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Identifying almost invariant sets in stochastic dynamical systems

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We consider the approximation of fluctuation induced almost invariant sets arising from stochastic dynamical systems. The dynamical evolution of densities is derived from the stochastic Frobenius–Perron operator. Given a stochastic kernel with a known distribution, approximate almost invariant sets are found by translating the problem into an eigenvalue problem derived from reversible Markov processes. Analytic and computational examples of the methods are used to illustrate the technique, and are shown to reveal the probability transport between almost invariant sets in nonlinear stochastic systems. Both small and large noise cases are considered. © 2008 American Institute of Physics. [DOI: 10.1063/1.2929748]

The effect of noise on dynamical systems can completely change the observed dynamics. In deterministic systems where multiple attractors have well defined basins of attraction determined by the global topology, stochastic perturbations may destroy such well defined behavior. For deterministic basins, any initial point starting within the basin remains there, and the resulting set of such points form an invariant set. In particular, there is no transport from one basin to another. Noise, on the other hand, destroys such distinct basins, and the sets are no longer invariant. However, for small noise, there may exist long periods of finite time in which the dynamics remains in a set before it leaves. We call these sets almost invariant sets, and they may be characterized by examining the probability of transport from one set to another. In this paper, we examine the tools to describe almost invariant sets in stochastic dynamical systems. In addition to using set theoretic methods to find the almost invariant sets, we also describe the transport between those sets for large and small noise cases. We apply the theory and tools to examine the dynamics of several examples of multistable systems.

I. INTRODUCTION

The effects of noise on nonlinear dynamical systems have become an important topic in recent years. Examples occur in noise induced instabilities arising in deterministic stable dynamics, such as escape from a potential,\textsuperscript{19} analysis of stochastic bifurcations including exploration of the interaction of noise and global bifurcation based on underlying chaotic saddles,\textsuperscript{14} noise induced escape using the Hamiltonian theory of large fluctuations,\textsuperscript{8} the theory of quasipotentials,\textsuperscript{12,13} or a variational formulation of optimal escape paths.\textsuperscript{7,16} It is well known that noise can excite unstable chaotic structures while destroying regular periodic dynamics, but most studies consider induced chaos occurring near a bifurcation and using small noise. Statistically, trajectories follow the chaotic saddle which exists just before a saddle-node point, or after a crisis of chaotic attractors. However, such techniques are restricted to limited parameter regimes and small noise. This paper will consider other dynamics excited by finite noise amplitudes and the tools we can use to quantify the underlying set topology of the stochastic attractors.

As an alternative to continuous stochastic differential equation analysis methods, set theoretic methods have been used to quantify the stochastic attractors, as well as transport. Transport in dynamical systems is an important topic of active research in that it applies to many problems in physical systems ranging from low to high dimensions. Most of the tools for handling deterministic transport from a geometric point view have come from modern dynamical systems, such as Ref. \textsuperscript{22}. In continuous dynamical systems with noise, one solves a master equation for the probability density function (PDF).\textsuperscript{19} While for discrete dynamical systems with noise, the benefit is that one can use the Frobenius–Perron operator (FP) formalism\textsuperscript{15} to determine density evolution. In Ref. \textsuperscript{4}, the idea of examining transition functions for deterministic chaotic systems generated the machinery for discretizing the FP operator, and was applied to kernels which correspond to deterministic systems. Using similar ideas, the FP operator was explicitly extended to stochastic kernels in Ref. \textsuperscript{1}, where it was applied to probabilistic transport in epidemic models and discrete low dimensional examples. A full treatment and analysis of the discretization of the FP operator is given in Ref. \textsuperscript{3}, as well as transport from one invariant region to another.

More recently, transport techniques have been extended to identify almost invariant sets, defined by structures in which the dynamics may remain for a long period of time prior to leaving the region.\textsuperscript{11} The techniques make use of results from graph theory on reversible, or detailed balance, Markov processes of the probabilistic transition dynamics.\textsuperscript{10} We follow a similar approach here for set theoretic dynamics generated by stochastic kernels. Previously, one would need a full analysis of the deterministic system to define basins and the underlying topology. Using the reversible transition matrix over a range of noise levels, one can identify almost...
invariant sets and how they change as a function of the noise level. In addition, we can use the method to identify regions of transport between the almost invariant basins for finite noise amplitudes. We examine this technique for analytic one-dimensional maps and a discretely sampled flow. Results from this technique agree with our previous findings in a two-dimensional bistable example from epidemiology. The method provides a faster and more sensitive tool that can be extended to higher dimensions and finer resolutions, depending on computational resources.

The layout of the remaining part of the paper is as follows: In Sec. III, we describe the general theory for the stochastic FP operator. Both continuous and discrete versions are given. Methods for approximating almost invariant sets and the transport between them are described. Section IV has three examples that we use to illustrate these methods. The first two are one-dimensional maps which clearly demonstrate a range of results derived from the reversible transition matrix. The last example is a two-dimensional application from epidemiology. Conclusions are summarized in Sec. IV.

II. GENERAL THEORY

A. Stochastic Frobenius–Perron operator

We define the Frobenius–Perron (FP) operator for a deterministic dynamical system the following way. Let \( F \) be map for a discrete dynamical system acting on a set \( M \subset \mathbb{R}^n \) defined by

\[
F: M \rightarrow M, \quad x \mapsto F(x). \tag{1}
\]

We define an associated dynamical system over the space of densities of ensembles of initial conditions,

\[
P_F: L^1(M) \rightarrow L^1(M), \quad \rho(x) \mapsto P_F[\rho(x)]. \tag{2}
\]

This FP operator \( P_F \) is defined by

\[
P_F[\rho(x)] = \int_M \delta(x - F(y)) \rho(y) dy,
\]

acting on PDFs \( \rho \in L^1(M) \), and \( \delta \) is the usual delta function.

Now consider the stochastically perturbed dynamical system \( F_\eta: \mathbb{R}^n \rightarrow \mathbb{R}^n, x \mapsto F(x) + \eta \), where \( \eta \) is a random variable having the PDF \( \nu(x) \). The effect of \( \nu(x) \) is to blur the deterministic contribution of the \( \delta \) function. The stochastic Frobenius–Perron (SFP) operator is defined to be

\[
P_{F_\eta}[\rho(x)] = \int_M \nu(x - F(y)) \rho(y) dy. \tag{4}
\]

For our applications, we will use a normal distribution in \( \mathbb{R}^n \) given by

\[
\nu(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(- ||x||^2/2\sigma^2) \tag{5}
\]

with mean \( \langle x \rangle = 0 \) and standard deviation \( \sigma \). For a discrete time system with constantly applied stochastic perturbation having normal distribution, it is clear that the SFP operator is

\[
P_{F_\eta}[\rho(x)] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_M e^{-||x - F(y)||^2/2\sigma^2} \rho(y) dy. \tag{6}
\]

B. The discrete theory

In order to compute finite dimensional approximations to the SFP operator, we approximate functions in \( L^1(M) \), with discretely indexed basis functions \( \{ \phi_i(x) \}_{i=1}^\infty \subset L^1(M) \). The finite dimensional linear subspace is generated by a subset of the basis functions, \( \Delta_n = \text{span} \{ \{ \phi_i(x) \}_{i=1}^n \} \), such that \( \phi_i \in L^1(M) \forall \phi_i \). One approximates density \( \rho(x) \) by the finite sum of basis functions, \( \rho(x) \approx \sum_{i=1}^n c_i \phi_i(x) \), where \( \phi_i(x) = \chi_{B_i}(x) \), \( \chi_B \) is an indicator defined on boxes \( \{ B_i \}_{i=1}^N \) covering \( M \). The transition matrix is approximated by the \( N \times N \) matrix,

\[
A(B_i, B_j) = A_{i,j} = \left[ P_{F_\eta}(\phi_i), \phi_j \right] = \int_M P_{F_\eta}[\phi_i(x)] \phi_j(x) dx
\]

for \( 1 \leq i,j \leq N \). Therefore, a transition matrix entry \( A_{i,j} \) value represents how mass flows from cell \( B_i \) to cell \( B_j \). Normalizing \( A_{i,j} \) by the measure of \( B_i \) yields the probability that a point in \( B_j \) has its image in \( B_j \), which is denoted by \( P_{i,j} \). \( P \) is therefore a stochastic, or probability, transition matrix.

Almost invariance can now be considered in terms of a given partition which covers the set \( M \). We can think of a set being almost invariant if its self-transition \( A(B_i, B_i) \) is close to 1. Therefore, we may adopt the following description of almost invariance. We say an almost invariant set consists of partition \( \mathcal{N}=\{B_1, \ldots, B_M\} \) such that \( \mathcal{N} = 1 / \max \|\text{span} \{B_i\}_{i=1}^M \| \text{max} \) is maximized over all partitions. In order to compute the almost invariant sets, we will require some modifications to the transition matrix.

C. Properties of the transition matrix

Given a stochastic transition matrix \( P \) where rows sum to unity, there are several quantities that can be easily computed from the Perron–Frobenius theorem:

1. Probabilities evolve discretely according to \( \pi^{(n+1)} = P^n \pi^{(n)} \).
2. There exists a left eigenvector \( \pi \) of \( P \) which approximates the invariant probability density of the dynamical system. It satisfies \( \pi^T = \pi^T P \), since it has an eigenvalue of 1.

To consider almost invariant sets of Markov chains, we will need to create a transition matrix that is reversible. The idea of using reversible Markov processes to find almost invariant sets comes from an algorithm in Ref. 10 which in turn uses results derived from ergodic graph theory ideas applied to image segmentation. \( 18 \) Let \( P \) be a stochastic primitive matrix. A matrix is called primitive if there is an integer \( k \geq 0 \) such that \( P^k > 0 \). For such a matrix, there exists a unique probability vector, such that it is a left eigenvector with eigenvalue 1, i.e., \( \pi^T = \pi^T P \). It also has a right eigenvector, \( 1^T = (1, 1, \ldots, 1)^T \). Since \( \Sigma \pi = 1 \), the left and right corresponding eigenvectors satisfy \( \pi^T 1 = 1 \). The interpretation of \( \pi \) is very useful, since each component \( \pi_i \) represents the fraction of time the dynamics spends in box \( B_i \).

We now consider the idea of a reversible Markov chain, or detailed balance. A reversible Markov chain is one which satisfies \( \pi_i P_{ij} = \pi_j P_{ji} \) for all \( i, j \). Such a condition implies \( \pi \) is
an invariant distribution of $P$. A reversible Markov chain in equilibrium is one whose forward sequence of events has the same probability of the reverse sequence, making it difficult to state which direction time is going in a real experiment. Letting $D=\text{diag}(\sqrt{\pi})$, it is easy to show that the detailed balance is equivalent to $D^2P=P^TD^2$.

Since in general $P$ is not reversible as we have defined it from Eq. (7), we construct a reversible Markov chain from our transition matrix. Let $R=(P+\hat{P})/2$, where $\hat{P}_{ij} = \pi_j / \pi_i P_{ij}$ and $\pi$ is an invariant probability density of $P$. Then the following properties may be shown to be true:

1. $D^2R=R^TD^2$, so $R$ is reversible.
2. $\sum_{j=1}^n R_{ij}=1$.
3. $\pi^T R = \pi^T$.
4. If $R$ is reversible, the eigenvalues are real and the eigenvectors are orthogonal.

Property 4 is especially important, since it will used to cluster the invariant sets.

In order to compute a collection of sets which are almost invariant, one may examine eigenvectors of $R$, which reveal the approximate almost invariant partitions. Our machinery based on the SFP operator defined in Eq. (4) allows us to do this quite easily. The basic idea behind generating the invariant sets via the eigenstructure of $R$ may be understood as follows.

Consider the deterministic case of Eq. (1) in the absence of noise, and assume there exist two basins of attraction, $A$ and $B$. Each of these sets are made up of aggregates of the appropriate union of sets from the partition $\mathcal{N}$. Given any nonempty index set $I$, define its characteristic vector $1_I$ by $1_{I,j}=1$ for all $j \in I$, and zero otherwise. Then the one step transition probability from a set $A$ to $B$ is given by

$$w_\pi(A,B) = \frac{\sum_{a \in A, b \in B} \pi_a P_{ab}}{\sum_{a \in A} \pi_a} = \frac{\langle 1_B, P 1_A \rangle \pi}{\langle 1_A, 1_A \rangle \pi},$$

(8)

where $\langle , \rangle_\pi$ is an inner product weighted by the distribution $\pi$; $\langle x,y \rangle_\pi = x^T \pi y$. [This should be compared with Eq. (7).]

Clearly, if $A$ is an invariant for the deterministic case, $w_\pi(A,A)=1$. Using the FP machinery, it is clear that if partition $\mathcal{N}$ may be decomposed into two disjoint invariant sets, then the stochastic case should be a perturbation of these sets. In the deterministic case, if $A$ and $B$ are invariant and uncoupled, then there exists right eigenvectors of $R$ corresponding to each diagonal block having eigenvalue of 1, with eigenvector entries equal to 1. That is, there exists an eigenspace spanned by the vectors

$$\chi_A = (1_A^T, 0) \quad \text{and} \quad \chi_B = (0, 1_B^T).$$

Using Eq. (9), we can write any basis of the eigenspace in terms of eigenvectors which are constant on the invariant sets. Perturbation theory states that we expect this structure to persist in the small noise cases as well, where the off-diagonal blocks represent sets in which transitions occur. In Ref. 1, we have exploited this idea using the graph theoretic notions presented in detail in Ref. 3.

Once we have the eigenstructure of $R$, the first few eigenvalues of $R$ will be clustered near unity and their associated eigenstates will be a mixture of almost invariant sets formed from the above basis elements. (In Ref. 10, they use clustering arguments based on fuzzy set theory, which we do not do here.) Since the first eigenvector of $R$ is a vector of ones, and Property 4 states that the second eigenvector must be orthogonal to the first, we use the second eigenvector to identify almost invariant sets emanating from basins of attraction. We illustrate this with three examples, which consist of multiple invariant attractors in the deterministic cases.

III. EXAMPLES

In this section we present three examples: a discrete map with three chaotic intervals, a bistable discrete map of periodic orbits, and a driven flow perturbed with a discrete noise source. We progress from basic one-dimensional examples to a two-dimensional application that can be studied only by numerical analysis. We show the advantages of using these tools in higher dimensions and suggest future projects, such as data-driven applications without models.
A map with three chaotic intervals

In this example, consider the one-dimensional piecewise linear map, \( f : [0, 1] \rightarrow [0, 1] \),

\[
  f(x) = \begin{cases} 
    2x & \text{if } 0 \leq x \leq 0.2 \\
    2x - \frac{2}{3} & \text{if } 0.2 < x \leq 0.4 \\
    \frac{3x}{2} - \frac{5}{3} & \text{if } 0.4 < x \leq 0.55 \\
    3x - \frac{5}{4} & \text{if } 0.55 < x \leq 0.6 \\
    2x - \frac{3}{5} & \text{if } 0.6 < x \leq 0.8 \\
    2x - 1 & \text{if } 0.8 < x \leq 1.
  \end{cases}
\]  

(10)

Depending on the initial condition, a trajectory can fill one of three disjoint chaotic intervals: \([0, 0.4], (0.4, 0.6], (0.6, 1]\). See the graph of Eq. (10) in Fig. 1.

In the absence of noise, the Markov partition of the domain is the same as the partition in the map definition. Therefore, the transition matrix is

\[
P = \begin{bmatrix}
  1 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
  \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
  0 & \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\
  0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}
\]  

(11)

and the three PDFs can be determined analytically from the left eigenvectors associated with the eigenvalue equal to 1.

The two end intervals have a constant PDF. The eigenvectors will reflect these values on the associated interval and have zeros elsewhere. The middle interval has a piecewise constant PDF of \(4/5\) on \((0.4, 0.55)\) and \(3/5\) on \((0.55, 0.6)\), with zeros elsewhere.

Using the SFP operator, we can monitor the transition matrix as a function of the standard deviation of the noise added to the system. In our simulations, we use MATLAB to approximate the stochastic transition matrix on a uniform mesh of 500 intervals in the domain. We used reflective boundaries to preserve the area and the errors for the row sums of the Markov matrix were less than \(10^{-8}\). See Fig. 2 for a contour map of the reversible transition matrix with standard deviation \(\sigma = 0.02\).

For small standard deviations of noise, we can easily detect the original disjoint basins from the piecewise constant values in the second right eigenvector of the transition matrix. This approach was studied for the deterministic case by Deuflhard et al.\(^5\) As shown in Fig. 3, the invariant intervals have nearly constant level patterns.

As the standard deviation of the noise increases, we see an increase in the switching between basins. As shown for \(\sigma = 0.02\) in the left graph of Fig. 4, trajectories remain in the middle basin for shorter time intervals and it becomes a transition region for the rest of the map. The right graph of Fig. 4 shows two PDF approximations, one using a time series of one million iterates (bars) and the other using the left eigenvector of the original transition matrix (curve). Notice that the frequency of the trajectory in the middle basin is higher than the right basin.

We can get a better idea of the dynamics from the second eigenvector of the reversible transition matrix. For example, Fig. 5 is a graph of the second eigenvector for \(\sigma = 0.02\). We see an increase in the variation in the piecewise constant intervals as the standard deviation of the noise increases. For intervals where there is stochastically induced transport, the second eigenvector values lie in between the piecewise constant functions. To show that this agrees with the reversible transition matrix in Fig. 2, the intervals with mass mapped outside of the deterministically defined basins in the matrix are graphed in black. Notice the large amount of transition associated with the middle interval. As expected, large noise

FIG. 3. A graph of the second eigenvector for the reversible transition matrix of the three basin map with a standard deviation of \(\sigma = 0.001\).

FIG. 4. On the left is a graph of a typical time series for the three basin map with Gaussian noise added with a standard deviation of \(\sigma = 0.02\). On the right is the PDF for this map. The curve represents the PDF calculated by the left eigenvector of the transition matrix and the bars are a histogram from a time series of one million iterations.
standard deviations decrease the stability of the middle almost invariant basin and the trajectories transition to the other basins fairly quickly from it.

The variation in the second eigenvector from the piecewise constant function indicates increased transport between almost invariant sets. Therefore, we can use the second eigenvector inversely to detect regions in the domain that have the highest probabilities for transport to other basins. As expected in this example, transport in the outer basins occurs in regions where the image is closest to another basin.

B. A bistable map

In this example, we consider the following bistable, one-dimensional, piecewise linear map,

$$f(x) = \begin{cases} 
0.8x + 0.8 & \text{if } 0 \leq x \leq 0.25 \\
1.8x - 0.25 & \text{if } 0.25 < x \leq 0.35 \\
0.8x + 0.1 & \text{if } 0.35 < x \leq 0.65 \\
1.8x - 0.55 & \text{if } 0.65 < x \leq 0.75 \\
0.8x - 0.6 & \text{if } 0.75 < x \leq 1. 
\end{cases} \quad (12)$$

Depending on the initial condition, a trajectory can asymptotically approach the fixed point $1/2$ or the period two orbit $\{1/9, 8/9\}$. The basin for $1/2$ is the open interval $(5/16, 11/16)$, which has unstable fixed points as endpoints. The rest of the domain is the basin for the period two orbit. See the graph in Fig. 6.

In the case of no noise, the PDF for each basin is a delta function on the attracting orbit. We continue with the small noise case with the standard deviation of $\sigma=0.01$. To perform the following analysis, one must take care to use a sufficiently large standard deviation for the noise to induce switching between basins. If not, the PDF will include infinitesimally small values and the matrix cannot be made reversible (to machine precision). Similar to the last example, we used MATLAB to calculate the transition matrix with a uniform mesh of 500 intervals on the domain. We used reflective boundaries to preserve the area and the error for the row sums of the Markov matrix were less than $10^{-8}$.

The second eigenvector of the reversible matrix for $\sigma=0.01$ correctly identifies the two almost invariant basins by piecewise constant functions, as shown in Fig. 7. The different levels identify the sets. Notice that most of the variation away from the constant functions occurs near the basin boundaries defined in the deterministic case, which indicates transport.

As we increase the standard deviation of the noise, there is increased switching between basins. In Fig. 8, we plot the time series and PDF of the map adding noise at a standard deviation of $\sigma=0.05$. Notice the agreement in the two PDF approximations, using a time series (bars) versus the left eigenvector of the transition matrix (curve).

As shown in Fig. 9, the second eigenvector of the reversible transition matrix clearly distinguishes the two almost invariant basins and the intervals where transitions between the two most likely occur. In this case, since there are two almost invariant basins, notice that they occur at the extremes of the range for the second eigenvector. We exploit this property in the following two-dimensional example.
C. Predicting almost invariant sets from epidemics

We consider a modified epidemic model for the spread of disease where the state variables are susceptibles $S$ and infectives $I$. (See Ref. 2 for a derivation and notation of parameters.) The modified SI model (MSI), given by

$$
\begin{align*}
S'(t) &= \mu - \mu S(t) - \beta(t)I(t)S(t), \\
I'(t) &= \left(\frac{\alpha}{\mu + \gamma}\right) \beta(t)I(t)S(t) - (\mu + \alpha)I(t), \\
\beta(t) &= \beta_0(1 + \delta \cos 2\pi t).
\end{align*}
$$

The parameter values are set to the standard used for measles, and are fixed at $\mu = 0.02$, $\alpha = 35.84$, $\gamma = 100$, $\beta_0 = 1575$, $\delta = 0.1$. In the deterministic case, period two and period three attractors exist. See Fig. 10 for a schematic of the periodic orbits and associated basins. Notice that the basin boundary is formed by the stable manifold of the period three saddle orbit, marked by the solid red curve. For a full description of the deterministic solutions, see Ref. 2.

Since the MSI model is periodically driven with period one and both $S$ and $I$ are fractions of the population, it may be viewed as a two-dimensional map of the unit box into itself. The stochastic model is also considered to be discrete for the purposes of this paper. That is, noise is added to the population rate equations at the same period as the forcing, having mean zero and standard deviation $\sigma$. The dynamics may then be represented as a map, $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, where

$$(S,I)(t+1) = F[(S,I)(t)] + \eta(t).$$

Here $\eta(t)$ is a discrete noise term.

In Ref. 2, it was shown that the existence of bi-instability is a sufficient condition for noise to excite chaos. That is, there exists a sufficiently large standard deviation in which new unstable orbits are created and sampled. For example, see Fig. 11 for a PDF at $\sigma = 0.035$. Trajectories do not simply switch between basins, but spend a significant amount of time on the almost heteroclinic tangle in between the attractors, emulating chaotic behavior. A study of the transport due to the addition of noise using Eq. (4) was originally reported in Refs. 1 and 3. Here, we extend the results to extract the almost invariant sets by analyzing the second eigenvector of the reversible transition matrix.

For this model we used a parallelized FORTRAN code to calculate the transition matrix with a uniform mesh of 100 $\times$ 100 boxes on the domain. We then used MATLAB to approximate the top eigenvalues and eigenvectors of this sparse matrix. The second eigenvectors have a length of 10,000 entries and are shown by mapping the data back to the original mesh on the domain and using a contour map of the values.

Consider the small noise almost case with $\sigma = 0.005$. In Fig. 12, the second eigenvector identifies the almost invari-
ant sets by the maximum and minimum values of the color scale. The red regions correspond to an almost invariant set close to deterministic period 3 attractor, while the dark blue regions correspond to the set close to the period 2 attractor. The light blue to yellow regions are the transition areas between the sets.

Fluctuation induced almost invariant sets may be considered at a larger noise amplitude. The second eigenvector of the reversible transition matrix for $\sigma=0.035$ clearly identifies the two underlying almost invariant sets and the transport regions, as shown in Fig. 13. Note that the splitting occurs without any sorting or processing of data. The bulk of the computation is in approximating the transport matrix to the scale needed for the application.

IV. CONCLUSIONS

We considered the approximation of fluctuation induced almost invariant sets arising in stochastic dynamical systems as a function of noise strength. We showed that by starting with the stochastic FP operator, a finite transition matrix may be computed in one step. Using similar techniques of high dimensional cluster data analysis, the problem of formulating an approximation to the almost invariant sets could be translated into finding the eigenstructure of a reversible Markov process. We showed the efficiency and advantages of using this method in both one- and two-dimensional applications in specific stochastic dynamical systems.

One of the problems in using the reversible approach for determining almost invariant sets from the second eigenvector stems from choosing a threshold. In many cases, due to
orthogonality of the second eigenvector with the PDF, there are sets which are separated by a sign change. However, in many instances, a nontrivial portion of the measure of the almost invariant region is indistinguishable from zero, and it makes set boundary detection difficult. More research into other methods to handle this problem is still needed.

The methods for locating clusters, or aggregates of sets, can be found in many problems stemming from identifying metastable states of stochastic problems having large data sets.20 (For example, see Ref. 21, and references therein.)

The methods used in Refs. 10 and 11 find their origins in image segmentation.18 In those works, as here, we based our cluster analysis on signed eigenstates of a reversible Markov process. Such algorithmic techniques rely on having a spectral gap in the transition matrices in order to define permutation operators that will create a block diagonal stochastic matrix. However, examples exist, where this technique is not robust due to the lack of a clear spectral gap, or mixing between almost invariant sets are not known a priori. In contrast to the above methods, diagonal dominance techniques exist, based on singular value decomposition methods.9 The idea is to use a sorting of the entries of the largest singular value to obtain a permuted transition matrix. There are also techniques that use the Laplacian matrix from graph partitioning and clustering.6,21 The results show a dual correspondence between the permutation techniques based on signed eigenstates and the graph clustering problem.

Understanding transport between almost invariant sets is very important in areas of stochastic control where noise amplitudes are finite. In these regions, it is expected that there is sufficient transition probability in which actuation is needed to restrict transition, or enhance it depending on the application.17 Therefore, the above tools and variations may be used for monitoring stochastic dynamical systems in which transport between almost invariant regions play a dominant role.

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