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Transition to Chaos in Continuous-Time Random Dynamical Systems

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We consider situations where, in a continuous-time dynamical system, a nonchaotic attractor coexists with a nonattracting chaotic saddle, as in a periodic window. Under the influence of noise, chaos can arise. We investigate the fundamental dynamical mechanism responsible for the transition and obtain a general scaling law for the largest Lyapunov exponent. A striking finding is that the topology of the flow is fundamentally disturbed after the onset of noisy chaos, and we point out that such a disturbance is due to changes in the number of unstable eigendirections along a continuous trajectory under the influence of noise.

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The interplay between chaos and noise has been a topic of interest in nonlinear dynamics and statistical physics [1–7]. A main question in this area concerns how chaos can arise under the influence of random noise. The pioneering work of Crutchfield *et al.* [1] established that, in the common route to chaos via period-doubling bifurcations, noise tends to smooth out the transition and induce chaos in parameter regimes where there is no chaos otherwise. The observability and scaling of fractal structures near the transition to chaos in random maps were addressed in Ref. [4]. Features of transition to chaos in noisy dynamical systems, such as intermittency and a smooth behavior of the Lyapunov exponents, were also found in the transition from strange nonchaotic to strange chaotic attractors in quasiperiodically driven systems [8] and in the bifurcation to chaos with multiple positive Lyapunov exponents in high-dimensional systems [9]. Recently, it was demonstrated [6] that the transition is closely related to the problem of noise-induced synchronization in chaotic systems [5]. Noise-excited chaos is also an important phenomenon in the dynamics of epidemic outbreaks [7].

The aim of this Letter is to address the characteristic features of transition to chaos in *continuous-time* random dynamical systems. A situation of interest [7] is where the system possesses a regular attractor, coexisting with a nonattracting chaotic saddle, as can arise when the deterministic system is in a periodic window, or in a period-doubling parameter region (not in a window) where the stable and unstable manifolds of some unstable periodic orbits can become tangent and then form a homoclinic or heteroclinic tangle. In the latter case, the presence of small noise can induce the homoclinic or heteroclinic tangencies and create a *stochastic chaotic saddle* [7]. In the absence of noise, the largest asymptotic Lyapunov exponent of the system is zero. As the noise is turned on and its amplitude becomes sufficiently large, there is a nonzero probability that a trajectory originally on the attracting set escapes it and wanders near the coexisting chaotic saddle. In this case, the largest Lyapunov exponent becomes positive, signifying chaos for trajectories starting from typical initial conditions [10]. But what are the dynamical characteristics of the transition?

The principal results of this paper are as follows: (i) Noise leads to trajectories that move in phase-space regions with distinct numbers of unstable eigendirections, i.e., noise induces *unstable-dimension variability* [11,12]. As a result, the transition is smooth in the sense that the largest Lyapunov exponent becomes positive continuously from zero as the noise amplitude is increased. (ii) After the transition, the topology of the flow is disturbed in a fundamental way: There is no longer a zero Lyapunov exponent, indicating that, for noisy chaos, there exists no neutral direction along which infinitesimal distances are conserved, in sharp contrast to deterministic chaotic flows [13]. To be more specific, let D be the noise amplitude and D_c be the critical value of the noise amplitude required for the onset of chaos [14]. Then, for $D > D_c$, there exists no zero Lyapunov exponent. However, for D sufficiently larger than D_c, one Lyapunov exponent becomes increasingly close to zero. The topological destruction of the neutral direction of the flow is, therefore, most severe for $D \gtrsim D_c$. (iii) Quantitatively, the largest Lyapunov exponent versus the noise amplitude obeys the following scaling law, for $D \geq D_c$:

$$\lambda_1 \sim (D - D_c)^{\alpha}, \tag{1}$$

with the exponent $\alpha = 1 - \frac{1}{2\lambda_1^s \tau}$, where τ and λ_1^s are the lifetime and the largest Lyapunov exponent of the original chaotic saddle, measured in units of T and $1/T$, respectively, and T is the average time between successive crossings of the chaotic flow through a Poincaré surface of section. The topological aspect of the transition, which is unique for continuous-time flows, and the scaling law (1) have not been previously observed. An additional feature of our paper is that, under noise, there can be unstable-dimension variability even in low-dimensional chaotic systems possessing at most one positive Lyapunov exponent, in contrast to the common belief that this type of nonhyperbolicity should usually occur in high-dimensional systems [12].

The interplay between noise and dynamics can be visualized by focusing on three-dimensional autonomous flows and examining a proper Poincaré surface of section transverse to the direction of the flow, as shown schematically in Fig. 1, where there are a periodic attractor and a coexisting, nonattracting chaotic saddle in the phase space. The circular region surrounding the periodic attractor specifies the effective range of the influence of noise of amplitude D, which can be conveniently called the *noisy basin* of the attractor. For clarity, the stable and unstable manifolds of the chaotic saddle are represented by lines, although they too are fattened by noise. For $D < D_c$, there is no overlap between the stable manifold of the chaotic saddle and the noisy basin of the periodic attractor, as shown in Fig. 1(a). In this case, a random initial condition leads to a trajectory that is confined in the vicinity of the periodic attractor, although there can be transient chaos initially, in the sense that the trajectory may move toward the chaotic saddle along its stable manifold, wander near the saddle for a finite amount of time, and leave it along its unstable manifold. For $D > D_c$, a subset of the stable manifold of the chaotic saddle is located in the noisy basin of the periodic attractor, as seen in Fig. 1(b). As a result, there is a nonzero probability that a trajectory near the periodic attractor is kicked out of the noisy basin and moves toward the chaotic saddle along its stable manifold. Because of the nonattracting nature of the saddle, the trajectory can stay in its vicinity for only a finite amount of time before leaving along its unstable manifold and then enter the noisy basin of the periodic attractor again, and so on. The enlarged noisy attractor, which contains both the periodic attractor and the chaotic saddle, must be chaotic. This can be argued by examining the Lyapunov spectrum.

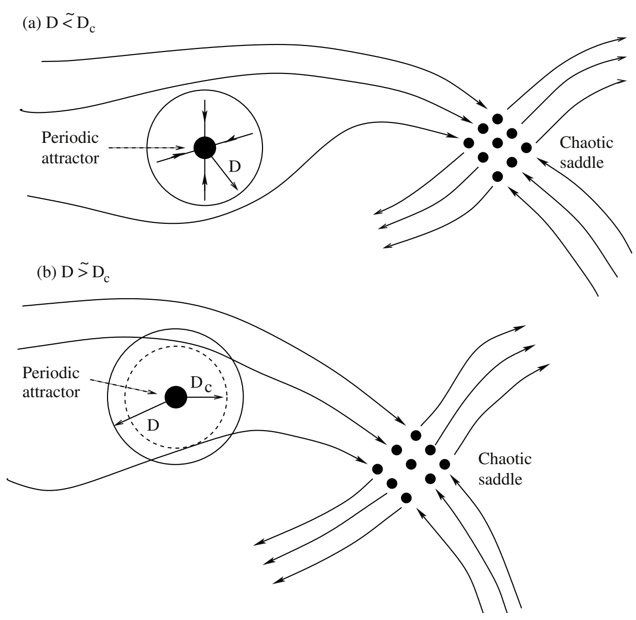

FIG. 1. Schematic illustration of the interplay between noise and dynamics on a Poincaré surface of section: (a) before and (b) after the transition to noisy chaos.

Let $\lambda_3^A \leq \lambda_2^A < \lambda_1^A = 0$ and $\lambda_3^S < \lambda_2^S = 0 < \lambda_1^S$ be the Lyapunov spectra of the periodic attractor and the chaotic saddle, respectively, in the absence of noise. Let $\lambda_3 < \lambda_2 < \lambda_1$ be the Lyapunov spectrum of the noisy attractor. For $D < D_c$, the noisy attractor is only a fattened version of the original periodic attractor. Thus, we have $\lambda_i = \lambda_i^A$ $(i = 1, 2, 3)$ [15]. In particular, there is still a null Lyapunov exponent, despite the presence of noise, indicating that the topology of the flow is preserved. For $D > D_c$, there is an intermittent hopping of the trajectory between regions that contain the original periodic attractor and the chaotic saddle. Let f_A and f_S be the fractions of time, asymptotically, that the trajectory spends in the corresponding regions. Then, we have

$$\begin{aligned} \lambda_1 &\approx f_A \lambda_1^A + f_S \lambda_1^S = f_S \lambda_1^S > 0, \\ \lambda_2 &\approx f_A \lambda_2^A + f_S \lambda_2^S = f_A \lambda_2^A < 0, \\ \lambda_3 &\approx f_A \lambda_3^A + f_S \lambda_3^S < 0. \end{aligned} \tag{2}$$

The remarkable feature is that the Lyapunov spectrum of the noisy attractor now contains no null exponent. Thus, immediately after the noise amplitude exceeds the critical value D_c, the noisy attractor becomes chaotic in the sense that its largest Lyapunov exponent is positive. This chaotic attractor is, however, fundamentally different in its flow topology from any deterministic chaotic attractors in that it no longer contains a neutral direction. We stress that this topological disturbance of the flow exists only for $D > D_c$. For $D < D_c$, the neutral direction of the flow is well preserved, despite the influence of noise.

At a fundamental level, the destruction of the neutral direction of the flow accompanying the onset of noisy chaos can be understood by noticing that the noisy chaotic attractor possesses unstable-dimension variability. In particular, the original periodic attractor contains no unstable direction, and the chaotic saddle possesses one unstable dimension. The role of noise, when it is sufficiently large ($D > D_c$), is to link these two dynamical invariant sets with distinct unstable dimensions. To see how unstable-dimension variability destroys the neutral direction, we examine the local eigenplanes that contain the neutral direction of the flow associated with the periodic attractor and the chaotic saddle, as shown schematically in Fig. 2. In the local eigenplane about the periodic attractor, there is a stable direction and a neutral direction. Let $\mathbf{v}$ be the eigenvector in the neutral direction. In the eigenplane of a point in the chaotic saddle, there is an unstable direction and a neutral direction. When a trajectory is driven by noise from the periodic attractor to the chaotic saddle along

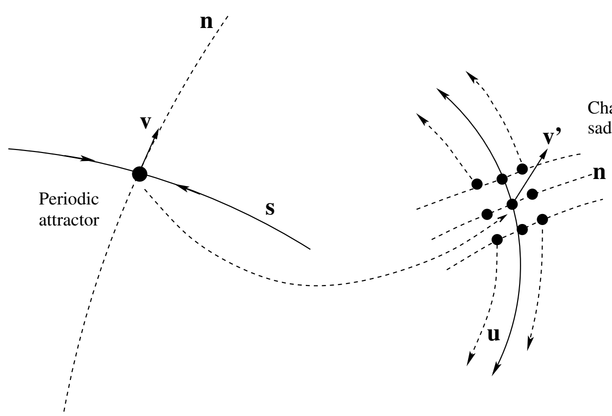

FIG. 2. Schematic illustration of the destruction of the neutral direction of the noisy chaotic flow due to unstable-dimension variability. The local planes about the periodic attractor and a point in the chaotic saddle do not coincide in general, and these planes are not in a Poincaré surface of section.

the stable manifold of the saddle, the eigenvector $\mathbf{v}$ maps to $\mathbf{v}'$ (see Fig. 2) and the vector $\mathbf{v}'$ can lie anywhere in the local eigenplane of the corresponding point in the chaotic saddle. After an amount of time, the vector will be aligned in the unstable direction, due to the expanding dynamics of the chaotic saddle. Distances along the neutral direction of the original periodic attractor can no longer be preserved. In general, a neutral vector associated with an invariant set can no longer be a neutral one, when the trajectory that "carries" the vector moves to another invariant set with more unstable directions. Thus, we see that unstable-dimension variability plays a fundamental role in determining the topology of the flow.

Based on the above picture, the scaling law (1) can be derived, as follows: From Eq. (2), we see that the largest Lyapunov exponent of the noisy chaotic attractor is proportional to f_S, the probability that a trajectory is kicked out of the noisy basin of the original periodic attractor and wanders near the chaotic saddle. This probability is proportional to the natural measure of the stable manifold of the chaotic saddle in the noisy basin of the periodic attractor, which is determined by the dimension of the stable manifold. For a two-dimensional ball of size ϵ on the Poincaré surface of section, the natural measure of the stable manifold [16] is proportional to $\epsilon^{D_s} = (\epsilon^2)^{D_s/2}$, where ϵ^2 is proportional to the area of the ball, D_s is given by $D_s = 2 - 1/(\lambda_1^S \tau)$, and τ is the lifetime of the chaotic saddle of the Poincaré map (τ is thus in the unit of T, the average time that a typical trajectory crosses the Poincaré section). From Fig. 1(b), we see that, for $D \gtrsim D_c$ [or $(D - D_c) \ll 1$], the area in which the stable manifold of the chaotic saddle penetrates the noisy basin of the periodic attractor is proportional to $(D^2 - D_c^2)$. We thus have $f_S \sim (D^2 - D_c^2)^{D_s/2} \sim (D - D_c)^{1-1/(2\lambda_1^S \tau)}$, which is the scaling relation (1).

We now provide numerical support with well-studied chaotic flows. Our first example is the Rössler system [17]: $dx/dt = -y - z + D\xi_x(t)$, $dy/dt = x + 0.2y + D\xi_y(t)$, and $dz/dt = 0.2 + z(x - c) + D\xi_z(t)$, where c is a bifurcation parameter and $\xi_{x,y,z}$ are independent Gaussian random variables of zero mean and unit variance [18]. We choose $c = 5.3$ so that the system is in a period-3 window. The set of stochastic differential equations is integrated by utilizing the standard second-order Milshtein method [19]. Figure 3(a) shows the first two Lyapunov exponents of the asymptotic attractor versus the noise amplitude D, where we identify the critical noise level for the onset of chaos: $D_c \approx 10^{-2.26}$. The absence of the null Lyapunov exponent for $D > D_c$ is unequivocal, indicating that the topology of the noisy flow for $D > D_c$ is fundamentally different from that for $D < D_c$ or $D = 0$ (the deterministic case). For $D \gtrsim D_c$, the scaling of the largest Lyapunov exponent of the noisy chaotic attractor is shown in Fig. 3(b), which is apparently algebraic. A least-squares fit between $\log_{10}\lambda_1$ and $\log_{10}(D - D_c)$ gives the slope of 0.94 ± 0.03. To determine the theoretical slope α, we use a large number of noiseless transient chaotic trajectories on the original chaotic saddle to determine λ_1^S and τ. We obtain $\lambda_1^S \approx 0.35$ and $\tau \approx 24.1$, which gives $\alpha \approx 0.94$. We see that there is an excellent agreement between the theoretical scaling law (1) and numerics.

We next consider the Lorenz system [20]: $dx/dt = 10(y - x) + D\xi_x(t)$, $dy/dt = \gamma x - y - xz + D\xi_y(t)$, and $dz/dt = -8/3z + xy + D\xi_z(t)$, where γ is the bifurcation parameter. Choosing $\gamma = 92.8$ results in a period-6 window in the Lorenz system. Figure 4(a) shows the first two Lyapunov exponents of the noisy attractor versus the noise amplitude D, where a behavior similar to that in Fig. 3(a) is observed. The transition to noisy chaos occurs at $D_c \approx 10^{-1.46}$, where the neutral direction of the flow is destroyed for $D > D_c$. The scaling of the largest Lyapunov exponent versus $(D - D_c)$ is shown in Fig. 4(b), where the numerical slope is 0.97 ± 0.03. The theoretical slope is computed to be approximately

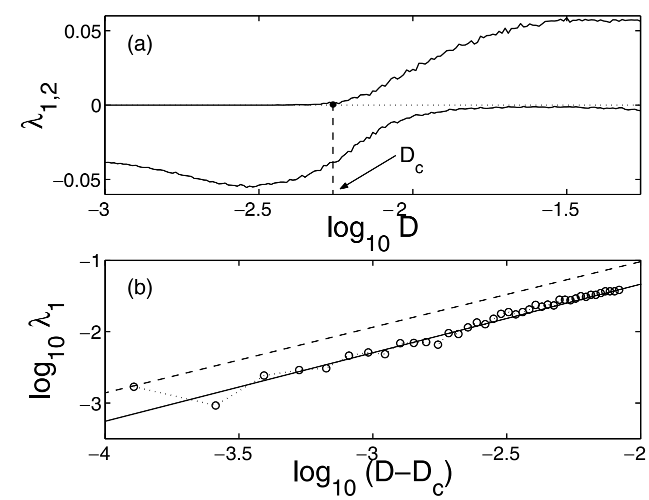

FIG. 3. For the Rössler system: (a) the first two Lyapunov exponents near the transition to noisy chaos; (b) algebraic scaling of the largest Lyapunov exponent after the onset of chaos. The dashed line indicates the theoretical slope.

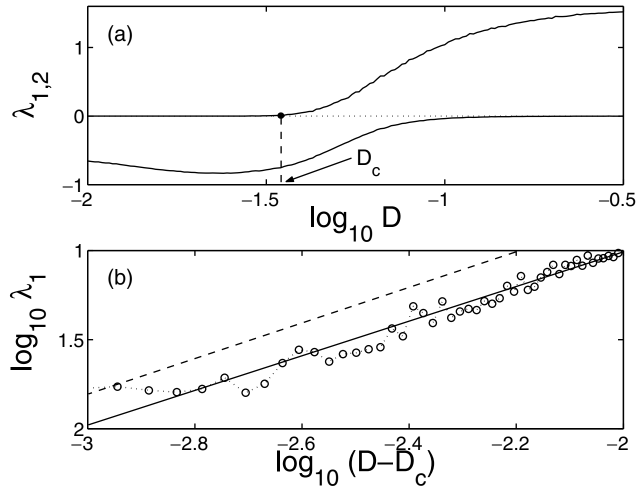

FIG. 4. For the Lorenz system: (a) the first two Lyapunov exponents near the transition to noisy chaos; (b) algebraic scaling of the largest Lyapunov exponent after the onset of chaos. The dashed line indicates the theoretical slope.

0.99 ($\lambda_1^S \approx 0.59$ and $\tau \approx 75.0$). Again, there is good agreement between the numerics and the theoretical scaling law (1).

In summary, we have uncovered features associated with noise-induced chaos in dynamical systems described by differential equations: The transition to chaos is accompanied by a destruction of the neutral direction of the flow. This topological change of the flow is argued to be due to unstable-dimension variability. We have also obtained a general scaling law of the largest Lyapunov exponent versus the noise amplitude, the validity of which is supported by numerical computations. The problem of noise-induced chaos is theoretically interesting and practically important for statistical physics, nonlinear science, and other disciplines such as computational biology as well, and we believe that we have provided some fundamental insights into the problem which were not noticed previously.

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[10] The Lyapunov exponents are the time-averaged stretching or contracting rates of infinitesimal vectors along a typical trajectory in the phase space, regardless of whether the system is deterministic or random.

[11] Unstable-dimension variability means that, along a typical trajectory, the number of local unstable directions can change. This clearly violates one of the basic defining conditions for hyperbolicity. The phenomenon was first conceived by R. Abraham and S. Smale [Proc. Symp. Pure Math. (AMS) **14**, 5 (1970)]. It was later realized that this type of nonhyperbolicity is fairly common in high-dimensional chaotic systems [12]. One of the results of our contribution is that noise can induce unstable-dimension variability even in low-dimensional chaotic systems and, hence, it is expected to occur more often than previously thought in realistic situations.

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[14] The value of D_c depends on the details of the systems.

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