term refers to heat conduction due to a temperature gradient. Using the Eq. (3), Horne et al. [13], ignoring the heat conduction term, have shown that steady flow of a viscous Newtonian fluid in some simple geometries is seen to coincide with the minimum of the entropy production, subject to a constant pressure gradient. They observe that this principle may be invalid when the system reaches a state which is far from equilibrium. Such an extremum (maximization) of σ_s has also been effectively employed recently by Malek and Rajagopal [19] in deriving constitutive models for inhomogeneous incompressible fluid-like materials. Of course, the extremum entropy principle is nothing new; the extremum (minimization and maximization) of entropy production in irreversible processes have been found to be valid in several physical contexts [1,5,15,19,20,24,30]. Martyushev and Seleznev [20] explain the difference in the maximum and minimum principles as arising simply from the variable with respect to which we choose to extremize P (of which Prigogine's principle [25] is merely a special case).

3. A variational formulation

For the fluid–structure interaction problem discussed in Section 1, the equation for local entropy production given in Eq. (3) needs to be modified slightly to account for the energy dissipation of the falling body as well and can be given by

$$\sigma_s = \frac{1}{T}\mathbf{T} : \mathbf{D} + \frac{1}{T}\rho_e \mathbf{g} \cdot \mathbf{U} + \mathbf{j}_q \cdot \nabla \left(\frac{1}{T}\right) \tag{4}$$

where ρ_e is the effective density experienced by the body. The first term in Eq. (4) corresponds to the viscous dissipation in the fluid and the second term on the right hand side emerges from the rate of change of potential energy of the fluid and the final term is the heat conduction term. In the rest of this paper we assume, as in [13] that the effect of the heat conduction term is negligible and also that the ambient temperature $T = T_0$ is a constant (see also [5]). Integrating the Eq. (4) over the entire unbounded fluid domain gives us the net entropy production, namely

$$P = \frac{1}{T_0}\int_\Omega \mathbf{T} : \mathbf{D} + \frac{m_e}{T_0}\mathbf{g} \cdot \mathbf{U} \tag{5}$$

$$= \frac{2\mu}{T_0}\int_\Omega \mathbf{D} : \mathbf{D} + \frac{m_e}{T_0}\mathbf{g} \cdot \mathbf{U} \tag{6}$$

where in the last equation the pressure term drops out of the stress tensor due to the divergence free nature of the velocity field and $m_e = (\rho_b - \rho_f)|\mathcal{B}|$ is the effective mass, where $|\mathcal{B}|$ represents the volume of the body. We identify the Eq. (5) to be a sum of the product of fluxes and forces as in Eq. (2) when written out appropriately. In order to study Eq. (6), we employ the linearization of the velocity field $\mathbf{u}$, motivated by Onsager's theory, in the following manner:

[1] See [25,17,23] for an introduction to the subject of non-equilibrium thermodynamics.

[2] The forces may originate from hydrodynamic viscosity, chemical reactions, thermal gradients etc.

$$\mathbf{u} = \mathbf{u}_s + \mathbf{w} \tag{7}$$

where the velocity field $\mathbf{u}_s$ is independent of inertial effects (and hence the Reynolds number) and satisfies the Stokes equations while the remainder term $\mathbf{w}$ depends on the Re (under appropriate non-dimensionalization) and contributes to the inertial effects in the system. If we now put Eq. (7) into Eq. (6), we obtain

$$\begin{aligned} P = {} & \frac{2\mu}{T_0}\int_\Omega \mathbf{D}(\mathbf{u}_s):\mathbf{D}(\mathbf{u}_s) + \frac{m_e}{T_0}\mathbf{g}\cdot\mathbf{U} \\ & + \frac{4\mu}{T_0}\int_\Omega \mathbf{D}(\mathbf{u}_s):\mathbf{D}(\mathbf{w}) + \frac{2\mu}{T_0}\int_\Omega \mathbf{D}(\mathbf{w}):\mathbf{D}(\mathbf{w}) \end{aligned} \tag{8}$$

$$\Rightarrow P = \frac{2\mu}{T_0}\int_\Omega \mathbf{D}(\mathbf{u}_s):\mathbf{D}(\mathbf{u}_s) + \frac{m_e}{T_0}\mathbf{g}\cdot\mathbf{U} + \mathcal{Q}(\mathbf{u}_s,\mathbf{w}) \tag{9}$$

where $\mathcal{Q}$ represents higher order contributions due to $\mathbf{w}$. In general $P = P(Re)$, however we make the approximation

$$P \approx \frac{2\mu}{T_0}\int_\Omega \mathbf{D}(\mathbf{u}_s):\mathbf{D}(\mathbf{u}_s) + \frac{m_e}{T_0}\mathbf{g}\cdot\mathbf{U}. \tag{10}$$

Let us now consider a rigid body of any shape moving in a fluid in its steady state. We can assume that in the steady state, the body will fall along a plane, i.e. $\mathbf{U} = (U_1, U_2, 0)$ with respect to a frame attached to the body and hence we can decompose the motion of the fluid from that of the body in the following manner:

$$\mathbf{u}_s = U_1\mathbf{h}^{(1)} + U_2\mathbf{h}^{(2)} \tag{11}$$

where $\mathbf{h}^{(1)}$ and $\mathbf{h}^{(2)}$ are auxiliary incompressible fields satisfying the steady Stokes equations with no slip conditions and respectively equal to $\mathbf{e}_1, \mathbf{e}_2$ as $\mathbf{x} \to \infty$ (see Eqs. (29)–(32) in the Appendix). These fields represent the fluid flow as the body translates in the $\mathbf{e}_1$ and $\mathbf{e}_2$ directions respectively (see [10] for a discussion about these auxiliary fields). It must be kept in mind that this approximation made in Eq. (7) is valid for very small Re.[3]

The approximation for the velocity field is so chosen since an exact analytical expression is available for the creeping flow field around bodies of certain shapes at this level [6] which allows us to effectively invoke Onsager's principle to study this fluid structure problem. Putting this form of $\mathbf{u}_s$ into Eq. (10) and simplifying, we have

$$T_0 P = (K_{11}U_1^2 + K_{22}U_2^2 + 2K_{12}U_1U_2) + m_e\mathbf{g}\cdot\mathbf{U} \tag{12}$$

where

$$K_{11} = 2\mu\int_\Omega \mathbf{D}(\mathbf{h}^{(1)}):\mathbf{D}(\mathbf{h}^{(1)}) \tag{13}$$

$$K_{22} = 2\mu\int_\Omega \mathbf{D}(\mathbf{h}^{(2)}):\mathbf{D}(\mathbf{h}^{(2)}) \tag{14}$$

$$K_{12} = 2\mu\int_\Omega \mathbf{D}(\mathbf{h}^{(1)}):\mathbf{D}(\mathbf{h}^{(2)}). \tag{15}$$

[3] We define $Re = \frac{\rho_f U d}{\mu}$ where ρ_f is the fluid density, U is the characteristic velocity of the fluid and d can be taken to be the maximum dimension of the body.

The Eq. (12), for entropy production can be written in indicial form as

$$T_0 P = \left(K_{ij}U_j + m_e g_i\right)U_i \tag{16}$$

where we employ the Einstein summation convention and $i, j = 1, 2$. Note that in the case of bodies with fore-aft symmetry, such as cylinders, disks and spheroids, the term $K_{12} = 0$ [28]. It is now clearly seen that in case of bodies with fore-aft symmetry, the appropriate phenomenological constants in this problem are $\frac{1}{T_0}K_{ij}$ and can be shown to satisfy the Onsager reciprocity relations.[4] The proof of the validity of the reciprocity relations for the first term of P follows since

$$\begin{aligned} K_{11} &= 2\mu\int_\Omega |\mathbf{D}(\mathbf{h}^{(1)})|^2 > 0, \\ K_{22} &= 2\mu\int_\Omega |\mathbf{D}(\mathbf{h}^{(2)})|^2 > 0, \\ K_{12} &= 2\mu\int_\Omega \mathbf{D}(\mathbf{h}^{(1)}):\mathbf{D}(\mathbf{h}^{(2)}) \\ &= 2\mu\int_\Omega \mathbf{D}(\mathbf{h}^{(2)}):\mathbf{D}(\mathbf{h}^{(1)}) = K_{21}. \end{aligned}$$

K_{ij} depends only on the auxiliary fields and is independent of U_i. Note that for a sedimenting body with $m_e > 0$, the body will always fall such that $\alpha := \mathbf{g}\cdot\mathbf{U} > 0$ (also see [8] for proof of this statement), where α is a scalar. In the case when $\mathbf{g}\cdot\mathbf{U} < 0$ it follows that $m_e < 0$ corresponding to a buoyant particle, therefore making the product $m_e\alpha > 0$. As a result, this term trivially satisfies the Onsager hypothesis.

A second, experimental fact, that bolsters our assumption in Eq. (7) is that the orientation phenomena is observed for extremely small Re, when the system is no longer in equilibrium. In fact, the Reynolds numbers in the experiments performed can be as small as 0.016 (see [10] and references cited therein).

In the rest of this paper, we restrict our discussion to the case of bodies with fore-aft symmetry. Based on Eq. (16) and the calculation outlined in the Appendix, an important observation follows immediately.

Remark 1. In the Stokes regime, when $Re = 0$, since inertial effects are absent, the balance of linear momentum yields (see Appendix)

$$K_{ij}U_j + m_e g_i = 0 \tag{17}$$

$$\Rightarrow P = 0. \tag{18}$$

The zero entropy production case is identified with reversible processes [27] and Eq. (18) indicates that sedimentation if slow enough, is a reversible process.[5] Further, since P vanishes, it is independent of $\mathbf{U}$ and hence also of the orientation of the falling body. In other words, we see that in the creeping motion regime, the sedimenting body can fall with any orientation which is consistent with the observations of Leal [18].

[4] The constants K_{ij} correspond to the force and flux terms, $Y = \frac{1}{T_0}K_{ij}U_i$ and $X = U_j$.

[5] This fact has also been observed by Horne et al. [13] in internal flow problems.

In the case where inertial effects appear and the linear momentum equation tells us that (see Eq. (39) in the Appendix)

$$K_{ij}U_j + m_e g_i \neq 0 \tag{19}$$

as a result of which $P \neq 0$. The corresponding expression for force, as shown in the Appendix displays the emergence of an inertial force term for non-vanishing inertial effects. As a consequence, there is an interesting transition in the behavior of the sedimenting body. In order to see the relation between the terminal orientation of the body with P clearly, we rewrite the vector $\mathbf{U} = (|U|\cos\theta, -|U|\sin\theta, 0)$ in polar coordinates where θ is the angle between the longer axis of the body and the direction of motion. As a result, we have

$$\begin{aligned} T_0 P &= (K_{11}|U|^2\cos^2\theta + K_{22}|U|^2\sin^2\theta) \\ &\quad + m_e|\mathbf{g}||\mathbf{U}|\cos\psi \end{aligned} \tag{20}$$

where ψ is the angle between the motion of the body and the direction of gravity and is independent of θ. Therefore, the extremum of P, now with respect to θ gives us

$$\frac{\mathrm{d}P}{\mathrm{d}\theta} = \frac{1}{T_0}|U|^2(K_{22} - K_{11})\sin 2\theta = 0 \tag{21}$$

provided $\theta = 0$ or $\pi/2$. The second variation of P is

$$\frac{\mathrm{d}^2P}{\mathrm{d}\theta^2} = \frac{2}{T_0}|U|^2(K_{22} - K_{11})\cos 2\theta. \tag{22}$$

We have shown, in an earlier paper [29] that the Eqs. (13) and (14) simply refer to the force coefficients corresponding to the motion of the body along the $\mathbf{e}_1$ or $\mathbf{e}_2$ directions, respectively. In the specific case of a prolate spheroid, the coefficients K_{11} and K_{22} have been computed numerically (with $\mu = 1$ and $T_0 = 1$ without loss of generality) for varying eccentricities and are shown in Fig. 1. The calculation has been performed by means of a finite volume formulation using body fitted coordinates. Validation for this method are shown in [29]. From Fig. 1 we see that $K_{22} > K_{11}$ for any eccentricity(also compare with the results of Chwang and Wu [6]).

We are therefore left to choose between two states, one corresponding to a minimum of the entropy production and the other to a maximum. In order to choose the stable one, we must resort to the physics literature and perhaps some intuition. Guided by the previous literature [5,19,20,24], we invoke the maximum entropy production principle and it follows that P has a maximum when $\theta = \pi/2$, which is in line with experimental observations. Another justification for this choice comes from the more recently proposed *Constructural theory*, which suggests that stable physical configurations, where a system can select from infinitely many choices, are governed by the shortest time taken to evolve [2,26]. It has also been proved that the contructural principle is related to entropy production.

This final result of this computation is in agreement with the results of experimental observations which show that $\theta = \pi/2$ is indeed the final orientation of a settling spheroid. A final remark ensues:

Remark 2. In the presence of inertia when the problem of particle sedimentation in a fluid becomes irreversible, but still near equilibrium, a long body such as an ellipsoid will fall in its steady state along its shortest axis, or in other words, along the position of maximum drag.

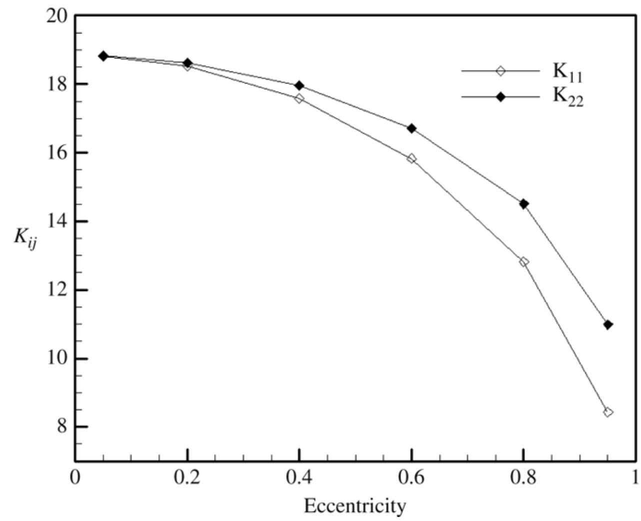

Fig. 1. Force coefficients, K_{11} and K_{22} versus eccentricity, computed numerically with $\mu = 1$, corresponding to the motion of a prolate spheroid along $\mathbf{e}_1$ and $\mathbf{e}_2$ directions respectively. We have shown that for the spheroid, $K_{ij} = 0, i \neq j$ due to the symmetry of the body and the flow field.

This maximum drag principle seems to be pertinent in sedimentation problems and has been also previously discussed (see [12] and also references cited therein) in the context of the settling speed of a sphere in a viscous fluid.

4. Discussion

In conclusion, we note that the problem of fluid structure interaction, in particular the terminal orientation of a body in a fluid during freefall can be determined by the laws of non-equilibrium thermodynamics. Specifically, we see that the extrema of entropy production of the system, determines the allowed stable states. In the creeping flow regime, when inertia is absent, the system has already reached a state of maximum entropy which the second law of thermodynamics dictates and also the rate of entropy production is in fact zero. This suggests that the body can take on any orientation which is determined by its initial state. When inertial effects emerge, the terminal state corresponds to one of maximum entropy production or one corresponding to maximum drag.

We note that when the extremum is evaluated with respect to U_i ($i = 1, 2, 3$), one observes only a minimum. However when we extremize with respect to θ, we observe that P is a minimum at $\theta = 0$ while P is a maximum when $\theta = \pi/2$. There are two arguments for choosing the latter variable as the appropriate one. Firstly, since we are seeking to find the terminal orientation of the falling body, θ is the natural and physically relevant variable to be considered. Secondly, we consider $\frac{\mathrm{d}P}{\mathrm{d}U}$ to not be meaningful since this represents change in Reynolds number (or inertial effects) with velocity, with a minimum occurring along both U_1 and U_2. Since the terminal orientation is a steady state, it occurs at a fixed Reynolds number and therefore it is not

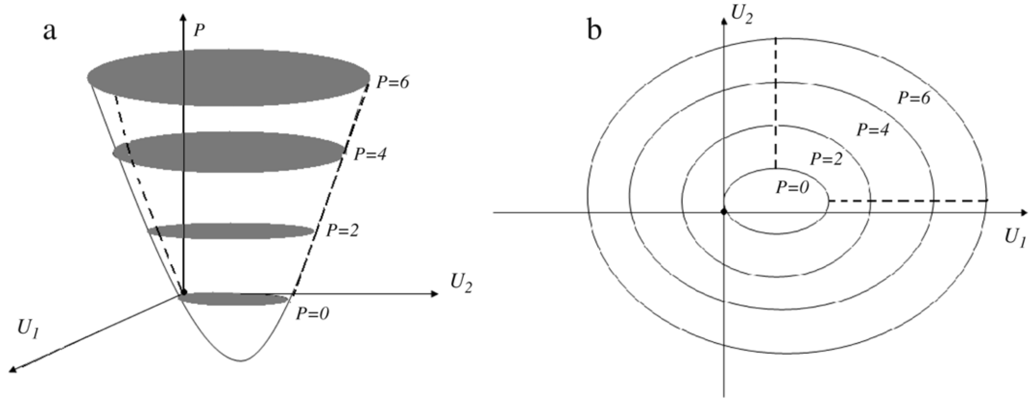

Fig. 2. A schematic of the variation of P with U_1 and U_2. In figure (a) we see the three dimensional plot of P versus the two velocity components. Different slices of this surface are shown for increasing inertia or increasing values of P. As inertial effects increase, P moves up along the paraboloid surface along the dashed lines. Figure (b) is the same as figure (a) seen from above. The concentric ellipses are nothing but contours of P on the U_1–U_2 plane. Once again, the dashed line indicates the direction along which the system can transition and there are only two ways of doing this which coincides with the equilibrium solutions of Eq. (21). In order to find the stable solution, the system will choose the path that allows for the shortest distance between any two contours, which happens to lie along U_2. In other words, a sedimenting body will choose to fall along its shorter axis in its terminal state.

relevant to speak of variations in U_i. The derivative $\frac{dP}{d\theta}$ displays two equilibrium points $\theta = 0$ and $\theta = \pi/2$ at any given Re, resulting in a minimum and maximum respectively. Therefore, whereas $\frac{dP}{dU}$ does not distinguish between U_1 and U_2, $\frac{dP}{d\theta}$ does distinguish between $\theta = 0$ and $\theta = \pi/2$.

Fig. 2 shows a schematic of the variation of P with U_1 and U_2. In figure (a) we see a plot of P versus the two velocity components. Figure (b) is the same as figure (a) seen from above; the concentric ellipses are the contours of P on the U_1–U_2 plane. The dashed line indicates the direction along which the system can transition and there are only two ways of doing this which coincides with the equilibrium solutions of Eq. (21). The important question is: Why the systems transitions from a $Re = 0$ case where any angle is allowed to a state with only one stable steady orientation ? In order to select the maximum entropy production state over the minimum, we look to previous optimization arguments in physics. As discussed above, they all seem to indicate that any physical system always chooses to minimize the time taken or path taken to transition between states. Using this approach, as explained in Fig. 2, we argue that maximum entropy production state should result in the stable terminal state, which also happens to coincide with experimental observations.

Even though our analysis is formulated for extremely small Re, the results of this analysis continues to hold for intermediate Re up to around 50. It is seen that as the Reynolds number increases and is in the range $57 < Re < 211$, the particle exhibits periodic motion about the stable steady state position. The oscillatory motion has been attributed to the phenomenon of vortex shedding[14]. It remains to be seen how the entropy argument will need to be adapted to explain time dependent motions of the sedimenting body. However, in the presence of inertia, as long as the wake structure of the flow remains symmetric, the body assumes a steady terminal orientation.

Several questions remain to be answered regarding this problem. The stability of the maximum entropy production state at large Re, which is equivalent to large values of P, is a significant and yet unanswered question even in the case of internal flows or bounded domains. Also, this paper does not treat the problem of a falling body in a non-Newtonian fluid which is currently being examined. We are, however, aware of experimental observations on the sedimentation of ellipsoids which indicate that when falling in a non-Newtonian fluid (for instance, a polymer) the body always seems to take the position of *minimum drag*. In viscoelastic fluids, the essential normal stress components emerge from the nonlinearities of the stress tensor. However, Onsager's linearization principle is unable to capture the significant effects of normal stress by the same decomposition of the velocity field. It therefore remains to be seen how additional constraint of viscoelasticity will present itself in the entropy production equation, leading to a different result.

Acknowledgments

We dedicate this article to Professor Giovanni Paolo Galdi in honor of his 60'th birthday. The authors wish to thank Professors Paolo Galdi, Roberto Camassa, Richard McLaughlin and Peter Mucha for interesting discussions. Author AV acknowledges the RTG NSF DMS-0502266 at the University of North Carolina. Part of this work was conducted at IST, Portugal during the summer of 2007. Author AV would also like to thank Prof. Ana Silvestre and Prof. F.L. Carapau for their invitation and support for this trip.

Appendix

Since the problem that we are solving involves a coupling of the fluid and the body, we must write down the governing equations for both. The equations for the fluid motion can be given by

$$\rho_f \mathbf{u} \cdot \nabla \mathbf{u} = \operatorname{div} \mathbf{T}(\mathbf{u}, p) + \rho_f \mathbf{g} \tag{23}$$

$$\text{div}\,\mathbf{u} = 0 \tag{24}$$

$$\mathbf{T}(\mathbf{u}, p) = -p\mathbf{I} + 2\mu\mathbf{D}(\mathbf{u}) \tag{25}$$

where $\mathbf{I}$ is the identity tensor, p is the pressure, $\mathbf{u}$ is the velocity field, ρ_f is the density of the fluid and $\mathbf{D}$ is the symmetric part of the velocity gradient. We write $\mathbf{g} = \text{grad}\,\phi$, so the stress tensor can be given by

$$\rho_f\mathbf{g} + \text{div}\,\mathbf{T}(\mathbf{u}, p) = \text{div}\,[(-p + \rho_f\phi)\mathbf{I} + 2\mu\mathbf{D}(\mathbf{u})] \tag{26}$$

$$= \text{div}\,\bar{\mathbf{T}}. \tag{27}$$

Also, the force acting on the body is given by

$$\mathbf{F}_{f-b} + m_e\mathbf{g} = 0 \tag{28}$$

where $\mathbf{F}_{f-b}$ is the force imposed by the fluid on the body, $\mathbf{g}$ is the gravity vector and m_e is the effective mass.

We introduce the non-dimensional translational Stokes auxiliary field $(\mathbf{h}^{(i)}, p^{(i)})$ which satisfies the equations

$$\Delta\mathbf{h}^{(i)} = \text{grad}\,p^{(i)} \tag{29}$$

$$\text{div}\,\mathbf{h}^{(i)} = 0 \tag{30}$$

$$\mathbf{h}^{(i)} = 0 \text{ on the boundary .} \tag{31}$$

$$\lim_{|\mathbf{x}|\to\infty} \mathbf{h}^{(i)} = \mathbf{e}_i \tag{32}$$

where $i = 1, 2$. The field corresponds to the translational motion of the body and has been analytically computed for certain shapes of the body such as spheres, prolate and oblate spheroids. The particular advantage of this field is that it allows for the velocity field $\mathbf{u}$ to be decomposed into the fluid motion and the body motion which can be very convenient for our upcoming analysis.

We now take the dot product of Eq. (23) by $\mathbf{h}^{(l)}$ then integrate over the fluid domain to get

$$\rho_f \int_\Omega \mathbf{u}\cdot\text{grad}\,\mathbf{u}\cdot\mathbf{h}^{(l)}\mathrm{d}\Omega = \int_\Omega \text{div}\,\bar{\mathbf{T}}\cdot\mathbf{h}^{(l)}\mathrm{d}\Omega \tag{33}$$

$$= \int_S \mathbf{n}\cdot\bar{\mathbf{T}}\cdot\mathbf{e}_j\mathrm{d}S - 2\mu\int_\Omega \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{h}^{(l)})\mathrm{d}\Omega \tag{34}$$

$$\Rightarrow \int_S \mathbf{n}\cdot\bar{\mathbf{T}}\cdot\mathbf{e}_j\mathrm{d}S = 2\mu\int_\Omega \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{h}^{(l)})\mathrm{d}\Omega + \rho_f\int_\Omega \mathbf{u}\cdot\text{grad}\,\mathbf{u}\cdot\mathbf{h}^{(l)}\mathrm{d}\Omega \tag{35}$$

where $j, l = 1, 2$. However, the left hand side of Eq. (35) is nothing but $\mathbf{F}_{f-b}$, therefore

$$2\mu\int_\Omega \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{h}^{(l)})\mathrm{d}\Omega = -m_e\mathbf{g} - \rho_f\int_\Omega \mathbf{u}\cdot\text{grad}\,\mathbf{u}\cdot\mathbf{h}^{(l)}\mathrm{d}\Omega. \tag{36}$$

We now write the velocity field $\mathbf{u} = \mathbf{u}_s + \mathbf{w}$ as described in Eq. (7) in the paper and further decompose the fluid motion from the body motion according to $\mathbf{u}_s = U_1\mathbf{h}^{(1)} + U_2\mathbf{h}^{(2)}$. Then Eq. (36) becomes

$$2\mu\int_\Omega \mathbf{D}(\mathbf{u}_s) : \mathbf{D}(\mathbf{h}^{(l)})\mathrm{d}\Omega = -m_e\mathbf{g} + \mathcal{H}(\rho_f, \mathbf{u}_s, \mathbf{w}) \tag{37}$$

where $\mathcal{H}$ contains the inertial terms. Writing

$$2\mu\int_\Omega \mathbf{D}(\mathbf{u}_s) : \mathbf{D}(\mathbf{h}^{(l)})\mathrm{d}\Omega = K_{ml}U_l \tag{38}$$

and comparing with Eq. (37), we get

$$K_{ml}U_l + m_e g_m = \mathcal{H}(\rho_f, \mathbf{u}_s, \mathbf{w}). \tag{39}$$

Therefore in the Stokes regime the right hand side of Eq. (39) vanishes, whereas in general the term $\mathcal{H}$ is non zero.

References

[1] M.A. Biot, Variational principles and irreversible thermodynamics with applications to viscoelasticity, Phys. Rev. 97 (6) (1955) 1463–1469.

[2] A. Bejan, Shape and Structure, from Engineering to Nature, Cambridge University Press, Cambridge, UK.

[3] K. Chiba, K. Song, A. Horikawa, Motion of a slender body in quiescent polymer solutions, Rheol. Acta 25 (1986) 280–388.

[4] K. Cho, Y.I. Cho, N.A. Park, Hydrodynamics of a vertically falling thin cylinder in non-newtonian fluids, J. Non-Newton. Fluid Mech. 45 (1992) 105–145.

[5] T. Christen, Application of the maximum entropy production principle to electrical systems, J. Phys. D: Appl. Phys. 39 (2006) 4497–4503.

[6] A.T. Chwang, T.Y. Wu, Hydromechanics of low-Reynolds-number flow. part 2. singularity method for stokes flows, J. Fluid Mech. 45 (1992) 105–145.

[7] B.D. Coleman, W. Noll, On the thermodynamics of continuous media, Arch. Ration. Mech. Anal. 3 (1959) 289–303.

[8] G.P. Galdi, A. Vaidya, Translational steady fall of symmetric bodies in Navier-Stokes liquid, with application to particle sedimentation, J. Math. Fluid Mech. 3 (2001) 183–211.

[9] G.P. Galdi, M. Pokorny, A. Vaidya, D.D. Joseph, J. Feng, Orientation of bodies sedimenting in a second-order liquid at non-zero Reynolds number, Math. Models Methods Appl. Sci. 12 (11) (2002) 1653–1690.

[10] G.P. Galdi, On the motion of a rigid body in a viscous fluid: A mathematical analysis with applications, in: Handbook of Mathematical Fluid Mechanics, Elsevier Science, Amsterdam, 2002, pp. 653–791.

[11] R. Ghesselini, Elastic free energy of an upper convected maxwell fluid undergoing fully developed planar poiseuille flow: A variational result, J. Non-Newton. Fluid Mech. 46 (1993) 229–241.

[12] V. Happel, H. Brenner, Low Reynolds Number Hydrodynamics, Prentice Hall, 1965.

[13] C. Horne, C.A. Smith, K. Karamcheti, Aeroacoustic and Aerodynamic Applications of the theory of nonequilibrium thermodynamics, NASA Technical Paper 3118, June 1991.

[14] H.H. Howard, D.D. Joseph, A.F. Fortes, Experiments and direct simulations of fluid particle motions, Int. Video J. Eng. Res. 2 (1992) 17–24.

[15] E.T. Jaynes, The minimum entropy production principle, Ann. Rev. Phys. Chem. 31 (1980) 579–601.

[16] D.D. Joseph, Y.J. Liu, Orientation of long bodies falling in a viscoelastic fluid, J. Rheol. 37 (1993) 961–983.

[17] H.J. Kreuzer, Nonequilibrium Thermodynamics and its Statistical Foundations, Clarendon Press, Oxford, 1981.

[18] L.G. Leal, The slow motion of slender rod-like particles in a second-order fluid, J. Fluid Mech. 69 (1975) 305–337.

[19] J. Malek, K.R. Rajagopal, On the modeling of inhomogeneous incompressible fluid-like bodies, Mech. Mater. 38 (2006) 233–242.

[20] L.M. Martyushev, V.D. Seleznev, Maximum entropy production principle in physics, chemistry and biology, Phys. Rep. 426 (2006) 1–45.

[21] L. Onsager, Reciprocal relations in irreversible processes, I., Phys. Rev. 37 (1931) 405–426.

[22] L. Onsager, Reciprocal relations in irreversible processes. II, Phys. Rev. 38 (1931) 2265–2279.

[23] H.C. Öttinger, Beyond Equilibrium Thermodynamics, Wiley Interscience, USA, 2005.

[24] H. Ozawa, A. Ohmaru, R.D. Lorenz, T. Pujol, The second law of thermodynamics and global climate system: A review of the maximum entropy production principle, Rev. Geophys. 41 (4) (2003) 1–24.

[25] I. Prigogine, Introduction to Thermodynamics of Irreversible Processes, Interscience Publishers, New York, 1955.

[26] H.A. Reis, Constructal Theory: From engineering to physics, and how flow systems develop shape and structure, Appl. Mech. Rev. 59 (2006) 269–282.

[27] J.S. Thomsen, The reversible process: A zero-entropy production limit, Am. J. Phys. 64 (5) (1996) 580–583.

[28] A. Vaidya, A note on the terminal orientation of symmetric bodies in a power-law fluid, Appl. Math. Lett. 18 (12) (2005) 1332–1338.

[29] A. Vaidya, B. Chung, On the slow motion of a sphere in fluids with non-constant viscosities, 2007 (submitted for publication).

[30] H-J. Woo, Variational formulation of nonequilibrium thermodynamics for hydrodynamic pattern formation, Phys. Rev. E 66 (2002) 066104-1–066104-5.

[31] H. Ziegler, An Introduction to Thermodynamics, North-Holland, Amsterdam, 1983.