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Fleurbaey–Michel conjecture on equitable weak Paretian social welfare order

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ABSTRACT

The paper examines the problem of explicit description of a social welfare order over infinite utility streams, which respects anonymity and weak Pareto axioms. It provides a complete characterization of the domains of one period utilities, for which it is possible to explicitly describe a weak Paretian social welfare order satisfying the anonymity axiom. For domains containing any set of order type similar to the set of positive and negative integers, every equitable social welfare order satisfying the weak Pareto axiom is non-constructive. The paper resolves a conjecture by Fleurbaey and Michel (2003) that there exists no *explicit* (that is, avoiding the axiom of choice or similar contrivances) description of an ordering which satisfies weak Pareto and indifference to finite permutations. It also provides an interesting connection between the existence of social welfare function and the constructive nature of social welfare order by showing that the domain restrictions for the two are identical.

1. Introduction

In this paper, we are concerned with the problem of finding an explicit description of a Paretian social welfare order, which satisfies equal treatment of all generations.

Early contributions to the related literature were concerned with the idea of a preference for advancing the timing of future consumption (which came to be known as *impatience*). The first modern axiomatic treatment of the notion of impatience is contained in Koopmans (1960). He came up with a minimal set of axioms, namely, continuity, sensitivity, non-complementarity and stationarity, which any non-trivial social welfare function should satisfy and showed that for such a social welfare function, "impatience[1] prevails at least in certain areas of the program space" Koopmans (1960, p. 288).

Subsequent research generalized this result in two directions. Koopmans et al. (1964), generalized the impatience criterion to a property called *weak time perspective*[2] and showed the existence of weak time perspective from which the earlier result on impatience can be obtained and extended to a larger part of the program space. Other authors, relaxed some of the axioms of Koopmans (1960) to show that the existence of impatience (including its variations) persists even in a weaker environment.

In order to review these contributions, we use the framework that has become standard in this literature. We consider the problem of defining social welfare orders on the set X of infinite utility streams, where this set takes the form of $X = Y^{\mathbb{N}}$, with Y denoting a non-empty set of real numbers and $\mathbb{N}$ the set of natural numbers.

It is generally agreed in the literature that any social preference should satisfy the following two axioms. The first axiom is "equal treatment" of all generations (present and future), which is formalized in the form of an *Anonymity Axiom*[3] on social preferences. It requires that society should be indifferent between two streams of well-being, if one is obtained from the other by interchanging the levels of well-being of any two generations. The other axiom is the *Pareto Axiom*. Society should consider one stream of well-being to be superior to another if at least one generation is better off and no generation is worse off in the former compared to the latter.

With respect to weakening the Pareto axiom, we can justifiably take the position that the so-called *Weak Pareto Axiom* is more compelling than the Pareto axiom; it requires that society should consider one stream of well-being to be superior to another if *every* generation is better off in the former compared to the latter. In the context of evaluating infinite utility streams, it is debatable whether in comparing two utility streams, society is always better

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[1] Koopmans defines impatience as "if in any given year the consumption of a bundle x of commodities is preferred over that of a bundle x', the consumption in two successive years of x, x', in that order, is preferred to the consumption of x', x".

[2] They described it as "As the timing difference between any two programs is made to recede into a more distant future, the utility difference between the programs either remains the same or diminishes".

[3] This definition applies to utility streams which can be obtained from any utility stream by carrying out finitely many permutations. It does not include the more general case of utility streams which can be obtained by carrying out infinitely many permutations.

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off if one generation is (or a finite number of generations are) better off and all other generations are unaffected, so the standard Pareto axiom might not be self-evident.

In a seminal contribution, Diamond (1965) showed that there does not exist any continuous social welfare order satisfying the equal treatment and Pareto axioms (where continuity is defined with respect to sup metric) when Y is the closed interval [0, 1]. A social welfare order satisfying the Pareto axiom and the continuity requirement is representable by a social welfare function which is continuous in the sup metric, when Y is the closed interval [0, 1]. Therefore, Diamond's result also implies that there does not exist any social welfare function satisfying Pareto and equal treatment axioms, which is continuous in the sup metric when Y is the closed interval [0, 1].

Basu and Mitra (2003) showed that this last statement can be refined as follows: there does not exist any social welfare function satisfying the Anonymity and Pareto axioms, when Y contains at least two distinct elements. Another way of stating this is that there does not exist any *representable* social welfare order satisfying the anonymity and Pareto axioms, when Y contains at least two distinct elements.

If one requires neither continuity of the social welfare order nor its representability, it is possible to show the existence of a social welfare order satisfying the anonymity and Pareto axioms. Svensson (1980) established this important result, using Szpilrajn (1930) Lemma, assuming Y to be the closed interval [0, 1].

These two results clearly bring out the dividing line between the social welfare function and the social welfare order as they imply the non-existence of an equitable Paretian social welfare function on any non-trivial domain as against the existence of an equitable Paretian social welfare order for all domains. Thus ethical Paretian social welfare order could turn out to be a useful policy instrument, if it is possible to construct one explicitly.

However, if the use of the Axiom of Choice is necessary in the proof of the existence result, then social welfare orders would be non-constructive objects. While seeking an explicit description of an equitable Paretian social welfare order, one may, following the approach of Svensson (1980), consider binary relations which are quasi-orders[4] satisfying the anonymity and Pareto axioms, for possible extension into an order. However, there seems to be a conflict between transitivity and completeness in this exercise.

Fleurbaey and Michel (2003) considered two approaches dealing with this problem: (i) extend an egalitarian Paretian quasi-order to an order, and (ii) extend a complete and quasi-transitive[5] binary relation to an equitable Paretian order. They observed the critical role of the existence of a *free ultrafilter*[6] in extending these binary relations to an order. This difficulty led them to make a conjecture that there exists no *explicit* (that is, avoiding the axiom of choice or similar contrivances) description of an ordering which satisfies weak Pareto and indifference to finite permutations.

Before we explain the contribution of this paper, we describe the existing literature on this subject. Zame (2007) shows that the existence of a social welfare order satisfying the anonymity and weak Pareto axioms entails the existence of a non-measurable set which is a non-constructive object, for the domain $Y = [0, 1]$. Also for domain $Y = \{0, 1\}$, Lauwers (2010) shows that a quasi-order in the set of infinite utility streams satisfying anonymity and intermediate Pareto (which assumes monotonicity in addition to infinite Pareto[7]) is either incomplete or contains a non-Ramsey set which is again a non-constructive object.

Notice that Zame (2007) and Lauwers (2010) have taken different domains and also different versions of Pareto axiom to prove their results. Our approach, in contrast, is to consider the domain Y itself to be a variable and examine the restrictions on Y for the Fleurbaey–Michel conjecture to hold.

We characterize the restrictions on domain Y for which any social welfare order, satisfying anonymity and weak Pareto axioms, is non-constructive. Using the order properties of subsets of the real line, we show that if the domain Y contains a subset of order type similar to the set of positive and negative integers,[8] it is not possible to describe any equitable social welfare order satisfying the weak Pareto axiom. We also explicitly construct a social welfare order for all domains which do not contain any subset of order type similar to the set of positive and negative integers.

This result leads to a complete resolution of the Fleurbaey–Michel conjecture. There exist non-trivial subsets of [0, 1] for which a social welfare order can be written explicitly. It refines Zame's result which holds for the interval [0, 1] but has now been shown to not hold for all subsets of [0, 1]. Also, it shows how the domain, for which a social welfare order can be explicitly constructed, expands when we weaken the intermediate Pareto axiom used by Lauwers to the weak Pareto axiom.

The second contribution of this paper is as follows. We are now in a position to consider following relationship between the existence of a social welfare function and the constructive nature of social welfare order combining the results in the literature with those established here.

For $Y = \{0, 1\}$, Basu and Mitra (2003) show that there does not exist any social welfare function on $X = Y^{\mathbb{N}}$ which satisfies the anonymity and Pareto axioms. Further Crespo et al. (2009) extend this result to the case where Pareto axiom is weakened to infinite Pareto axiom. Using the results of Zame (2007, p. 200) and Lauwers (2010, p. 37), it is clear that the domain restrictions for the existence of a social welfare function and the constructive nature of social welfare order satisfying Pareto or infinite Pareto and anonymity axioms, are identical. This lets us explore the possibility of a similar result holding in cases where we apply the weak Pareto and anonymity axioms.

For Y being of order type μ, Dubey and Mitra (2011) show that there exists no social welfare function on $X = Y^{\mathbb{N}}$ which satisfies the anonymity and weak Pareto axioms. This domain coincides with the set Y for which there does not exist an explicit description of a social welfare order as shown in Theorem 1.

We are thus led to the result that domain restrictions are identical for (i) non-existence of equitable weak Paretian social welfare function, and (ii) non-constructive nature of equitable weak Paretian social welfare order. Thus for any domain Y, either there exists an equitable Paretian social welfare function or no equitable Paretian social welfare order is constructive. In cases where a social welfare function exists, it has a simple form – a linear combination of inf and sup of the sequence of one period utilities.

2. Basic framework

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{N}$ the set of positive integers, $\mathbb{Q}$ the set of rational numbers and $\mathbb{I}$ the set of positive and negative integers. Suppose $Y \subset \mathbb{R}$ is the set of all possible utilities that any

4 A binary relation is a quasi-order if it is reflexive and transitive. A quasi-order is an order if it is complete.

5 A binary relation is quasi-transitive if for all $x, y, z, x \succ y$ and $y \succ z$ imply $x \succ z$.

6 A brief overview of the notions of a filter, an ultrafilter and a free ultrafilter can be found in Fleurbaey and Michel (2003, p. 790–791). A free ultrafilter is a non-constructive concept. See Halpern (1964).

7 Infinite Pareto axiom postulates sensitivity in each set of infinitely many elements of the utility streams.

8 Such ordered sets have been termed as sets of order type μ in Dubey and Mitra (2011). Please refer to Section 2 for a precise definition.

generation can achieve. Then $X = Y^{\mathbb{N}}$ is the set of all possible utility streams.

If $x \equiv \langle x_n \rangle \in X$, then $\langle x_n \rangle = (x_1, x_2, \ldots)$, where for all $n \in \mathbb{N}$, $x_n \in Y$ represents the amount of utility that the generation of period n earns. For all $y, z \in X$, we write $y \geqslant z$ if $y_n \geqslant z_n$, for all $n \in \mathbb{N}$; $y > z$ if $y \geqslant z$ and $y \neq z$; and $y \gg z$ if $y_n > z_n$ for all $n \in \mathbb{N}$.

If Y has only one element, then X is a singleton, and the problem of ranking or evaluating infinite utility streams is trivial. Thus, without further mention, the set Y will always be assumed to have at least two distinct elements.

A social welfare order (SWO) is a binary relation $\succcurlyeq$ on X which is complete and transitive. We write $x \sim y$ if both $x \succcurlyeq y$ and $y \succcurlyeq x$ hold; and $x \succ y$ when $x \succcurlyeq y$ and $y \not\succcurlyeq x$ hold. A social welfare function (SWF) is a mapping, $W : X \to \mathbb{R}$.

2.1. Definitions

2.1.1. Pareto and anonymity axioms

The first axiom is the weak Pareto axiom; this is a version of the Pareto[9] axiom that has been widely used in the literature, and could possibly be even more compelling than the standard Pareto axiom.

Weak Pareto axiom: For all $x, y \in X$, if $x \gg y$, then $x \succ y$.

The next axiom is the one that captures the notion of "intergenerational equity". We call it the "anonymity axiom".

Anonymity axiom: For all $x, y \in X$, if there exist $i, j \in \mathbb{N}$ such that $x_i = y_j$ and $x_j = y_i$, and for every $k \in \mathbb{N} \setminus \{i, j\}$, $x_k = y_k$, then $x \sim y$.[10]

2.1.2. Non-constructive weak Paretian egalitarian SWO

First we define the non-Ramsey set. Let T be an infinite set and let n be a positive integer. Let $\mathscr{T}(n) \equiv [T]^n$ be the collection of all the subsets of T with exactly n elements. Ramsey (1928) showed that for each subset $\mathscr{S}$ of $\mathscr{T}(n)$, there exists an infinite set $\bar{T} \subset T$ such that either $\bar{\mathscr{T}}(n) \equiv [\bar{T}]^n \subset \mathscr{S}$ or $\bar{\mathscr{T}}(n) \bigcap \mathscr{S} = \varnothing$.

Ramsey Theorem fails when n is replaced by countable infinity. There exists a subset $\mathscr{U}$ of $\mathscr{T} \equiv [T]^\infty$ such that for each infinite subset $\bar{T}$ of T, the class $\bar{\mathscr{T}} \equiv [\bar{T}]^\infty$ intersects both $\mathscr{U}$ as well as its complement $\mathscr{T} \setminus \mathscr{U}$. Such a set $\mathscr{U}$ is said to be a non-Ramsey set. Observe that non-Ramsey set is a non-constructive object, a fact established in Mathias (1977).

If the existence of a non-constructive mathematical object is a necessary condition for the existence of a weak Paretian SWO satisfying the anonymity axiom, then we say that such a SWO is non-constructive.

2.1.3. Domain types

We recall a few concepts from the mathematical literature dealing with *types* of spaces, which are strictly ordered by a binary relation.

We will say that the set A is *strictly ordered* by a binary relation $\mathscr{R}$ if $\mathscr{R}$ is *connected* (if $a, a' \in A$ and $a \neq a'$, then either $a\mathscr{R}a'$ or $a'\mathscr{R}a$ holds), *transitive* (if $a, a', a'' \in A$ and $a\mathscr{R}a'$ and $a'\mathscr{R}a''$ hold, then $a\mathscr{R}a''$ holds) and *irreflexive* ($a\mathscr{R}a$ holds for no $a \in A$). In this case, the strictly ordered set will be denoted by $A(\mathscr{R})$. For example, the set $\mathbb{N}$ is strictly ordered by the binary relation $<$ (where $<$ denotes the usual "less than" relation on the reals).

We will say that a strictly ordered set $A'(\mathscr{R}')$ is *similar* to the strictly ordered set $A(\mathscr{R})$ if there is a one-to-one function f mapping A onto A', such that:

$$a_1, a_2 \in A \quad \text{and} \quad a_1\mathscr{R}a_2 \Rightarrow f(a_1)\mathscr{R}'f(a_2).$$

We now specialize to strictly ordered subsets of the reals. With Y a non-empty subset of $\mathbb{R}$, let us define some *order types* as follows. We will say that the strictly ordered set $Y(<)$ is:

(i) of order type ω if $Y(<)$ is similar to $\mathbb{N}(<)$;
(ii) of order type σ if $Y(<)$ is similar to $\mathbb{I}(<)$;
(iii) of order type μ if Y contains a non-empty subset Y', such that the strictly ordered set $Y'(<)$ is of order type σ.

The characterization of these types of strictly ordered sets is facilitated by the concepts of a cut, a first element and a last element of a strictly ordered set.

Given a strictly ordered set $Y(<)$, let us define a *cut* $[Y_1, Y_2]$ of $Y(<)$ as a partition of Y into two non-empty sets Y_1 and Y_2 (that is, Y_1 and Y_2 are non-empty, $Y_1 \cup Y_2 = Y$ and $Y_1 \cap Y_2 = \varnothing$), such that for each $y_1 \in Y_1$ and each $y_2 \in Y_2$, we have $y_1 < y_2$.

An element $y_0 \in Y$ is called a *first element* of $Y(<)$ if $y < y_0$ holds for no $y \in Y$. An element $y^0 \in Y$ is called a *last element* of $Y(<)$ if $y^0 < y$ holds for no $y \in Y$.

The following result can be found in Sierpinski (1965, p. 210).

Proposition. *A strictly ordered set $Y(<)$ is of order type σ if and only if the following two conditions hold:*

(i) *Y has neither a first element nor a last element.*
(ii) *For every cut $[Y_1, Y_2]$ of Y, the set Y_1 has a last element and the set Y_2 has a first element.*

3. Fleurbaey–Michel conjecture

We analyze the conjecture in the following steps. In Lemma 1, we show that the existence of any SWO satisfying anonymity and weak Pareto axioms for infinite utility streams in $X \equiv \mathbb{I}^{\mathbb{N}}$ implies existence of a non-Ramsey set using technique similar to that used in Lauwers (2010).

In order to show that the result holds for all ordered sets of order type μ, we prove an invariance result in Lemma 2 for subsets Y of $\mathbb{R}$, for which there exists monotone one-to-one map from $\mathbb{I}$ onto some subset Y' of Y. The general claim for all sets of order type μ is proved in Theorem 1.

3.1. Weak Pareto and anonymity imply non-Ramsey set

First we construct a pair of utility streams consisting of integers from any arbitrary infinite subset of natural numbers. Let $N \equiv \{n_1, n_2, \ldots\}$ be an infinite subset of $\mathbb{N}$ such that $n_k < n_{k+1}$ for all $k \in \mathbb{N}$. Given N, we construct an infinite utility stream $x(N) \equiv \langle x_n \rangle$ as follows.

(i) Using N, we create the following subsets of $\mathbb{N}$,

$$\overline{N} = \{n_1, n_1 + 1, \ldots, n_2 - 1, n_3, n_3 + 1, \ldots, n_4 - 1, \ldots\},$$

and

$$\underline{N} = \mathbb{N} \setminus \overline{N}.$$

Note that both $\overline{N}$ and $\underline{N}$ are infinite subsets of $\mathbb{N}$.

(ii) We rewrite the above two subsets $\overline{N}, \underline{N}$ as under,

$$\overline{N} \equiv \{m_1, m_2, \ldots\}, \quad m_1 < m_2 < \cdots,$$

and

$$\underline{N} \equiv \{l_1, l_2, \ldots\}, \quad l_1 < l_2 < \cdots.$$

9 The standard Pareto axiom is, *Pareto axiom*: For all $x, y \in X$, if $x > y$, then $x \succ y$. It is also referred to as strong Pareto axiom in the literature.

10 In informal discussions throughout the paper, the terms "equity" and "anonymity" are used interchangeably.

Note that,

$m_1 = n_1, \ldots, m_{n_2-1} = n_2 - 1, \quad m_{n_2} = n_3, \ldots;$

and

$l_1 = 1, \ldots, l_{n_1-1} = n_1 - 1, \quad l_{n_1} = n_2, \ldots.$

Observe also that $\bar{N} \bigcup \underline{N} = \mathbb{N}$ and $\bar{N} \bigcap \underline{N} = \varnothing$.

(iii) The sequence $\langle x_n \rangle$ is defined as follows.

$$x_n = \begin{cases} r & \text{if } n = m_r \text{ for some } r \in \mathbb{N}, \\ -s & \text{if } n = l_s \text{ for some } s \in \mathbb{N}. \end{cases} \tag{1}$$

We also construct sequence $y(N) \equiv \langle y_n \rangle$ using the subset $N \setminus n_1$ in place of subset N, in identical fashion.

Observe that each of the two sequences contains the entire set of integers, the positive integers occurring in increasing order and the negative integers occurring in decreasing order.

Example 1. Let $N = \{2, 4, 6, \ldots\}$. Then,

$\overline{N} = \{2, 3, 6, 7, \ldots\}$,

and

$\underline{N} = \{1, 4, 5, 8, \ldots\}$

would imply that $x(N) = \{-1, 1, 2, -2, -3, 3, 4, \ldots\}$.

Corresponding sequence y would be $y(N) = \{-1, -2, -3, 1, 2, -4, -5, \ldots\}$.

For the sequences x, y constructed as above, we state the following two claims (proofs are given in the Appendix) which would be used in the proof of Lemma 1.

Claim 1. *For x, y, if $x_n \geqslant y_n$ for all $n \in \mathbb{N}$ and $x_n > 0 > y_n$ for infinitely many $n \in \mathbb{N}$, then there exists a finite permutation $\langle x_n(\pi) \rangle$ of $\langle x_n \rangle$ such that $x(\pi) \gg y$.*

Claim 2. *For x, y, if $x_n < 0 < y_n$ for a finite number of n, $x_n > 0 > y_n$ for an infinite number of n, and x_n, y_n have same sign for all the remaining n, then there exists a finite permutation $\langle x_n(\pi) \rangle$ of $\langle x_n \rangle$ such that $x(\pi) \gg y$.*

Next we present Lemma 1 (Proof in Appendix) showing that the existence of non-Ramsey set is necessary for the existence of egalitarian weak Paretian SWO for $X \equiv \mathbb{I}^{\mathbb{N}}$.

Lemma 1. *The existence of a social welfare order satisfying weak Pareto and anonymity axioms for all $x, y \in X$ entails the existence of a non-Ramsey set.*

Since non-Ramsey set is a non-constructive object, Lemma 1 shows that Fleurbaey–Michel conjecture is true for the domain $Y = \mathbb{I}$.

3.2. An invariance result

In this subsection, we will see that the result in Lemma 1 actually holds for all non-empty domains $Y \subset \mathbb{R}$ of order type μ because of an *invariance* result, which states that any possibility result is invariant with respect to monotone transformations of the domain.

Lemma 2. *Let Y be a non-empty subset of $\mathbb{R}$, $X \equiv Y^{\mathbb{N}}$, and let there exist a social welfare order, $\succcurlyeq_X$, satisfying the weak Pareto and anonymity axioms. Suppose f is a monotone (increasing or decreasing) function from $\tilde{Y}$ to Y, where $\tilde{Y}$ is a non-empty subset of $\mathbb{R}$. Then, there exists a social welfare order, $\succcurlyeq_{\tilde{X}}$, satisfying the weak Pareto and anonymity axioms, for $\tilde{X} = \tilde{Y}^{\mathbb{N}}$.*

Proof. It is easy to prove this proposition by adapting the technique of the proof of Proposition 2 in Dubey and Mitra (2011). □

3.3. SWO for Y of order type μ satisfying weak Pareto and anonymity is non-constructive.

We can now state the non-constructive SWO result for general domains of order type μ.

Theorem 1. *Let Y be a non-empty subset of $\mathbb{R}$ such that $Y(<)$ is of order type μ. Then, the existence of a weak Paretian social welfare order for $X = Y^{\mathbb{N}}$, satisfying anonymity axiom implies existence of a non-Ramsey set.*

Proof. Since $Y(<)$ is of order type μ, Y contains a non-empty subset Y' such that $Y'(<)$ is of order type σ. That is, there is a one-to-one mapping, g, from $\mathbb{I}$ onto Y' such that:

$a_1, a_2 \in \mathbb{I} \quad \text{and} \quad a_1 < a_2 \Rightarrow g(a_1) < g(a_2).$

Thus, g is an increasing function from $\mathbb{I}$ to Y'. Using Lemma 2, there exists a SWO on $\tilde{X}$ satisfying the weak Pareto and anonymity axioms, where $\tilde{X} = \mathbb{I}^{\mathbb{N}}$. But then by Lemma 1, the existence of a SWO on $\tilde{X}$ satisfying the weak Pareto and anonymity axioms, where $\tilde{X} = \mathbb{I}^{\mathbb{N}}$, implies existence of a non-Ramsey set. □

4. Equivalence of SWF and constructible SWO

In this section, first we use the possibility result in Dubey and Mitra (2011) on the existence of an equitable SWF satisfying weak Pareto axiom to show that it is possible to construct an equitable SWO satisfying weak Pareto axiom for domains which are not of order type μ.

4.1. Example of SWO

Let $\hat{Y}$ be a non-empty subset of [0, 1], with $\hat{X} \equiv \hat{Y}^{\mathbb{N}}$ and suppose that $\hat{Y}(<)$ is not of order type μ. Then the function $W(\cdot)$, defined below, from Proposition 1 in Dubey and Mitra (2011), satisfies both weak Pareto and anonymity axioms.

Example 2. For all $x \in \hat{X}$, consider the SWF,

$$W(x) = \theta \inf\{x_n\}_{n \in \mathbb{N}} + (1 - \theta) \sup\{x_n\}_{n \in \mathbb{N}}, \tag{2}$$

where $\theta \in (0, 1)$ is a parameter.

Using this $W(\cdot)$, we can describe following SWO $\succcurlyeq$, for all $x, y \in \hat{X}$,

$x \succcurlyeq y \quad \text{if and only if} \quad W(x) \geqslant W(y).$

Observe that the SWO satisfies weak Pareto and anonymity axioms since $W(\cdot)$ does.

The possibility result in Example 2 is stated for domains $\hat{Y} \subset [0, 1]$. In Proposition 3 in Dubey and Mitra (2011), the existence of equitable weak Paretian SWF for $Y \subset \mathbb{R}$, which are not of order type μ, has been proved using an invariance result similar to Lemma 2. It is possible to construct a SWO for domains $Y \subset \mathbb{R}$, provided they are not of order type μ, by using the SWF mentioned above. The possibility result for general domain is stated as follows.

Lemma 3. *Let Y be a non-empty subset of $\mathbb{R}$ and let $X \equiv Y^{\mathbb{N}}$. There exists an explicit description of social welfare order, $\succcurlyeq$, satisfying the weak Pareto and anonymity axioms if $Y(<)$ is not of order type μ.*

Proof. In Proposition 3 in Dubey and Mitra (2011), existence of an SWF $W(\cdot)$ satisfying weak Pareto and anonymity has been established by construction. Hence, the SWO,

$x \succcurlyeq y \quad \text{if and only if} \quad W(x) \geqslant W(y),$

also satisfies weak Pareto and anonymity axioms. □

Using Lemma 3 and Theorem 1, we are in a position to completely characterize the domain for which SWO can be explicitly written down. It coincides with the domain for which there exists a weak Paretian SWF satisfying anonymity axiom.

Theorem 2. *Let Y be a non-empty subset of $\mathbb{R}$ and $X = Y^{\mathbb{N}}$. The following two statements are equivalent.*

(a) *There does not exist a weak Paretian social welfare function $W : X \to \mathbb{R}$ satisfying anonymity axiom.*
(b) *For all $x, y \in X$, existence of a weak Paretian social welfare order satisfying anonymity axiom implies existence of a non-Ramsey set.*

Proof. By Theorem 1 in Dubey and Mitra (2011) (a) holds if and only if $Y(<)$ is of order type μ. To prove equivalence, we need to show (b) also holds if and only if $Y(<)$ is of order type μ.

(b) *If*: Theorem 1. *Only if*: Lemma 3. □

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Appendix

Proof (of Claim 1). Let $N \equiv \{n_1, n_2, \ldots\}$ with $n_1 < n_2 < \cdots$ be the infinite subset of $\mathbb{N}$, for which the inequality $x_{n_k} > 0 > y_{n_k}$ holds, for all $k \in \mathbb{N}$. There are two cases.

Case (a). $n_1 = 1$. Then $x_1 = 1 > -1 = y_1$ and $x_n > y_n$ for all $n > 1$. In this case, we take $\langle x_n(\pi)\rangle = \langle x_n\rangle$.

Case (b). $n_1 > 1$. Then $x_k \geqslant y_k + 1$, for all $k \geqslant n_1$ and $x_k = y_k$, for all $k < n_1$. Let $n_1, \ldots, n_{n_1-1}$ be the (n_1-1) smallest elements of N. Define $\langle x_n(\pi)\rangle$ to be the utility stream obtained by interchanging the kth entry of x with the n_kth entry of x for $k < n_1$ and leaving all other entries unchanged. For $k < n_1$, if $x_k = y_k < 0$, then $x_k(\pi) = x_{n_k} > 0 > y_k$ and $x_{n_k}(\pi) = x_k = y_k > y_{n_k}$, and if $x_k = y_k > 0$, then $x_k(\pi) = x_{n_k} \geqslant x_k + 1 > y_k$ and $x_{n_k}(\pi) = x_k > 0 > y_{n_k}$. Hence, for all $n \in \mathbb{N}$, $x_n(\pi) > y_n$.

Therefore, $x(\pi) \gg y$ in both cases. □

Proof (of Claim 2). The proof consists of two steps. First we show that there exists a finite permutation $\langle x_n(\mu')\rangle$ of $\langle x_n\rangle$ such that $x_n(\mu') \geqslant y_n$ for all $n \in \mathbb{N}$ and $x_n(\mu') > 0 > y_n$ for infinitely many $n \in \mathbb{N}$. Then we use Claim 1 to prove existence of a finite permutation $\langle x_n(\mu)\rangle$ of $\langle x_n(\mu')\rangle$ such that $x(\mu) \gg y$. The desired permutation would be $\pi \equiv (\mu) \circ (\mu')$.

Step 1. Let $m = \max\{n \in \mathbb{N} : x_n < 0 < y_n\}$, which is finite. Observe that first m elements of $\langle x_n\rangle$ has p (which can at most be m) fewer positive elements compared to the first m elements of $\langle y_n\rangle$. However, since there are infinitely many instances of $x_n > 0 > y_n$, it is possible to make up this shortfall by reaching further out in the sequence to say m', such that there are p instances of $x_n > 0 > y_n$.

Then first m' elements of $\langle x_n\rangle$ have same number of positive and negative elements as the first m' elements of $\langle y_n\rangle$. In other words, they consist of elements which are identical subset of integers. Hence permutation (μ') is such that it transforms the first m' elements of the sequence $\langle x_n\rangle$ to give the first m' elements of $\langle y_n\rangle$ without changing any other element of $\langle x_n\rangle$.

Step 2. Observe that $x_n(\mu') \geqslant y_n$ for all $n \in \mathbb{N}$ and there remain infinitely many $n > m'$ for which $x_n(\mu') > 0 > y_n$ holds. Denote $\langle x_n(\mu')\rangle$ by $\langle z_n\rangle$. By Claim 1, there exists a finite permutation μ such that $z_n(\mu) > y_n$ for all $n \in \mathbb{N}$. We take $\pi = (\mu) \circ (\mu')$ which permutes only finitely many elements of $\langle x_n\rangle$ to complete the proof. □

Proof (of Lemma 1). Let the SWO satisfy weak Pareto and anonymity axioms. We claim that for all infinite subsequences N of $\mathbb{N}$, set $\mathscr{P} \equiv \{N | x(N) \prec y(N)\}$ is a non-Ramsey set. We need to show that for each infinite sequence $T = \{t_1, t_2, \ldots\}$ of $\mathbb{N}$, denoting the collection of infinite subsequences of T by $\mathscr{T}$, there exists an element $S \in \mathscr{T}$ such that either $T \in \mathscr{P}$ or $S \in \mathscr{P}$ with the either, or being exclusive. As the binary relation is complete, either $x(T) \prec y(T)$ or $x(T) \succ y(T)$ or $x(T) \sim y(T)$ must hold.

Case (a). Let $x(T) \prec y(T)$ or $T \in \mathscr{P}$. We drop t_1 and t_{4n+1}, t_{4n+2} for all $n \in \mathbb{N}$ from T to obtain $S = \{t_2, t_3, t_4, t_7, t_8, t_{11}, \ldots\}$. Hence $S \in \mathscr{T}$. Denote the set of coordinates $\{t_{4n+1}, \ldots, t_{4n+2} - 1 : n \in \mathbb{N}\}$ by $\hat{T}$.

Observe that for all $t \in \hat{T}$, $x_t(S) > 0 > y_t(T)$; $x_t(T) > 0 > y_t(S)$, and there are infinitely many coordinates in $\hat{T}$. Further, $x_t(T) > y_t(S)\ \forall t \geqslant t_1$, $x_t(T) = y_t(S)\ \forall t < t_1$ and $x_t(S) > y_t(T)\ \forall t \geqslant t_5$, $x_t(S) = y_t(T)\ \forall t < t_5$.

Then using Claim 1 there exist finite permutations π and μ such that $\langle x(T)\rangle \gg \langle y(\pi)(S)\rangle$ and $\langle x(\mu)(S)\rangle \gg \langle y(T)\rangle$. Then $y(\pi)(S) \prec x(T)$ and $x(\mu)(S) \succ y(T)$ by weak Pareto; and $y(\pi)(S) \sim y(S)$ and $x(\mu)(S) \sim x(S)$ by anonymity axiom. Since $x(T) \prec y(T)$, we get

$$y(S) \sim y(\pi)(S) \prec x(T) \prec y(T) \prec x(\mu)(S) \sim x(S) \Rightarrow S \notin \mathscr{P}.$$

Case (b). Let $x(T) \succ y(T)$. So $T \notin \mathscr{P}$. We drop t_1 and t_{4n}, t_{4n+1} for all $n \in \mathbb{N}$ from T to obtain $S = \{t_2, t_3, t_6, t_7, t_{10}, t_{11}, \ldots\}$. Hence $S \in \mathscr{T}$. Denote the set of coordinates $\{t_{4n}, \ldots, t_{4n+1} - 1 : n \in \mathbb{N}\}$ by $\hat{T}$.

Observe that for all $t \in \hat{T}$, $y_t(S) > 0 > x_t(T)$; $y_t(T) > 0 > x_t(S)$, and there are infinitely many such coordinates. Further, $y_t(T) > x_t(S)\ \forall t \geqslant t_4$ and $y_t(T) = x_t(S)\ \forall t < t_4$. Hence by Claim 1, there exist a finite permutation π such that $\langle y(\pi)(T)\rangle \gg \langle x(S)\rangle$.

Also (i) for $t \in \{t_1, \ldots, t_2 - 1\}$, $y_t(S) < 0 < x_t(T)$; (ii) for all coordinates $t \in \hat{T}$, $y_t(S) > 0 > x_t(T)$; and (iii) for all the remaining elements (i.e., for $t \in \mathbb{N} \setminus \{\hat{T} \bigcup \{t_1, \ldots, t_2 - 1\}\}$) both $y_t(S)$ and $x_t(T)$ have same sign. Hence by Claim 2, there exists a finite permutation μ such that $\langle y(\mu)(S)\rangle \gg \langle x(T)\rangle$.

Then $y(\pi)(T) \succ x(S)$ and $y(\mu)(S) \succ x(T)$ by weak Pareto and $y(\pi)(T) \sim y(T)$; $y(\mu)(S) \sim y(S)$ by anonymity axiom. Since $x(T) \succ y(T)$,

$$y(S) \sim y(\mu)(S) \succ x(T) \succ y(T) \sim y(\pi)(T) \succ x(S) \Rightarrow S \in \mathscr{P}.$$

Case (c). Let $x(T) \sim y(T)$ or $T \notin \mathscr{P}$. We drop t_2, t_3 and t_{4n+2}, t_{4n+3} for all $n \in \mathbb{N}$ from T to obtain $S = \{t_1, t_4, t_5, t_8, t_9, \ldots\}$. Hence $S \in \mathscr{T}$. Denote the set of coordinates $\{t_2, \ldots, t_3 - 1\} \cup \{t_{4n+2}, \ldots, t_{4n+3} - 1 : n \in \mathbb{N}\}$ by $\hat{T}$ as before.

Observe that for all $t \in \hat{T}$, $x_t(S) > 0 > x_t(T)$ and $y_t(T) > 0 > y_t(S)$ and there are infinitely many coordinates in $\hat{T}$. Further, for all coordinates $t \geqslant t_2$, $x_t(S) > x_t(T)$, and $y_t(T) > y_t(S)$ and for all coordinates $t < t_2$, $x_t(S) = x_t(T)$, and $y_t(T) = y_t(S)$.

Hence by Claim 1, there exist finite permutations π and μ such that $\langle x(\pi)(S)\rangle \gg \langle x(T)\rangle$ and $\langle y(T)\rangle \gg \langle y(\mu)(S)\rangle$. Then $x(\pi)(S) \succ x(T)$; and $y(\mu)(S) \prec y(T)$ by weak Pareto and $x(\pi)(S) \sim x(S)$; $y(\mu)(S) \sim y(S)$ by anonymity axiom. Since $x(T) \sim y(T)$, we get

$$y(S) \sim y(\mu)(S) \prec y(T) \sim x(T) \prec x(\pi)(S) \sim x(S).$$

We use result in (*Case* (b)) to get $\bar{S} \in \mathscr{S} \subset \mathscr{T}$ such that

$$y(\bar{S}) \succ x(S) \succ x(T) \sim y(T) \succ y(S) \succ x(\bar{S}),$$

where $\mathscr{S}$ is the collection of infinite subsequences of S. Hence $\bar{S} \in \mathscr{P}$. □

References

Basu, K., Mitra, T., 2003. Aggregating infinite utility streams with intergenerational equity: the impossibility of being paretian. Econometrica 71 (5), 1557–1563.

Crespo, J.A., Núñez, C., Rincón-Zapatero, J.P., 2009. On the impossibility of representing infinite utility streams. Economic Theory 40 (1), 47–56.

Diamond, P., 1965. The evaluation of infinite utility streams. Econometrica 33 (1), 170–177.

Dubey, R.S., Mitra, T., 2011. On equitable social welfare functions satisfying the weak pareto axiom: a complete characterization. International Journal of Economic Theory, forthcoming (doi:10.1111/j.1742-7363.2011.00161.x).

Fleurbaey, M., Michel, P., 2003. Intertemporal equity and the extension of the Ramsey criterion. Journal of Mathematical Economics 39 (7), 777–802.

Halpern, J., 1964. The independence of the axiom of choice from the boolean prime ideal theorem. Fundamenta Mathematicae 55, 57–66.

Koopmans, T., 1960. Stationary ordinal utility and impatience. Econometrica 28 (2), 287–309.

Koopmans, T., Diamond, P., Williamson, R., 1964. Stationary utility and time perspective. Econometrica 32 (1), 82–100.

Lauwers, L., 2010. Ordering infinite utility streams comes at the cost of a non-Ramsey set. Journal of Mathematical Economics 46 (1), 32–37.

Mathias, A., 1977. Happy families. Annals of Mathematical Logic 12 (1), 59–111.

Ramsey, F.P., 1928. On a problem of formal logic. Proceedings of the London Mathematical Society 2 (1), 264–286.

Sierpinski, W., 1965. Cardinal and Ordinal Numbers. PWN-Polish Scientific Publishers.

Svensson, L., 1980. Equity among generations. Econometrica 48 (5), 1251–1256.

Szpilrajn, E., 1930. Sur l'extension de l'ordre partiel. Fundamenta Mathematicae 16, 386–389.

Zame, W., 2007. Can utilitarianism be operationalized? Theoretical Economics 2, 187–202.