1996

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Published Citation

Billings, L., & Curry, J. H. (1996). On noninvertible mappings of the plane: Eruptions. Chaos, 6(2), 108-120. doi:10.1063/1.166158

On Noninvertible Mappings of the Plane: Eruptions

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April 30, 1996

Abstract

This article is concerned with the dynamics of noninvertible transformations of the plane. Three examples are explored and possibly a new bifurcation, or "eruption", is described. A fundamental role is played by the interactions of fixed points and singular curves. Other critical elements in the phase space include periodic points and an invariant line. The dynamics along the invariant line, in two of the examples, reduces to one dimensional Newton's method which is conjugate to a degree two rational map. We also determine, computationally, the characteristic exponents for all of the systems. An unexpected coincidence is that the parameter range where the invariant line becomes neutrally stable, as measured by a zero Lyapunov exponent, coincides with the merging of a periodic point with a point on a singular curve.

1 Introduction

"Bairstow's Methods" are best known as numerical tools that find roots of high degree polynomials. Developed by Leonard Bairstow in 1916 [3], they decompose polynomials into products of quadratic factors, thus finding roots without using complex numbers. The methods define algorithms that have fixed points corresponding to quadratic factors. While no general proof of convergence to these fixed points exists. There is value in exploring these mappings because of their richness as dynamical systems.

This article focuses on families of noninvertible maps and the role singularities play in their dynamics. Such maps always present themselves whenever Newton's method is used to find the stationary solutions to an evolution equation. As we will see, there is an "eruption", possibly a new type of bifurcation, that produces a large locally attracting set that is an attractor in the sense of Milnor [19]. Noninvertible dynamical systems, in general, have not been widely studied. However, let us note in this regard, the ground-breaking work of C. Mira and his colleagues [20] and [11] (see specifically [11] for extensive references). Recent work has been carried out by E. N. Lorenz [17], Y. Kevrekidis [1], and R. McGehee [18].

We will be concerned, for the most part, with Bairstow's methods applied to the normalized one-parameter family of polynomials, $P_a(x)$, given by $x^3 + (a-1)x - a$. This family includes all cubic polynomials having real coefficients except x^3. This family makes the phenomenon presented, and the dynamics, somewhat more accessible than a higher degree family of polynomials, however, we will also consider a subset of a quintic family of polynomials $x^5 + ax + b$. For this second family we will only explore a one-parameter slice through the space of quintic polynomials.

*Supported in part by DOE grant DOE DE-FG03-94ER25194, and ONR grant ONR N00014-92-J-1274

The next two sections of this article focus on some of the critical elements and qualitative properties of the family of mappings, B_a. We end those sections with a fairly lengthy discussion and summary of our findings for the given family. The paper then proceeds with the introduction and discussion of two more related transformations. In section eight we present computations of the characteristic exponents for the three maps that we focused on. Section nine is a general discussion of our findings for all three maps. In the appendix we provide explicit derivations of the maps from the factorization methods that were discovered by Bairstow and Grau.

2 Qualitative Properties of $\mathbf{B_a}$

Bairstow's method (B_a) for factoring $P_a(x)$ produces the following noninvertible transformations mapping the plane into itself:

$$B_a \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2u^2+v} \begin{pmatrix} u^3 + u(v-a+1) + a \\ v(u^2+a-1) + 2au \end{pmatrix}, \tag{1}$$

where $P_a(x) = x^3 + (a-1)x - a$. This transformation is designed to find u and v such that $x^2 + ux + v$ is a factor of $P_a(x)$. For a detailed derivation, see Appendix A.1.

Our dynamical systems are noninvertible, a consequence of using Netwon's method. This characteristic is clear from the Jacobian matrix defining the two-dimensional iterative method. It has determinant zero when $2u^2 + v = 0$. Let us also note the preimages of points under the action of B_a are determined by the two transformations:

$$B_a^+ \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u + \sqrt{u^2-v+a-1} \\ \frac{(u+\sqrt{u^2-v+a-1})(a-uv-v\sqrt{u^2-v+a-1})}{-u^2+v-a+1-u\sqrt{u^2-v+a-1}} \end{pmatrix}, \tag{2}$$

$$B_a^- \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u - \sqrt{u^2-v+a-1} \\ \frac{(u-\sqrt{u^2-v+a-1})(a-uv+v\sqrt{u^2-v+a-1})}{-u^2+v-a+1+u\sqrt{u^2-v+a-1}} \end{pmatrix}. \tag{3}$$

B_a has several prominent features, detailed below.

2.1 Singularities

An obvious aspect of the map B_a is the singular set of $\{(u,v)|\ 2u^2+v=0\}$. While we still do not fully understand the role played by the singular set for noninvertible maps in guiding the overall dynamics, we will see in a later section that when fixed points and singular sets merge, the consequences can be unexpected and lead to, possibly, new dynamical behaviors. (See [10] for an algebraic characterization of singular sets for Bairstow's method.)

2.2 Invariant lines

Perhaps the most interesting phenomenon of these maps is the presence of invariant lines. Such lines are associated with the existence of linear factors in the underlying polynomial. (See [7], [4] and [5] for more details.) To explore the dynamical behavior on these lines, specific values of a must be chosen. As the parameter a increases through the value $\frac{1}{4}$, the number of invariant lines drops from three to two to one. But here, it is sufficient to note that one invariant line, $v = -u - 1$, is always present for B_a whether fixed points lie on it or not. Understanding the dynamical behavior on this line is a focus of this article.

2.3 Fixed Points

The fixed point set consists of the following three phase space points:

$$r_1 = \begin{pmatrix} 1 \\ a \end{pmatrix}, \ r_2 = \begin{pmatrix} \frac{-1+\sqrt{1-4a}}{2} \\ \frac{-1-\sqrt{1-4a}}{2} \end{pmatrix}, \text{ and } r_3 = \begin{pmatrix} \frac{-1-\sqrt{1-4a}}{2} \\ \frac{-1+\sqrt{1-4a}}{2} \end{pmatrix}.$$

Any phase point on the vertical line through a fixed point will collapse to that fixed point in one iterate. Also note that r_2 and r_3 are real for values $a \leq \frac{1}{4}$.

Stability of the the fixed points is determined by examining the eigenvalues of the Jacobian matrix of (1). The Jacobian matrix of B_a is:

$$J_{B_a}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{2u^4+(2a-2+v)u^2-4au+(1-a)v+v^2}{(2u^2+v)^2} & \frac{(a+u+u^2)(u-1)}{(2u^2+v)^2} \\ \frac{2(-2au^2+(2v-2av+v^2)u+av)}{(2u^2+v)^2} & \frac{2u(a+u+u^2)(u-1)}{(2u^2+v)^2} \end{pmatrix}. \tag{4}$$

An immediate conclusion is that the fixed points of B_a are contractive when they exist since all entries of the Jacobian matrix are identically zero when evaluated at such points.

3 An "Eruption" of Periodic Points for $\mathbf{B_a}$

In what follows, we focus on the dynamical behavior only for $a > \frac{1}{4}$. (For a study of the parameter $a < \frac{1}{4}$ see [5].)

3.1 $\mathbf{B_a}$ Restricted to the Invariant Line

A simple exercise shows that the restriction of Bairstow's method to the invariant line $v = -u - 1$ is a one-dimensional Newton's method applied to the polynomial $u^2 + u + a$ (see [24] or [26]). When a exceeds $\frac{1}{4}$, and initial conditions are chosen on this line, the iteration does not converge. An analogous case for one-dimensional Newton's method would be attempting to determine the $\sqrt{-1}$ by iterating from a "real" initial condition. While this reduction to one dimensional Newton's method makes some of the dynamics more sensible, it is imperative that we keep in mind that we are studying a two-dimensional noninvertible dynamical system.

It is possible to linearize the iteration in a neighborhood of the invariant line. Making the substitution $v \to -u - 1$, we have for B_a and its Jacobian matrix:

$$B_a\begin{pmatrix} u \\ -u-1 \end{pmatrix} = \begin{pmatrix} \frac{u^2-a}{2u+1} \\ \frac{-u^2-2u-1+a}{2u+1} \end{pmatrix} \tag{5}$$

$$J_{B_a}\begin{pmatrix} u \\ -u-1 \end{pmatrix} = \frac{a+u+u^2}{(2u+1)(2u^2-u-1)}\begin{pmatrix} 2u-1 & 1 \\ 2 & 2u \end{pmatrix}. \tag{6}$$

The eigenvalues of the linearization at any point u along the line:

$$\lambda_1(u) = \frac{a+u+u^2}{(2u+1)(u-1)} \quad \text{and} \quad \lambda_2(u) = \frac{2(a+u+u^2)}{(2u+1)^2}.$$

When a exceeds $\frac{1}{4}$ there is a bifurcation, or an "eruption", which produces an infinite number of periodic and aperiodic orbits. In the following sections, we will provide numerical evidence of this, and then show that B_a is conjugate on the invariant line to a piecewise linear mapping of the interval. These observations will guide us in understanding the global effects of the "eruption".

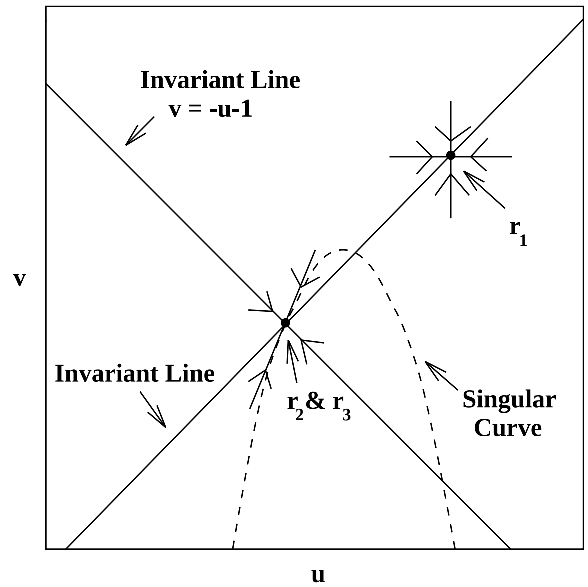

Figure 1: A schematic of the two fixed points of B_a merging into the singularity at $(-\frac{1}{2}, -\frac{1}{2})$ when the parameter $a = \frac{1}{4}$. Included are the invariant lines and the singular curve $2u^2 + v = 0$.

3.2 Eigenvalues and Eigenvectors for the critical Fixed Point

Our analysis of B_a restricted to the invariant line continues with the consideration of the behavior of r_2 and r_3 as they merge into a point that is on the singular curve $2u^2+v=0$. (See Figure 1 for a schematic diagram.) The eruption occurs at $(-\frac{1}{2}, -\frac{1}{2})$ when the parameter a just exceeds $\frac{1}{4}$. The other stationary solution, r_1, apparently plays no role in the eruption. A linear stability analysis at the merged fixed points shows that the two eigenvalues are $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = 0$. This confirms that the mapping is still locally a contraction. Examining the resultant eigenvectors shows that one eigenvector lies along the invariant line $v = -u - 1$ and the other one has a slope of two. From (6) and $u = -\frac{1}{2}$, we obtain the eigenvectors $e_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

3.3 Periodic Points On the Invariant Line

In this subsection we continue to analyze the dynamics in a neighborhood of the invariant line by examining some of the low order periodic points. The location of these orbits can be determined explicitly, leading to a better understanding of the nature of the "eruption" that produces them. (These orbits only exist when $a > \frac{1}{4}$.)

3.3.1 Period Two

The single period two cycle, defined by the formulas:

$$Per2_1(a) = -\frac{1}{2} - \frac{\sqrt{4a-1}}{2\sqrt{3}} \quad \text{and} \quad Per2_2(a) = -\frac{1}{2} + \frac{\sqrt{4a-1}}{2\sqrt{3}},$$

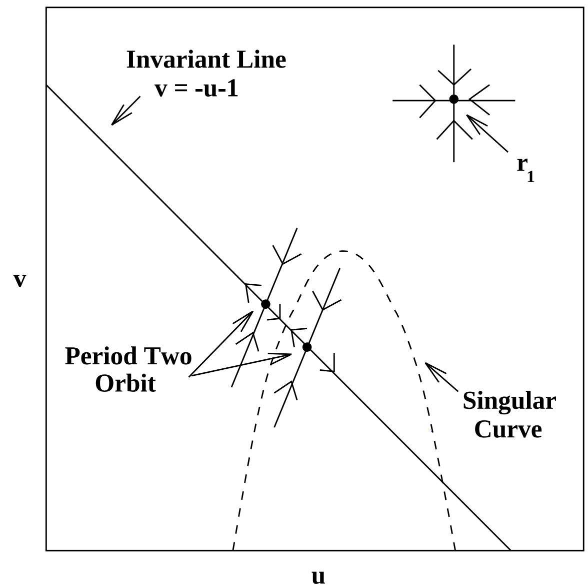

Figure 2: A schematic showing the period two points when the parameter $a > \frac{1}{4}$. Included are the invariant line and the singular curve $2u^2 + v = 0$.

has eigenvalues $\lambda_1 = 4$ and $\lambda_2 = \frac{-1+4a}{a-7}$. These formulas indicate that the period two cycle is "born" with a constant λ_1, associated with the eigenvector lying along the invariant line, while λ_2, associated with the transverse eigenvector, approaches 0 as a approaches $\frac{1}{4}$. Repeating the eigenvector calculation, the transverse eigenvector has a slope of 2 for any value of a greater than $\frac{1}{4}$. Recall that this was also the slope of the transverse eigenvector associated with the merged fixed point at the critical parameter value $a = \frac{1}{4}$. (See Figure 2 for a schematic.)

3.3.2 Period Three

There are two periodic orbits of period three. A tabulation of the behavior of the eigenvalues is summarized in Table 1. The eigenvalue associated with the eigenvector lying along the invariant line is λ_1 and has a constant value of 8 for both orbits. It is independent of a. The eigenvalue associated with the transverse eigenvector, λ_2, depends on a and tends to zero as a approaches $\frac{1}{4}$ for both orbits.

The transverse eigenvectors have a slope of 2 for both period three orbits, as was also noted for the period two orbit. This leads us to speculate that the local behavior of the stable manifold, in the neighborhood of an orbit of period n is inherited from the merged fixed point. Further, the period four orbits also mimicked the prominent characteristics of the period two and three orbits.

3.4 A Conjugacy on the Invariant Line

In the previous sections we have provided evidence that there is an "eruption" which produces periodic orbits. This is not only reasonable, but fact, since we show in this section there is a conjugacy between B_a and a well known degree two rational mapping when the dynamics are restricted to the invariant line.

a	Orbit 1 λ_1	λ_2	Orbit 2 λ_1	λ_2
0.3	8.	-0.007077	8.	0.01592
0.29	8.	-0.005233	8.	0.01076
0.28	8.	-0.003531	8.	0.006571
0.27	8.	-0.002014	8.	0.003335
0.26	8.	-0.0007588	8.	0.001082
0.255	8.	-0.0002811	8.	0.000361
0.254	8.	-0.0002036	8.	0.0002546
0.253	8.	-0.0001341	8.	0.0001627
0.252	8.	-0.00007421	8.	0.00008691
0.251	8.	-0.00002682	8.	0.00003000

Table 1: Table of eigenvalues for the period three orbits on the invariant line $v = -u - 1$ in the map B_a. λ_1 is constant and λ_2 approaches zero as a approaches $\frac{1}{4}$.

Denote the u component of B_a restricted to the invariant line, (5), by the function $L_B : \mathbb{R} \to \mathbb{R}$,

$$L_B(u) = \frac{u^2 - a}{2u + 1},$$

and $a > \frac{1}{4}$. Use the following transformation,

$$f(z) = \frac{z\sqrt{4a - 1} - 1}{2},$$

to determine the conjugacy:

$$f^{-1} \circ L_B \circ f(z) = \frac{1}{2}\left(z - \frac{1}{z}\right).$$

This function is conjugate to the map $g(x)$ on the interval $I : [-\frac{\pi}{2}, \frac{\pi}{2}]$ where,

$$g(x) \quad = \quad \begin{cases} 2x + \frac{\pi}{2}, & x \in \left[-\frac{\pi}{2}, 0\right] \\ 2x - \frac{\pi}{2}, & x \in \left[0, \frac{\pi}{2}\right] \end{cases} \quad . \tag{7}$$

$g(x)$ is conjugate to $L_B(u)$ for all values of $a > \frac{1}{4}$. The dynamics of $g(x)$ are well understood from both a topological and measure theoretical point of view. Further, as long as the derivative of g exceeds one in absolute value, a result due to Bowen [6] allows us to conclude that there is an invariant ergodic measure associated with the transformed family of mappings.

3.5 Discussion of the dynamics of B_a

The behavior of B_a as a exceeds $\frac{1}{4}$ is somewhat more sensible after an examination of the low order periodic orbits, their associated eigendata, and the existence of a conjugating map. For a just slightly above $\frac{1}{4}$ the formulas for the periodic orbits and both numerical and symbolic computation indicate that there is an "eruption" of periodic and aperiodic orbits all of which emerge from a point on the singular curve. The periodic orbits are "born" with a zero eigenvalue associated with the eigenvector that is transverse to the invariant line $v = -u - 1$. The existence of infinitely many periodic orbits on the invariant line is assured because there is a conjugacy between one dimensional Newton's method

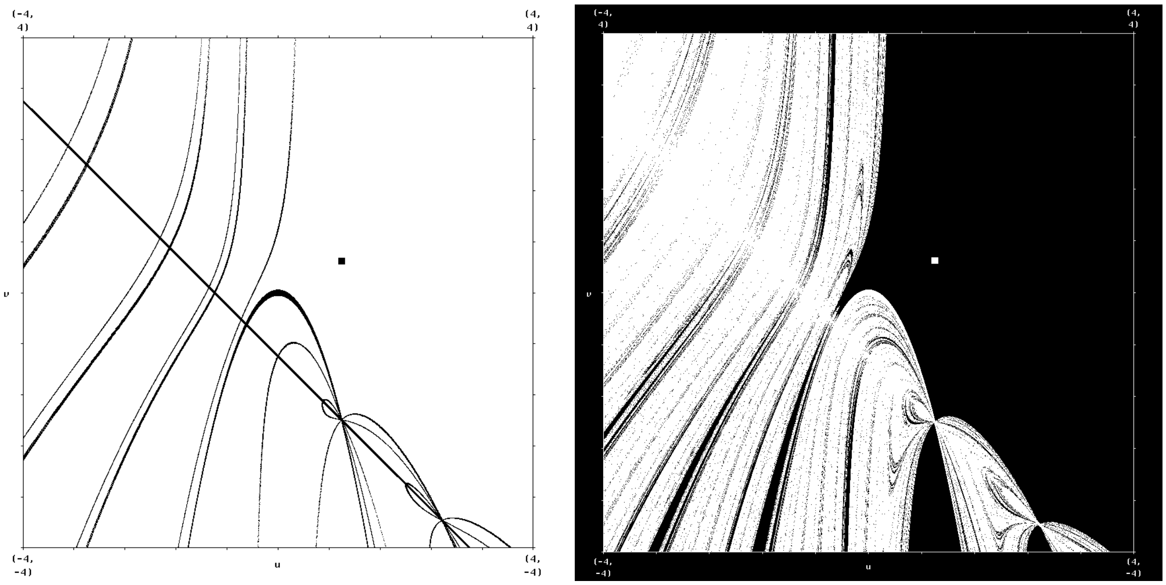

Figure 3: The left picture shows the first few preimages of the singular curve for the map $B_{0.5}$. The right picture is the basin map associated with $B_{0.5}$. The white points have not converged to the fixed point $(1, 0.5)$ in 200 iterations. It is clearly visible that the preimages outline the regions of points in the basin map that do not converge to the fixed point.

(a degree two rational mapping) and a piecewise linear transformation of an interval presented in the previous subsection.

We have not yet mentioned the role played by preimages of the singular curve. Their role is more evident from the images displayed in Figure 3. The left image displays the singular curve and a few of its preimages. Recall that each point has two preimages, defined by (2) and (3), so a binary tree forms as we iterate the map backwards. The basin map on the right shows a black region corresponding to points converging to the fixed point located at $(1, a)$ and a white region that is attracted to the invariant line $v = -u - 1$.

When the images in the figure are juxtaposed, we see that components of the basin of attraction of the only attractive fixed point touch the invariant line and are bounded by preimages of various components of the singular curves. In particular, the immediate basin boundary of the attractive fixed point is composed of such curve segments. These components, may make, the basin for the fixed point "riddled" [2] because any neighborhood containing points which converge to the invariant line also seem to contain points that must converge to the fixed point.

We also note that as the parameter increases the density of preimages of the singular curve in any region appears to decrease. (See Figure 4.) Why this happens is not clear. But we do have an explanation of the decrease in density of points converging to the invariant line, (see the section on characteristic exponents.)

We were surprised that the eigenvalues associated with the transverse eigenvector should approach zero as a approaches $\frac{1}{4}$. Further, we have not seen this type of bifurcation or "eruption," one apparently coming from a singularity, reported in the literature. While we have presented only limited evidence, we conjecture that at the "parameter of birth" all of the periodic orbits on the invariant line come into existence having a zero eigenvalue, which guarantees that the periodic points are all hyperbolic and that the invariant line is locally an attractor in the sense of Milnor.

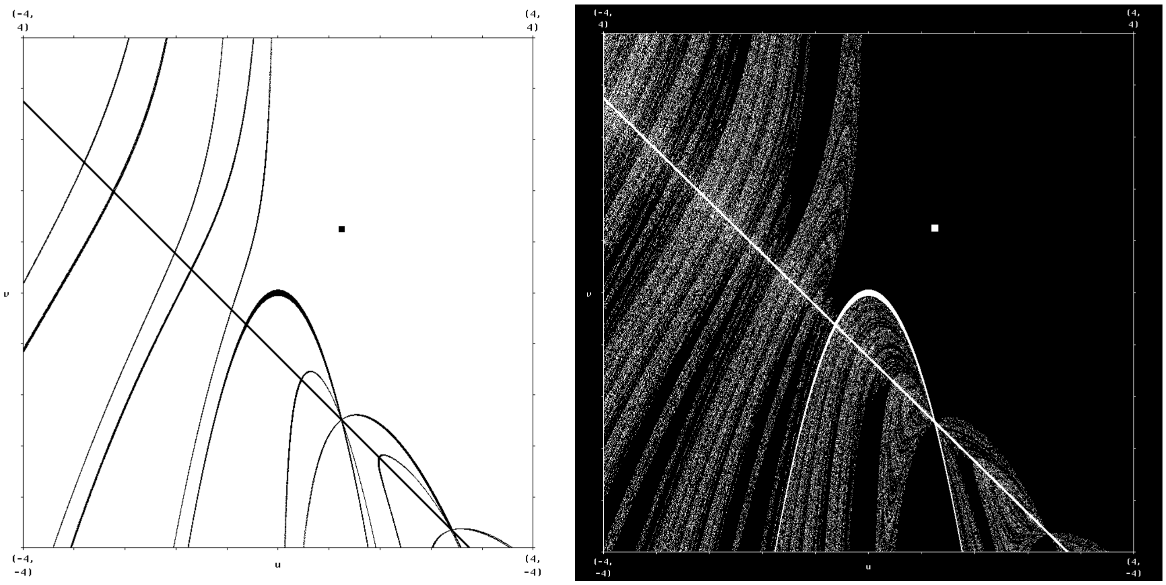

Figure 4: The left image shows the first few preimages of the singular curve for the map $B_{1.0}$. The right picture is the basin mapping associated with $B_{1.0}$. The white points have not converged to the fixed point $(1, 1)$ in 200 iterations. The preimages still outline the region of points in the basin map that do not converge to the fixed point. Notice the change of density in the basins.

There is also evidence that further bifurcations of the periodic orbits occur as a increases. It particular, the period two orbit bifurcates into a new period four orbit (a source) at $a = \frac{8}{5}$. This new cycle does not lie on the invariant line. We will not study such bifurcations further in this article.

The dynamics along the invariant line are sensible because we know that that map is conjugate to a piecewise linear map. A deeper mystery is why the periodic orbits on the invariant line erupt as saddles. Indeed, it is certainly conceivable that the "eruption" could have produced sources or a neutrally stable invariant set. It is reasonable to conjecture that there is some type of "transfer of stability" involved where the fixed points that merge into the singular point $(-\frac{1}{2}, -\frac{1}{2})$ have transferred their stability to all the saddle periodic points along the invariant line.

We shall see, in the next sections, that the "eruption" is present in other factorization methods. Additional rich dynamical structure is also present so the discussion is not simply a repeat of what has already been said.

4 Qualitative Properties of M_a

Other strategies for generating polynomial factorization methods were developed by Grau [13]. The family which we will now consider is a generalization of Bairstow's method and shares some of its properties. Much of the analysis of this transformation parallels the discussion of B_a so we shall provide a more concise discussion than the one given for that mapping. Just as before, we shall focus on the dynamics as a exceeds $\frac{1}{4}$. For a derivation of this mapping we refer the reader to Appendix A.2. Unlike Bairstow's method, this iteration function restricted to the invariant line is not simply one dimensional Newton's method applied to $x^2 + x + a$. However, it does generate a degree two rational map when it

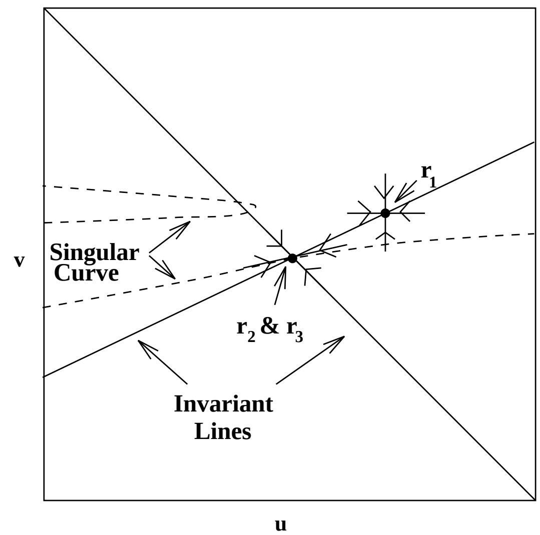

Figure 5: A schematic of the the singular curve, invariant lines, and fixed points when the parameter $a = \frac{1}{4}$.

is restricted to the invariant line. M_a transforms pairs of points as follows:

$$M_a\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{v^3 + auv + a^2}\begin{pmatrix} a(au + v - av + 2v^2) \\ v(2a^2 + auv - v^2 + av^2) \end{pmatrix}. \tag{8}$$

This map is also noninvertible. Its preimages are governed by the two transformations:

$$M_a^+\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{1-a+v}{u} + \frac{\sqrt{a}(u^2-v+a-1)}{u\sqrt{a-uv}} \\ \frac{a-\sqrt{a}\sqrt{a-uv}}{u} \end{pmatrix}, \tag{9}$$

$$M_a^-\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{1-a+v}{u} - \frac{\sqrt{a}(u^2-v+a-1)}{u\sqrt{a-uv}} \\ \frac{a+\sqrt{a}\sqrt{a-uv}}{u} \end{pmatrix}. \tag{10}$$

M_a has several prominent features similar to those present in the B_a family of maps.

4.1 Singularities and Invariant Lines

Members of this family has a variable singular set $\{(u,v)|\ u + \frac{a}{v} + \frac{v^2}{a} = 0\}$, which represents a curve with two branches. (See Figure 5.) Just as in the case of B_a there are invariant lines and one of them is always $v = -u - 1$; and other invariant lines connect two or more fixed points.

4.2 Fixed Points

The three stationary solutions corresponding to quadratic factors are identical to those of B_a,

$$r_1 = \begin{pmatrix} 1 \\ a \end{pmatrix},\ r_2 = \begin{pmatrix} \frac{-1+\sqrt{1-4a}}{2} \\ \frac{-1-\sqrt{1-4a}}{2} \end{pmatrix},\ \text{and } r_3 = \begin{pmatrix} \frac{-1-\sqrt{1-4a}}{2} \\ \frac{-1+\sqrt{1-4a}}{2} \end{pmatrix}.$$

Here we encounter a new phenomena, a line of neutrally stable fixed points corresponding to the u axis. So for $v = 0$, any point $(u, 0)$ maps to itself.

Convergence to r_1, r_2, and r_3 is much the same as before. However, there is an interchange of the roles played by horizontal and vertical lines through the fixed points. That is, any point on the horizontal line through a fixed point will map to that fixed point in one iteration, excluding the line of fixed points mentioned above.

We will use the Jacobian matrix in a neighborhood of the invariant line as a tool for analyzing the dynamics of M_a. The Jacobian is as follows:

$$J_{M_a}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{a^2(a^2-v^2+av^2-v^3)}{(a^2+auv-v^3)^2} & \frac{a^3(1-a-u^2+4v)-a^2uv^2+2av^3(a-1-v)}{(a^2+auv-v^3)^2} \\ \frac{av^2(-a^2+v^2-av^2+v^3)}{(a^2+auv-v^3)^2} & \frac{2a^4+2a^3uv+a^2v^2(u^2-3+3a-4v)+auv^3(2a-v-2)}{(a^2+auv-v^3)^2} \end{pmatrix}. \qquad (11)$$

5 An "Eruption" of Periodic Points for M_a

5.1 M_a Restricted to the Invariant Line

Along the invariant line $v = -u - 1$, M_a and the Jacobian (11) become

$$M_a\begin{pmatrix} u \\ -u-1 \end{pmatrix} = \frac{1}{-u^2-2u+a-1}\begin{pmatrix} a(1+2u) \\ (1+u)(1-2a+u) \end{pmatrix} \text{ and} \qquad (12)$$

$$J_{M_a}\begin{pmatrix} u \\ -u-1 \end{pmatrix} = \frac{a(a+u+u^2)}{(u+a+1)(-u^2-2u+a-1)}\begin{pmatrix} a & -2(u+1)-a \\ -(1+u)^2 & -u^2+2a+1 \end{pmatrix}. \qquad (13)$$

Unlike B_a, the u component of M_a restricted to the invariant line is not simply one dimensional Newton's method applied to $u^2 + u + a$. The neutrally stable fixed point on this line is located at $u = -1$, and two of the singularities on the line depend on a as follows: $u = -1 \pm \sqrt{a}$. From (13) we can find the eigenvalues at any point u along the invariant line from the formulas:

$$\lambda_1 = \frac{a(u^2+u+a)}{(u+a+1)(a-(u+1)^2)} \quad \text{and} \quad \lambda_2 = \frac{2a(u^2+u+a)}{(a-(u+1)^2)^2}.$$

5.2 Periodic Points On the Invariant Line

5.2.1 Period Two

The single period two cycle (for $a > \frac{1}{4}$) is always of the form

$$Per2_1(a) = \frac{2-5a+a\sqrt{3}\sqrt{-1+4a}}{2(a-1)} \quad \text{and} \quad Per2_2(a) = \frac{2-5a-a\sqrt{3}\sqrt{-1+4a}}{2(a-1)}$$

The eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = \frac{-1+4a}{a-7}$, just the same as in B_a. (See Figure 6.)

5.2.2 Period Three

Included in Table 2 are the eigenvalues associated with the two periodic orbits of period three. Notice that the eigenvalues for the corresponding periodic orbits are identical to those corresponding to the equivalent orbits for B_a. (See Table 1.) This was unexpected. The behavior of the eigenvalues for the periodic orbits of period four behave similarly and are also identical to the eigenvalues associated with the period four orbits for B_a.

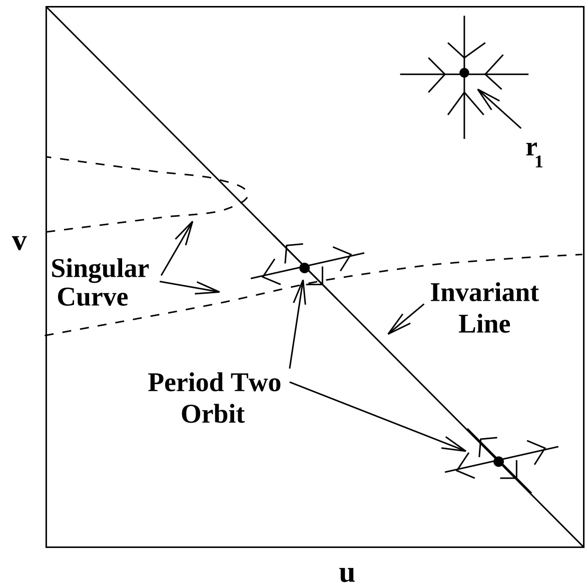

Figure 6: A schematic of the the singular curve, invariant line, and periodic points when the parameter $a = \frac{1}{2}$.

5.3 Another Conjugacy on the Invariant Line

In section (3.4) we proved there is a conjugacy between B_a and a degree two rational mapping when the dynamics are restricted to the invariant line. This is also true of M_a.

Denote the u component of M_a restricted to the invariant line, (12), as the function $L_M : \mathbb{R} \to \mathbb{R}$,

$$L_M(u) = \frac{a(2u+1)}{a-(u+1)^2},$$

and $a > \frac{1}{4}$. Use the following transformation,

$$h(z) = -1 + \frac{2a}{1+\sqrt{4a-1}\,(z)},$$

to conjugate L_M to:

$$h^{-1} \circ L_M \circ h(z) = \frac{1}{2}\left(z - \frac{1}{z}\right).$$

Thus, L_M is also conjugate to $g(x)$ defined in equation (7).

6 Discussion of the Dynamics of M_a

As a exceeds $\frac{1}{4}$ the dynamics of M_a parallels those of the family B_a, especially for periodic orbits and their associated eigenvalues. We also noted the existence of a persistent invariant line of neutrally stable fixed points. However, the key phenomena of study, the eruption of periodic and aperiodic orbits from the merged fixed points is present and again leads to a gross change in the behavior of the dynamical system. A further striking feature is presented in the tables of eigenvalues for the periodic orbits. We noted that the locations of the periodic orbits varies but the associated eigenvalues are the same as those

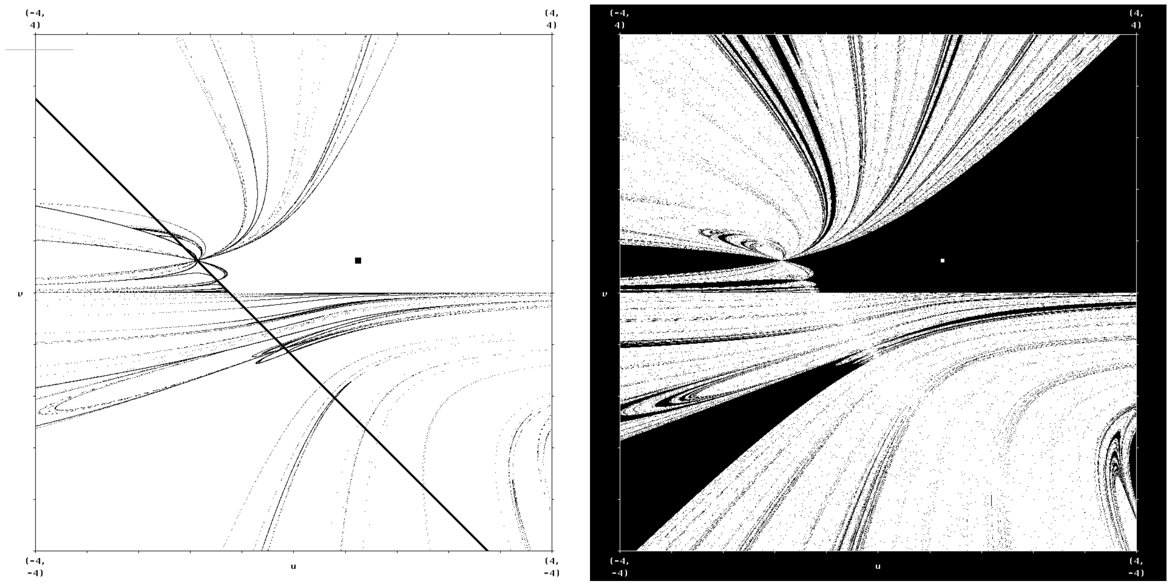

Figure 7: The left image shows the first few preimages of the singular curves for the map $M_{0.5}$. The right picture is the basin mapping associated with $M_{0.5}$. The white points have not converged to the fixed point in 200 iterations. The preimages still outline the region of points in the basin map that do not converge to the fixed point. Notice the change in density in the basins as reflected by those points converging to the fixed point.

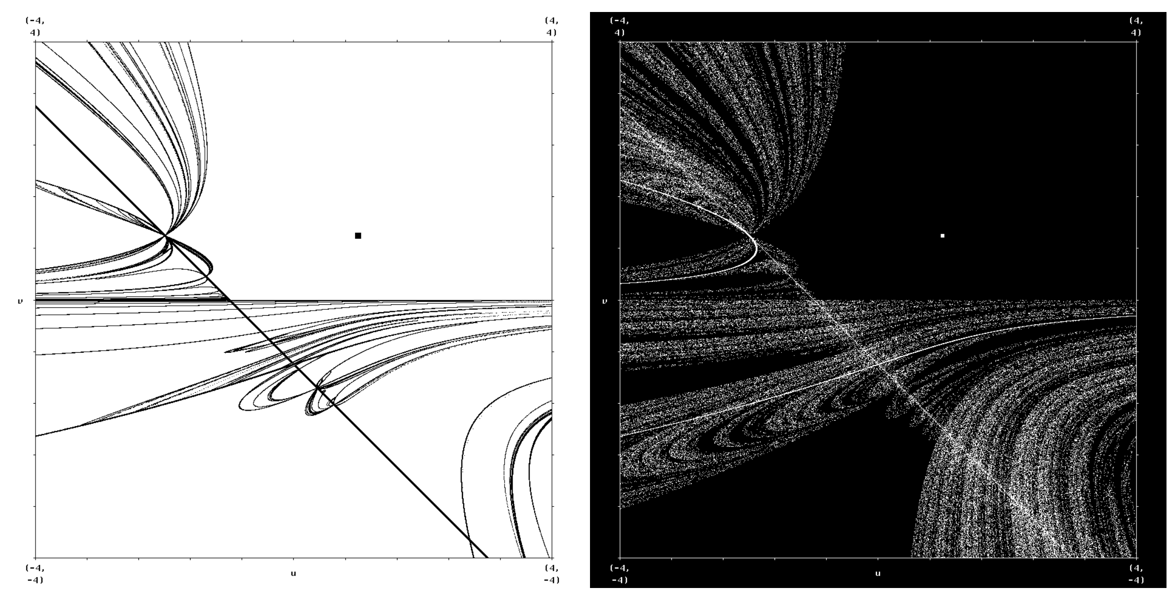

Figure 8: The left image shows the first few preimages of the singular curve for the map $M_{1.0}$. The right picture is the basin mapping associated with $M_{1.0}$. The white points have not converged to the fixed point in 200 iterations. The preimages still outline the region of points in the basin map that do not converge to the fixed point. Notice the change of density in the basins.

a	Orbit 1 $\lambda 1$	$\lambda 2$	Orbit 2 $\lambda 1$	$\lambda 2$
0.3	8.	-0.007077	8.	0.01592
0.29	8.	-0.005233	8.	0.01076
0.28	8.	-0.003531	8.	0.006571
0.27	8.	-0.002014	8.	0.003335
0.26	8.	-0.0007588	8.	0.001082
0.255	8.	-0.0002811	8.	0.000361
0.254	8.	-0.0002036	8.	0.0002546
0.253	8.	-0.0001341	8.	0.0001627
0.252	8.	-0.00007421	8.	0.00008691
0.251	8.	-0.00002682	8.	0.00003000

Table 2: Table of eigenvalues for the period three orbits on the invariant line $v = -u - 1$ in the map M_a. λ_1 is constant and λ_2 approaches zero as a approaches $\frac{1}{4}$.

determined for B_a and for the same parameter values. We also note that M_a restricted to the invariant line is conjugate to the same one dimensional map as B_a when restricted to its invariant line.

We can consider the preimages of singular curves. In the M_a family such preimages either accumulate at infinity or at the neutrally stable line of fixed points along the horizontal axes. But the behavior of the preimages of singular curves seems much more complicated than before.

7 Qualitative Properties of Q_a

The final family considered is the most complicated of all and our understanding of it is by no means complete. However, just as with the previous two families there are certain critical elements that we shall mention, including, fixed points, singular sets and the invariant line. We shall focus on the "eruption" discussed above. The new twist is the presence of an additional fixed point in the phase space. (See Figure 9).

In what follows we will be concerned with the one parameter slice through the family of quintic polynomials given below. For a derivation of Bairstow's method see appendix A.3 of this article. In what follows we will be concerned with the one parameter family

$$x^5 + ax^3 - ax^2 + (a-1)x - a.$$

The family has two natural factors:

$$(x^2+1)(x^3 + (a-1)x - a),$$

therefore one Bairstow fixed point always has coordinates $(0, 1)$.

7.1 Fixed Points

The fixed points of Q_a are

$$r_1 = \begin{pmatrix} 1 \\ a \end{pmatrix},\ r_2 = \begin{pmatrix} \frac{-1+\sqrt{1-4a}}{2} \\ \frac{-1-\sqrt{1-4a}}{2} \end{pmatrix},\ r_3 = \begin{pmatrix} \frac{-1-\sqrt{1-4a}}{2} \\ \frac{-1+\sqrt{1-4a}}{2} \end{pmatrix},\ \text{and } r_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

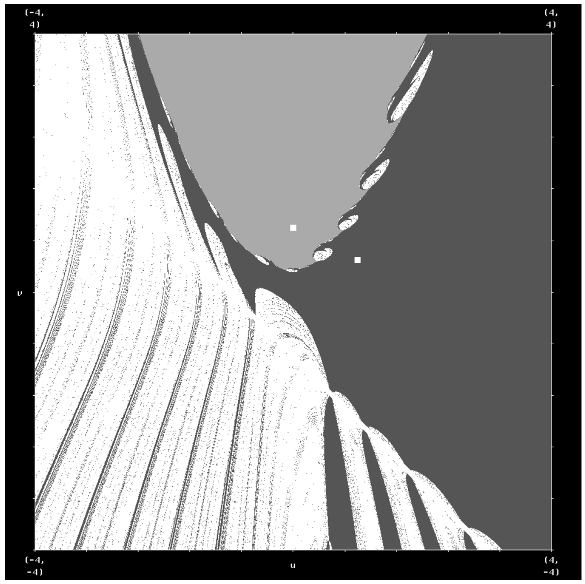

Figure 9: The basin map of $Q_{0.5}$ showing three distinct basins. One each for the fixed points and the invariant line.

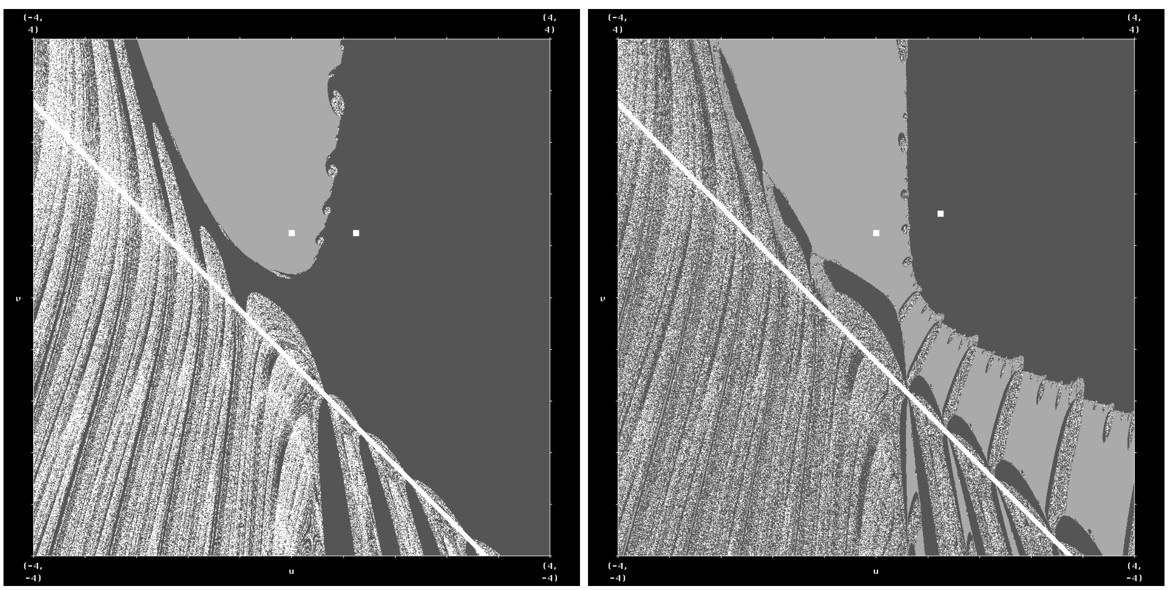

Figure 10: Two basin maps.The left is the basin map of $Q_{1.0}$ while the right is the basin map of $Q_{1.3}$ The light grey points converge to the fixed point $(0,1)$, the dark grey point converge to the fixed point $(1,a)$, and the white points do not converge in 200 iterations of the map. The second figure results after a collision between the basin of a fixed point and a point whose preimage is on the singular curve.

The equation defining the singular set is a sixth degree polynomial in u and v and is given in the appendix.

The Jacobian is too long, and dense, to write down in closed form, but restricting the map to the invariant line $v = -u - 1$ (the restriction is one dimensional Newton's method applied to the polynomial $(u^2+1)(u^2+u+a)$) simplifies the process of determining the parallel and transversal eigenvalues:

$$\begin{aligned}\lambda_1 &= \frac{(u^2+u+a)(u^2+2u+2)(3u^2+2u+a+2)}{(u^3+u^2+au+2u-2)(4u^3+9u^2+2au+8u+2a+2)} \\ \lambda_2 &= \frac{2(u^2+u+a)(u^2+2u+2)(6u^2+9u+a+4)}{(4u^3+9u^2+2au+8u+2a+2)^2}\end{aligned}$$

Table 3 shows the location and behavior of the eigenvalues for the period two cycle for the Q_a family. These data confirm our expectation that an "eruption" takes place, and it produces periodic points. Further, since Q_a is generated by Bairstow's method its restriction to the invariant line is one dimensional Newton's method applied to a quartic polynomial equation that has no real roots. So that iteration must also fail to converge to any point on the invariant line. We expect that there is a conjugacy between this restriction and a piecewise linear mapping on an interval, just as we saw for B_a, but we do not explore this possibility here.

a	p_1	p_2	λ_1	λ_2
0.3	-0.68056	-0.41686	-0.03296	4.22838
0.29	-0.65753	-0.42212	-0.02593	4.16178
0.28	-0.63210	-0.42885	-0.01908	4.10457
0.27	-0.60332	-0.43806	-0.01245	4.05803
0.26	-0.56863	-0.45240	-0.00608	4.02304
0.255	-0.54626	-0.46433	-0.00300	4.01002
0.254	-0.54086	-0.46763	-0.00240	4.00778
0.253	-0.53488	-0.47150	-0.00179	4.00566
0.252	-0.52798	-0.47627	-0.00119	4.00365
0.251	-0.51934	-0.48280	-0.00059	4.00177

Table 3: Table of eigenvalues for the period two orbits on the invariant line $v = -u - 1$ in the map Q_a. λ_2 approaches zero as a approaches $\frac{1}{4}$.

8 Characteristic Exponents of B_a, M_a, and Q_a

In a 1977 article D. Boyd [7] noted in his study of Bairstow's method that for a "modification" of that method, an invariant line exhibited a type of "stability." That is, an initial condition chosen in a neighborhood of the invariant line seemed to remain in a neighborhood of that line. In this section we revisit Boyds' observation, see also the paper of Fiala and Krebsz [10].

Since we have explicit formulas for both the transversal and parallel growth rates associated with the linearizations along the invariant line $v = -u - 1$ it is a straight forward numerical computation to determine the qualitative behavior of those two eigenvalues as a is varied. The results for the maps are presented in Figure 11 and Figure 12.

In two of the images the horizontal line across the top portion of the graph is the characteristic exponent associated with the rate of expansion along the invariant line. That value is constant and equal to $log(2)$, while in the third figure it fluctuates about that value.

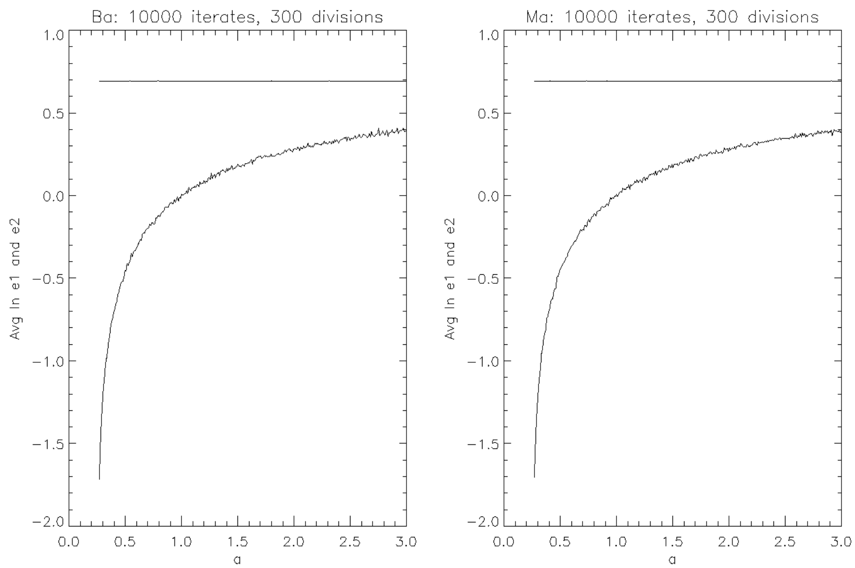

Figure 11: B_a and M_a characteristic exponents

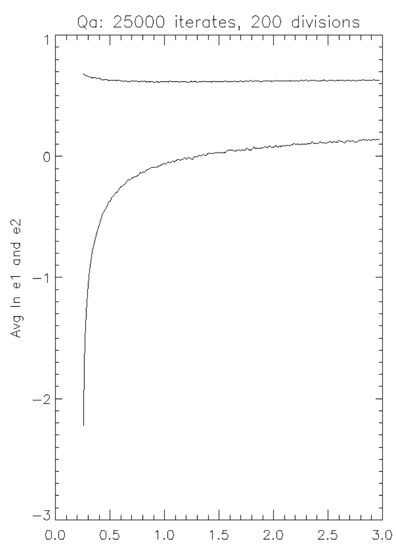

Figure 12: Q_a characteristic exponents

In all figures the other curve is associated with the transversal growth rate along an orbit and is roughly increasing. These curves indicate a decrease in stability of the invariant line to transverse perturbations. (Noted by Boyd.) As indicated in Table 4, there is a critical parameter value near $a = 1$, for both B_a and M_a, for which the invariant line is, on average, neutrally stable to transverse perturbations. However, a closer examination of one of these graphs also shows, possibly significant, fluctuations in the behavior of the transverse characteristic exponent.

a	B_a	M_a
0.963267	-0.009214	-0.011412
0.972400	-0.030905	-0.017269
0.981533	-0.010235	-0.005603
0.990667	-0.014621	-0.004807
0.999800	0.009469	0.007598
1.008933	0.002723	-0.005361
1.018067	0.002855	0.006396
1.027200	0.013738	0.021446
1.036333	0.012223	0.025175
1.045467	0.030567	0.019560

Table 4: Table of Lyapanov exponents for B_a and M_a in a neighborhood of $a = 1$.

The lost of stability to transverse perturbations does not necessarily imply that saddles no longer exist along the invariant line. For example, we know that the period two orbit, discussed for B_a, does not become a source until $a = 1.40$, well above the threshold value indicated for neutral stability. (This also applies to M_a.)

For Q_a the invariant line loses its stability at approximately $a = 1.30$. Notice also that the characteristic exponent on the invariant line fluctuates around $log(2)$. This means that while we constructed the family Q_a by simply adding one additional Bairstow quadratic factor, the results for the dynamics along the invariant line was to add, apparently, many new orbits that are not present in either the B_a or M_a families. However, even with this added complexity, we do have the same qualitative behavior, an eruption, as a exceeds $\frac{1}{4}$.

The computations presented in the graphs are consistent with the the existence of an ergodic invariant measure associated with the dynamics on the invariant line. (See [6] and [25].) Let us also note that the recent preprint of P. Ashwin, Jorge Buescu, and Ian Stewart appears to address some of the behavior of the characteristic exponents detailed in this section.

9 Discussion and Summary

In this article we have reported the results of our study of three noninvertible dynamical systems in the plane. All of the maps arise from the study of factorization algorithms. The goal of the factorization algorithms is to produce quadratic factors that correspond to fixed points in phase space. We have examined two test cases associated with factoring a family of cubic polynomials. This family contains all cubics except x^3. We have also examined a one parameter subset of the family of all quintic polynomials.

Our algorithms are based on applying two-dimensional Newton's method to a pair of functions $F(u,v)$ and $G(u,v)$ to determine their simultaneous zero set. For B_a the two functions are polynomials in two variables while for M_a they are rational functions. Both of these iterations have similar qualitative properties. Two of them share the same fixed points, invariant lines and characteristic exponent behavior.

They are even conjugate to the same map when restricted to the invariant line. The third family, Q_a, has almost all of those critical elements plus an additional contractive fixed point.

In all cases we have observed a, possibly new, "eruption" or bifurcation which produces infinitely many periodic and aperiodic orbits as a critical parameter is exceeded. These orbits are born as a consequence of two fixed points merging into a singularity for a critical parameter value. For values of the parameter just below the critical parameter value two contractive fixed points approach a point on a singular curve. As the parameter exceeds the critical value there is a chaotic "eruption" along an invariant line. The dynamics along the invariant line reduces in two cases to one dimensional Newton's method applied to polynomials having no real roots.

Chaotic regions in the plane are bounded by preimages of the singular curves. On the invariant line these are preimages of the critical points. While off of the line there are sets of, planar, positive measure that are attracted to the invariant lines and the fixed points. This indicates that the attractors share properties with attractors defined in the sense of Milnor [19].

There is some evidence that $a = 1$ is a special parameter for both B_a and M_a. This may be associated with one of the points on the periodic two cycle having a point on a singular curve as a "preimage". Indeed, the mapping seems to be degenerate for this parameter value but we have little understanding of the nature of this degeneracy. We hope to more fully explore its nature in a future article.

We have presented illustrations showing the behavior of the characteristic exponents for our systems. We observed two types of behavior, a constant value of $log(2)$ for the parallel characteristic exponent along the invariant line, line and a roughly monotone increasing behavior for the other exponent. And we can certainly associate this behavior with the lost of transverse stability of the invariant line. For two of the mappings we have determined an explicit conjugacy that shows that there must be chaotic dynamics present.

This article introduces a class of noninvertible dynamical systems that arise naturally when attempting to factor polynomials. These mappings have intrinsic interest because in certain cases they reduce to one dimensional Newton's method and the one dimensional dynamics, certainly influences, but may also "govern" critical transitions. Understanding the role played by singular sets of such functions seems to be important.

10 Acknowledgments

This work was reported on and benefited from discussions at the Workshop on Noninvertible Mappings held at the Geometry Center located at the University of Minnesota. In this regard, special thanks to Professor C. Mira and Dr. Fournier-Prunaret for several very insightful discussions and observations concerning noninvertible mappings. Thanks also to Jim Meiss for helpful discussions and Robert MacKay for bringing the work of P. Ashwin, J. Buescu, and I. Stewart to our attention.

LB also thanks the Geometry Center and the Mid-West Dynamical Systems Group for their hospitality and support.

A portion of this article was prepared while JHC was a sabbatical visitor at the National Center for Atmospheric Research, he thanks them for their hospitality.

A Derivations of Bairstow's Factorization Methods

A.1 Derivation of B_a

In what follows we shall only consider a cubic family of polynomials with real coefficients:

$$P_a(x) = x^3 + (a-1)x - a.$$

Now divide $P_a(x)$ by the quadratic $D(x)$ defined as

$$D(x) = x^2 + ux + v \tag{14}$$

with real coefficients u and v. The roots of $D(x)$ are roots of $P(x)$ if and only if $D(x)$ divides $P(x)$ without a remainder. We can write $P(x)$ in the general form

$$P_a(x) = D(x)Q(x) + F(u,v)x + G(u,v)$$

where $Q(x)$ is a first order polynomial. This can be used to find the partial derivatives of F and G with respect to u and v for Newton's Method:

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} \frac{\partial F(u,v)}{\partial u} & \frac{\partial F(u,v)}{\partial v} \\ \frac{\partial G(u,v)}{\partial u} & \frac{\partial G(u,v)}{\partial v} \end{pmatrix}^{-1} \begin{pmatrix} F(u,v) \\ G(u,v) \end{pmatrix}. \tag{15}$$

Now, equation (15) takes the following form:

$$\begin{aligned} B_a \begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} 2u & -1 \\ v & u \end{pmatrix}^{-1} \begin{pmatrix} a-1-v+u^2 \\ -a+uv \end{pmatrix} \\ &= \frac{1}{2u^2+v} \begin{pmatrix} u^3 + u(v-a+1) + a \\ v(u^2+a-1) + 2au \end{pmatrix}. \end{aligned}$$

A.2 Derivation of M_a

Again we consider the cubic $P_a(x)$ with real coefficients:

$$P_a(x) = x^3 + (a-1)x - a.$$

Now divide $P_a(x)$ by the quadratic $D(x)$, (14) to obtain $P_a(x)$ in the general form

$$P_a(x) = D(x)Q(x) + F(u,v)x^2 + G(u,v)x$$

where $Q(x)$ is a first order polynomial. Note that F and G are multiplied by x^2 and x respectively. Again, using Newton's Method equation (15) takes the following form:

$$\begin{aligned} M_a \begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} -1 & -\frac{a}{v^2} \\ \frac{a}{v} & -1-\frac{ua}{v^2} \end{pmatrix}^{-1} \begin{pmatrix} -u+\frac{a}{v} \\ a-1-v+\frac{au}{v} \end{pmatrix} \\ &= \frac{1}{v^3+auv+a^2} \begin{pmatrix} a(au+v-av+2v^2) \\ v(2a^2+auv-v^2+av^2) \end{pmatrix}. \end{aligned}$$

A.3 Solving the Quintic Using Bairstow's Method

Consider the quintic $T_a(x)$ with real coefficients:

$$T_a(x) = x^5 + Bx + A.$$

Multiplying the quadratic $x^2 + cx + d$, and the cubic $x^3 + (a-1)x - a$, we can write $T_a(x)$ in the general form

$$T_a(x) \quad = \quad x^5 + cx^4 + (d + a - 1)x^3 + (ac - c - a)x^2 + (ad - ac - d)x - ad.$$

Solving for a, c, and d such that the coefficients of x^4, x^3, and x^2 are zero results in the following: $a = 0, c = 0, d = 1$. Keeping the parameter a, the coefficients result in the following:

$$T_a(x) \quad = \quad x^5 + ax^3 - ax^2 + (a-1)x - a.$$

In this case, the map takes the following form:

$$w = \left[\left(1 + 3u + 2u^2 + v\right)a^2 + \left(5u^3 + 6u^4 - 10uv - 8u^2v - 4v^2\right)a + \left(4u^6 - 13u^4v + 12u^2v^2 + 4v^3\right)\right]^{-1},$$

$$\begin{aligned} Q_a\begin{pmatrix} u \\ v \end{pmatrix} \quad = \quad & \begin{pmatrix} -u + u^2 + u^3 + uv \\ 1 + 2u + v + uv + u^2v \end{pmatrix} wa^2 + \\ & \begin{pmatrix} 1 + u + 3u^2 - u^3 + 3u^4 + 4u^5 - 2v + 4uv - 6u^2v - 4u^3v + v^2 - 5uv^2 \\ 4u^3 - v - 6uv + 3u^2v + 3u^3v + 4u^4v - 2v^2 - 4uv^2 - 6u^2v^2 - v^3 \end{pmatrix} wa + \\ & \begin{pmatrix} u^3 + 3u^7 - 4uv - 9u^5v + 7u^3v^2 + 4uv^3 \\ -3u^2v + 3u^6v + 2v^2 - 10u^4v^2 + 9u^2v^3 + 2v^4 \end{pmatrix} w. \end{aligned}$$

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