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Financial instability contagion: a dynamical systems approach

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We build a multi-agent dynamical system for the global economy to investigate and analyse financial crises. The agents are large aggregates of a subeconomy, and the global economy is a collection of subeconomies. We use well-known theories of dynamical systems to represent a financial crisis as propagation of a negative shock on wealth due the breakage of a financial equilibrium. We first extend the framework of the *market instability indicator*, an early warning signal defined for a single economy as the spectral radius of the Jacobian matrix of the wealth dynamical system. Then, we formulate a quantitative definition of instability contagion in terms thereof. Finally, we analyse the mechanism of instability contagion for both single and multiple economies. Our contribution is to provide a methodology to quantify and monitor the level of instability in sectors and stages of a structured global economic model and how it may propagate through its components.

Keywords: Sovereign credit; Systemic risk; Contagion; Multi-economy model; Market instability indicator

JEL Classification: C00, G00

1. Introduction

In the past decade and half or so, the global economy seems to have entered a regime of recurring instability. The prudence in the financial markets arising from the crash of the equity bubble in 2000 quickly abated thanks to powerful global trends and domestic policies, both fiscal and monetary, only to rekindle real estate bubbles in many developed and developing countries. Such a trend continued and climaxed in early 2007. In the US the hallmark of the bubble was the extreme leveraging attained through tranches of Structured Investment Vehicles that popularized securitization of loans, asset backed securities (ABS) and collateralized debt obligations. When the brewing crisis finally erupted in September 2008, financial markets nearly froze in the face of systemic uncertainty as to both the size and the complexity of such assets and the role they played in the balance sheet of major institutions.

The ensuing crisis induced the US Federal Reserve (the Fed) in conjunction with the Treasury to implement extreme measures: injecting equity into the financial sector and buying their illiquid assets in unprecedented amounts. In 2009, such action was hailed by the financial markets as the only way to resolve the insolvency of major banks, and other governments followed suit. Credit markets resumed functioning, panic abated and most asset classes rebounded in value, but it soon transpired that as in the squeezing of the proverbial toothpaste tube, the bulk of the risk had simply been shifted from banks to governments.

Beginning in early 2010, credit markets started focusing on the risk of the fiscally precarious members of the European Monetary Union (EMU) (BBC 2012, Wikipedia 2010a). It is widely agreed that the EMU was put in place without systemic fiscal institutions. Therefore, individual countries, while still responsible for their economic policy, have a weak control of monetary policy. At the same time, sovereign debt was considered virtually riskless regardless of issuance, which led banks and other institutions to chase the higher yields of peripheral debt thereby increasing their leverage. In the fall of 2011, the first brush with 'contagion,'§ from the smaller peripheral economies to Italy and Spain, seemed a fait accompli. Italian 10-year yields broke the threshold of sustainability of 7% (Trading Economics 2012, Yahoo! Finance 2012). The emergency lending implemented by the ECB stanched the liquidity haemorrhage (*The Financial Times* 2011). However, just as in the spring of 2009 within the US banking system, this was much more an issue of solvency. At the same time,

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§Such 'contagion' follows the qualitative and common usage of the word. It denotes the spreading of distress in financial, and specifically debt markets and is different from the quantitative definition we introduce later.

austerity alone is unlikely to allow debtor nations to satisfy their creditors, as Greece has demonstrated (Wikipedia 2010b).

Our goal is to provide a practical way of measuring the dynamic instability of the financial system, in particular financial risk contagion like the current Eurozone crisis as outlined above. The second author already jointly carried out such research in the case of one economic system having in sight the subprime crisis (Choi and Douady 2012, 2013). In this paper, we extend that framework to the case of multiple economies as part of a global economic system, and provide a definition of contagion that is consistent with the previous concepts of instability. As was the case of the single economy model, we use the theories of dynamical systems (Brin and Stuck 2003, Robinson 1999) to investigate financial market instability. We first divide an economy into n economic aggregates called agents (cf. Hommes 2008), then we construct a dynamical system based on the evolution of the corresponding wealths: $f : \mathbb{R} \times X \longrightarrow X$, $(t, \mathbf{w}(t_0)) \longmapsto f^t(\mathbf{w}(t_0)) = \mathbf{w}(t_0 + t)$, where $X \subset \mathbb{R}^n$, $\mathbf{w}(t_0) = (w_1(t_0), w_2(t_0), \ldots, w_n(t_0))$, and $w_i(t_0)$ is the initial wealth† of agent i.

The original contribution of this paper is to first extend the usage of the market instability indicator $I(t)$, introduced in Choi and Douady (2012) for a single economy system, to a system of multiple economies, then give a quantitative definition of a 'contagion' (of market instability) in terms of $I(t)$. Finally, we analyse the mechanism of contagion in three categories: a sector-to-sector contagion within a single economy, cross-border contagion due to counterparty risk and cross-border contagion due to a fear factor. We do not mention the impact of monetary interventions on the market (in)stability, which is beyond the scope of the present analysis.

The balance of the paper is organized as follows. In section 2, we extend the economic assumptions that underline the model in a dynamical systems framework stressing how this generalizes previous work. In section 3, we introduce the definition of *contagion*, which is the most original contribution of this paper. Further, in section 4 we analyse the mechanism of instability contagion for three common scenarios. In the appendices we recall and adapt previously established constructs to the present situation.

2. Assumptions on economies

We adopt the multi-agent model (Choi and Douady 2012, 2013) and keep the same assumptions on the economy. Details can be found in appendix A.1, where we focus on the features necessary to extend the previous model to a system of multiple economies.

2.1. *Flow of funds among economic agents*

The goal of this article is to model a sovereign credit crisis and related systemic risk, so we consider the economies of multiple sovereign nations. We extend the five agent ($n = 5$ in the above notation) model proposed by Choi and Douady (2013) to s economies.‡ We represent the economy of each country as the interaction of four domestic aggregates: consumers, firms, banks and the government. We consider investors as portfolio managers dealing with both domestic and foreign clients, therefore assume that they interact with agents domiciled in all countries. Consistently with Choi and Douady (2013), we call these five aggregates 'agents.' To distinguish the domestic agents of different countries, we use superscripts which match the notation of the economy under consideration. Thus, for economy i, we have the following five agents:

- C^i Consumers (the general public)
- F^i Firms (producers of goods and services, corporations)
- B^i Banks (lenders in general)
- G^i Government (the public authority in general, but excluding the central bank)
- I^i Investors (asset managers such as pension and other funds) restricted to economy i §

In Choi and Douady (2012, 2013) the central bank and the government were classified as one agent because they consistently coordinate their emergency interventions. In this article, however, we separate the two on the grounds of their distinct balance sheets and natures of their policy tools.¶ In the present instantiation of our model, we consider central banks as super-systemic agents that monitor and intervene on the global economic system to prevent instability. Other versions of the model can encapsulate central banks as new agents if necessary—for example to monitor their economic and financial role during and after a financial crisis.

These agents interact through cash flows, of which the data can be found in the flow of funds accounts. Within a country, the cash flow are divided into two groups by their nature, whether they are at-will (variable) or scheduled. The following are typical examples of flow of funds in each category: 1–4 for at-will cash flows; 5–8 for scheduled ones.

(1) Equity investment
  - (a) C^i to C^i: consumers invest in houses and other goods, and sell those to one another
  - (b) F^i to F^i: companies invest in each other
  - (c) I^i to B^i and F^i: investors buy bank and corporations stocks
  - (d) G^i to B^i and F^i: the government acts as an investor

(2) Debt investment (Loan)
  - (a) B^i to C^i: home mortgage, credit cards and other financing
    B^i to B^i: interbank lending, market for securitized assets
    B^i to F^i: bank loans to companies
  - (b) I^i to G^i, B^i and F^i: investors buy bonds issued by G^i, B^i and F^i

(3) Dividends and Distributions
  - (a) B^i and F^i to I: investors earn dividends from the B and F stocks
  - (b) I^i to C^i: investors pay distributions to their clients

(4) Consumption

†We assume all wealth is rescaled with respect to a base year by a deflator.
‡Hence this model specializes to Choi and Douady's (2013) for $s = 1$.
§By 'investors restricted to economy i', we mean the part of the global **I** that manages the flow of funds from and to economy i.
¶A difference that is all the more significant in the Eurozone.

   - (a) C^i and F^i to F^i: consumers and companies buy goods and services
   - (b) G^i to F^i: the government buys goods and services, launches expense programmes

(5) Coupons

   - (a) C^i to B^i: mortgage and other financing payments, credit card debt payments
   - (b) B^i to B^i and I^i: coupons for MBS and ABS and other fixed income securities, also included are CDS premiums
   - (c) F^i to B^i and I^i: companies pay coupons to the bond holders
   - (d) G^i to I^i and B^i: coupons of government bonds such as the US Treasury bonds, German Bunds and British Gilts

(6) Salary, pension: B^i, F^i, I^i and G^i to C^i

(7) Contributions (e.g. pension fund): C^i to I^i

(8) Tax: C^i, F^i, B^i, I^i to G^i

A third category of cash flows is neither at will, nor scheduled, but *contingent*, in the sense that they are mandatory and their level depends on the state of certain economic variables at specific points in or periods of time (often, but not necessarily, prescribed at the inception of the corresponding contract). Most notable examples are quantitative easing (central bank's buying the government's debt, so G^i to G^i if the central bank is grouped with the government, or increase in money supply in the system with an invisible source if it is considered super-systemic) and derivative pay-off, such as CDS payouts in case of a credit event (B^i to B^i, I^i). Figure 1 is an illustration of the cash flows among agents.

When a pair of economies i and j interact in an international market, there are cash flows between agents from each country in addition to the ones above:

(9) International investment

   - (a) B^i to/from B^j: interbank lending and investment
   - (b) B^i to/from G^j: B^i invests in G^j's bonds and G^j pays B^i interest and principal.

(10) International consumption and trade

   - (a) C^i to F^j: direct consumption by C^i of goods and services produced by F^j, such as tourism
   - (b) F^i to/from F^j: companies do business one another, such as import and export

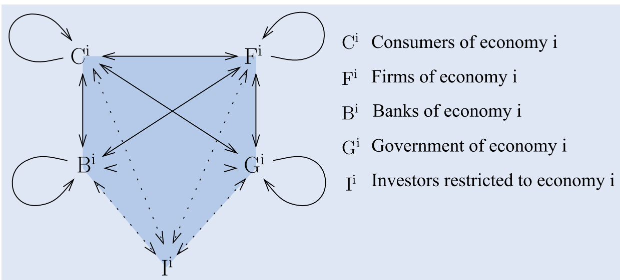


Figure 1. Combined cash flows among five agents in economy i.

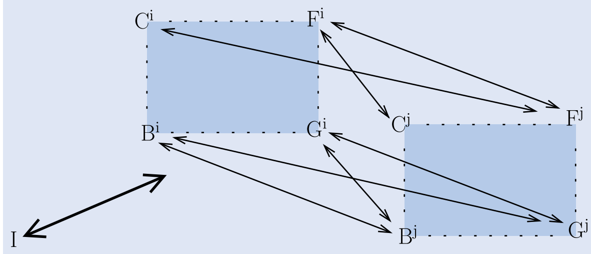


Figure 2. Cash flows between domestic agents of economy i and j. The supernational I interacts with all agents from economies i and j.

In the above classification, we consider foreign branches of a parent company as part of the economy where the branch is domiciled. For example, suppose company A is headquartered and registered in country i and has a branch in country j, then the branch is part of F^j and its local employees are part of C^j. The consumption of company A's goods and service by C^j is considered as a cash flow from C^j to F^j, and not to F^i, even when the consumed goods are produced in country i, for they will be imported by the company's local branch in country j. The import/export cash flows between the headquarter and the local branch are considered as a trade between F^i and F^j. On the other hand, if tourists from country i spend money in country j, then it is a direct consumption of goods produced by F^j by C^i.

Lending at private level is treated the same manner. If consumers or firms in country i borrow money from a bank of country j through its branch in i, then the cash flow here is B^i to C^i or B^i to F^i. However, at *sovereign* level, we assume *only* direct lending between banks and the governments, hence if the government of country i sells bonds to a bank headquartered in country j, then it is a cash flow from B^j to G^i, whether the local branch of the bank in country i is involved in the transaction or not.

Private investment is classified in the same manner. If consumers in country i invest in securities of corporations in country j (here i is not necessarily different from j), then there are two cash flows involved, one between C^i and I^i, and the other between I^j and F^j. This assumption implies that even within one economy, consumers invest in other agents through asset managers. Note that there exist private investment from C^i to C^j, such as investment in real estate and other goods, but they are not significant compared with other cash flows, so we ignore them and assume there is no interaction between C^i and C^j for $i \neq j$.

Figure 2 is an illustration of the cash flows between agents.

There are many other kinds of cash flows and our selection is just representative ones. Nevertheless, it would be enough to explain the interactions economic agents which is a key component of our construction and analysis of a dynamical system of wealth.

3. Spreading financial risk: contagion

3.1. Intuition for contagion

A qualitative definition of financial contagion is not hard to formulate. For example, one could define financial contagion as the dynamics, whereby a crisis in a given economic system (e.g. a country) spreads to other systems. Among the most recent examples, we would like to point out the current Eurozone sovereign credit crisis as well as the 2007–2009+ US credit crisis.

To investigate the spread of credit and economic distress within the conceptual framework laid out in the previous section, we divide Eurozone countries into two groups, Group 1 of fiscally weaker economies in the periphery that have experienced the sovereign debt crisis, e.g. Greece, Ireland, Portugal, Cyprus and possibly Spain, and Italy, and Group 2 of their creditor countries, i.e. the home countries of the banks which are heavily exposed to the sovereign debts of the former. Consider a peripheral economy i (e.g. Greece) and a creditor economy j (e.g. France). The main way in which the 'Eurozone contagion' is taking place is through the perception that the sovereign of economy i cannot fulfil its obligation to the banks of economy j, i.e. that the cash flow from G^{i} to B^{j} falls below its obligatory level. In such a event, the equity level of B^{j} would plunge and it is very likely that several banks would experience bank runs or even go bankrupt. This can trigger a severe economic instability or even a financial crisis in economy j (figure 3).

Once this happens, there can be a further contagion to the rest of the world (outside the Eurozone) which we call Group 3. Consider an economy k whose banks are the major counterparties of B^{j} (e.g. the US). Troubles in B^{j} affects the credit conditions of not only other banks in the Eurozone but also of B^{k}, which would drastically reduce lending in both the financial and real sectors of the relevant economy. This could lead to a global credit crunch and subsequent shrinkage of the global economy, similar to the crisis that ensued from Lehman Brothers' bankruptcy, but far greater in magnitude (figure 4).

Difficulties immediately arise when one attempts to formulate a definition of contagion that would adequately model the scenario outlined above. Here are a few questions it is natural to ask (cf. also Karolyi 2003)

(a) How does one detect a crisis in an economic system?
(b) How does one determine causation between one crisis and another? Is succession in time sufficient?

Below, we will address these questions along with a quantitative definition of contagion within our dynamical systems framework. To do so, we will devote the next section to establishing a dynamical systems of the wealth of the global market economy.

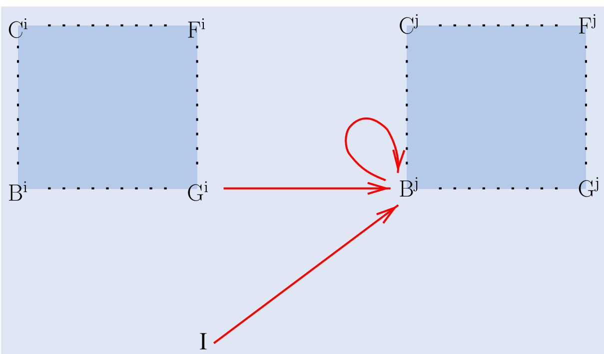


Figure 3. Contagion of financial crisis from a debtor economy i to a creditor economy j.

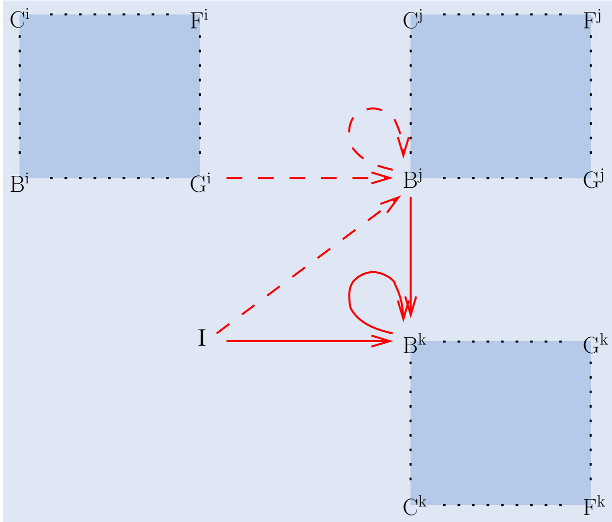


Figure 4. Contagion spilled to outside of the Eurozone, from economy j to its non-Eurozone counterparty k.

3.2. Dynamical system of wealth for multiple economies

In this section, we extend the multi-agent dynamic model for a single economy proposed by Choi and Douady (2012, 2013) to a system of multiple economies. Their results and mathematical details are summarized in appendix A.1, and in this section we use them with only necessary modifications.

In order to generalize that framework is sufficient enough to accommodate the joint modelling of multiple economic systems, we slightly modify the underlying assumptions as follows. Our model assumes a continuous-time dynamical system (X, f)

$$\begin{aligned} f : \mathbb{R} \times X &\longrightarrow X \\ (t, x) &\longmapsto f^t(x) \end{aligned} \tag{1}$$

where $\mathbb{R}$ thereby acts as a group on the group of endomorphisms of X. For the rest of this article we will assume that $X \subset \mathbb{R}^n$,† that it is open and connected, and the action is continuous and differentiable up to a set of isolated points‡ S so that, in particular, the maps f^t admit Jacobians everywhere except for S. At every $x \in S$, the differentiability is relaxed to admit only left and right partial derivatives. Further, we assume that $\bar{M} \subset X$, where $\bar{M}$ is the set of feasible wealths (cf. appendix A.1). To model the partition of the global system into subeconomies, we also assume such system can be decomposed into the product of s factors, namely (cf. Brin and Stuck (2003, Sec. 1.1)),

$$\begin{aligned} f = \prod_{k=1}^{s} f_k : \mathbb{R} \times \prod_{k=1}^{s} X_k &\to \prod_{k=1}^{s} X_k \\ (t, x_1, \ldots x_s) &\mapsto (f_1^t(x_1), \ldots, f_s^t(x_s)) \end{aligned} \tag{2}$$

Each factor space X_k is a subset of $\mathbb{R}^{n_k}$, $X_k \subset \mathbb{R}^{n_k}$.§ Most of our study will be focusing on orbits of the dynamical system of wealths. Such an orbit will be denoted with $f^t(x_0) = \mathbf{w}(t_0+t)$,

†If more attention needs to be devoted to the orbit topology or similar problems, one may let X be a topological or smooth manifold.
‡The topology on X is that induced by the standard Euclidean one generated by n-dimensional balls.
§Note that the X_k's will be also open and connected, which follows from the openness of the canonical projections.

where the initial point $x_0 = \mathbf{w}(t_0)$, is left unspecified.† The components of the orbit are denoted as $f_k^t(x_0) = \mathbf{w}^k(t_0+t)$.‡ Finally, since most of our attention is devoted to discrete orbits, we clarify that these are obtained by canonically embedding the integers into the reals, $\mathbb{Z} \hookrightarrow \mathbb{R}$, that is, a discrete dynamical system is obtained by considering $f(c, x) = f^c(x)$ and the corresponding factors for all $c \in \mathbb{Z}$.

In our model, the evolution of the financial market is the forward orbit $\{f^t(\mathbf{w}(t_0))\}_{t>0}$ of $\mathbf{w}(t_0)$ for some t_0, market equilibria are the fixed points of f, and a financial crisis§ is a propagation of a negative wealth shock throughout the system. When an agent, say i, experiences an abrupt change Δw_i (a shock) on its wealth, the original dynamical system f is perturbed to produce a new one. The scope of Δw_i induces a one-parameter family of maps $\{f_\mu\}$, and the leverage of agent i determines the size of the perturbation: if its leverage is small enough, then f_μ stays close to f¶; if too big, then some equilibria may not withstand the perturbation and their stability types change, i.e. a bifurcation has taken place. All agents are interconnected via cash flows, and a negative wealth shock on a highly leveraged agent spreads to others, lowering the wealth level of all agents. Therefore $\mathbf{w}(t)$, which used to be in the basin of attraction of a stable equilibrium $\mathbf{p}$ of f,‖ now repels away from an unstable equilibrium p_μ of f_μ. Because of the compactness of $\bar{M}$, the forward orbit $\{f_\mu^t(\mathbf{w}(t_0))\}_{t>0}$ would eventually enter the basin of attraction of another stable equilibrium, say $\mathbf{q}_\mu$, that represents much lower wealth level than $\mathbf{p}$ for all agents. In financial terms this means that the market has entered a recession and the wealth levels of all agents do not go up without an intervention by monetary (and quite often fiscal as well) authorities. Thus, the monetary intervention by authorities is to perturb the dynamical system to break the stable equilibrium $\mathbf{q}_\mu$ and create an instability. The subspace N of the phase space X of f that corresponds to an ideal economy, i.e. where the authorities want the wealth $\mathbf{w}(t)$ to be,†† is locally invariant, $f(N) \subset N$, therefore dynamic instability within N necessarily produces a chaotic behaviour. At a stable equilibrium, the Jacobian matrix df has all eigenvalues absolute value less than 1 and the same is true for points in its basin of attraction. This the rationale why *the market instability indicator* $I(t)$ (Choi and Douady 2012) is defined as the spectral radius of the Jacobian matrix of the dynamical system of wealth.

We consider a collection of s economies such that the economy k is divided into n_k aggregates which we call 'economic agent'. The number of economies s and those of the agents may differ from case to case. In our Eurozone example in section 2.1, $n_k = 5$ for all $1 \le k \le s$ while s was not specified. At each time t, we observe $\mathbf{w}(t) = (w_1(t), \ldots, w_n(t)) \in \mathbb{R}^n$, the *global wealth vector* of the agents where $n = \sum_{k=1}^s n_k$. More precisely, $\mathbf{w}(t)$ is a canonical embedding of the respective wealth vector of each economy, $\mathbf{w}^k(t) = (w_1^k(t), w_2^k(t), \ldots, w_{n_k}^k(t))$, where $w_j^k(t)$ is the wealth of agent j of economy k at time t. Therefore $w_i(t) = w_j^k(t)$ if

$$i = N(k) + j, \qquad N(k) = \sum_{l=1}^{k-1} n_l \tag{3}$$

As a result, $w_i(t)$ for each i inherits the equity–debt wealth decomposition of $w_j^k(t)$ such that

$$w_i(t) = E_i(t) + D_i(t) = E_j^k(t) + D_j^k(t) \tag{4}$$

and the liquidity-invested asset decomposition

$$w_i(t) = L_i(t) + K_i(t) = L_j^k(t) + K_j^k(t) \tag{5}$$

The results of appendix A.1 apply to multi-economic system as well, yet we need to make some modification to accommodate the differentiable dynamical system we are considering. The original 'elasticity coefficient' (Choi and Douady 2012, 2013) was defined to be ‡‡:

$$a_{ij} = \frac{\partial F_{ij}}{\partial w_j} \tag{6}$$

or equivalently,

$$\Delta F_{ij}(t) = a_{ij}(t)\Delta w_j(t) \tag{7}$$

for all $1 \le i, j \le n$, including $j = i$. This definition assumes the existence of the partial derivative regardless of the sign of the wealth shock Δw_j, and works well when analysing the default mechanism of agents on the brink of and during a financial crisis. But on other occasions we can observe discontinuities of elasticity coefficients.§§ This is graphically illustrated in figure 5.

For our differentiable dynamical system, we will need one-sided elasticity coefficient as follows to accommodate this problem¶¶:

†This could be interpreted as the wealth in a base year as for many other macroeconomic variables.

‡Unless otherwise specified, we take our base year to be the origin of time, $t_0 = 0$. Accordingly, we will write $\mathbf{w}(t)$ and $\mathbf{w}^k(t)$ for the generic elements of a global and component orbit, respectively.

§This term is applied broadly to a variety of situations in which some financial assets suddenly lose a large part of their nominal value (Wikipedia 2005).

¶We use the C^0-topology on X defined by $d_{0,X}(f, g) = \sup\{|f(x) - g(x)| \mid x \in X\}$ for some metric $|\cdot|$ on X.

‖Usually wealth appreciation slows down after an exuberant growth phase (bubble phase), so we can assume $\mathbf{p}$ is stable.

††Hence this N is a subset of $\bar{M}$, the set of feasible wealths, mentioned above and in appendix A.1.

‡‡In Choi and Douady (2012, 2013) the differentiability of f was not specified but $\frac{\partial F_{ij}}{\partial w_j}$ was used in the sense of $\frac{\Delta F_{ij}}{\Delta w_j}$. In appendix A.1 is specifically mentioned that f is piecewise linear, and the elasticity a_{ij} is defined to be $\frac{\Delta F_{ij}}{\Delta w_j}$.

§§For example, when major US banks were bailed out after the collapse of Lehman Brothers and the following credit crunch, the lending to consumers and firms did not increase as much as the policymakers expected, for the banks hoarded cash instead of lending. Another example is the prevalent high unemployment rate: when firms' wealth decrease, they compensate it by laying off employees, however they are reluctant to hire people even when their profits go up. This behaviour was observed even during the bubble time: when consumers had extra disposable income due to house price appreciation and wage increase for example, they did not make more debt reimbursement than needed or increase savings. They instead spent the money on goods and services, increasing the cash flows C to F. When the bubble burst, many people could not pay-off their mortgage and other debt payments, thus the cash flows from C to B decreased.

¶¶Recall that we relaxed the differentiability condition on the set of isolated points $S \subset X$. The introduction of these lateral elasticities further specifies this condition.

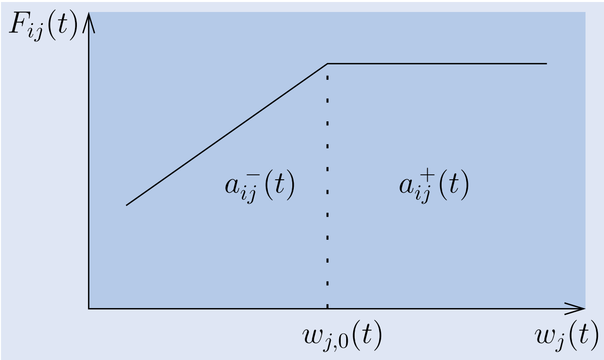


Figure 5. When the wealth $w_j(t)$ falls below the target wealth $w_{j,0}(t)$, the cash outflow $F_{ij}(t)$ from j to i decreases, but it stays flat when $w_j(t) > w_{j,0}(t)$.

$$\begin{cases} a_{ij}^{+}(t) = \lim_{\Delta w_j(t)\to 0^+} \dfrac{F_{ij}(w_j(t)+\Delta w_j(t)) - F_{ij}(w_j(t))}{\Delta w_j(t)} \\ a_{ij}^{-}(t) = \lim_{\Delta w_j(t)\to 0^-} \dfrac{F_{ij}(w_j(t)+\Delta w_j(t)) - F_{ij}(w_j(t))}{\Delta w_j(t)} \end{cases} \tag{8}$$

or equivalently,

$$\begin{cases} \Delta F_{ij}(t) = a_{ij}^{+}(t)\Delta w_j(t) & \text{if } \Delta w_j(t) > 0 \\ \Delta F_{ij}(t) = a_{ij}^{-}(t)\Delta w_j(t) & \text{if } \Delta w_j(t) < 0 \end{cases} \tag{9}$$

Depending on the context we choose an appropriate one-sided limit as the elasticity and write $a_{ij}(t)$ without the sign.

Like the case of a single economy, the entries of the Jacobian matrix b_{ij} are related to the elasticities a_{ij} of the dynamical system as follows (see also equations (58) and (59)):

$$b_{ij} = a_{ij} + \delta_{ij}\left(1 - \sum_{k\neq j} a_{kj}\right) \tag{10}$$

which in matrix form reads

$$B = A + I - A^{\sharp} \tag{11}$$

where $A^{\sharp} = \operatorname{diag}\left(\sum_{k\neq 1} a_{k1}, \sum_{k\neq 2} a_{k2}, \ldots \sum_{k\neq n} a_{kn}\right)$.

We extend the indexing in equation (3) to show how the global system and individual economies are related, and indeed the individual subsystem can be naturally embedded into the global system.

The cash flow $F_{ij}(t)$ moves from agent j to i at time t. When the agents belong to the same economy, say k, and we need to focus on that particular economy, we specify it by an upper index:

$$F_{N(k)+i,N(k)+j}(t) = F_{ij}^{k}(t) \tag{12}$$

We do the same thing for the Jacobian matrix and elasticity matrix:

$$b_{N(k)+i,N(k)+j}(t) = b_{ij}^{k}(t) \tag{13}$$

and

$$a_{N(k)+i,N(k)+j}(t) = a_{ij}^{k}(t) \tag{14}$$

where $B^{(k)}(t) = \left(b_{ij}^{k}(t)\right)$ is the Jacobian matrix of economy k and $A^{(k)}(t) = \left(a_{ij}^{k}(t)\right)$ is the elasticity matrix of economy k.

To track the interaction between agents from different economies, we use both upper index (for the economy) and lower index (for agents). Thus, we denote the cash flow from agent j of economy l to agent i of economy k at time t by $F_{ij}^{kl}(t)$, which is in terms of the global system can be written as†

$$F_{ij}^{kl}(t) = F_{N(k)+i,N(l)+j}(t) \tag{15}$$

Therefore, we get the elasticity $a_{ij}^{kl}(t)$ for discrete f as

$$\begin{aligned} a_{ij}^{kl}(t) &= \frac{F_{ij}^{kl}\left(w_j^l(t) + \Delta w_j^l(t)\right) - F_{ij}^{kl}\left(w_j^l(t)\right)}{\Delta w_j^l} \\ &= \frac{\Delta F_{ij}^{kl}(t)}{\Delta w_j^l(t)} \end{aligned} \tag{16}$$

where $w_j^l(t)$ is the wealth of the jth agent in the lth economy. The elasticities for continuous f can be obtained by taking one-sided limits of equation (16) as $\Delta w_j^l(t) \to 0$.

In a global economy with $n = \sum_{k=1}^{s} n_k$ agents, the wealth of each agent is defined using the cash flows between all n agents,

$$w_i(t+1) = w_i(t) + \sum_{j=1}^{n} F_{ij}(t) - \sum_{\substack{k=1\\k\neq i}}^{n} F_{ki}(t), \tag{17}$$

and this wealth level is inherited by each subeconomy. In other words, we do not use only the agents in a subeconomy to calculate the wealths of agents in that subeconomy. As a result, the $w_i(t)$ in equation (17) is used to calculate the local elasticity matrix $A^{(k)}(t)$ and local Jacobian matrix $B^{(k)}(t)$ as well as the global elasticity matrix $A(t)$ and global Jacobian matrix $B(t)$. Therefore, we have the following canonical embedding of local matrices into the global ones:

$$A(t) = \left(\begin{array}{c:c:c:c} A^{(1)}(t) & A^{(12)}(t) & \ldots & A^{(1s)}(t) \\ \hdashline A^{(21)}(t) & A^{(2)}(t) & & \\ \hdashline \vdots & & \ddots & \\ \hdashline A^{(s1)}(t) & \ldots & & A^{(s)}(t) \end{array}\right) \tag{18}$$

and

$$B(t) = \left(\begin{array}{c:c:c:c} B^{(1)}(t) & A^{(12)}(t) & \ldots & A^{(1s)}(t) \\ \hdashline A^{(21)}(t) & B^{(2)}(t) & & \\ \hdashline \vdots & & \ddots & \\ \hdashline A^{(s1)}(t) & \ldots & & B^{(s)}(t) \end{array}\right) \tag{19}$$

†Notice that not to overburden notation we have not adopted the double superscripts for a single economy, so that we are implicitly defining $F_{ij}^{kk}(t) = F_{N(k)+i,N(k)+j}(t) = F_{ij}^{k}(t)$. A similar stylistic remark applies to the elasticity and Jacobian matrices.

where

$$A^{(kl)}(t) = \left(a_{ij}^{kl}(t)\right)_{\substack{1\le i\le n_k\\ 1\le j\le n_l}}. \tag{20}$$

3.3. *Quantitative definition of contagion*

When modelling an economic system through theories of dynamical systems, Choi and Douady (2012) introduced a market instability indicator, namely the spectral radius of the Jacobian matrix of the wealth dynamical system. In symbols this market instability indicator can be defined as

$$I(t) := \rho\,(B(\mathbf{w}(t))) \tag{21}$$

where $\rho(M)$ denotes the spectral radius of a matrix $M \in \mathbb{R}^{n\times n}$, and B is the Jacobian matrix of the dynamical system f at $\mathbf{w}(t)$. (See appendix A.2 for details.)

Note that this indicator is an early warning system of a financial crisis, viz., a given economic system is liable to enter a crisis iff $I(t) > 1$ (cf. also Hollo *et al.* 2012).†

Consistently with the above observations, we assume that for contagion to take place it is necessary that

$$B(t) \neq \oplus_{k=1}^{s} B^{(k)}(t), \tag{22}$$

i.e. $B(t)$ is not a block-diagonal matrix, for otherwise a given system would be insensitive to any other. The reducibility of the global Jacobian matrix to a direct sum of its constituent's subsystems can be regarded as an analogue of the independence of random risk factors. In order to account for the dynamic feature of contagion (i.e. the fact that a system passes from a state of normalcy to one of crisis), we demand that initially no instability is observed at the local (subsystemic) as well as global level. Then at least one of the subsystems enters a state of crisis, finally, at yet another time, instability propagates to the entire system, and this occurs when the transmission of risk is possible, i.e. the Jacobian matrix is not totally reducible to its components. This motivates the following definition:

Definition 3.1 (i-Contagion) We say that *the contagion of financial instability*, or *i-contagion*, in a global economic system occurs if given two time instants t_0, t_1 with $0 < t_0 < t_1$,‡ one has

(i) At time $t < t_0$, $\max_k \rho(B^{(k)}(t)) < 1$ and $\rho(B(t)) < 1$.
(ii) At time $t \in (t_0, t_1)$, $\max_k \rho(B^{(k)}(t)) > 1$ and $\rho(B(t)) < 1$.
(iii) At time $t > t_1$ $B(t) \neq \oplus_{k=1}^{s} B^{(k)}(t)$ and $\rho(B(t)) > 1$.

Remark 3.1 The last condition in the definition is meant to model the causal feature of contagion, that is, the fact that a global crisis ensues from risk transmission from one subsystem to another. We rule out simultaneous crises that are unrelated. Notice that when $B(t) = \oplus_{k=1}^{s} B^{(k)}(t)$ one has $\rho(B(t)) = \max_k \rho(B^{(k)}(t))$, hence without the last condition a global crisis could arise from the independent occurrence of subsystemic crises.

Remark 3.2 Since we chose to leave stability undetermined for a spectral radius precisely equal to 1, we omit the end point of the time interval from our definition in case the Jacobian matrix evolves continuously.

Remark 3.3 The definition of i-contagion applies to a system of economies from different currency zones by converting the cash flows to a major currency such the US dollar or the Euro. It can also be used to investigate contagion in a single economy by considering partitions of agents as subsystems. The next section provides more details about this i-contagion in a single economy.

4. Mechanism of instability contagion

In this section, we will study the contagion of instability in different settings. For all cases we will use the well-known result that the trace of a square matrix is the sum of its eigenvalues:

$$\mathrm{tr}\,(M) = \sum_{i=1}^{n} \lambda_i, \tag{23}$$

which implies

$$\frac{|\mathrm{tr}\,(M)|}{n} \le \max_{\lambda_i\in\sigma(M)} |\lambda_i| = \rho(M) \tag{24}$$

where $\sigma(M)$ denotes the spectrum of M. Applying this to equation (10), we get the following lower bound of the market instability indicator $I(t) = \rho(B(t))$,

$$\left|1 + \frac{1}{n}\sum_{i=1}^{n} a_{ii}(t) - \frac{1}{n}\sum_{i\neq j} a_{ij}(t)\right| \le \rho(B) \tag{25}$$

When an agent is highly leveraged, a sudden negative shock on its wealth can affect the outgoing cash flow of the agent, and as we witnessed during the US subprime crisis, the financial distress can transmit across both the real and financial sectors within an economy, eventually leading to a financial crisis. This kind of instability contagion may happen cross-border, and this is the main concern about the Eurozone sovereign credit crisis. We will show that the sign and size of the diagonal entries of the Jacobian matrix of the wealth dynamical system play a crucial role in the emergence of an instability contagion.

First, we start with a lemma that can be applied to both growth phase and crisis period of an economy.

Lemma 4.1 *Let $f : I \to \mathbb{R}$ be a function defined on an open set $U \subset \mathbb{R}$ and assume its first derivative exists on an interval $I \subset U$ and select t_0 and t_1 from I with $t_0 < t_1$, then*

(a) *If f is strictly increasing or strictly decreasing, then* $\frac{f'(t_1)}{f'(t_0)} > 0$
(b) *If f is strictly increasing and convex, then* $\frac{f'(t_1)}{f'(t_0)} > 1$
(c) *If f is strictly decreasing and concave, then* $\frac{f'(t_1)}{f'(t_0)} > 1$

Proof (a): Throughout we pick an arbitrary $\theta > 0$ such that $t + \theta$ stays in I. If f is increasing in t, then both $f'(t)$ and $f'(t+\theta)$ are positive. Hence $\frac{f'(t+\theta)}{f'(t)} > 0$. If f is decreasing in t, then both $f'(t)$ and $f'(t+\theta)$ are negative, which yields the same result.

(b) and (c): If f is increasing and convex, then $f'(t+\theta) > f'(t) > 0$, hence $\frac{f'(t+\theta)}{f'(t)} > 1$. If f is decreasing and concave, then $f'(t+\theta) < f'(t) < 0$, hence $\frac{f'(t+\theta)}{f'(t)} > 1$. □

The convexity for the increasing case is to model the accelerating wealth of (some) agents, which can be observed

†The behaviour of an economic system with spectral radius precisely equal to 1 is left undefined.
‡We assume the origin of time in our model is 0.

during a growth phase, whereas the concave decreasing case can model a crisis period when the wealth level decreases with acceleration. We will use the decreasing case repeatedly to establish a lower bound of the market instability indicator on the brink of or during a financial crisis.

4.1. Contagion within an economy

Consider a 5-agent model with consumers (C), firms (F), banks (B), government (G) and investors (I). The central bank is considered as a super-systemic agent and excluded from the system.† The wealth of agent i is

$$w_i(t+1) = w_i(t) + \sum_{j=1}^{5} F_{ij}(t) - \sum_{k\neq i}^{5} F_{ki}(t) \tag{26}$$

where F_{ij} is a cash inflow from j to i (e.g. investment, wage) and $-F_{ki}$ is a cash outflow from i to k (e.g. consumption, withdrawal). Often this F_{ij} is a voluntary, at-will investment while $-F_{ki}$ is necessary or required expenditure. Assume that consumers are highly leveraged, i.e. borrowed heavily using their invested assets K_1 as collateral when it was appreciating rapidly.‡ Further assume that the growth of K_1 passed its peak and K_1 has been declining. We make the same assumption for w_1.

By Lemma 4.1 and chain rule, $b_{11}(t) = \dfrac{\partial w_1(t+1)}{\partial w_1(t)} = \dfrac{w_1'(t+1)}{w_1'(t)} > 0$ where the prime $'$ means differentiating with respect to time t. If $w_1(t)$ is concave in t as well, then $b_{11}(t) > 1$ by Lemma 4.1 part (c). The size of the cash outflow from 1 to other agents gives further information on the lower bound of the spectral radius of the Jacobian, i.e. the market instability indicator.

Recall that $b_{11} = 1+a_{11}-\sum_{k=2}^{5} a_{k1}$ and $a_{k1}(t) = \dfrac{\partial F_{k1}(t)}{\partial w_1(t)}$. As the wealth of consumers decreases, the outgoing cash flow may change accordingly as the following analysis suggests:

(a) a_{21}: F_{21} is what consumers should pay to firms, i.e. consumption. As consumers' wealth decreases consumption goes down as well, but it will level soon because there are basic expenditures. Hence a_{21} becomes almost 0 eventually.

(b) a_{31}: F_{31} is consumers' loan payment including interests and fees.§ When consumers' wealth decreases so does the value of the collateral used to take the loan, hence the payment F_{31} increases. Delinquent loans will increase the interest and penalty to make F_{31} increase even more. Therefore $a_{31} < 0$.

(c) a_{41}: F_{41} is consumers' tax paid to the government. When consumers' wealth decreases, their tax, which consists mostly of income tax and property tax, to the government decreases. This implies $a_{41} > 0$.

(d) a_{51}: F_{51} is what consumers are obliged to pay the investors.¶ Since investors function to consumers as portfolio managers rather than lenders, there is no particular change in F_{51}. This implies $a_{51} \sim 0$.

The figures in figure 6 show representative relations between F_{k1} and $w_1(t)$.

Then

$$\begin{aligned} b_{11} &= 1 + a_{11} - a_{21} - a_{31} - a_{41} - a_{51} && (27)\\ &\approx 1 + a_{11} - 0 - a_{31} - a_{41} - 0 && (28)\\ &= 1 + a_{11} - a_{31} - a_{41}. && (29) \end{aligned}$$

As defined in equation (57) $a_{11} = \dfrac{\partial(\gamma_1(t)K_1(t))}{\partial w_1(t)}$, and by equation (42),‖

$$a_{11}(t) = \frac{K_1'(t+1) - K_1'(t) - \tilde{\Delta}K_i'(t)}{w_1'(t)}. \tag{30}$$

If $K_1(t)$ is decreasing and concave in t, then $K_i'(t+1) < K_i'(t)$. The conversion $\tilde{\Delta}K_1(t)$ of liquidity $L_1(t)$ to invested assets $K_1(t)$ would be negative at the beginning since consumers would liquidate their invested assets to make increased payments to banks, but there is a limit to doing so and it will level off eventually. Therefore $\tilde{\Delta}K_i'(t) \sim 0$. Hence, the numerator of equation (30) is negative, and the denominator $w'(t)$ is negative by our assumption, so $a_{11}(t) > 0$. In equation (27), a_{31} and a_{41} are of the opposite sign, yet the debt payment to banks tends to increase faster than the tax savings due to decreased wealth. As a result, if both $w_1(t)$ and $K_1(t)$ are decreasing and concave, $b_{11}(t)$ is not only greater than 1, but its size is pushed up as $|a_{31}(t)|$ increases.

By equation (24) the market instability indicator $\rho(B(t))$ is bounded below by $\dfrac{|\mathrm{tr}\,(B(t))|}{n}$, and a large b_{11} alone does not guarantee that the indicator goes above 1. Yet when we partition the economy into two groups, Consumers-Banks (Partition 1) and Firm-Government-Investors (Partition 2), the lower bound of Partition 1's market instability indicator $\rho(B^{(1)}(t))$ is $\dfrac{|b_{11}+b_{33}|}{2}$ and it is very possible that a large b_{11} can make the lower bound greater than 1 regardless of the size of b_{33}, making $\rho^{(1)}(B(t)) > 1$ while $\rho(B(t)) < 1$. This does not automatically imply a financial crisis in the entire economy, though. As long as consumers can make loan payments to banks despite decreased wealth, Partition 1 will maintain an unstable equilibrium. However, if they mass default on their payments, hence the wealth w_3 of banks gets reduced, there can be a contagion of instability from Partition 1 to the entire economy.

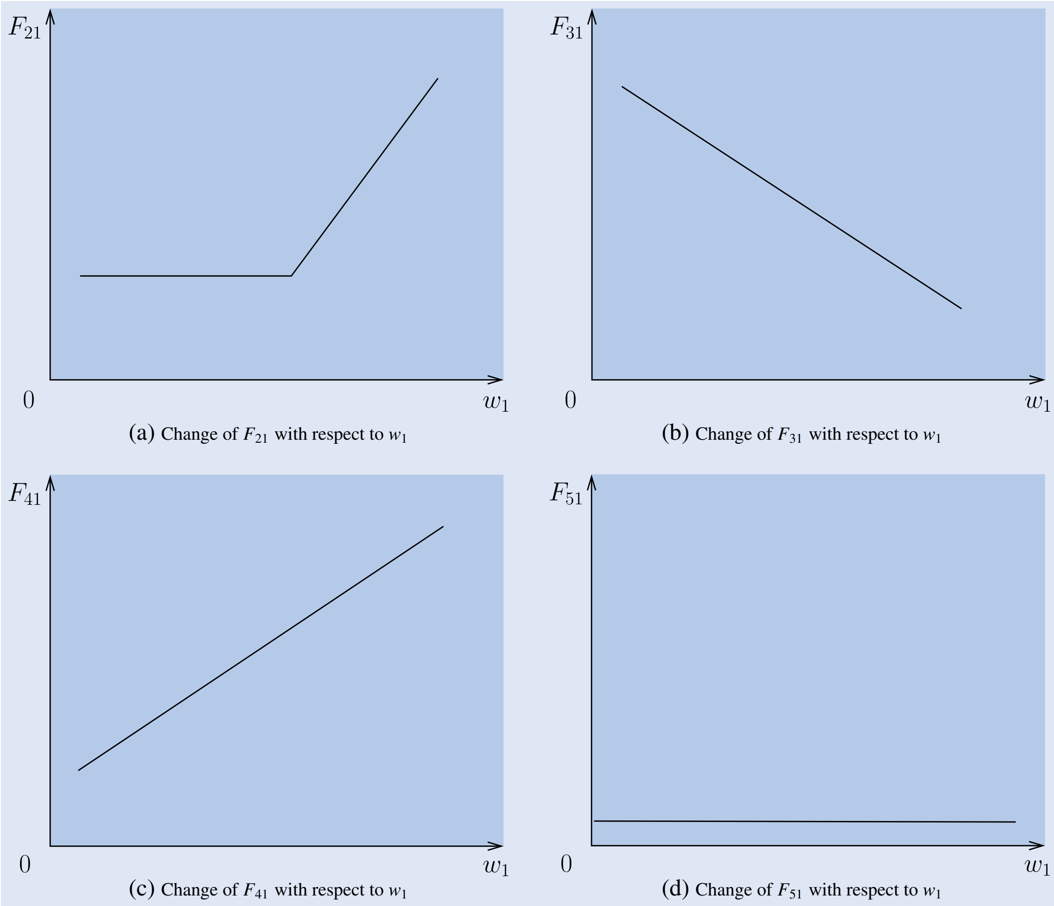
The diagonal entry of banks is

$$b_{33} = 1 + a_{33} - a_{13} - a_{23} - a_{43} - a_{53}. \tag{31}$$

The elasticities a_{13}, a_{23} and a_{53} are related to consumers', firms', and investors' withdrawal from banks, F_{13}, F_{23} and F_{53},

†One can see that, as a contagion develops, the economy indeed needs a super-systemic intervention to avoid a total collapse.
‡This was the case of the US, Spain and Ireland at the peak of their respective real estate bubbles.
§Consumers' investment in banks (deposit) is represented by the off-diagonal element $a_{31} = b_{31}$ of B.

¶Like bank deposit, consumers' investment is represented by the off-diagonal a_{51}.
‖The dynamics of K_i is defined to be discrete in equation (42), and when we assume continuous f, equations (41)–(43) should be modified accordingly. Since the mechanism of instability contagion is our main focus in this article, we leave out the mathematical details on the dynamics of D_i, K_i and L_i in continuous case.



(a) Change of F_{21} with respect to w_1

(b) Change of F_{31} with respect to w_1

(c) Change of F_{41} with respect to w_1

(d) Change of F_{51} with respect to w_1

Figure 6. Representative cash outflows from agent 1 as functions of its wealth.

respectively, and are negative in sign since those withdrawals increase as the banks' wealth decreases (and sometimes leads to a bank run). The only positive element in equation (31) is a_{43} which is related to the tax paid to the government. Once the banks' asset started to fall due to a mass delinquency by their debtors, its decline accelerates, i.e. $w_3''(t) < 0$. If loan delinquency continues, very likely the banks' invested asset $K_3(t)$ that includes all debt writedowns becomes decreasing and concave in t, and as is the case of consumers, liquidating invested assets has a limit, hence $a_{33} > 0$. Even if $K_3(t)$ is not concave, the size of $|a_{13}|$, $|a_{23}|$, and $|a_{53}|$ can push up b_{33}.† At this stage the economy could be in a state of financial crisis and other agents' wealths have been declining as well. Hence $b_{ii} > 0$ for all $i \neq 3$, so it is very possible that the market instability indicator $\rho(B(t))$ becomes greater than 1.

As was mentioned in the case of consumers, as long as banks manage to pay their interbank loan payments and continue lending at normal level despite their wealth reduction, the economy will maintain an unstable equilibrium, but in reality, at this stage many banks would have a serious liquidity problems and may face bankruptcy. This is indeed the case of Bear Stearns and Lehman Brothers. With or without bankruptcy, severe credit squeeze, represented by decreased $b_{i3} = a_{i3}$ in the off-diagonal of $B(t)$, would be inevitable as banks reduce lending. Reduced credit means reduced operating cost for firms, which leads to mass lay-off of workforce and eventually mass bankruptcy. This would result in severe reduction of wealth and cash flows of every agent, a total collapse of the economy. This is contagion of instability to the entire economy. It was at this stage during the US subprime crisis that the government and the central bank stepped in to bailout banks and started various monetary policies and operations some of which still continue to date (Wikipedia 2008).

4.2. Contagion within multiple economies

4.2.1. Contagion by default. Now we investigate contagion in more than one economies. We consider two economies only, for more economies can be dealt with in a similar manner. Consider two economies 1 and 2, each of which is modelled with the same five agents as in section 4.1. We index the agents as in equation (3), hence consumers in Economy 2 is assigned 6, firms 7, banks 8, government 9 and investors 10.

Assume that the government of Economy 1 (agent 4) has difficulty in paying back loans to the banks in Economy 2 (agent 8). The diagonal entry of agent 4 is

$$b_{44} = 1 + a_{44} - \sum_{k \neq 4}^{10} a_{k4}. \qquad (32)$$

The elasticities $-a_{14}$, $-a_{24}$, $-a_{64}$, $-a_{74}$, $-a_{94}$ and $-a_{10,4}$ are related to $-F_{14}$, $-F_{24}$, $-F_{64}$, $-F_{74}$, $-F_{94}$ and $-F_{10,4}$, respectively, which are the money that consumers and firms

†These elasticities are stochastic, but their probability distributions depend on the specific economy under consideration.

of Economy 1, consumers, firms, government and investors† of Economy 2 'take' from the government of Economy 1, and are almost zero. Thus, $-a_{14}$, $-a_{24}$, $-a_{64}$, $-a_{74}$, $-a_{94}$, and $-a_{10,4}$ are almost zero. The remaining terms in equation (32) are $-a_{34}$, $-a_{54}$, and $-a_{84}$ which are related to the obligatory payments of the government of Economy 1 to its banks and investors, and the banks in Economy 2, respectively. As shown in section 4.1, the loan payment of an agent increases as its wealth decreases. So a_{34}, a_{54}, and a_{84} are all negative and large in magnitude. When the government has trouble in paying back its loans, the other agents in the economy are very likely to have experienced wealth decrease, therefore $b_{ii} > 0$ for all $1 \le i \le 5$. This means that very likely the local instability indicator $\rho(B^{(1)}(t)) > 1$.

If the government of Economy 1 indeed defaults on its payment to the banks of Economy 2, then the wealth of agent 8 is reduced and many of its panicked investors would take out their investment, possibly causing a bank run. Then, the same analysis as in section 4.1 applies to show that $\rho(B^{(2)}(t))$ could go above 1. The global instability indicator $\rho(B(t))$ is bounded below by $\frac{|\mathrm{tr}\,(B(t))|}{n}$ and if b_{ii} for $i \neq 8$ are all positive or some are negative but small in absolute value compared with b_{88},‡ then very likely the global instability indicator $\rho(B(t))$ would go above 1. By definition the contagion of instability from Economy 1 to the global economy has taken place.

This cross-economy contagion analysis can be applied to model the current Eurozone sovereign credit crisis. Default by the governments of peripheral economies would cause serious write-offs for banks in creditor countries that could eventually cause bank runs and credit freeze. However, unlike the case of the US subprime crisis, the Eurozone governments do not have the power to implement monetary policies on their own. This problem has been in the center of the crisis and how to resolve it is still under debate.

4.2.2. Contagion by fear factor. Now we add a third economy, Economy 3, to the global system mentioned in the previous section. Assume that there is no interaction between Economy 1 and Economy 3, and the banks in Economy 2 have invested in the sovereign debt of both Economies 1 and 3. Since we add five more agents from Economy 3 we need to use the following equation to recalculate the wealth $w_i(t)$ for all 15 agents in the three economies,

$$w_i(t+1) = w_i(t) = \sum_{j=1}^{15} F_{ij}(t) - \sum_{k \neq i}^{15} F_{ki}(t) \qquad (33)$$

and the Jacobian matrix $B(t)$ and the elasticity matrix $A(t)$ should be recalculated using the new wealth. Yet the wealth $w_i(t)$ for $1 \le i \le 5$, the local Jacobian matrix $B^{(1)}(t)$, and elasticity matrix $A^{(1)}(t)$ remain unchanged because there is no interaction between Economies 1 and 3 by assumption.

†Recall that the investors are an international entity, and when we consider a single economy, we consider the part restricted to that economy. Therefore, investors in Economy 2—agent 10—interacts only with agents in Economy 2.

‡It is very probable that the diagonal entries of $B^{(1)}(t)$ are still high due to high the leverage of the agents in Economy 1.

Suppose the government of Economy 1 (agent 1) was eventually bailed out by international monetary authorities and their loans from the banks in Economy 2 (agent 8) have been restructured, and its local market instability indicator $\rho(B^{(1)}(t))$ is now less than 1. We further assume that the market fears that the government of Economy 3 (agent 14), although it currently is not going through any macroeconomic change, is following the path of Economy 1 and may cause another round of loan restructuring and write-offs. This fear would drive up the yield of the government bond of Economy 3 to a very high level. It was at this stage in the Eurozone sovereign credit crisis when the European Central Bank (ECB) stepped in: after the government bond yield of Spain and Italy rose to an unsustainable level, the ECB President Mario Draghi pledged that the ECB was ready to do 'whatever it takes' to preserve the Euro (*The Financial Times* 2012). After this announcement the market calmed down and the Spanish and Italian yields went back to a sustainable level.

However, we want to investigate what would happen if the sovereign bond yield of an economy remains high and its government cannot borrow enough at an affordable rate. Assume that the banks of Economy 2 (agent 8), the banks and investors of Economy 3 (agent 13 and 15, respectively) indeed reduce their lending to agent 14. This means reduced cash flows $F_{14,8}$, $F_{14,13}$, $F_{14,15}$ in equation (33) for $i = 14$. The government bond yield is a lower bound of other domestic interest rates, so the consumers and firms in Economy 3 (agent 11 and 12, respectively) have to pay higher interest on their loans. This means lower consumption and productivity, and eventually a lower tax revenue to the government which is represented by reduced $F_{14,11}$ and $F_{14,12}$ in equation (33) for $i = 14$. hence $\Delta w_{14}(t) < 0$. On the other hand, the government's payment to its lenders increase due to higher interest rate, which means $a_{ki} < 0$ for $i = 14$ and $k = 8, 13, 15$. Troubled real sector could result in unemployment and bankruptcies, so the government's benefit payment to consumers and firms would go up, which means $a_{ki} < 0$ for $i = 14$ and $k = 11, 12$. The diagonal element $b_{14,14}$ is formulated as (recall the assumption on no interaction between Economies 1 and 3)

$$b_{14,14} = 1 + a_{14,14} - \sum_{\substack{i=1 \\ i \neq 14}}^{15} a_{k,14} = 1 + a_{14,14} - \sum_{\substack{i=6 \\ i \neq 14}}^{15} a_{k,14} \quad (34)$$

and $a_{k,14} = 0$ for $k = 6, 7, 9, 10$ since there is hardly any payment obligation from the government of Economy 3 to the consumers, firms, government and investors of Economy 2, respectively. By Lemma 4.1 $b_{14,14} > 0$ and is very likely to be big due to the magnitude of $\sum_{\substack{i=6 \\ i \neq 14}}^{15} a_{k,14}$. Also by the same Lemma, it is very possible that $b_{ii} > 0$ for all other agents in Economy 3, hence $\rho(B^{(3)}(t)) > 1$. If $\rho(B) < 1$ still, this is Stage (ii) in the definition of i-Contagion. If $\rho(B)$ becomes greater than 1 after some time, for example because of a fear of default by agent 14 or because of the impact of the turmoil in Economy 3 to other economies, by definition, there has been a contagion of instability from Economy 3 to the global one and the cause of the local instability is a fear factor.

5. Conclusion

In this article, we modelled financial crises and contagion in a multi-agent system comprising multiple economies. We used theories of dynamical systems to give a quantitative definition of financial instability contagion via the *market instability indicator*—the spectral radius of the Jacobian matrix of a dynamical system of wealths—of both the subeconomies and the global one. Further, we analysed the mechanism of contagion, from one domestic partition to another or from one subeconomy to another.

Our model can be applied to the ongoing Eurozone sovereign credit crisis. Traditionally sovereign debts have been considered risk-free, but the Eurozone crisis has changed the paradigm. Global investments and trades have intertwined countries inside and outside of the Eurozone, and even distress in a relatively small economy can trigger a global-scale financial crisis through contagion.

More generally, the mathematical model we introduced can be used to analyse the structure of economic systems, track the conduits of risk transmissions and suggest methods to cordon off crises to prevent contagion. Testing the theory using actual data is the next step of our research.

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Appendix A. Results on one economy

A.1. Dynamical system of wealth

Recently Choi and Douady (2012, 2013) proposed a multi-agent model of an economy to analyse a financial crisis as breakage of economic stability (Benhabib 1992, Benhabib and Nishimura 1979). Their articles focus on the time period during which a crisis is seeded, ripens and fully emerges. This section refines their results and extends the timeline to analyse the aftermath of a crisis. Also defined in this section are notations used in this article. We keep their original approach and analyse the evolution of a dynamical system of wealth. Yet, our construction of the wealth dynamical system differs from theirs. We will construct a map directly by observing historical flow of funds such as (Flow of Funds Accounts of the United States 2012, National Income and Product Accounts Tables 2012), which is more realistic for verifying the model with real-life data (cf. Bê Duc and Le Breton 2009).

Given an economy, its participants are classified into n large aggregates called *agents*. The vector $\mathbf{w}(t) = (w_1(t), w_2(t), \ldots, w_n(t))$ is the wealth of the economy at time t where $w_i(t)$ be the wealth of the agent i at t. The global wealth $S(\mathbf{w}(t))$ is the sum of all wealths:

$$S(\mathbf{w}(t)) = \sum_{i=1}^{n} w_i(t) \tag{35}$$

Two assumptions on the economy are made:

Minimality A minimum number of agents are selected for the economy to function such that any removal of an agent would make the system collapse. Mathematically, this means that there is a minimum weight† $c > 0$ of each agent in the overall economy so that $w_i(t)/S(\mathbf{w}(t)) \geq c, \forall i$.

†Note that $w_i(t)/S(\mathbf{w}(t))$ represent the relative wealth or 'weight' of the ith agent with respect to total wealth.

Boundedness The economy is based on limited resources and market participants, therefore the production, consumption and the total wealth of the economy is bounded above and below. Mathematically, this means that there is time adjustment factor† $\alpha(t)$, some $C', C > 0$ such that $C' \leq S(\mathbf{w}(t))\alpha(t)^{-1} \leq C$.

Hence, the *normalized wealth vector* $\bar{w}(t) = \alpha(t)^{-1}w(t)$ stays inside a compact and convex subset of $\mathbb{R}^n$,

$$\bar{M} = \left\{ \bar{\mathbf{w}} \in \mathbb{R}^n \;\middle|\; C' \leq \sum_{i=1}^{n} \bar{w}_i \leq C, \quad w_i \geq c\,C' \quad \forall\, i = 1, \ldots, n \right\} \tag{36}$$

The wealth $w_i(t)$ of the agent i at t is defined to be the sum of the equity and debt,

$$w_i(t) = E_i(t) + D_i(t) \tag{37}$$

and also the sum of liquidities $L_i(t)$ (essentially equivalent to the monetary base, M0 in the case of the US) and invested assets $K_i(t)$ (financial securities, property, human resources etc.),

$$w_i(t) = L_i(t) + K_i(t) \tag{38}$$

As such, $L_i(t)$ produce no income, while $K_i(t)$ can produce capital gains. It is assumed that during the time period $[t, t+1]$, only $K_i(t)$ has an internal (i.e. independent of incoming or outgoing cash flows) growth that is measured by the *internal rate of return (IRR)* $\gamma_i(t)$ on the investment.

The increase in the liquidities is money-in less money-out plus cash raised from liquidation less new investment, therefore

$$\Delta L_i(t+1) = \sum_{j\neq i}^{n} F_{ij}(t) - \sum_{k\neq i}^{n} F_{ki}(t) - \tilde{\Delta} K_i(t+1) \tag{39}$$

where $F_{ij}(t)$ is the fund transferred from agent j to agent i at time t and $\tilde{\Delta}K_i(t+1)$ is equal to new investments less liquidating part of $K_i(t)$ to raise cash. The invested asset at the next time is

$$K_i(t+1) = (1+\gamma_i(t))\,K_i(t) + \tilde{\Delta}K_i(t+1) \tag{40}$$

where $\gamma_i(t)$ is the return on $K_i(t)$.

We observe in flow of funds data such as (Flow of Funds Accounts of the United States 2012) $L_i(t)$, $K_i(t)$, and $D_i(t)$. The $D_i(t)$, $K_i(t)$ and $L_i(t)$ evolve as follows‡ :

$$D_i(t+1) = (1+r_i(t))D_i(t) + \tilde{\Delta}D_i(t+1) \tag{41}$$

$$K_i(t+1) = (1+\gamma_i(t))K_i(t) + \tilde{\Delta}K_i(t+1) \tag{42}$$

$$L_i(t+1) = L_i(t) + \Delta L_i(t+1) \tag{43}$$

where

- $\tilde{\Delta}D_i(t+1)$ is equal to new net loans, i.e. new loans less payments.
- $r_i(t)$ is the average interest that applies to $D_i(t)$.

With equations (38), (39), and (43),

$$w_i(t+1) = w_i(t) + \gamma_i(t)K_i(t) + \sum_{j\neq i}^{n} F_{ij}(t) - \sum_{k\neq i}^{n} F_{ki}(t). \tag{44}$$

The internal return $\gamma_i(t)K_i(t)$ of the invested asset $K_i(t)$ can be interpreted as a result of 'self-investment', hence replaced by

$$F_{ii}(t) = \gamma_i(t)K_i(t), \tag{45}$$

hence equation (44) becomes

$$w_i(t+1) = w_i(t) + \sum_{j=1}^{n} F_{ij}(t) - \sum_{k\neq i}^{n} F_{ki}(t) \tag{46}$$

†This can be thought of as a deflator, such as the GDP deflator or the consumer price index (CPI).

‡The notation $\tilde{\Delta}$ denotes a contribution that is different from the ordinary time-increments, e.g. $\Delta D_i(t+1) = D_i(t+1) - D_i(t)$.

Remark A.1 Care should be taken on the nature of the flow of funds in the above equation. Both F_{ij} and $-F_{ki}$ are 'at-will', or 'required' flow of funds from i to k that change the wealth level of w_i. The investment of i in another agent k is a fund transfer from $L_i(t)$ to $K_i(t)$ and expressed as $\tilde{\Delta}K_i(t+1)$. This is an asset reallocation that does not change the wealth $w_i(t)$, and is not part of F_{ki} in equation (46).

Let $\mathbf{w}(t) = (w_1(t), w_2(t), \ldots, w_n(t))$ be the wealth of the economy. Now, we build a wealth dynamical system f that tracks the evolution of the $\mathbf{w}(t)$. First, we convert the observed variables $w_i(t)$ and $F_{ij}(t)$ to constant dollars by $\alpha(t)$ to obtain rescaled wealth $\bar{w}_i(t) = \alpha(t)^{-1}w_i(t)$ and rescaled flow of funds $\bar{F_{ij}} = \alpha(t)^{-1}F_{ij}$. Then define a map§

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^n \tag{47}$$

$$(\bar{w}_1(t), \bar{w}_2(t), \ldots, \bar{w}_n(t)) \longmapsto (\bar{w}_1(t+1), \bar{w}_2(t+1), \ldots, \bar{w}_n(t+1)). \tag{48}$$

By connecting $\mathbf{w}(t)$ and $\mathbf{w}(t+1)$ for each t in the period under consideration, we obtain a piecewise linear map. Since the set $\bar{M}$ defined in equation (36) represents all feasible wealth, we can consider $(\bar{M}, f)$ as our wealth dynamical system.

To ease the notation, we omit the 'bar' and write $w_i(t)$ for $\bar{w}_i$, $F_{ij}(t)$ for $\bar{F}_{ij}$ etc.

Given a continuous dynamical system

$$g : \mathbb{R}^n \longrightarrow \mathbb{R}^m \tag{49}$$

$$(x_1, x_2, \ldots, x_n) \longmapsto (g_1(x_1, x_2, \ldots, x_n), \ldots, g_m(x_1, x_2, \ldots, x_n)), \tag{50}$$

the Jacobian J of g is defined to be $\left(\dfrac{\partial g_i}{\partial x_j}\right)_{\substack{1\leq i\leq m \\ 1\leq j\leq n}}$. We apply this concept to the piecewise linear f to define a Jacobian matrix $df(t) = B(t) = \big(b_{ij}(t)\big)_{1\leq i,j\leq n}$ where

$$b_{ij}(t) = \frac{\Delta w_i(t+1)}{\Delta w_j(t)} \tag{51}$$

and $\Delta w_i(t)$ is a change to w_i at t.¶ A shift $\Delta\mathbf{w}(t) = (\Delta w_1(t), \Delta w_2(t), \ldots, \Delta w_n(t))$ of wealth from $\mathbf{w}(t)$ at time t induces a shift $\Delta\mathbf{w}(t+1) = (\Delta w_1(t+1), \Delta w_2(t+1), \ldots, \Delta w_n(t+1))$ at $t+1$. This means we have the following approximate equality

$$\Delta\mathbf{w}(t+1) = B(t)\Delta\mathbf{w}(t) \tag{52}$$

or equivalently, for any agent i:

$$\Delta w_i(t+1) = \sum_{j=1}^{n} b_{ij}(t)\Delta w_j(t) \tag{53}$$

The *elasticity coefficient* between two agents i and j is defined as the change rate of outgoing cash flow with respect to the wealth of the payer,

$$a_{ij}(t) = \frac{\Delta F_{ij}(t)}{\Delta w_j(t)} \tag{54}$$

or equivalently,

$$\Delta F_{ij}(t) = a_{ij}(t)\Delta w_j(t) \tag{55}$$

with the assumption

$$\frac{\Delta F_{ik}}{\Delta w_j} = 0 \quad \text{if} \quad k \neq j. \tag{56}$$

§The map f is defined to be deterministic, but in Choi and Douady (2012, 2013), the wealth dynamical system was allowed to be random yet estimable. Appendix C of Choi and Douady (2012) explains how the stability of an equilibrium of such a map should be understood.

¶Since $w_i(t)$ is a function of time $\Delta w_i(t) = w_i(t+\Delta t) - w_i(t)$ for some time Δt.

To reflect the internal change of wealth, we define the 'self-elasticity' a_{ii} of agent i as

$$a_{ii}(t) = \frac{\Delta\,(F_{ii}(t))}{\Delta w_i(t)} = \frac{\Delta\,(\gamma_i(t)K_i(t))}{\Delta w_i(t)}, \tag{57}$$

hence we have $a_{ij}(t)\Delta w_j(t) = \Delta F_{ij}(t)$ for all $1 \leq i, j \leq n$, including $j = i$.

It was shown in Choi and Douady (2012) that the Jacobian $B = (b_{ij})$ and the elasticity matrix $A = (a_{ij})$ are related such that

$$b_{ii} = 1 + a_{ii} - \sum_{k \neq i}^{n} a_{ki} \quad \text{and} \tag{58}$$

$$b_{ij} = a_{ij} \qquad \text{for } i \neq j \tag{59}$$

This relation will be used in section 4 to investigate the mechanism of instability contagion.

A.2. Market instability indicator

The 'market instability indicator' ρ is defined as the spectral radius of the Jacobian matrix $B(t)$ of f,

$$I(t) = \rho(B(t)). \tag{60}$$

The higher the indicator, the more unstable the market. In stable market conditions, the equilibrium point $\tilde{w}$ is an attractor, the eigenvalues of $B|_{\tilde{w}}$ have modulus less than 1 and, when the market is close enough to the equilibrium and, as a consequence, in its basin of attraction, the instability indicator $I(t)$ is also below the critical value 1.

When $I(t) < 1$ then perturbations of the system tend to be absorbed and disappear. On the contrary, when $I(t) > 1$ then most of the perturbations contain a component that will increasingly propagate within the system, either as a propagation of contraction of payments, or simply as an increase of leverage making liquidity constraints tighter and tighter and reactions to variations of income stronger and stronger.