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Kevin C. Moore
University of Georgia

Jason Silverman
Drexel University

Teo Paoletti
Montclair State University, paolettit@mail.montclair.edu

Dave Liss
California State Polytechnic University Pomona

Stacy Musgrave
University of Georgia

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Kevin C. Moore[a,*], Jason Silverman[b], Teo Paoletti[c], Dave Liss[d], Stacy Musgrave[e]

[a] University of Georgia, United States
[b] Drexel University, United States
[c] Montclair State University, United States
[d] University of Wisconsin–La Crosse, United States
[e] Cal Poly Pomona, United States

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ABSTRACT

In this paper, we use relevant literature and data to motivate a more detailed look into relationships between what we perceive to be conventions common to United States (U.S.) school mathematics and individuals' meanings for graphs and related topics. Specifically, we draw on data from pre-service (PST) and in-service (IST) teachers to characterize such relationships. We use PSTs' responses during clinical interviews to illustrate three themes: (a) some PSTs' responses implied practices we perceive to be conventions of U.S. school mathematics were instead inherent aspects of PSTs' meanings; (b) some PSTs' responses implied they understood certain practices in U.S. school mathematics as customary choices not necessary to represent particular mathematical ideas; and (c) some PSTs' responses exhibited what we or they perceived to be contradictory actions and claims. We then compare our PST findings to data collected with ISTs.

1. Introduction

In discussing mathematics curriculum and learning, Hewitt (1999), 2001) differentiatedbetween necessary information students can deduce for themselves, such as making quantitative comparisons, and information that students need to be informed about by an external source due to its arbitrariness yet establishment in mathematical communities, such as the canonical name of an object. In addressing graphs and coordinate systems, Hewitt (1999) noted:

> These are some aspects of where mathematics lies within the topic of co-ordinates, rather than with the practising of conventions. I am not saying that the acceptance and adoption of conventions is not important within mathematics classrooms, but that it needs to be realised that this is not where mathematics lies. So I am left wondering about the amount of classroom time given over to the arbitrary compared with where the mathematics actually lies. (p. 5)

Whereas we imagine mathematicians and mathematics educators largely agree with Hewitt's distinction and concern, the extent to which students and teachers hold meanings consistent with his description is an open question. That is, what aspects of representing a mathematical concept do students and teachers perceive as conventional and what aspects do they perceive as necessary?

In this report, we present findings from investigating this question with attention to pre-service teachers' (PSTs') and in-service teachers' (ISTs') mathematical meanings. We describe the extent that what we perceive to be graphing conventions associated with

* Corresponding author at: Department of Mathematics and Science Education, University of Georgia, 105 Aderhold Hall, Athens, GA 30602, United States.

E-mail address: kvcmoore@uga.edu (K.C. Moore).

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secondary mathematics concepts—conveying *x*-values or a function's input values along a Cartesian axis oriented horizontally—are perceived as conventions by PSTs and ISTs. In the sections that follow, we first describe our theoretical grounding, including a lens to characterize individuals' meanings with attention to the aforementioned conventions. We then describe our methods, including the design of the tasks we used to investigate teachers' meanings. We present the results of our analyses by detailing PSTs' and ISTs' responses to two tasks and highlighting themes in their responses and the nature of their meanings as inferred from those responses. We close with a discussion of our results, including potential avenues of future research and the implications these findings have for teaching.

2. Theoretical grounding and background

We have left the term *convention* undefined to this point. Like most terms in mathematics education, the term convention can take on different definitions and loci depending on the perspective taken. We consider a cognitive perspective most appropriate for achieving this study's aims, and thus describe a convention in terms of an individual constructing a concept in relation to an image of the practices of some perceived community.

2.1. Radical constructivism and the subjectivity of knowledge

We adopt a radical constructivist perspective of knowing and thus approach knowledge as actively constructed by the cognizing subject in order to organize her or his experiential world (von Glasersfeld, 1995). von Glasersfeld (1984) argued that defining cognition as an adaptive function implies that knowledge is not approached as a reflection of some objective reality, nor is knowledge construction a movement toward a more accurate picture of some reality as it exists independent of the knower. von Glasersfeld (1984) explained,

> Any cognitive structure that serves its purpose in our time, therefore, proves no more and no less than just that – namely, given the circumstances we have experienced (and determined by experiencing them), it has done what was expected of it. Logically, that gives us no clue as to how the "objective" world might be; it merely means that we know *one* viable way to a goal that we have chosen under specific circumstances in our experiential world. It tells us nothing – and cannot tell us anything – about…how that experience which we consider the goal might be connected to a world *beyond* our experience. (emphasis in original, p. 5)

Because knowledge is adaptive and actively constructed, an individual's knowledge is idiosyncratic and no person can have access to a body of knowledge, concept, object, etc. as it exists independent of her or his knowing.

Approaching knowledge as idiosyncratic has important implications for how we define conventions and investigate teachers' meanings with attention to what we perceive to be conventional among a relevant community. Before discussing such implications, we highlight a radical constructivist perspective does not deny the importance of social interactions to mathematical thought. Echoing Piaget, von Glasersfeld and numerous mathematics educators have repeatedly contended the opposite (Steffe & Thompson, 2000a, 2000b; Thompson, 2000; von Glasersfeld, 1995). An individual's social interactions provide stimuli that occasion opportunities for assimilation and accommodation. Important to our work, through repeated reciprocal acts of assimilation, interacting individuals can construct "intersubjective knowledge" (Steffe & Thompson, 2000a). Intersubjective knowledge is established in an interaction when "no one sees a reason to believe others think differently than he or she presumes they do" (Thompson, 2000).

Claiming individuals have established intersubjective knowledge is not a claim of equivalence or homogeneity of meanings. Individuals can maintain different or contradictory meanings for an idea, concept, or conversation, yet interact in ways that leave no reason for either individual to conclude their meanings are different or contradictory. To be clear, the meanings some individual attributes to another are personal meanings. An individual cannot escape her or his personal experience to provide an objective account of social interactions, communicated meanings, or the meanings held by another. An individual can, however, construct images of interacting others that enable organizing her or his experiences with those others. These images may come to entail concepts and associated representational practices the individual understands as conventional within that particular group of interacting individuals.

2.2. Conventions and habits

Conventions play an important role in mathematics, with notable examples including notational systems and order of operations. A primary reason interacting individuals establish or adopt a convention is that they perceive some practice to afford consistent, simplified, or efficient ways to capture or convey aspects of ideas and reasoning. The conventions established by a collection of interacting individuals, however, typically do not originate at the collective level. Conventions predominantly emerge through a process of negotiation, wherein interacting individuals collectively adopt, reject, or modify the ways in which an *individual originally* attempts to convey logico-mathematical aspects of her or his thinking (Ball, 1893; Cajori, 1993; Eves, 1990; Menninger, 1969; Thompson, 1992; von Glasersfeld, 1995). For this reason, the emergence and use of conventions cannot be reduced strictly to issues of notation and communication.

Negotiation of conventions is not only about a choice of physical notation or representation. It is also a negotiation of intended interpretation and meaning between interacting individuals (Thompson, 1992, 1995). That is, the construction of intersubjective knowledge is critical to the emergence of conventions, and thus an individual's understanding of a convention is shaped via the meanings he attributes to others through social interactions. When an individual enters a community, he or she understanding any

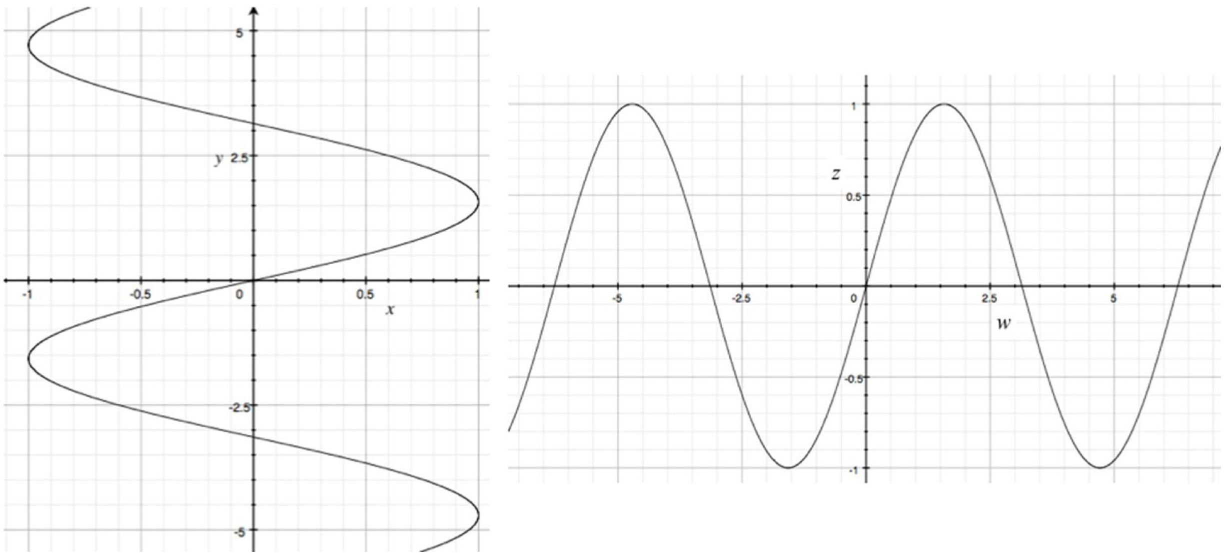


Fig. 1. Two viable graphs of the sine function.

idea requires that the individual address issues of notation or representation while simultaneously constructing implied constraints on interpretations and meanings (Thompson, 1992), and thus understanding something as a convention of that community requires both as well. Although it is often the case that an individual speaks of conventions as existing in a community independent of him or herself, a convention is a personal construct that an individual has externalized as if it is a property of some community. We claim that a person has constructed a *convention* when that person has in mind a concept, a community of individuals, and some representational practice that he perceives as a choice in that community among a variety of equally valid choices.

The extent some practice is understood as a convention can vary from individual to individual and also be dependent upon the community held in mind. Relevant to our work here, consider the practice of graphically representing a function's input values along the Cartesian horizontal axis. On one hand, each member in our research group perceives this to be a convention among most pre-calculus and calculus textbook writers. Textbook writers often identify this as a choice and use the Cartesian vertical axis as a viable alternative for a function's input values. Graphically representing a function's input values along the Cartesian horizontal axis is also a convention within our research group. Within our research group, we regularly construct a function's graph using different orientations (Fig. 1), understanding each orientation as equally viable because the logico-mathematical actions remain invariant among different axes orientations (Moore, 2014b; Thompson, 1992; Zazkis, 2008). With respect to Fig. 1, changes in orientation change perceptual features, but the underlying quantitative structures including input-output pairs and their covariation remain equivalent; a logico-mathematical property between x and y is necessarily a property between z and w. On the other hand, we have inferred from research discussed in a subsequent section and our previous work (Moore, 2014b; Moore, Silverman, Paoletti, & LaForest, 2014) that representing a function's input values along the Cartesian horizontal axis is not necessarily a convention to students or secondary teachers. Rather, and as we illustrate in the present article, such a representational practice can instead be an indissociable aspect of their meanings

Approaching conventions as personal constructs presents an issue in our attempt to describe another individual's meanings with attention to what *we* consider a convention of a relevant community; it is not necessarily the case that another individual understands that practice as a convention. To address this issue we draw on Thompson (1992) distinctions in a person's use of notation and representational systems,

> A person's meaningful use of notation can be highly idiosyncratic, it can be creative expression constrained by convention, or it can be a [habitual] use of convention. In the first case the individual is engaging in personal expression. In the second case the individual is conforming to convention with the awareness of conforming. In the third case the individual is using convention unthinkingly—perhaps unknowingly…To understand a convention *qua* convention, one must understand that approaches other than the one adopted could be taken with equal validity. It is this understanding that separates convention from ritual. (pp. 124–125)

Reflecting our focus on individuals who are not engaging in initial acts of personal expression and learning (i.e., the first case in the above quote), we situate this study with respect to the latter two cases identified by Thompson. We claim that an individual's use of graphs entails the *habitual use of "convention"* if her or his actions indicate some practice that we understand to be a convention is instead, for the individual, an essential or inherent aspect of her or his meanings for graphs and associated topics. It is a misnomer to name that aspect of an individual's meaning a convention because it is not a representational choice among a variety of equally valid choices. We thus use quotation marks to indicate when we are speaking of something that we or the reader might understand to be a

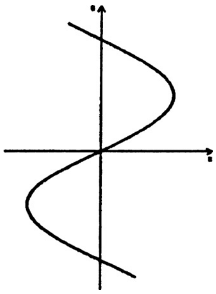


Fig. 2. Graph from Breidenbach et al. (1992), p. 281).

convention but that our participant does not. We claim that an individual understands "a convention *qua* convention"[1] if he or she understands some representational choice as one way to represent some concept among other equally valid choices.

3. Related literature

Speaking on various conventions practiced in U.S. and international school mathematics, Mamolo and Zazkis (Mamolo & Zazkis, 2012; Zazkis, 2008) argued that students (and teachers) are not supported in understanding certain conventions as customary choices if educators unquestionably maintain particular conventions. Mamolo and Zazkis hypothesized that a potential outcome of educators unquestionably maintaining conventions is that students do not develop meanings that enable them to understand novel and unconventional situations (e.g., alternative coordinate systems). Mamolo and Zazkis's stance echoes Thompson (1992) claim, "to ignore convention in our teaching can lead students to think of mathematics ritualistically" (1992, p. 125).

International and U.S. education researchers who have investigated students' meanings for function and other related areas have reported findings that are compatible with Mamolo, Zazkis, and Thompson's sentiments. Researchers (Akkoc & Tall, 2005; Even, 1993; Montiel et al., 2008; Oehrtman, Carlson, & Thompson, 2008) have documented that students' meanings for function in graphical contexts foregrounds the ritual application of the *vertical line test.*[2] As case in point, Montiel et al. (2008) identified students who applied the vertical line test when investigating relationships in the polar coordinate system. Doing so resulted in those students claiming that relationships such as $r = 2$ do not define a function. As another example suggesting students' ritualistic application of the vertical line test, Breidenbach et al. (1992) illustrated that only 11 of 59 students understood the graph in Fig. 2 as representing a function (i.e., the quantity's values represented along the horizontal axis as a function of the quantity's values represented along the vertical axis). In these examples, the researchers (Breidenbach et al., 1992; Montiel et al., 2008) posed graphs that they understood to be representative of functions, yet the students' meanings for functions and their graphs did not result in their assimilating the graphs as representative of functions.

Our purpose here is not to rehash the well-documented claim that students often understand function in unsophisticated ways constrained to the application of the vertical line test (see Leinhardt, Zaslavsky, and Stein (1990) and Oehrtman et al. (2008) for more extensive reviews). Rather, our purpose is to draw attention to a particular feature of students' meanings that, as we illustrate in subsequent sections, is more deep-rooted than researchers have previously reported. Namely, we infer that one explanation for the students' actions in our colleagues' studies is that the students drew on meanings in which a particular coordinate system and what we perceive to be conventions of that coordinate system had become inseparable from those meanings (i.e., habitual use of "convention"). For instance, what we perceive to be the convention of representing a function's input along the Cartesian horizontal axis did not appear to be a convention to those students reported on by Breidenbach and colleagues.

We interpret Sajka (2003) description of a student's use of function notation to imply another example of the habitual use of "convention". Sajka argued that, to the student, function notation was more about what "we usually write" (2003, p. 247) than about using the notation to represent her ideas and reasoning. Using Thompson's (1992) language, the student was more focused on a ritualistic use of notation than on using notation as an act of personally expressing meaning and concepts. A consequence of this was that the student deemed incorrect those examples that did not conform to her image of what "we usually write." Or, the student assimilated examples in ways that were consistent with her image of what "we usually write" but inconsistent with or inattentive to the researcher's intent. Sajka noted that by conflating what "we usually write" and essential aspects of a mathematical idea, the student produced numerous inconsistencies in her use of function notation, some of which the student was aware of and others that were only inconsistencies from the researcher's perspective.

[1] We use *qua* to mean "acting in the capacity of", and thus the phrase "convention qua convention" underscores that we infer an individual understands a practice as a convention when there is evidence the individual understands it as a customary choice among a variety of equally viable choices.

[2] The vertical line test is a technique commonly taught in several countries in the context of functions and their graphs. The technique involves imagining sweeping a vertical line horizontally across a graph displaying two variable values. If, at any moment of the sweep, the vertical line intersects the graph at more than one point, the graph does not represent a function. This technique is pervasive in U.S. instruction.

4. Methods and task design

We interpret the collection of findings discussed above to indicate the need for a deeper examination of the extent individuals' meanings entail the habitual use of "convention" versus a convention *qua* convention. In cases that secondary teachers' meanings entail the habitual use of "convention", we would expect their meanings to become problematic in situations that an observer considers breaking particular conventions. We would also expect them to (consciously or subconsciously) impose particular "conventions" in order to make sense of their experiences and tasks. In order to explore and better understand these phenomena in the context of teachers' meanings for graphs, we designed and conducted task-based semi-structured clinical interviews (Ginsburg, 1997) with PSTs and on-line open-ended surveys with ISTs. In this section, we first describe our participants and methods. We then describe our task design relative to our stated intentions.

4.1. Participants and setting

Our work with PSTs involved 31 participants enrolled at a large state university in the U.S. Each PST was entering her or his first semester in a four-semester preparation program for secondary mathematics teachers. Each PST began the program during her or his junior year (in credits), and each PST had completed at least two mathematics courses past Calculus II (e.g., multi-variable calculus and introduction to proof) before beginning the program. We chose the PSTs by asking for volunteers from their initial meetings of a secondary mathematics content course. We drew participants from four different sections of the course. Because all interviews took place outside of class, we chose participants from the volunteer pool whose schedules aligned with the researchers' schedules.

In order to better understand practicing teachers' meanings, we gathered similar data from ISTs. We adapted our PST interview tasks for an on-line survey completed by 45 ISTs. The ISTs were enrolled in a fully online graduate mathematics course offered to ISTs by a private U.S. research university as part of a master's degree program in mathematics education. The ISTs were geographically distributed across the U.S. They all had more than three years of experience teaching middle or secondary mathematics and had completed at least one mathematics course beyond Calculus III. All ISTs were invited to complete the survey during their third quarter in the program.

We worked with both PSTs and ISTs for several reasons. This study is a natural extension of our ongoing research agenda that focuses on understanding students' and teachers' reasoning about quantities and relationships between quantities in the context of functions and their graphs (Moore, 2014a, 2014b; Paoletti, Stevens, Hobson, Moore, & LaForest, 2018; Silverman & Thompson, 2008). This study also reflects our interest in understanding relationships between individuals' meanings and what we perceive to be conventions common to U.S. instruction of secondary mathematics. Because the chosen teacher populations had completed at least 14 years of mathematics schooling and identified a career in teaching, we conjectured we would gain insights into the extent teachers' meanings entail the habitual use of "convention" or convention *qua* convention in the context of concepts and practices relevant to the secondary mathematics education community.

The present study began with a focus on PSTs. After collecting and analyzing PST data, we grew curious as to whether the themes identified were specific to the PST participants or if the themes could explain IST activity. Namely, we were interested in understanding if the themes identified with PSTs would be ameliorated or otherwise affected by middle or high school teaching experience. We thus extended our work to include ISTs in order to explore if the themes identified with PSTs were similar to those of the ISTs. We chose an on-line survey format for two primary reasons. First, the IST population enrolled in the fully online graduate mathematics course had a diverse range of home locations, and thus it was not feasible to travel to and conduct clinical interviews with the participants. Second, we were interested in the extent that the identified themes could viably explain ISTs' responses to an on-line survey, as an on-line survey could offer us a mechanism by which to collect and analyze more expansive data in follow-up studies.

4.2. Data collection and analysis

In the initial study with PSTs, we conducted task-based semi-structured clinical interviews (Ginsburg, 1997) during which the PSTs worked on tasks we had designed as discussed in the next section. Each PST participated in one interview lasting approximately 90–120 minutes, with the interview occurring during the first two weeks of the course from which they were pulled. During the interviews, a member of the author team asked that the PSTs verbalize their thinking as much as possible. Although we designed each interview task with particular purposes, the clinical interviews were semi-structured in that we asked questions formulated in the moment and on the basis of our interpretations of a PST's response (Merriam & Tisdell, 2005). We posed follow-up questions for the purpose of gaining deeper insights into the PST's thinking while also attempting to minimize shifts in the PST's thinking due to the researchers' questioning (Goldin, 2000; Hunting, 1997).

We videotaped each interview and digitized all written work. We analyzed the data following a selective open and axial analysis approach (Strauss & Corbin, 1998) for the purpose of modeling the PSTs' thinking on the basis of their utterances and observable actions, which Thompson (2008) described as a conceptual analysis. This process first involved identifying instances of PST activity that offered insights into his or her meanings. We used these instances to develop hypothesized models of the PST's meanings. With these initial models of each PST's thinking developed, we compared a PST's activity across instances and tasks in order to test and improve our interpretations of her or his activity, including identifying themes across instances and tasks. Lastly, we compared across PSTs in order to identify compatible and contrasting aspects of their meanings. The research team met throughout the data analysis phase in order to discuss data analysis efforts, including differences and uncertainties in interpretations of PSTs' activity. These meetings included (re)watching clips, reviewing data analyses, and developing or refining models of PSTs' meanings as a group. As

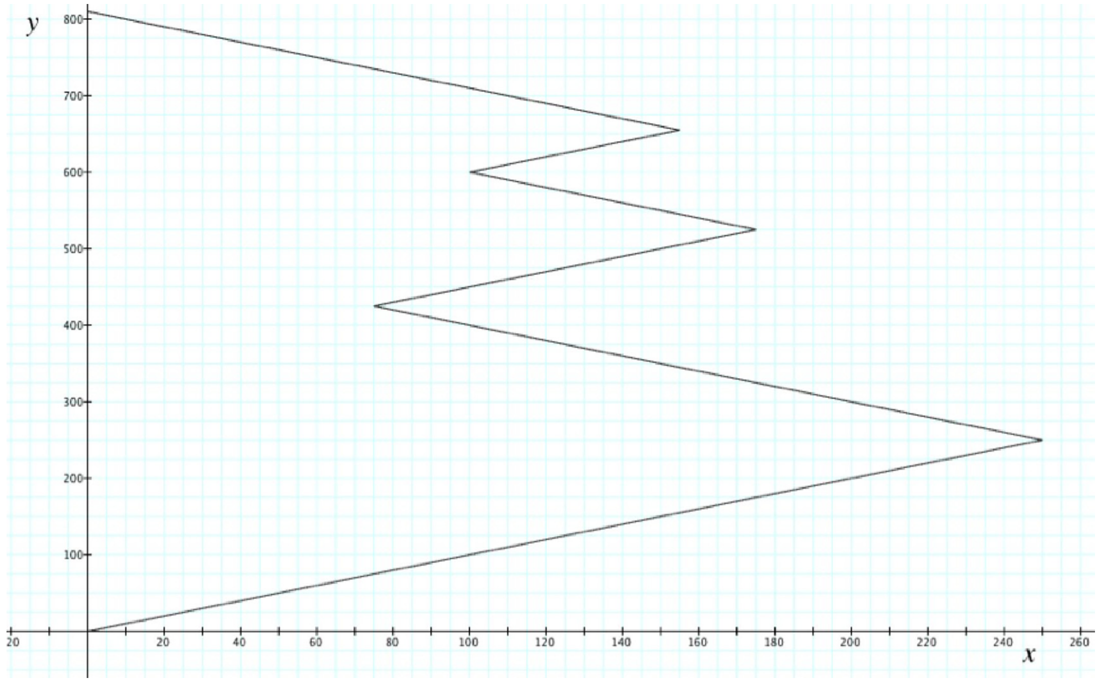


Fig. 3. Does this graph represent a function?

differences were reconciled and themes identified, the research team revisited related instances of PST activity in order to further refine models of PSTs' meanings and clarify themes in the PSTs' meanings and uses of graphs.

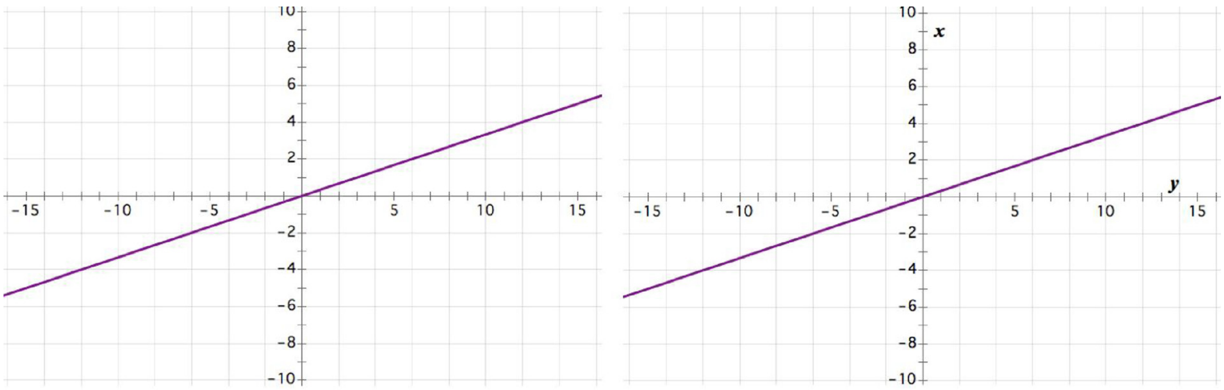
IST responses to the online survey items were analyzed through an iterative process that began with a first review of the entire data corpus and the subsequent development of a coding scheme heavily informed by our work with the PSTs. Members of the research team analyzed a subset of the ISTs' responses and we met to discuss our observations, identify commonalities across ISTs' responses, and adapt or create new codes to capture more ISTs' responses. We iterated this process four times as we refined our codes to capture all ISTs' responses. After obtaining final codes (see Table 5), a second researcher recoded approximately 65% of the data to check for inter-rater reliability. We obtained Cohen Kappa values of 0.78 and 0.85 for the two tasks described, indicating a high level of agreement.

4.3. Task design

In order to examine teachers' meanings in relation to what we perceived to be specific conventions of graphs and related topics, we designed tasks to include a feature we considered unconventional with respect to the use of graphs in U.S. secondary school mathematics. Tasks were unconventional in two ways, including switching axes orientations with respect to a (stated or implied) function's input and output or using unconventional axes labeling with respect to letters designated for variable values. We designed these tasks to be unconventional, but we intended each task to include a mathematically viable graph as presented with respect to a particular claim; we did not intend that the tasks be 'unsolvable' or 'incorrect'.

We did not expect the teachers to spontaneously interpret the given tasks as entailing unconventional aspects, and thus each task included a series of prompts that transitioned from open-ended statements or questions to specific claims that identified features we intended to be unconventional. We first provided open-ended prompts to determine how the teachers initially assimilated the tasks including their capacity to attribute viable meanings to a hypothetical individual who constructed a given graph. Second, we included subsequent prompts that made claims resting on unconventional choices to explore the extent that a "convention" was a habitual aspect of the teachers' meanings. We expected the given claims to possibly contradict how the teachers first responded to the tasks, thus providing insights into whether or not they would conceive the given graphs and associated claims as uncustomary yet mathematically viable. Furthermore, by introducing tasks in terms of hypothetical student work, we hoped to increase the likelihood the participants identified themselves as (current or future) secondary mathematics teachers and responded in ways sensitive to such a role and community of peers or students.

To illustrate our task design, consider two graphs (Figs. 3 and 4) that we presented the PSTs during their interviews. We first presented Fig. 3 to the teachers with the question, "Does this graph represent a function?". We returned each teacher to the graph later in the interview and asked, "Is there a way that we (or a student) could consider this graph as representing a function?" After the teacher responded and had indicated he or she exhausted potential responses, and if necessary, we posed, "What about a student who claims that x is a function of y?"



(a) (b)

Fig. 4. A hypothetical student's work to graphing $y = 3x$.

With respect to Fig. 4a, we presented the graph as the work of a hypothetical student graphing $y = 3x$. We first asked the teachers to describe how the hypothetical student might have been thinking when creating the graph. After their explanations for the first graph, we provided a follow-up prompt that included the same graph but with the axes labeled by the hypothetical student in order to clarify his graph of $y = 3x$ (Fig. 4b). We then asked the teachers to comment on the student's graph. Both tasks illustrate our decision to transition from not posing explicit, unconventional interpretations of a graph to posing explicit claims involving (from our perspective) such interpretations. The tasks also illustrate our designing graphs that can be conceived mathematically viable as presented with respect to the given prompts and claims.

The IST online survey was modeled after the PST interview protocol using virtually identical prompts. Due to the temporal nature of the images and prompts, multiple part items were displayed on multiple pages. For example, the online analog for the item in Fig. 2 is shown in Fig. 5. Following the format of the PST interview, online survey participants were subsequently posed the student

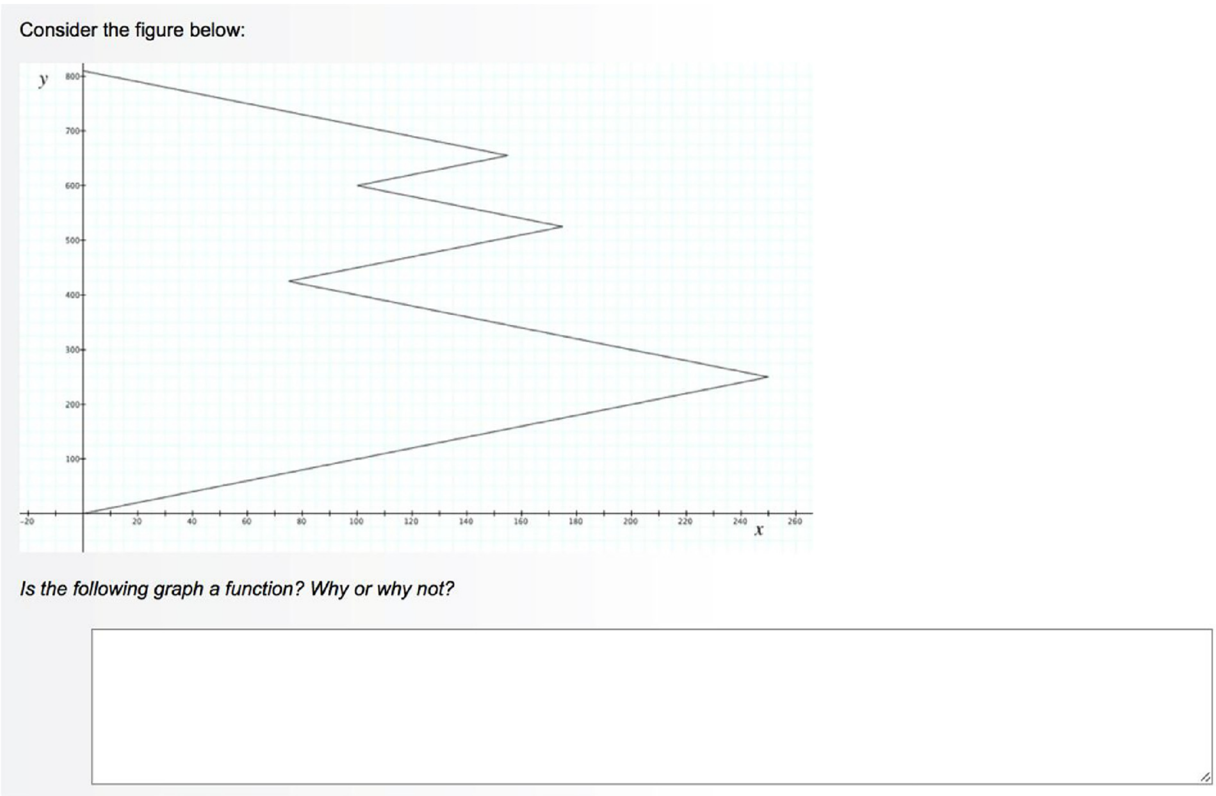


Fig. 5. Is x a function of y? (online version).

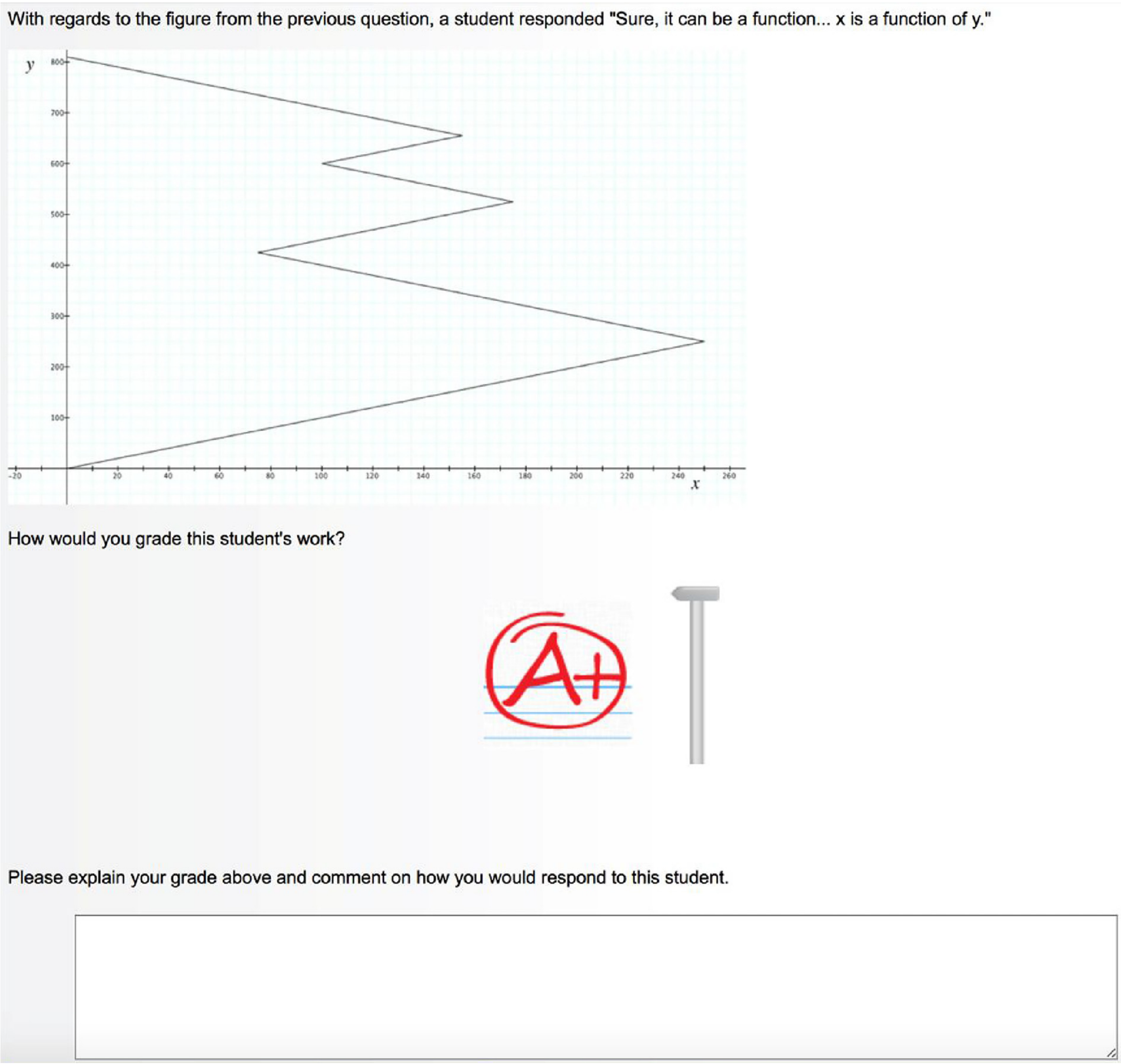


Fig. 6. Hypothetical student response to “Is x a function of y?” provided to ISTs (online version).

response, “Sure, it can be a function… x is a function of y”, and asked to grade that student response and explain why they assigned that grade to the student (see Fig. 6).

5. Results

We structure the results section around the teachers’ responses to the two aforementioned tasks due to the teachers’ responses being salient representations of their responses to other tasks. We provide excerpts to illustrate themes in the PSTs’ responses to each task. We also present summative PST data for each task to offer the reader a sense of the variety of PST responses. Recall that we use *convention qua convention* and variants of this phrasing to refer to those instances in which an individual’s actions imply he or she understands a practice as a customary, but not necessary, choice in viably (or correctly) representing particular mathematical concepts. We use *habitual use of “convention”* and variants of this phrasing to refer to those instances in which we infer that a practice we perceive to be a convention is either maintained apparently unknowingly by an individual or the individual considers the practice a rule to be unquestionably followed, thus not acknowledging other alternatives as viable and correct representations of a mathematical concept.

We also draw attention to PST instances that we term *contradicting actions and claims*. We provide a separate focus on these instances for three reasons. First, although such instances are not disjoint from the habitual use of “convention”, they are a distinct phenomenon that occurred in select instances of PSTs’ habitual use of “convention”. Second, we clarify the perspectives from which

Table 1
PST responses to the question, “Is there some way that we or a student could consider the graph as representative of a function?”.

PST Response Category	# out of 25
Yes, if rotated counter-clockwise 90-degrees	6
Yes, if rotated counter-clockwise 90-degrees and axes relabeled so that y and x were represented along the vertical and horizontal axes, respectively, in the new orientation	5
Did not determine how a hypothetical student might claim that the graph represents a function; maintained that the graph does not represent a function	14

Table 2
PST responses to the statement, “x is a function of y.”.

Category code	PST Response Category	# out of 25
Habitual use of “convention”	Unsure	1
	Not true	9
	True, if rotated counterclockwise 90-degrees and axes relabeled so that y and x were represented along the vertical and horizontal axes, respectively, in the new orientation	1
	True, if graph is rotated counterclockwise 90-degrees	7
Convention *qua* convention	True	7

we claim PSTs’ actions and claims are contradictory. At times the contradictions in PSTs’ actions and claims were contradictions from a potential observer’s (e.g., a student or our own) perspective but not from the PSTs’ perspectives. At other times, the PSTs experienced states of perturbation due to their perceiving contradictions or inconsistencies in their meanings. Third, we give explicit attention to these instances due to their implications for when teachers’ students are the observers of the teachers’ responses, a point we detail in the discussion section.

5.1. *x is a function of y*

Due to interview timing constraints, we asked 25 of the 31 PSTs the entire sequence of prompts associated with Fig. 3 and thus include only the results for those 25 participants. On the initial pass—“Does this graph represent a function?”—24 of 25 PSTs claimed that the graph did not represent a function because of the vertical line test, the graph failing to have a unique y value for each x value, or a combination of the two. The remaining PST claimed that the graph did not represent a function because, “I don’t like it [*referring to the cusps*].” With respect to the subsequent prompt—“Is there some way that we or a student could consider the graph as representative of a function?”—we provide a summary of the PST responses in Table 1. No PST provided a viable way to think about the graph as representative of a function in the *given* orientation. Table 2 presents a summary of the PST responses to the claim, “x is a function of y.” In the sections that follow we discuss themes in the PST responses to this claim.

5.1.1. *Convention qua convention*

We interpreted 7 of the 25 PSTs’ responses to suggest that they did not require x or the horizontal axis to represent the possible values for a function’s input. We note that 5 of these 7 PSTs described that they had a tendency to imagine the graph oriented so that the values defined as the function’s input were represented along the horizontal axis. Ultimately, each of the seven PSTs understood the graph *as given* to be consistent with the claim “x is a function of y” (*Excerpts 1*), thus suggesting that they understood representing a function’s input values along the vertical axis (denoted by the variable y) as a viable practice.

Excerpts 1. x is a function of y; convention *qua* convention.[3]

S1:I want to look at this and say this is a function y of x because that’s how I would traditionally view a graph but I think it’s valid to view it as x of y. And then you’re still [*pause*] obeying what a function is. But you just have to be cognizant that your axes have changed so I guess it’s like, valid.

S13:Rather than y being a function of x…Yeah I guess if you do it this way [*writes* ‘$x(y)$’ *on paper*]…for every y there is exactly one x. And for every y [*puts marker on vertical axis on graph and moves it horizontally to a point where it hits the curve*] yeah, there’s exactly one x…I’ve never thought about it that way but yeah, he’s right…awesome way of thinking about that.

5.1.2. *Habitual use of “convention”*

Two notable characteristics emerged from our analyses of the PST responses to the sequence of prompts associated with Fig. 3: 16 of the PSTs either maintained x as representing input values *or* maintained the horizontal axis as representative of input values. These

[3] For space purposes, we use “…” to indicate the removal of spoken words and actions that we did not interpret to alter our interpretation of the PSTs’ activity.

Table 3
PST responses to the prompt and graph associated with Fig. 4a.

PST Response Category	# out of 31
Hypothetical student held some misunderstanding of slope (e.g., 'rising 1 and running 3')	16
Hypothetical student graphed $x = 3y$, $y = (1/3)x$, or interpreted the equation to mean x is three times as large as y (e.g., variables as literal objects)	13
Hypothetical student graphed y on the horizontal axis and x on the vertical axis	13

characteristics of the PSTs' meanings were most apparent when we posed the claim, "x is a function of y."

Collectively, 11 of the 25 (Table 2, first three response categories) PST responses suggested they assimilated the phrase "x is a function of y" no differently than "y is a function of x" (*Excerpts 2*).[4] These PSTs maintained that the graph does not represent a function because the graph does not pass the vertical line test, because there exists x-values for which there is not a uniquely associated y-value, or a combination of the two. To each of these 11 PSTs, we infer that "function" drew to mind an action that entailed implicitly or explicitly conceiving x and the values represented along the horizontal axis as the input values, thus implying their habitual use of "convention". We note that a PST conceiving the requirement of an input value having a unique output (see S7, S24) is a more sophisticated meaning for function than merely imagining the vertical line test (Carlson, 1998; Oehrtman et al., 2008), but we underscore that the PSTs' actions were in response to the claim "x is a function of y" paired with the graph presented in Fig. 3. None of these 11 PSTs' actions indicated their considering y or the values represented along the vertical axis as a viable input quantity, despite repeated prompts and associated graphs intended to indicate otherwise.

Excerpts 2. x is a function of y; habitual use of "convention".

S7:Okay. Um [*pause*] x is a function of y. [*long pause*]…Well you know something's not a function if [*placing her marker in a vertical line over the given graph*], two different outputs can give you the same, I mean if two different inputs can give you the same output… Which you have here obviously that, you know, these one two three four five six x-values give you different y-values [*using her marker to mark points on the graph in a vertical line*]. I mean these, the same x-value can give you six different y-values.

S24:[*laughing*] Oh gosh, um, well [*pause*] if x is a function of y, well you can't [*pause*] for it to be a function you can't have more than one y-value for the x [*motioning the marker over the graph indicating points vertically from one another*]. So, like if I wanted to know what, umm [*pause*] f of one hundred was, or something, like I would get a bunch of different [*begins to mark points on the graph for a specific x-value*], I mean, yeah I would get a bunch of different y-values for it, you know [*has marked multiple points on the graph with the same x-value*]…you can't get more than one y-value per x-value. It's not a function.

Turning our attention to the remaining 7 PSTs, they maintained that the statement is true only on the condition that the graph is rotated 90-degrees counterclockwise and axes *not* relabeled. We infer these PSTs to have understood the phrase "x is a function of y" in two parts. The phrase both defined the axes orientation and provided a claim, each to be considered with respect to a relationship between paired values (*Excerpts 3*). Because they understood the given phrase to *necessitate* a particular axes orientation—an orientation in which input values are represented horizontally, thus requiring that they rotate the given graph to consider the validity of the claim—we characterize their responses as the habitual use of "convention".

Excerpts 3. x is a function of y; habitual use of "convention".

S4:I guess she doesn't understand what graphs represent. Or what is meant by [*pause*], so she said x is a function of y. That'd be, that'd be looking at it this way [*turning the paper 90-degrees counterclockwise*] and saying look there's no [*motioning hand over the graph as if doing the vertical line test*], there's no crossing…So, I mean that's true, but you'd have to flip the whole graph…[*redraws graph in rotated orientation, labeling the horizontal axis as y and the vertical axis as x*] That'd be y and that'd be x. So x is a function of y. And that's a function…[*Interviewer returns S4's attention to the graph in its given orientation*] No, because x isn't a function of y. This is the graph of y as a function of x [*motioning to her sketch*].

S14:x is a function of y. y is a function of x. Yeah, but like, okay so x is a function of y. That's trueeee [*turning the paper 90-degrees counterclockwise*]. [*Turning the paper back to the given orientation*] But y is not, y is not a function of x…That's what we're looking at here…So you want y is a function of x. Is that what you said to me, no you said x is a function of y…That's backwards [*laughing*]… because like x is a function of y, so that, I think of that as, like the graph like kinda this way [*turning the paper 90-degrees counterclockwise*]…[*motioning over the horizontal–now y–axis*] like if this is our horizontal that's true. Because for every y [*pointing to y-label on horizontal axis of turned graph*] there is one unique x [*pointing to x-label on vertical axis of turned graph*] but [*turns the paper back to given orientation*] for every x [*pointing to x-label on horizontal axis*] there is not [*pointing to y-label on vertical axis*] one unique y… she's incorrect because it's like backwards…that's not what we're looking at [*referring to graph in given orientation*]. [*Interviewer asks S14 about the given statement with respect to the rotated graph*] I would agree.

[4] The interviewer repeated the phrase several times when questioning the PSTs, often until the PST repeated the phrase as stated, to ensure that he or she 'heard' the phrase (see S7 and S24, both of which repeated the phrase).

Table 4
PST responses to the prompt and graph associated with Fig. 4b.

Category code	PST Response Category	# out of 31
Habitual use of “convention”	Hypothetical student did not construct a correct graph	5
	Hypothetical student constructed a graph that is both correct and incorrect	11
	Uncertain if the hypothetical student constructed a correct graph	4
Convention *qua* convention	Hypothetical student unquestionably constructed a correct graph	11

5.2. A graph of $y = 3x$

We asked all 31 PSTs the sequence of prompts associated with Fig. 4. With respect to the initial prompt asking the PST to describe how the hypothetical student might have been thinking when creating the graph (Fig. 4a), we provide a summary of their responses in Table 3. No PST encountered difficulty attributing a viable approach to producing the hypothetical solution. Explaining an accumulative count exceeding 31, some PSTs provided multiple explanations as to how a student might create the graph.

When we presented Fig. 4b with the axes labeled and asked the PSTs to interpret the hypothetical solution, we asked them to comment on the correctness of the solution (i.e., “Does the graph represent $y = 3x$?”) in order to gain insights into the extent to which they considered the graph a viable representation of $y = 3x$. Recall that by our definition of convention *qua* convention, a response under this category would acknowledge that the graph is a correct representation of $y = 3x$, albeit uncustomary. We summarize their responses in Table 4. In the sections that follow we discuss themes in the PST responses to the graph and prompts.

5.2.1. Convention qua convention

Eleven PSTs maintained that the graph as oriented in Fig. 4b is unquestionably a viable representation of $y = 3x$ (*Excerpts 4*). These PSTs identified the graph’s departure from convention, and specifically its departure from a customary axes orientation. They also claimed that the departure does not influence the correctness of the represented relationship between x and y, suggesting their use of a convention *qua* convention.

Excerpts 4. A graph of $y = 3x$; convention *qua* convention.

S18:It’s the same $y = 3x$, correct? I just said that as long as his axes are labeled, it would be right…[*S18 then expresses worry that a fellow teacher might deem the solution incorrect because it is not conventional*].

S21:Ohhhh…this graph is saying…y is three times bigger than x…so where x is one, y is three times bigger [*checking graph*]. Yes. Where x is two, y is three times bigger [*checking graph*]. So this graph is correct… y is three times bigger than x.

S29:Oh. It’s clever. We have a clever kid over here. OK, so it now technically is y equals three x…you are clever…it’s just not the standard way of doing it…[*S29 then claims that they could not take off a point if grading the solution because it is correct*] They see the relationship between x and y.

S30:He graphed it completely right. That’s y equals three x…he’s not wrong. He just has a different perspective than the traditional x-y…that’s just counter to tradition and normal classroom settings. But I think it’s smart of him to understand that it’s not glued.

5.2.2. Habitual use of “convention”

We interpreted 20 PSTs (Table 4, first three response categories) who deemed Fig. 4b incorrect or who expressed uncertainty about the hypothetical solution to hold meanings that entailed the habitual use of “convention”. These “conventions” included assigning x-values to the horizontal axis, maintaining particular axes directions for positive and negative values (which arose after rotating the graph to obtain x-values oriented horizontally), using the horizontal axis to represent a function’s “input” (and inferring the given graph contradicted an equation defining x values as “input”), or a combination of these (*Excerpts 5*). In some cases, PSTs discarded the hypothetical student’s solution or deemed the solution incorrect because of its departure from these “conventions”, thus treating “conventions” as unquestionable rules of a coordinate system and graphing (see S19, S20, S25). In other cases, PST responses to the hypothetical solution suggested they drew on meanings for slope or rate that entailed the habitual use of a particular Cartesian orientation (see S23, S25). For instance, after rotating the graph 90-degrees counterclockwise so that the x-axis was oriented horizontally, some PSTs understood the slope as negative because the line is directed downward left-to-right. This argument is valid under conventional Cartesian orientations, yet the rotated graph entailed positive x-values oriented to the left. Importantly, we did not have evidence of the PSTs acknowledging this orientation of positive values, nor did the PSTs discuss how slope (or rate of change) associations are dependent on the orientation of positive values. To these PSTs, a positive slope required a line sloping upwards left-to-right and a line sloping downward left-to-right was necessarily a negative slope, thus suggesting their habitual use of “convention”.

Excerpts 5. A graph of $y = 3x$; habitual use of “convention”.

S19:I feel like you should know your x and y, and like, know which one is which. And, yeah, you’re going to get it all wrong I think.

S20:The horizontal axis should always be x and the vertical axis should always be y.

S23: Because if you turn it this way [*referring to graph rotated 90-degrees counterclockwise*] then this [*traces left to right along the x-axis which is now oriented horizontally*] and this [*traces top to bottom along the y-axis*] and it would be still not right though…this [*laying the marker on the line which is sloping downward left-to-right*] is negative slope. So I would…show them like the difference between positive and negative slopes also. Because that's something that, like, when I was in middle school we, like, learned kind of like a trick to remember positive, negative, no slope, and zero [*making hand motions to indicate a direction of line for each*]. Like where the slopes were…it's important to know which direction they're going.

S25:They messed up the placement of x and y…They are looking at it like this right now [*rotating graph 90-degress counter-clockwise*]…If you are looking at it this way, it's a negative slope [*tracing graph*] and it should be a positive slope [*tracing imagined graph upward left-to-right*]…slope is wrong.

5.3. PSTs' contradicting actions and claims

Returning to the task associated with Fig. 3, we draw attention to S4 and S14′s responses (*Excerpts 3*). S4 and S14 claimed the graph as given *was not* such that x is a function of y, and they claimed the rotated graph *was* such that x is a function of y. From our perspective, there is a potential contradiction that exists in these two claims: each y-value has a uniquely associated x-value regardless of the orientation of the paper. However, the PSTs' claims were not a contradiction from their perspective. As we described above, we infer that these PSTs' meanings for functions and graphs were such that the ways they conceived x–y pairings were dependent on the axes orientations (i.e., habitual use of "convention"), even when presented with prompts and interviewer questions asking them to consider otherwise.

More generally, we observed several examples in which the PSTs' habitual use of "convention" included their exhibiting contradicting actions and claims. At times, these were potential contradictions only from our perspective as observers, and at other times the PSTs conceived some contradiction in their actions and claims. We focus on the task associated with Fig. 4 to provide additional illustrations of this phenomenon, and we return to its implications in the discussion section. First, we note the potential contradiction an observer can perceive with the claim that rotating the given graph changes the represented relationship or slope (see *Excerpts 5*, S23 and S25). Regardless of orientation, an observer can understand the graph so that each y-value is three times as large as the associated x-value and that any variation in y is three times as large as the corresponding variation in x. Hence, an observer can viably claim that the graphed relationship, no matter the rotation of the paper, has a slope of three. S23′s and S25′s actions indicate that slope was as much, or more, an indicator of direction constrained to particular Cartesian "conventions" than a multiplicative comparison between covarying quantities. Thus, no contradiction existed with respect to their system of meanings when they claimed that the "slope" changes as the given graph and paper is rotated.

Second, numerous PSTs claimed that the graph in Fig. 4b was *both* correct and incorrect in its representation of $y = 3x$. Compatible with the aforementioned examples, in some cases a PST claiming that the graph is both correct and incorrect was only a potential contradiction from the perspective of an observer; the graph represents $y = 3x$ regardless of the paper's orientation and is thus correct. Some PSTs held meanings that enabled them to claim the graph is both correct and incorrect without perceiving a contradiction (*Excerpts 6*, S2, S6, S9). For instance, the PSTs understood that the graph as given entailed coordinate points satisfying $y = 3x$. At the same time, they held meanings for coordinate systems that entailed the habitual use of "convention" in the form of axes orientations, thus requiring those orientations in order to claim a graph is how it "should be written." An axes orientation different than "convention" was not an equally valid choice, and hence incorrect to some extent.

In other cases, PSTs experienced a perturbation that stemmed from a perceived contradiction in claiming that the graph satisfies the equation $y = 3x$ and that it is incorrect due to its orientation (*Excerpts 6*, S17, S19). These PSTs did not resolve their perturbation during the interviews, which led to each PST expressing uncertainty about whether particular axes orientations must be maintained. These PSTs explained that they did not know whether the departure from a particular coordinate orientation implied that the solution is mathematically incorrect (e.g., "I don't know. I've always just done what I was told. I don't really know why it has to be that way" and "honestly I never really thought about it").

Excerpts 6. A Graph of $y = 3x$; contradicting actions and claims.

S2:[*S2 is addressing how he would respond to the student who produced* Fig. 4*b*]…how to correct it for next time or like just on what he did wrong. Um, I mean I would tell him that this is the correct graph because it technically is. But I would just explain to him, and I don't know how I would explain but how, like when graphing functions y is always going to be the vertical axes and x is always going to be the horizontal axis… explain to him that next time he needs to change his axis. And why [the graph] is right but wrong at the same time.

S6:It represents x equals three y. No. Yeah. It still does but it's kinda like the inverse I guess… So it still represents y equals three x…I'd say [the student] did everything right. Um, however, [the student] got [his] x and y mixed up…Next time, we'll work on getting your x and your y in the right spot.

S9:It's wrong with like how we normally write graphs…So he should lose points because he wrote the graph in like really incorrectly to what, how the graph should be written. Like the horizontal axis should always be x and the vertical axis should always be y. But if you're looking at it based on did he understand that, when y equals three, x equals one, like he understood that, um, relationship between x and y.

S17:He got them mixed up…he got the thought process correct…So he did the problem correctly…he didn't understand how the graph worked…that the x is always on this [*referring to the horizontal axis*], and y is always on the vertical axis…[his graph] is correct [*making finger quotations surrounding correct*] but it's not mathematically correct. [*S17 then draws canonical graph and illustrates how the graph is correct using points*]…it's not wrong, it's just not what graphs are supposed to be…I don't know. I've always just done what I was told. I don't really know why it has to be that way…I never really questioned why x is there and y is there.

S19:It's kinda hard to explain why this has to be your x [*turns paper 90-degrees counterclockwise and points to each axis*] and that has to be your y. Like you know what I mean? I feel like it's just like engrained in your brain now where you know that. But like why couldn't it be y and x [*turns the paper to original orientation*] you know…Um, honestly I never really thought about it.

5.4. IST responses

For brevity's sake, we do not present the IST responses to each task as they are compatible with the PST responses. Table 5 provides codes we created to characterize the IST responses to the last stage of hypothetical student work for both tasks, example responses to the hypothetical student prompt associated with Fig. 3, and counts of the number of IST responses coded within each category for each task. 12 of the 45 IST responses for the task associated with Fig. 3 indicate the ISTs understood a convention *qua* convention. 25 IST responses for the task associated with Fig. 4b indicate the ISTs understood a convention *qua* convention. The remaining 33 and 20 ISTs, respectively, maintained meanings which entailed a habitual use of "convention".

Table 5
Codes description, counts, and sample responses of IST online survey (n = 45).

Convention category	Code	Example Responses to the task in Fig. 3	Fig. 3	Fig. 4b
Convention *qua* convention	The student's mathematical statement is correct despite breaking from conventions.	That's great! I am so glad you were able to apply the "vertical line test" in a horizontal orientation and realize that you would have a function. You are correct in saying that x is a function of y.	12	25
Habitual use of "convention"	The student's mathematical statement is true but the student is incorrect because he/she broke from conventions. (contradicting actions and claims)	I think the student is understanding that x can be a function of y but they are not displaying it correctly through the graph.	9	7
	The student's mathematical statement is incorrect or the IST did not address the student's mathematical statement.	It was not a good explanation and x is not a function of y, y is a function of x. The value of y depends on x. They also did not describe what would make it a function.	24	13

5.5. Comparing PST and IST responses

To identify similarities between PST and IST responses—and consider the possible persistence of PST meanings into teachers' professional careers—we present the number and percentage of PSTs and ISTs whose responses we coded in each of the two categories (Convention *qua* convention and habitual use of "convention") for each problem. Tables 6 and 7 provides the average scores for PSTs and ISTs across the tasks in Figs. 3 and 4, respectively. We used a chi-squared test for independence for each task with the null hypothesis that there is no relationship between PST and IST responses in either category *Convention qua convention* or *Habitual use of "convention"*. Analysis indicates we have no reason to reject the null hypothesis for either task ($p > 0.05$ in both cases); a possible characterization of this result is that PSTs and ISTs do not provide different responses that differ with respect to our convention coding categories in relation to the tasks used in our studies.

6. Discussion, implications, and future research

By adopting a cognitive perspective, we provided analyses of teachers' meanings as they responded to tasks we designed to be unconventional. Using the unconventional nature of these tasks, along with our attention to clarifying from whose perspective a particular practice is a convention, we identified the extent that certain graphing practices we perceive to be conventions were

Table 6
Number (and percent) of PST and IST responses we coded in each category for the task in Fig. 3 and p-value obtained from a chi-squared test.

	Convention qua convention	Habitual use of "convention"
PSTs	7 (28%)	18 (72%)
ISTs	12 (27%)	33 (73%)
p-value	0.9043	

Table 7
Number (and percent) of PST and IST responses we coded in each category for the task in Fig. 4 and p-value obtained from a chi-squared test.

	Convention qua convention	Habitual use of “convention”
PSTs	11 (35%)	20 (65%)
ISTs	25 (56%)	20 (44%)
p-value	0.0850	

instead indissociable aspects of teachers’ meanings. In this section, we clarify the significance of our results including the potential implications of teachers’ meanings for their interactions with students.

6.1. Significance of findings

Our results support the claims of Montiel et al. (2008); Breidenbach et al. (1992), and Sajka (2003), who provided data that we interpret to imply their participants’ meanings of function and their graphs entailed the habitual use of particular coordinate systems, orientations, or variable symbols. Our work extends their work in an important way. We are not aware of other researchers who have used task-based clinical interviews or on-line surveys (as opposed to an instructional setting) to present participants explicit claims—through written or spoken words—about scenarios that are unconventional with respect to the notation and coordinate orientations used here. Our findings in this regard shed insights into the extent some teachers have (or have not) dissociated logico-mathematical aspects of mathematical concepts from what we perceive to be conventions for representing those concepts. Most notably, we show that despite providing explicit claims and concepts represented in ways compatible with these claims, many of the teachers assimilated the situations in ways that implied their habitual use of “convention”. The persistence of some teachers’ habitual use of “convention” after a repeated sequence of interview questions and explicit prompts addressing the “conventions” is particularly noteworthy.

At the most fundamental level, it is significant that both PSTs and ISTs—who have completed advanced mathematics courses beyond the undergraduate calculus sequence and many of whom have several years of teaching experience—have developed mathematical meanings that, at best, limit their ability to attribute mathematical viability to school mathematics concepts presented unconventionally. Further, our data indicates teachers’ meanings for graphing conventions (or “conventions”) may persist into their professional career, suggesting that experience teaching does not necessarily support shifts in teachers’ conceptions of graphing conventions (or “conventions”). Also significant is that some teachers (or soon be teachers) held meanings that led to claims and actions that, although potentially internally viable to the teachers, were potentially contradictory from the perspective of an observer.

Reflecting on the teachers’ responses, and specifically those responses that were internally viable yet entailed some sort of contradiction (whether perceived or not by the teacher), we cannot help but think of Erlwanger’s (1973) characterization of a twelve-year old student’s mathematical activity. Erlwanger illustrated that the student, Benny, had constructed a mathematical world involving rules “that worked.” In the case that Benny’s rules did not work, he modified the rules or developed ad hoc new rules. Benny’s mathematics thus became a collection of “100 different kinds of rules” which, when applied in the appropriate context, produced correct answers despite any judgment of sensibility to an observer. Each rule, to Benny, was internally consistent relative to its appropriate context, but each rule was arbitrary in that a rule need not be mathematically consistent with other rules. Hence, rules that led to different answers (i.e., contradictions) were not contradictory to Benny, as the “correctness” of a rule was not judged by its generalizability.

Both Erlwanger’s description of Benny’s mathematical activity and Sajka (2003) description of a student’s use of function notation are similar to some of the characterizations we provide in the current study—particularly those involving the habitual use of “convention”. In fact, based on the nature of their responses, we hypothesize that many of the teachers in this study would describe the origins of “conventions” in a way similar to Benny describing the origin of his rules or Sajka’s student describing notation. Benny explained that his rules were created “by a man or someone who was very smart. … It must have took this guy a long time … about 50 years … because to get the rules he had to work all of the problems out like that…” (Erlwanger, 1973, p. 54). Sajka’s student emphasized what “we usually write” over personal meaning. This sentiment of external creation and rules was echoed by several teachers including S17 (*Excerpts 6*): “I don’t know. I’ve always just done what I was told. I don’t really know why it has to be that way…I never really questioned why x is there and y is there.”

One of the key implications in Erlwanger’s study, namely that analysis of students’ mathematical work can highlight important and sometimes hidden aspects of even successful students’ mathematical activity, applies to our teachers. In our study, it was clear that when conventions (or “conventions”) were not violated, the PSTs and ISTs were able to respond in ways that were sensitive to the mathematical viability of students’ solutions. However, this study illustrates that being able to respond appropriately in one context does not necessarily indicate logico-mathematical coherence in teachers’ meanings. Specifically, in tasks that included hypothetical student work that we designed to violate particular conventions, we had the opportunity to infer aspects of teachers’ meanings that can persist without indication or perturbation in cases of canonical representations, resulting in their habitual use of “convention” in situations designed to be noncanonical. We argue that a contribution of this paper is that it provides a mechanism to identify previously invisible aspects of learners’ meanings of core mathematical ideas. This mechanism is important, as identifying such aspects of meanings enable educators to support learners'—including teachers—development of flexible and generative mathematical meanings (Moore et al., 2014; Thompson, 2013; Zazkis, 2008).

6.2. Implications for teaching

In this article, we documented three characterizations that helped us organize PSTs' and ISTs' responses to various tasks and the nature of their meanings for mathematical ideas. These three characterizations were convention *qua* convention, habitual use of "convention", and contradicting claims or actions, with the third being an example of the habitual use of "convention". The first category, in which individuals' responses differentiate aspects that are customary choices from those logico-mathematical aspects necessary to representing a concept, indicates a more robust and generative meaning of a concept. For teachers, however, understanding convention *qua* convention has additional implications.

Consider a teacher who has a student who is able to reason multiplicatively and covariationally about the relationship conveyed by $y = 3x$, and, either in class or in written work, graphs this relationship unconventionally (i.e., with the x-axis oriented vertically and the y-axis oriented horizontally). As opposed to simply remediating an "incorrect" response, based on our results we hypothesize that teachers who understand axes orientations as a convention *qua* convention are more likely to emphasize the mathematical viability and generativity of the student's response. In such a case, the teacher is likely to find ways to affirm the students' meaning and to help the student consider how her or his work connects to more standard representational conventions. In other words, we assert that teachers who understand conventions *qua* conventions are likely positioned to identify kernels of mathematical viability and sophistication, thus focusing student conversations and interactions on mathematical concepts, even when student's work includes a variety of unconventional aspects (Thompson, 1995). Furthermore, such a teacher may make addressing conventions qua conventions an explicit part of their practice so that all students in their classroom are afforded the opportunity to differentiate conventions from those aspects essential to representing mathematical concepts. They could do so by either making a student's unconventional work an explicit point of classroom conversation or by introducing unconventional work in order to raise for conversation the role of conventions in representing concepts.

On the other hand, and as indicated by our data (see *Excerpts 5* and *Excerpts 6*), a teacher holding a meaning that entails the habitual use of "convention" is likely to focus on remediating an "incorrect" response so that the student follows the "convention". At best, and as illustrated by Thompson (1995) when discussing educators' tendencies to conflate convention and mathematical concepts, the teacher can convey a confusing message by identifying the student's work as both correct and incorrect. We do not have empirical data to comment on the effects of an interaction in which a teacher makes these comments to a student who produced an unconventional graph of $y = 3x$ in an instructional setting. However, it is not a stretch to imagine the student left wondering what he or she did wrong. Furthermore, the student is likely to perceive his teacher as a mathematical expert and infer that axes-variable pairs and positive number orientations are necessary features of a mathematical concept or unquestionable rules that must be followed. Regardless, such interactions could result in students conflating logico-mathematical aspects of a concept with those choices that can be perceived as arbitrary, and ultimately giving privilege to what should be arbitrary choices of representation over the logico-mathematical aspects to be represented (see Sajka, 2003).

6.3. Future work

In the present work, we were not interested in investigating or engendering student or teacher learning. Given our teachers' responses and their propensity to exhibit contradictory actions and claims, we conjecture that a productive line of inquiry will involve researchers investigating teachers' and students' *learning* with greater attention to their construction of conventions. One suggestion is for researchers to work with students who are yet to have sustained experiences with instruction or curricula in order to understand their highly idiosyncratic initial acts of expression (e.g., students being introduced to the Cartesian coordinate system). Such research could provide insights into students' activities and reasoning prior to their constructing meanings that entail particular conventions qua conventions or the habitual use of "conventions", including how these students might come to naturally choose conventions by which to organize their activity (see diSessa et al., 1991). By supporting students in an authentic act of negotiating conventions, particularly through aiding their transition from an idiosyncratic use of notation to notation negotiated through interactions in a community and the construction of intersubjective knowledge, researchers will contribute insights into the accommodations that occur as students construct meanings that differentiate conventions from those logico-mathematical aspects necessary to representing mathematical concepts.

Our current work indicates that a number of PSTs' meanings entail the habitual use of "convention", with some of these cases involving their exhibiting contradicting claims or actions. This trend is consistent with the participating ISTs. If we accept that teachers (and students) understanding mathematical concepts in ways that entail conventions *qua* conventions is desirable, then an important question for teacher education is *how might the desired meanings develop?* We believe that our work, in combination with that by previous researchers (Mamolo & Zazkis, 2012; Thompson, 1995; Zazkis, 2008), provides initial guidance in this area. Specifically, for teachers holding meanings that entail the habitual use of "convention", we hypothesize that one way to support the transition to understanding convention *qua* convention is to develop instruction that affords teachers the opportunity to raise and reconcile potential contradictions between claims and actions. A teacher educator who intentionally violates conventions (or "conventions") in ways that maintain mathematical validity may support teachers in examining their meanings for certain representational features. In doing so, teachers have an opportunity to become reflectively aware of particular aspects of their meanings and thus differentiate those aspects that enable them to assimilate a wider range of representations from those aspects that are practices of representation among a variety of viable options. Examples provided in this paper provide some viable contexts for these conversations. Moore et al. (2014) and Johnson (2015) shared additional strategies that speak to Thompson's (1995) suggestion of placing an emphasis on synthesizing issues of convention, quantitative reasoning, and notation. Mamolo and Zazkis (Mamolo & Zazkis, 2012; Zazkis, 2008) provided other examples that include using unfamiliar coordinate systems. Each of these strategies can be used as design and implementation principles for teacher educators and researchers interested in supporting and understanding PSTs' and ISTs' development of meanings that are consistent with convention *qua* convention.

Before closing, we acknowledge the limitations of using on-line survey data to draw inferences about ISTs' meanings. We also acknowledge that our work with PSTs was at one university, thus limiting the diversity of the participant pool. There is a need for additional studies into both PST and IST populations, and we suggest that investigations of populations include other qualitative methods including classroom observations. Our data does not allow us to make definitive claims about PST or IST *teaching*. Our participants' responses could have been influenced by the research setting, thus influencing their actions relative to the conventions we focused on. In other words, what our participants consider a convention could change in the setting of their teaching and the associated emergence of intersubjective knowledge. We also suggest that future studies seek to provide more detailed insights into students' and teachers' meanings when confronted with contradicting actions and claims. Most PSTs in our study did not reconcile these contradictions when they became aware of them. It remains to be seen if other PST and IST populations do so or if interventions are necessary to support their reconciliation of these contradictions.

7. Closing

We close by underscoring that we do not intend to discredit conventions, nor to convey that conventions are unimportant for mathematical reasoning. We hope we have been clear that a convention is important to the extent that an individual understands it as a convention *qua* convention. We also do not contend that educators can realistically address every convention they perceive to be part of their mathematical community. We argue, however, that our results respond to and strengthen calls for a more detailed consideration of how educators and researchers treat and understand conventions (or "conventions").

We agree with Zazkis (2008), who explained, "Conventions are choices made and agreed upon, within a certain group, to assure successful communication. Of course, conventions are to be respected…but there is a need to become aware of them" (p. 138).[5] We also agree with Thompson's claim, "…[we] can attempt to make explicit those conventions that are assumed and treated as given, those conventions that are assumed and presented as conventions, and those conventions that are meant for students to recreate or to create in some idiosyncratic form" (1992, p. 125). Educators and researchers must be sensitive to the negotiation of conventions among students, teachers, and any member of a perceived community. Given the complexity of teaching and the idiosyncratic nature of learning, understanding what this sensitivity might look like will take concerted efforts at many levels with particular emphasis given to students' meaningful creation and use of notation and representations (e.g., diSessa et al., 1991; Meira, 1995; Thompson, 1995; Tillema & Hackenberg, 2011). In short, if students and teachers are to understand a convention *qua* convention, then they need opportunities to come to understand mathematical concepts in ways that include the negotiation of customary choices within the context of those ideas. A productive negotiation of conventions should occur in conversations where the logico-mathematical aspects of a concept—which are understood as remaining invariant among several choices of representation—are foregrounded.

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[5] It is more appropriate to say that conventions *aid* or *expedite* successful communication, as establishing and maintaining conventions certainly does not assure successful communication of mathematical ideas and meanings.

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Declaration of Competing Interest statements were not included in the published version of the following articles that appeared in previous issues of Journal of Mathematical Behavior

The appropriate Declaration/Competing Interest statements, provided by the Authors, are included below.

1 "When itâ€™s on zero, the lines become parallel: Preservice elementary teachersâ€™ diagrammatic encounters with division by zero" [Journal of Mathematical Behavior, 2020; 58C: MATBEH_2018_209] 10.1016/j.jmathb.2020.100760 Declaration of competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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9 “A quadratic growth learning trajectory” [Journal of Mathematical BehaviorJournal of Mathematical BehaviorJournal of Mathematical Behavior, 2020; 59C: MATBEH_2019_188] 10.1016/j.jmathb.2020.100795 Declaration of competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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