1-1-2015

Modeling Contagion in the Eurozone Crisis Via Dynamical Systems

Giuseppe Castellacci
New York University

Youngna Choi
Montclair State University, choiy@mail.montclair.edu

Journal of Banking & Finance 50 (2015) 400–410

Contents lists available at ScienceDirect

Journal of Banking & Finance

journal homepage: www.elsevier.com/locate/jbf

Modeling contagion in the Eurozone crisis via dynamical systems

Giuseppe Castellacci [a], Youngna Choi [b,*]

[a] New York University, United States
[b] Department of Mathematical Sciences, Montclair State University, Montclair, NJ 07043, United States

ARTICLE INFO

Article history:
Received 10 September 2013
Accepted 6 May 2014
Available online 2 June 2014

JEL classification:
E10
G01

Keywords:
Multi-agent models
Dynamical systems
Bifurcation
Market instability indicator
Sovereign credit
Systemic risk
Contagion

ABSTRACT

We recently (Castellacci and Choi, 2013) formulated a theoretical framework for the modeling of financial instability contagion using the theories of dynamical systems. Here, our main goal is to model the Eurozone financial crisis within that framework. The underlying system comprises many economic agents that belong to several subsystems. In each instantiation of this framework, the hierarchy and nesting of the subsystems is dictated by the nature of the problem at hand. We describe in great detail how a suitable model can be set up for the Eurozone crisis. The dynamical system is defined by the evolution of the wealths of the individual agents and can be estimated by solving a nonlinear programming problem that incorporates features of prospect theory. Contagion is formulated in terms of how the market instability indicators for the different subsystems and the global system behave. We present several scenarios tailored to recent financial developments in the Eurozone and discussed within our model. These all point to the key role played by the elasticity coefficients of the wealth dynamical system. Accordingly, we put forward general recommendations on how regulators or other super-systemic agents may act to prevent and forestall the spreading of financial distress.

1. Introduction

Since the bursting of the NASDAQ bubble in the year 2000, the global economy seems to have entered a regime of recurring instability. In the long-term alternation of risk aversion and risk appetite, the hiatus between crises induces participants in the financial markets (and to a certain extent, any economic agent) to gradually forget the lessons of previous crises and correspondingly indulge in the illusion that *This Time is Different* (Reinhart and Rogoff, 2008).

In the early 2000s the widespread prudence that pervaded the financial markets in the wake of the crash of the equity bubble and the spectacular collapse of major public corporations on both sides of the Atlantic, quickly abated thanks to several powerful global trends and domestic fiscal and monetary policies. These kindled a bubble in real estate in many developed and developing countries. Such trend climaxed likely some time in early 2007. The weakest links in the speculation chain that inflated the bubble in the US were also the canary in the mine: subprime mortgage loans and the securities they backed. The epitome of this bubble's inflation was likely the extreme leveraging attained through loss layers (tranches) of structured investment vehicles (SIV) and especially collateralized debt obligations (CDOs) comprising asset-backed securities (ABS), whose "compound tranching" structure led to a dramatic perception of low risk, as sanctioned by all rating agencies, in the senior tranches. When the brewing crisis finally erupted in September 2008, financial markets were confronted with systemic uncertainty as to both the size and the complexity of such asset and the role they played in the balance sheet of major institutions.[1] This time, the same generation of market participants that had experienced the last crisis was still investing into and managing financial assets. For many years, the markets had trusted the framework and agencies government had put in place in the wake of major crises (such as the Federal Reserve in the US) with their ability to protect the value of mainstream assets with "puts." The panics of the XIX century appeared consigned to history. As the fall of 2008 approached this sense of security seemed itself a thing of the past.

The ensuing crisis induced the US Federal Reserve (the Fed) in conjunction with the Treasury to implement extreme measures, some of which required legislation. This set in motion a new phase of the crisis: governments began absorbing private debt in unprecedented fashion. In 2009, the financial markets hailed such sweeping action as the only way in which a floor could be put under the insolvency of major banks. Panic abated, and financial assets

* Corresponding author. Tel.: +1 9177107942.
E-mail addresses: castel@alum.mit.edu (G. Castellacci), choiy@mail.montclair.edu (Y. Choi).

[1] Anecdotal reporting revealed that even the size of the Troubled Asset Relief Program (TARP) was determined rather haphazardly by the Treasury secretary.

http://dx.doi.org/10.1016/j.jbankfin.2014.05.013
0378-4266/

rebounded, but as it turned out this was a Pyrrhic victory. The bulk of the risk had simply been shifted. Emblematic in this regard was the decision of the Irish government to bail out the country's major bank thereby taming the national debt unmanageable.

Credit risk aversion reared its head when, in early 2010, it became apparent that the fiscal conditions of the peripheral economies[2] in the European Monetary Union (EMU, a.k.a. Eurozone), were much more precarious than previously perceived. The economic history of these countries is different, and correspondingly diverse is the way in which they led their national balance sheet into a danger zone. Common became the perception that the sovereign debt of these countries is unmanageable and default was increasingly likely. Furthermore, while the debt of smaller peripheral economies (Greece, Portugal, and Ireland) was deemed salvageable, Italian public debt is way beyond the arsenal of bail out funds available to the ECB and other monetary institutions.

Why did international credit market focus on the Eurozone problems? Arguably, the main reason is that the monetary union does not correspond to any systemic fiscal institution, unlike conventional nation states (e.g., the Fed and the Treasury in the US). Therefore individual countries have a very indirect and weak control of monetary policy while still responsible for their economies. And even the ECB is heavily constrained in managing the money supply in its mandate. At the same time, regulation on risk capital encouraged investing in sovereign debt by considering it virtually riskless regardless of issuance. Banks and other institutions chased the higher yields of PIIGS' debt thereby accumulating unprecedented leverage.[3] After spreading from Greece to the other small peripheral economies (Portugal and Ireland), in the fall of 2011 "contagion"[4] pushed Italian 10-year yields above the threshold of sustainability of 7%. The emergency lending implemented by the ECB stanched the liquidity hemorrhage. However, just as in the spring of 2009 within the US banking system, this is much more an issue of solvency. At the same time, austerity alone is unlikely to allow debtor nations to satisfy their creditors. Greece, which has endured the longest recession in economic history, is most emblematic in this regard.

Dynamical systems is a relatively new field of mathematics that studies the evolution of time-dependent systems. Regardless of the system under consideration, the predominant goal of the field is to investigate the stability of the state at a given time and its asymptotic behavior, which is often realized by studying the trajectory of a point in the system. Although the laws, physical or otherwise, that govern the system remain unaltered over time, the dynamical system (e.g. function or vector fields on the system) itself can be perturbed to yield a dynamic behavior that is totally different from the original one. This is why we believe that the theory of dynamical systems can be fruitfully applied to the modeling of financial crises, not only the 2007–09+ US subprime crisis and the Eurozone sovereign credit crisis but also general ones, from the analysis of the etiology and containment to prevention.

Indeed, the second author already carried out such research in the case of one economic system (Choi and Douady, 2012). Choi and Douady used multi-agent dynamical systems to model, first the 2007–09+ US financial crisis, then financial crises in a single economy in general. An economic system is divided into aggregates called "agent," and a dynamical system is constructed to track the wealth of the agents. High leverage and borrowing capacity constraints of the agents induce a bifurcation and subsequent change of the stability type of economic equilibria. Near an unstable equilibrium, a negative shock on wealth can propagate through the system via the feedback loop created by inter-agent cash flows, and due to the nonlinearity of the system, the shock can intensify while propagating, incurring wealth drops of all affected agents. If among the affected agents are banks, due to the interconnected of the banking system and extremely high leverage from the esoteric financial tools widely used these days, not only their wealth would drop with acceleration but also would follow bank runs and mass bankruptcies. The entire economy would be severely hit and very likely a financial crisis would emerge. The incorporation of bifurcation mechanism and theories on stability to explain financial crises is the major difference between Choi and Douady (2012) and other multi-agent based models.

In order to apply such ideas to the Eurozone sovereign debt crisis, we deemed it necessary to extend that framework to the case of multiple economies as part of a global economic system, and this is the major accomplishment of Castellacci and Choi (2013), the prequel of this article. While Castellacci and Choi (2013) studies financial crises and contagion in a multi-economy system in general, our focus in this article is its application to the Eurozone. Besides, we incorporate prospect theory by Kahneman and Tversky to explain agents' psychology in decision making that directly affects the flow of funds in the system.

The construction of this paper is as follows. In Section 2 is introduced that a structured global economy comprises many agents that are members of different subsystems as well as of the global system. Furthermore, we extend the construction of wealth dynamical systems via a nonlinear programming problem (NLP) by integrating concepts from Prospect Theory that reflect changing risk appetites. Then in Section 3, which is the core of the paper, we further motivate the notion of instability contagion that we have recently introduced (Castellacci and Choi, 2013). This is illustrated with and applied to scenarios that reflect recent developments in the Eurozone crisis. These give us the opportunity not only to analyze the mechanism of contagion, but also to put forth suggestions on how governments and independent, possibly super-national authorities may intervene to preempt or forestal the spreading of financial distress. Appendices recall and extend the framework first laid out in Choi and Douady (2012) and Castellacci and Choi (2013)) to adapt it to the problems at hand. The first author would like to thank Fedor A. Bogomolov and the Courant Institute of Mathematical Sciences for their kind support.

2. The dynamical system of wealth

We will follow the work of Choi and Douady (2012) in building a dynamical system of wealth. We assume that an economy is structured as a system of n agents, and a time-dependent deterministic dynamical system for the scaled (converted in constant base currency – the euro in our case) wealth vector $\mathbf{w}(t) = (w_1(t), \dots, w_n(t)) \in \mathbb{R}^n$ is constructed, where $w_i(t)$ is the wealth of agent i at time t. Then we extend this framework to more than one economy in the spirit of Castellacci and Choi (2013). We will leave the details to Appendixes A.1, A.2, and A.2, and in this section, focus on inter-agent cash flows and agents' investment decision making.

2.1. Economic agents and global wealth

According to the framework put forward in Castellacci and Choi (2013), we consider a global economic system that consists of s subeconomies. To reflect the Eurozone structure, we assume that each subeconomy is sovereign and fiscally independent, and the global economy is a monetary union with super-national monetary authorities (MA). As in Choi and Douady (2012), we represent each subeconomy by five *agents* (economic aggregates): consumers, firms, banks, government, and investors. Following Castellacci

[2] The so-called PIIGS, namely Portugal, Ireland, Italy, Greece, Spain.

[3] There are reports of leverage ratios exceeding 400.

[4] Here we are employing the common language sense of this word and not the quantitative definition we will be applying later in the article.

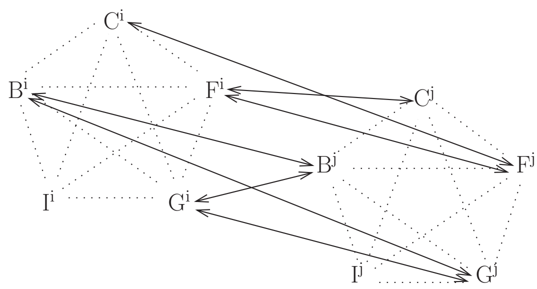

Fig. 1. Cash flows between subeconomies i and j. The dotted lines are domestic cash flows, the solid lines international ones.

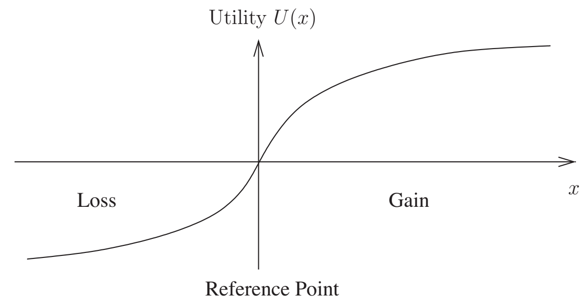

Fig. 2. A typical value function from prospect theory.

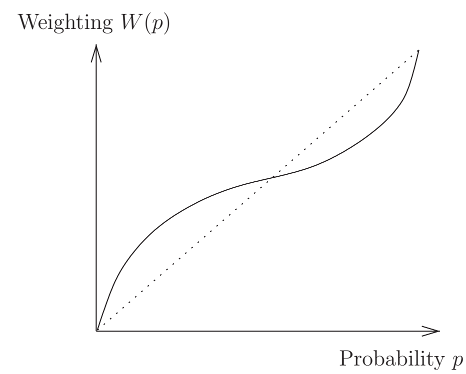

Fig. 3. A typical weighting function that distorts probability p.

and Choi (2013) we use an upper index to denote the subeconomy, thus the five agents in subeconomy i are C^i, F^i, B^i, G^i, and I^i.[5] The cash flows between agents, which drive the market and determine the global wealth, are broadly classified into four groups: at-will, scheduled, contingent, and international. Tables 1–4 summarize typical ones (more details can be found in Castellacci and Choi (2013)).

When considering international cash flows, we classify the nationality of firms and banks by their physical location, hence a multinational company can belong to several countries. As such, trades or fund transfers between its headquarter and local branches are considered as international cash flows. Lending at private level, regardless of the nationality of the lending banks, is considered as a domestic cash flows for the same reason. However we assume direct transactions between governments and banks, hence consider investment in foreign sovereign bonds as international cash flows as schematized in Fig. 1.

2.2. Optimal investment and financial crisis

An economically cogent technique to construct a wealth dynamical system is to require that each agent optimizes the utility of its investment in all other agents in the system. This problem can be solved constructively using a nonlinear programming problem (NLP), which we proceed to expose.

We choose an optimal cash flow F_{ji} from i to j that optimizes some utility of i of receiving $F_{ij}(s)$ for all $s > t$ subject to liquidity and solvency constraints. In doing so, we refine Choi and Douady (2012) approach with regards to the class of agent utility functions, which is based on the *Prospect Theory* by Kahneman and Tversky (1979),Tversky and Kahneman (1992). In this theoretical framework, classical utility (Friedman and Savage, 1948) is replaced by a *value function* $U : [a, b] \longrightarrow \mathbb{R}$ where $a < 0 < b$, along with a weighting function (cf. Weber (1994)). The value function $U(x)$ is increasing, convex for losses, and concave for gains, namely, $U'(x) > 0$, $U''(x) \geqslant 0$ for $x < 0$, and $U''(x) \leqslant 0$ for $x > 0$ (see Fig. 2). The weighting function W reflects the investor's attitude toward risk, and it is possible that $\int_{\mathbb{R}} W'(F(x))\, dF(x) > 1$. In our case, we use separate weighting functions, W^- for loss and W^+ for gain, (defined over intervals containing $[a, 0)$ and $(0, b]$, respectively) each of which looks like the one in Fig. 3, and write $W = \mathbb{1}_{[a,0)}\ W^- + \mathbb{1}_{(0,b]}\ W^+$.[6]

Expectations are taken with respect to a probability measure $\mathbb{P}$, which defines a cumulative distribution function (CDF) $F(x) := \mathbb{P}[X \leqslant x]$ for each random variable X. Then expected utility can be written as

$$E[U(X)] = \int_{\mathbb{R}} U(x)\ dF(x) \tag{1}$$

and the subjective utility (SU, defined in Rieger and Wang (2006) as a generalization of Tversky and Kahneman (1992)) is

$$\mathrm{SU}[X] = \int_{-\infty}^{0} U(x)\frac{d}{dx}W^-(F(x))dx + \int_{0}^{\infty} U(x)\frac{d}{dx}W^+(F(x))dx \tag{2}$$

$$= \int_{\mathbb{R}} U(x)\ W'(F(x))\ dF(x) \tag{3}$$

We apply and slightly extend prospect theory to suit our needs as follows. Each agent i is endowed with a value function satisfying the above properties. We assume that the probability measure can vary with time, but at each time t the probability measure $\mathbb{P}_t$ is common to all agents.[7] On the other hand, we assign a possibly different weighting function $W_{i,t}(x)$ to each agent. Thus, value functions reflect the same ordered preferences in time while the probability measures and weighting functions can model the evolution of the agents' risk appetites. Then, the subjective utility for i during $[t, t+1)$ is defined as

$$\mathrm{SU}_{i,t}[X] = \int_{-\infty}^{0} U_i(x)\frac{d}{dx}W^-_{i,t}(F_t(x))dx + \int_{0}^{\infty} U_i(x) \times \frac{d}{dx}W^+_{i,t}(F_t(x))dx \tag{4}$$

where F_t is the CDF of X defined by $\mathbb{P}_t$.[8]

[5] This agent represent international portfolio managers who manage assets of agents in subeconomy i.

[6] $\mathbb{1}_A$ denotes the indicator function of a set. Note that since this function will be used only as part of integrands, the fact that it not defined at $x = 0$ is not problematic. Ditto for its derivative.

[7] The reader concerned with the formalism of stochastics, may wish to think of this family of probability measures as part of stochastic base satisfying the usual conditions. In particular the filtration of σ-algebras is increasing so as to model the accretion of information.

[8] Notice that these random variables X will be taken to be fund flows between agents at time $t, F_{ji}(t)$. Thus, the initially stochastic flows give rise to a deterministic dynamical system via expectation. Incidentally, the notation for the CDF has nothing to do with that for the cash flows.

For an investment $F_{ji}(t)$ by i, we discount its future returns $F_{ij}(s)$ by a discount factor $D(t,s)$[9] and sum over a finite number of "resets" within the finite horizon $(t,T]$ to get the net present value (NPV) of the investment, $\sum_{t<s_l\leqslant T} D(t,s_l)F_{ij}(s_l) - F_{ji}(t)$. The net subjective utility (NSU) of the agent i for the investment $F_{ji}(t)$ is thus:

$$\mathrm{NSU}_{i,t}(F_{ji}(t)) = \mathrm{SU}_{i,t}\left[\sum_{t<s_l\leqslant T} D(t,s_l)F_{ij}(s_l) - F_{ji}(t)\right] \tag{5}$$

The optimal F_{ij}s are found by solving a system of NLP:

$$\mathrm{NLP}: \qquad \max \quad z_i = \sum_{j=1}^{n} \mathrm{NSU}_{i,t}(F_{ji}(t)) \tag{6}$$

$$\text{subject to} \qquad L_i(t) \geqslant 0 \tag{7}$$

$$|\widetilde{\Delta} K_i(t+1)| \leqslant \kappa_i(t+1)K_i(t) \tag{8}$$

$$\widetilde{\Delta} D_i(t+1) \leqslant D_{i\max}(t+1) - (1+r_i(t))D_i(t) \tag{9}$$

$$1 \leqslant i \leqslant n, \ t \geqslant 0.$$

The origin of time $t = 0$ is the beginning of the economic period under consideration. Each agent chooses an optimal $F^*_{ji}(t), 1 \leqslant i, \ j \leqslant n$ from the NLP (6) for each t, and optimally select its debt level $D_i(t+1)$ (within borrowing capacity limits) and allocates its new wealth $w_i(t+1)$ between $L_i(t+1)$ and $K_i(t+1)$. Details on the constraints can be found in Appendix A.1.

This system of NLP is an n-player coordination game with mixed strategy, cf. Von Neumann and Morgenstern (1944). It is a coordination game because each agent is well-informed on its investment conditions, ample macroeconomic information is publicly available, and they are happy to collaborate with one another to maximize their wealths. It is a mixed strategy game since changes in constraints result in different optimal solutions. As such, a mixed strategy Nash equilibrium exists (Straffin, 1993).[10] Once we find an optimal solution, a wealth dynamical system f can be constructed as in Choi and Douady (2012). This f is a predictable process, which, if there is no exogenous random influence on the system, becomes deterministic. Choi and Douady (2012) explains how Predictable process can be treated like a deterministic map in terms of equilibria and stability, so without loss of generality, we will assume our dynamical system f is deterministic.

3. Contagion in the Eurozone crisis

3.1. Origin of financial crises in this modeling framework

The modeling framework introduced in Choi and Douady (2012) connects all economic agents in a single economy with cash flows, and if the borrowing capacity of an agent is reduced while its leverage is already high, then a "default"[11] is inevitable. Furthermore, when the default is too sizable, or other agents are exceedingly leveraged with respect to the defaulting cash flows, the ensuing shock spreads through the economic system, causing systemic risk and possibly a financial crisis. The authors suggested that policy makers monitor the market instability indicator (Appendix A.3) especially as it crosses certain judiciously preset thresholds and intervene proactively, for instance, by deleveraging the market through close control of the elasticity coefficients (Appendix A.2). This can be attained by identifying the amount of liquidities to inject in each economic segment, using in each case the appropriate means, such as quantitative easing for banks, tax credits for investors and corporations, investment and expense programs for corporations, recovery of consumption for consumers, etc. However the authors' recommendation apply to a single economy with the assumption that the government has the means to implement such policies, if necessary by increasing the money supply. In the case of the sovereign crisis in the Eurozone, governments who don't have sovereign monetary authority are at greater risk of default, so their model does not apply as is.

The fears for a sovereign credit crisis in the Eurozone started developing in late 2009. In the next two years the crisis became reality, having resulted in bailouts of three countries, Greece (twice), Ireland, and Portugal. Nevertheless the situation worsened and the sovereign credit risk spread to Spain[12] and Italy, with the yields on their government bonds rising sharply. This prompted the European Central Bank (ECB) to purchase Spanish and Italian bonds to bring the yield down. On October 26, 2011, European leaders agreed on a deal that reduces the existing Greek debt, in which private investors take "voluntary" haircut of 50% (BBC timeline). On the news European stock markets soared (MarketWatch, 2011) and the crisis seemed to be contained without further contagion. The rally did not last long, and during the month following the Greek debt restructuring agreement, bank stocks in Europe and in the US dropped sharply (Yahoo! Finance) while the borrowing cost of the Eurozone countries rose to dangerous levels (Trading Economics), and the fear for "contagion" has emerged.

3.2. Defining contagion

But what is this "contagion?" In the economic context, this intuitively suggests spreading of asset price crash (burst of bubbles) in time and space. To mathematically define this concept, immediate questions one may ask (cf. Karolyi, 2003) are:

(a) How does one detect a crisis in an economic system?
(b) How does one determine causation between one crisis and another? Is succession in time sufficient?

To answer these questions, Castellacci and Choi (2013) recently gave quantitative definition of financial instability contagion.

Definition 3.1. (Instability Contagion). Consider a (global) economic system comprising s subsystems. Let $B(t)$ be the Jacobian matrix of the global system at time t (cf. Eq. (38)) and $B^{(k)}(t)$ the corresponding matrix for the k-th subsystem.[13] We say that *(instability) contagion* occurs if given two time instants t_0, t_1 with $0 < t_0 < t_1$[14] the market instability indicators of the global system, $\rho(B(\cdot))$, and of the subsystems, $\rho(B^{(k)}(\cdot))$, satisfy the following properties

(i) At time $t < t_0$,

$$\max_{1\leqslant k\leqslant s} \rho(B^{(k)}(t)) < 1 \quad \text{and} \quad \rho(B(t)) < 1 \tag{10}$$

(ii) At time $t \in (t_0, t_1)$,

$$\max_{1\leqslant k\leqslant s} \rho(B^{(k)}(t)) > 1 \quad \text{and} \quad \rho(B(t)) < 1 \tag{11}$$

[9] This can be thought of as the price of a zero coupon bond at t which delivers \$1 at s.

[10] In theory there are infinitely many optimal solutions due to the random constraints, but in reality, there are limited range of macroeconomic choices, and after rounding to common currency unit, such as billions of dollars, we can make the number of strategies finite.

[11] Choi and Douady (2012) defined "default" such that an agent i at a given time t is in *default* if it has no choice but violating its borrowing capacity constraint, i.e. it cannot meet its obligation because of borrowing capacity constraint. As a result, cash flows originating from agent i abruptly decrease.

[12] At the time of writing, Spain called for a bailout for its banks (The Financial Times, 2012). That of the government itself is likely to follow.

[13] For the relationships between these matrices, cf. Appendix A.2 as well as Castellacci and Choi (2013). Notice also that ρ denotes the spectral radius of a matrix consistently with Appendix A.3.

[14] We assume the origin of time in our model is 0.

(iii) At time $t > t_1$

$$B(t) \neq \oplus_{k=1}^{s} B^{(k)}(t) \quad \text{and} \quad \rho(B(t)) > 1. \tag{12}$$

Notice that the last condition is to capture the causal nature of contagion. Indeed, if $B(t) = \oplus_{k=1}^{s} B^{(k)}(t)$, then $\rho(B(t)) = \max_k \rho(B^{(k)}(t))$, which means a global crisis could arise from the independent occurrence of sub-systemic crises.

Example 1. In this simple example, we illustrate the mere mathematical aspects of our definition of contagion. Consider the following one-parameter family of matrices that represent the Jacobian matrix of a global system that consists of two mono-agent subeconomies[15]

$$B(t) = \begin{pmatrix} -t + 3/2 & 1/2 \\ -1/2 & -t/2 + 1/2 \end{pmatrix} \quad \text{for } t \in [0, 2]. \tag{13}$$

Notice that in this case the scalar economies' wealth decline linearly for $B^{(1)}(t) = (-t)$ and $B^{(2)}(t) = (-t/2)$ so that

$$B^{(1)}(t) \oplus B^{(2)}(t) = \begin{pmatrix} -t & 0 \\ 0 & -t/2 \end{pmatrix} \quad \text{for } t \in [0, 2], \tag{14}$$

so that $\max_k \rho(B^{(k)}(t)) = t$, whereas

$$\rho(B(t)) = \max \left| \frac{\operatorname{tr}(B(t)) \pm \sqrt{\operatorname{tr}(B(t))^2 - 4|B(t)|}}{2} \right| = \frac{2 - \frac{3t}{2} \pm \sqrt{t\left(\frac{t}{4} + t\right)}}{2}. \tag{15}$$

For $t \in [0, 1)$ $\max_k \rho(B^{(k)}(t)) < 1$ while for $t \in (1, 2]$ $\max_k \rho(B^{(k)}(t)) > 1$. On the other hand $\rho(B(t))$ has a more interesting behavior. For $t \in [0, 1/8)$, $\rho(B(t)) > 1$. For $t \in \left(1/8, \frac{13-\sqrt{41}}{4}\right)$, $\rho(B(t)) < 1$. Finally, $t \in \left(\frac{13-\sqrt{41}}{4}, 2\right)$, $\rho(B(t)) > 1$. This can be interpreted as there being initial systemic instability while the component economies are stable. The global instability abates and the system becomes stable after $t = 1/8$. Both component economies experience linearly declining wealth. Then the first economy enters a period of instability $t \in (1, 2)$. This gradually spills over the entire system, which eventually becomes unstable after $\frac{13-\sqrt{41}}{4}$. We depict this behavior in Fig. 4.

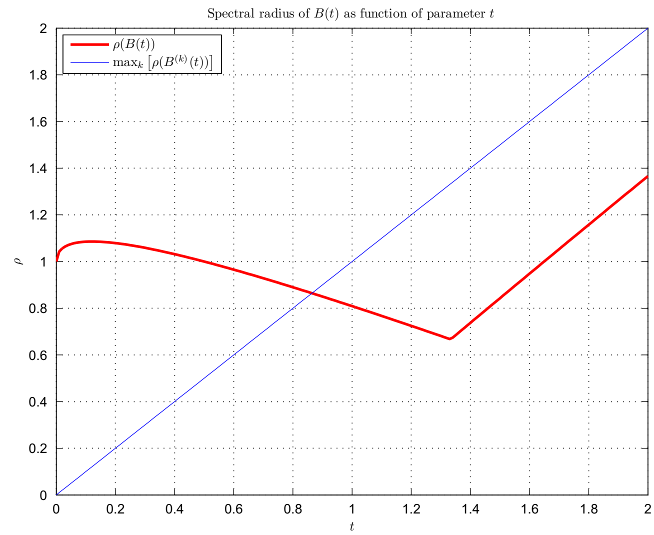

Fig. 4. The spectral radius of a one-parameter family of Jacobian matrices, which is consistent with (53).

3.3. Case studies

Here we illustrate the economic mechanism of contagion (Castellacci and Choi, 2013) with scenarios that pertain to the ongoing Eurozone crisis. Since contagion is defined via the market instability indicator (see Appendix A.3 for details), a lower bound thereof provides a sufficient condition for a contagion of instability. To this end, recall that the trace of a square matrix M is the sum of its eigenvalues:

$$\operatorname{tr}(M) = \sum_{i=1}^{n} \lambda_i, \tag{16}$$

which implies

$$\frac{|\operatorname{tr}(M)|}{n} \leqslant \max_{\lambda_i \in \sigma(M)} |\lambda_i| =: \rho(M), \tag{17}$$

where $\sigma(M) := \{\lambda_1, \ldots, \lambda_n\}$ is the matrix spectrum. Applying this to Eq. (54), we get the following lower bound of the market instability indicator $I(t) = \rho(B(t))$,

$$\left| \frac{1}{n} \sum_{i=1}^{n} b_{ii} \right| = \left| 1 + \frac{1}{n} \sum_{i=1}^{n} a_{ii}(t) - \frac{1}{n} \sum_{i \neq j} a_{ij}(t) \right| \leqslant \rho(B). \tag{18}$$

The mechanism of different types of contagion is analyzed in Castellacci and Choi (2013). We apply two of them, cross-border contagion from default and cross-border contagion from fear factor, to the sovereign crisis in the Eurozone to explore the scenarios in which the crisis can evolve. Although this study is based on a purely mathematical model, it may become useful in establishing the possible outcomes of the current situation and build appropriate strategies for each case.

When a "contagion" takes place, the spread of the instability starts within a subset of the global economy and develops in stages, hence we do not need to consider all the subeconomies in the monetary union all at once. Therefore, rather than modeling all the 17 countries of the Eurozone, we will focus on a "mini Eurozone" which consists of four countries which we consider the core of the crisis: the two peripheral members, Greece and Italy, and their two major creditors, France and Germany. The peripheral economies have different sizes and economic backgrounds, yet both are susceptible to sovereign credit conditions; the creditor countries, albeit being the 4th and the 5th in GDP size of the world economy and main driving forces of the Eurozone, have different current accounts and unemployment rates. We index with the integers from 1 to 4 to Greece, Italy, France, and Germany in that order, total 20 agents in the global economy (the mini Eurozone). For each country we assign 1 to 5 for the five agents, consumers (C), firms (F), banks (B), government (G), and investors (I), in that order. We follow the notations and indexing as we defined in Section A.2. Then we have the following elasticity matrix $A(t)$ and Jacobian matrix $B(t)$:

$$A(t) = \begin{pmatrix} A^{(1)}(t) & A^{(12)}(t) & A^{(13)}(t) & A^{(14)}(t) \\ A^{(21)}(t) & A^{(2)}(t) & A^{(23)}(t) & A^{(24)}(t) \\ A^{(31)}(t) & A^{(32)}(t) & A^{(3)}(t) & A^{(34)}(t) \\ A^{(41)}(t) & A^{(42)}(t) & A^{(43)}(t) & A^{(4)}(t) \end{pmatrix}$$

where $A^{(k)}(t) = \left(a_{ij}^{k}(t)\right)_{\substack{1 \leqslant i \leqslant 5 \\ 1 \leqslant j \leqslant 5}}$ is the elasticity matrix of subeconomy k, and $A^{(kl)}(t) = \left(a_{ij}^{kl}(t)\right)_{\substack{1 \leqslant i \leqslant 5 \\ 1 \leqslant j \leqslant 5}}$,

[15] Notice the $B^{(i)}(t) (i = 1, 2)$ are scalar (1-dimensional block matrices).

$$B(t) = \begin{pmatrix} B^{(1)}(t) & A^{(21)}(t) & A^{(31)}(t) & A^{(41)}(t) \\ A^{(21)}(t) & B^{(2)}(t) & A^{(61)}(t) & A^{(61)}(t) \\ A^{(31)}(t) & A^{(32)}(t) & B^{(3)}(t) & A^{(34)}(t) \\ A^{(41)}(t) & A^{(42)}(t) & A^{(43)}(t) & B^{(4)}(t) \end{pmatrix}$$

where $B^{(k)}(t) = \left(b^k_{ij}(t)\right)_{\substack{1 \leqslant i \leqslant 5 \\ 1 \leqslant j \leqslant 5}}$ is the Jacobian matrix of subeconomy k. We call $A^{(k)}(t)$ and $B^{(k)}(t)$ local matrices, and $A(t)$ and $B(t)$ global ones. We proceed to formulate the recent history of the Eurozone sovereign debt crisis in terms of our model. This will provide both a description of these events and policy recommendations.[16]

As is well known this originated in Greece, which correspond to the subeconomy $i = 1$. Eurozone political and monetary authorities, having acknowledged Greek financial turmoil, have enacted a number of measures aimed at cordoning off that instability and forestall contagion. In terms of the elasticity matrix $A(t)$, contagion is prevented if the entries in the block $A^{i1}(t)$ are small enough to keep $\rho(B(t)) < 1$ for all t[17]. The most drastic such measure was an agreement with private bondholder that resulted in a writedown over 50% and the debt swaps between the Greek government and private debt holders in March 2012 (Riskdata, 2012; Wikipedia). Despite such efforts, the market widely speculated the "Grexit," the Greek exit of the Eurozone, while Spain, a much bigger economy in the zone, struggled with bank bailouts (The Financial Times, 2012), a newly high cost of borrowing and credit rating downgrade (The Financial Times, 2012). Rather than following what has happened, we will provide scenarios to show how our model can be applied in each case. Therefore, the countries mentioned in the following scenarios can be replaced by any debtor and creditor countries in the Eurozone. In analyzing the contagion scenarios, we will use the following results from Castellacci and Choi (2013) on the behavior of the wealth dynamical system.

Result 1. *The wealth $w_i(t)$ of agent i at time t satisfies the following:*

(a) If $w_i(t)$ is strictly increasing or strictly decreasing in t, then $b_{ii}(t) > 0$.
(b) If $w_i(t)$ is strictly increasing and convex, then $b_{ii}(t) > 1$.
(c) If $w_i(t)$ is strictly decreasing and concave, then $b_{ii}(t) > 1$.

Result 2. *For $t_0 < t_1$, the invested asset $K_i(t)$ of agent i at time t satisfies the following:*

(a) If $K_i(t)$ is strictly increasing or strictly decreasing, $\frac{K'_i(t_1)}{K'_i(t_0)} > 0$.
(b) If $K_i(t)$ is strictly increasing and convex, $\frac{K'_i(t_1)}{K'_i(t_0)} > 1$.
(c) If $K_i(t)$ is strictly decreasing and concave, $\frac{K'_i(t_1)}{K'_i(t_0)} > 1$.

Scenario 1. *There is a credit event for Greek sovereign debt:*

Here "credit event" means any kind of debt reduction for creditors, should it be an agreed restructuring or downright default[18]. This would reduce the obligatory flow of funds from the Greek government (agent 4) to its creditor, notably Greek banks and their French and German counterparts. Outside Greece, this means both $F^{31}_{34} = F_{13,4}$ (payments to French banks) and $F^{41}_{34} = F_{18,4}$ (payments to German banks) would go down. This will reduce the wealth of French and German banks, w_{13} and w_{18}, respectively, and reduced banks' wealth can trigger further asset price drops, for example stock price decrease, increased withdrawals for fear of bank runs, and increased interest rate for interbank borrowing. Mathematically speaking, this means that for French banks,

$$b_{13,13} = 1 + a_{13,13} - \sum_{\substack{k=1 \\ k \neq 13}}^{20} a_{k,13}. \tag{19}$$

The cash flows corresponding to the withdrawals by French consumers, firms, and investors, and German firms, and banks are $F_{11,13}, F_{12,13}, F_{15,13}, F_{17,13}, F_{18,13}$, respectively, and are positive. The wealth shock $\Delta w_{13}(t)$ on French banks is negative, therefore the elasticities $a_{11,13}$, $a_{12,13}, a_{15,13}, a_{17,13}$, and $a_{18,13}$ are negative. French banks will pay less tax to the French government due to reduced wealth, hence $\Delta F_{14,13} < 0$ and subsequently $a_{14,13} > 0$. This tax saving, however, will be much smaller than the panic-driven withdrawals. French banks are not directly related with other agents, for example to Greek and Italian consumers or firms, hence the corresponding flows of funds, $F_{i,13}$ for $i \neq 11, 12, 13, 15, 17, 18$ change little as $w_{13}(t)$ goes down, and resulting $a_{i,13} \sim 0$. By Eqs. (35) and (52) $a_{13,13} = \frac{\partial(\gamma_{13}(t)K_{13}(t))}{\partial K_{13}(t)}$, and by Eq. (31),

$$a_{13,13}(t) = \frac{K'_{13}(t+1) - K'_{13}(t) - \widetilde{\Delta} K'_{13}(t)}{w'_{13}(t)}. \tag{20}$$

If $K_{13}(t)$ is decreasing and concave in t, i.e. $K_{13}(t)$ decreases with acceleration, then $K'_{13}(t+1) < K'_{13}(t)$ by Result 3.3. The conversion $\widetilde{\Delta} K_{13}(t)$ of liquidity $L_{13}(t)$ to invested assets $K_{13}(t)$ would be negative at the beginning since banks would liquidate their invested assets to cope with increased withdrawals and payments for interbank lending, but there is a limit to doing so and it will level off eventually. Therefore $\widetilde{\Delta} K_{13'}(t) \sim 0$. Hence the numerator of Eq. (20) is negative, and the denominator $w'_{13}(t)$ is negative by our assumption, so $a_{13,13}(t) > 0$.

If the economic situation in the Eurozone has so deteriorated as to induce a sovereign credit event, it is highly probable that many other agents' wealth has been declining, hence by Result 3.3 $b_{ii} > 0$. Thus in Eq. (18), the hikes of $b_{13,13}$ and $b_{14,14}$ contribute to the rise of $\frac{1}{20}\sum_{i=1}^{20} b_{ii}$, a lower bound of $\rho(B(t))$. The instability indicator $\rho(B^{(1)}(t))$ is likely to have gone above 1 before the credit event, i.e. it already experienced a domestic financial crisis, and if the global instability indicator $\rho(B(t))$ goes above 1, then by the definition, a contagion of instability has taken place in the mini Eurozone, and the cause is a credit event.

The Fed and the US government stepped in and bailed out banks when Lehman Brothers bankrupted and other banks were having severe liquidity shortage, which mathematically means lowering b_{33}, hence the lower bound $\frac{1}{5}\sum_{i=1}^{5} b_{ii}$.[19] Note that the ECB lending to European banks already sharply increased (The Financial Times, 2011) and European banks had already been borrowing dollars from central banks (Wall Street Journal, 2011). This probably would also happen if a Greek credit events takes place.

Scenario 2. *Fear factor:*

We have witnessed that, independently, markets lose confidence in sovereign bonds whose yields take turns to rise to unsustainable levels (Trading Economics). In this case is not caused by risk transmission from one subsystem to another but by the "fear factor," i.e. by investors' loss of confidence in the sovereign bond market. This "over-reaction" is reflected by the distorted probability by a weighting function $W(t)$ from Section 2.2. We use a simple example to explain such a loss of confidence. (see Tables 1–4)

[16] As our model fundamentally capture the dynamics of economic systems, it can be used both to describe features and to prescribe actions that can affect their evolution.

[17] The instability indicator for the Greek economy is likely to have crossed the threshold of 1, for the "default" of the Greek government has spread to other agents in the form of austerity measures, increased taxes, reduced wages for or even laying off civil servants etc. As a result, the country plunged into a deep recession.

[18] We are not concerned about CDS trigger, since the lending banks and the CDS issuing banks are classified as a single agent B, and their cash flows net out.

[19] Here we consider only the single US economy and agent 3 represents the US banks.

Table 1
Selected at-will cash flows among five agents in subeconomy *i*.

Fund type	From	To	Activities
Equity investment	C^i	C^i	Trade houses and other goods
	F^i	F^i	Invest into each other
	G^i, I^i	F^i, B^i	Invest in corporate and bank stocks
Debt investment	B^i	C^i	Mortgages, credit cards, other financing
		F^i	Bank loans to companies
		B^i	Interbank lending, securitization
	I^i	F^i, B^i, G^i	Invest in bonds
Dividends	F^i, B^i	I^i	Pay dividends on their stocks
	I^i	C^i	Pay dividends and pensions
Consumption	C^i	F^i	Consumes goods and services

Table 2
Selected scheduled cash flows among five agents in subeconomy *i*.

Fund type	From	To	Activities
Coupons	C^i	B^i	Mortgages, other loan payments
	B^i	B^i, I^i	Securitized debt, CDS premiums
	F^i	B^i, I^i	Coupons to bond holders
	G^i	B^i, I^i	Coupons to sovereign bond holders
Salary	F^i, B^i, G^i	C^i	Wages and benefits
Contributions	C^i	I^i	Contribution to retirement fund

This includes the minimum credit card payment.

Table 3
Examples of contingent cash flows in subeconomy *i*. Although MA (monetary authorities) is not an agent, it plays an active role during a time of emergency, thus included in the table.

Fund type	From	To	Activities
Quantitative easing	MA	G^i	Purchase sovereign bond
Derivative payoff	B^i	B^i, I^i	CDS payout

Example 2. Consider an uncertain economy consisting of 9 states of the world, $\Omega := \{\omega_1, \ldots, \omega_9\}$. We identify these states with the possible percentage returns on an investment as follows: $\omega_i = -100\% + i25\%$. Further, consider three probabilities on Ω, P_1, P_2, and P_3, which represent optimism, pessimism, and no-risk taking, respectively. We assume that an investor chooses one of the three probabilities at a time while keeping the same utility function all the time. The weighting function is the identity function, hence the traditional utility function and the subjective utility we adopt for this article are the same. We lay out the numerical details in Table 5.

The subjective utilities expected utilities $E_1[U(R)]$, $E_2[U(R)]$, and $E_3[U(R)]$ under the probabilities P_1, P_2, and P_3 are respectively:

$$E_1[U(R)] = 0\cdot 0 + 0\cdot\frac{1}{4} + 0\cdot\frac{1}{2} + \frac{1}{10}\cdot\frac{5}{8} + \frac{1}{10}\cdot\frac{3}{4} + \frac{2}{10}\cdot\frac{27}{32} + \frac{3}{10}\cdot\frac{30}{32} + \frac{2}{10}\cdot\frac{31}{32} + \frac{1}{10}\cdot 1 = \frac{282}{320} = 0.88125.$$

$$E_2[U(R)] = 0\cdot 0 + \frac{1}{10}\cdot\frac{1}{4} + \frac{2}{10}\cdot\frac{1}{2} + \frac{3}{10}\cdot\frac{5}{8} + \frac{2}{10}\cdot\frac{3}{4} + \frac{1}{10}\cdot\frac{27}{32} + \frac{1}{10}\cdot\frac{30}{32} + 0\cdot\frac{31}{32} + 0\cdot 1 = \frac{200}{320} = 0.640625.$$

$$E_3[U(R)] = 1\cdot\frac{3}{4} = 0.75.$$

Therefore an investor would invest when he feels optimistic about the market return, but hold the money when feels pessimistic. The choice of probability could be purely psychological and independent of the actual market performance.

Suppose the Greek government (agent 4) was eventually bailed out by international monetary authorities and the loans from French and German banks (agent 13 and 18, respectively) have been restructured, and the market fears that the Italian government (agent 9) may have difficulty in paying back its loans due to various domestic problems, although it currently is not going through any macroeconomic change. We further assume that this fear has increased the Italian bond yield to an unsustainable level, and the Italian government cannot refinance its loan at an affordable rate.[20]

Mathematically this means that the net subjective utility of the Italian sovereign bond investors has decreased, more precisely:

[20] In reality, when the government bond yield of Spain and Italy rose to an unsustainable level, the ECB President Mario Draghi pledged that the ECB was ready to do “whatever it takes” to preserve the Euro (The Financial Times, 2012). After this announcement the market calmed down and the Spanish and Italian yields went back to a sustainable level.

- If Italian B (agent 8) and I (agent 10) lose confidence in Italian sovereign debt:
  - $\mathrm{NSU}^2_{3,\ t}\left(F^2_{43}(t)\right)$ decreases $\Rightarrow$ $F^2_{43} = F_{98}$ decreases
  - $\mathrm{NSU}^2_{5,\ t}\left(F^2_{45}(t)\right)$ decreases $\Rightarrow$ $F^2_{45} = F_{9,10}$ decreases
- If French B (agent 13) loses confidence in Italian sovereign debt:
  $-\mathrm{NSU}^3_{3,\ t}\left(F^{23}_{43}(t)\right)$ decreases $\Rightarrow$ $F^{23}_{43} = F_{9,13}$ decreases
- If German B loses confidence in Italian sovereign debt:
  $-\mathrm{NSU}^4_{3,\ t}\left(F^{24}_{43}(t)\right)$ decreases $\Rightarrow$ $F^{24}_{43} = F_{9,18}$ decreases

By Eq. (36) the wealth w_9 of the Italian government is given by

$$w_9(t+1) = w_9(t) = \sum_{j=1}^{20} F_{9,j}(t) - \sum_{\substack{k=1\\k\neq 9}}^{20} F_{k,9}(t), \tag{21}$$

and reduced $F_{98}, F_{9,10}, F_{9,13}$, and $F_{9,18}$ imply reduced wealth $w_9(t)$. High government bond yield implies higher domestic interest rates, so Italian consumers and firms (agent 6 and 7, respectively) have to pay higher interest on their loans. This means lower consumption and productivity, and eventually a lower tax revenue to the government which is represented by reduced $F_{96}(t)$ and $F_{97}(t)$ in Eq. (21)[21], hence $\Delta w_9(t) < 0$. On the other hand, the Italian government's payment to the bond holders increase due to higher yield, hence $a_{k,9} < 0$ for $k = 8, 10, 13, 18$. Very likely the Italian economy would continue declining amid high cost of borrowing for all agents, which would result in mass unemployment and bankruptcies. The government's benefit payment to consumers and firms would increase while its

[21] The events happen progressively and we use *t* for general time without specifying the exact time of the event.

Table 4
Samples of international flow of funds between subeconomies i and j.

Fund type	From	To	Activities
Debt investment	B^i	B^j	Interbank lending
		G^j	Purchase sovereign bond
Coupons	B^i	B^j	Coupons to bond holders
	G^i	B^j	Coupons to sovereign bond holders

Table 5
Utility and probabilities for return x.

x	−1	−0.75	−0.5	−0.25	0	0.25	0.5	0.75	1
$U(x)$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{27}{32}$	$\frac{30}{32}$	$\frac{31}{32}$	1
P_1	0	0	0	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{2}{10}$	$\frac{1}{10}$
P_2	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{2}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	0	0
P_3	0	0	0	0	1	0	0	0	0

own wealth has decreased. This means $a_{ki} < 0$ for $i = 9$ and $k = 6, 7$. The diagonal element b_{99} is formulated as

$$b_{99} = 1 + a_{99} - \sum_{\substack{i=1 \\ i\neq 9}}^{20} a_{k,9} = 1 + a_{99} - \sum_{\substack{i=1 \\ i\neq 14}}^{20} a_{k,9} \tag{22}$$

and $a_{k,9} \sim 0$ for $1 \leqslant k \leqslant 5$ since there is hardly any payment obligation from the Italian government to the Greek agents. Same is true for $11 \leqslant k \leqslant 20, k\neq 13, 18$, which represents obliged payment by the Italian government to the French and German agents except for banks. The wealth $w_9(t)$ is likely to have declined, hence $b_{99} > 0$ by Result 1, and is very likely to be big due to the magnitude of $\sum_{\substack{i=1 \\ i\neq 9}}^{20} a_{k,14}$. Also by the same Result, it is very possible that for all other agents in Italy, $b_{ii} > 0$ and possibly bigger, hence $\rho(B^{(2)}(t) > 1$. If the global indicator $\rho((B))$ is still less than 1, the global economy is Stage (ii) in the definition of the instability contagion. If the Italian government cannot afford the high cost of borrowing yet cannot be bailed out by monetary authorities due to the size of its economy, hence defaults on some of its payments to French and German banks, then as seen in Scenario 1, $b_{13,13}$ and $b_{18,18}$ would rise, this time even more than the Greek credit event case, due to the size of the Italian economy. Then $\rho((B))$ could become greater than 1, and by definition, there has been a contagion of instability from Italy to the global economy and the cause of the local instability is a fear factor.

We have examined the two most common – or most worried about – causes of a cross-border instability contagion for a scaled-down Eurozone. This approach can be applied other subsets of the Eurozone or other monetary unions. Care should be taken, however. Our definition of "contagion" should be distinguished from "domino effect," which could be used to describe a situation where subeconomies become unstable in turns, and the chain of instability is linked by causation.

Scenario 3. *Domino Effect*

We revisit the 4-subeconomy mini Eurozone from Scenarios 1 and 2. Assume that the Greek economy becomes unstable at t_1, hence $\rho(B^{(1)}(t_1)) > 1$. The Greek government eventually defaults on its loans to the French banks, and as a result the Greek economy stabilizes and in turn, the French economy becomes unstable, i.e. $\rho(B^{(1)}(t_2)) < 1$ and $\rho(B^{(3)}(t_2)) > 1$ for some $t_2 > t_1$. Then Italian economy, having been affected by the French instability, becomes unstable while France has managed to stabilize its economy, hence $\rho(B^{(2)}(t_3)) > 1$ and $\rho(B^{(3)}(t_3)) < 1$ for some $t_3 > t_2$. This is a domino effect in the sense described above. In terms of instability contagion, this is for the global economy to stay in state (ii) while the global contagion never takes place.

A major advantage of the quantitative definition of contagion is that it can be applied to partitions of subeconomies, should it be domestic or international. Hence we can concentrate on the origin and path of a financial instability contagion without accessing the entire data set of the involved countries. For example, we can partition the (mini) Eurozone into Greek government – French banks – German banks, French consumers – French firms, German consumers – German firms etc. to investigate the contagion from the Greek government to French/German banking sector, then from respective French banking sector to French real economy and German banking sector to German real sector. "Local" data for this analysis are far easier to obtain, for example from the central banks of each country, the International Monetary Fund, The World Bank, than building the entire global matrices $A(t)$ and $B(t)$, which will be very difficult if not impossible, even for the mini Eurozone, due to data availability and frequency. Using real life data to calibrate and validate our instability contagion model is the goal of the next project.

4. Conclusion

The recently developed (Castellacci and Choi, 2013) concept of instability contagion is applied in this article to model the Eurozone sovereign credit crisis. This definition of contagion crucially relies on the *market instability indicator* applied to several subsystems of a global economic system consisting of judiciously selected agents. Given that the instability indicator is the spectral radius of Jacobian matrices that correspond to the subsystems and the global system, the *elasticities* of flows of funds between agents play a key role in (in)stability.

In this setting it becomes apparent that the elasticities corresponding to fund flow internal to a subsystem are responsible for individual economies instability, while those corresponding to fund flows between agents in different subsystems may cause global instability, hence contagion. We illustrate this mechanism with several scenarios that describe possible developments in the still ongoing Eurozone crisis. This investigation emphasizes the interconnected structure of the economies within and without the Eurozone, tracks the routes of risk transmission, and suggests methods that can be used to prevent the spreading of fiscal and financial distress.

Appendix A. Previous results

A.1. Dynamical system of wealth

Recently Choi and Douady (2012) proposed an multi-agent financial crisis model for a single economy, and Castellacci and Choi Castellacci and Choi, 2013 extended it to multi-economy systems. This section summarizes their results and defines the notations used in this article.

We consider a collection of s economies such that the economy k is divided into n_k aggregates which we call "economic agent." The number of economies s and those of the agents may differ from case to case. In our Eurozone example in Section 3.3, $s = 4$ and $n_k = 5$ for all $1 \leqslant k \leqslant 4$. At each time t, we observe $\mathbf{w}(t) = (w_1(t), \ldots, w_n(t)) \in \mathbb{R}^n$, the *global wealth vector* of the agents, where $n = \sum_{k=1}^{s} n_k$. Each $\mathbf{w}(t)$ is a canonical embedding of the wealth vector of the respective economy, $\mathbf{w^k}(t) = \left(w_1^k(t), w_2^k(t), \ldots, w_{n_k}^k(t)\right)$. Therefore $w_i(t) = w_j^k(t)$ if

$$i = N(k) + j, \qquad N(k) = \sum_{l=1}^{k-1} n_l. \tag{23}$$

The global wealth $S(w(t))$ is the sum of all wealths:

$$S(w(t)) = \sum_{i=1}^{n} w_i(t) \tag{24}$$

where $n = \sum_{k=1}^{s} n_k$.

Two assumptions on the economy are made:

Minimality A minimum number of agents are selected for the economy to function such that any removal of an agent would make the system collapse. Mathematically, this means that there is a minimum weight[22] $c > 0$ of each agent in the overall economy so that $w_i(t)/S(w(t)) \geqslant c, \forall i$.

Boundedness The economy is based on limited resources and market participants, therefore the production, consumption, and the total wealth of the economy is bounded above and below. Mathematically, this means that there is time adjustment factor [23] $\alpha(t)$, some $C', C > 0$ such that $C' \leqslant S(w(t))\alpha(t)^{-1} \leqslant C$. Hence the *normalized wealth vector* $\bar{\mathbf{w}}(t) = \alpha(t)^{-1}\mathbf{w}(t)$ stays inside a compact and convex subset of $\mathbb{R}^n$,

$$\overline{M} = \left\{ \bar{\mathbf{w}} \in \mathbb{R}^n | C' \leqslant \sum_{i=1}^{n} \bar{w}_i \leqslant C, \quad \bar{w}_i \geqslant c\ C' \quad \forall\ i = 1, \ldots, n \right\} \tag{25}$$

The wealth $w_i(t)$ of the agent i at t is defined to be the sum of the equity and debt,

$$w_i(t) = E_i(t) + D_i(t) \tag{26}$$

and also the sum of liquidities $L_i(t)$ (equivalent to the monetary base, M0 in the case of the U.S.) and invested assets $K_i(t)$ (the rest – financial securities, property, human resources etc.),

$$w_i(t) = L_i(t) + K_i(t) \tag{27}$$

As a result, $w_i(t)$ for each i inherits the equity-debt wealth decomposition of $w_j^k(t)$ such that

$$w_i(t) = E_i(t) + D_i(t) = E_j^k(t) + D_j^k(t) \tag{28}$$

and the liquidity-invested asset decomposition

$$w_i(t) = L_i(t) + K_i(t) = L_j^k(t) + K_j^k(t) \tag{29}$$

The liquidities $L_i(t)$ produce no income, while $K_i(t)$ can produce capital gains. It is assumed that during the time period $[t, t+1)$, only $K_i(t)$ has an internal (i.e. independent of incoming or outgoing cash flows) growth that is measured by the *internal rate of return (IRR)* $\gamma_i(t)$ on the investment.

The $D_i(t), K_i(t)$ and $L_i(t)$ evolve as follows:[24]

$$D_i(t+1) = (1 + r_i(t))D_i(t) + \widetilde{\Delta} D_i(t+1) \tag{30}$$

$$K_i(t+1) = (1 + \gamma_i(t))K_i(t) + \widetilde{\Delta} K_i(t+1) \tag{31}$$

$$L_i(t+1) = L_i(t) + \Delta L_i(t+1) \tag{32}$$

where

- $\widetilde{\Delta} D_i(t+1)$ is equal to new loans less payments.
- $r_i(t)$ is the average interest that applies to the debt $D_i t$.
- $\widetilde{\Delta} K_i(t+1)$ is equal to new investments less liquidation (i.e., converting part of $K_i(t)$ to cash).
- $$\Delta L_i(t+1) = \sum_{j \neq i}^{n} F_{ij}(t) - \sum_{k \neq i}^{n} F_{ki}(t) - \widetilde{\Delta} K_i(t+1) \tag{33}$$

where $F_{ij}(t)$ is the fund transferred from agent j to agent i at time t.

During normal (i.e. non-crisis) times, we can assume that the rates $r_i(t), \gamma_i(t)$, and the residuals $\widetilde{\Delta} K_i, \widetilde{\Delta} D_i$, and ΔL_i are continuous, hence $L_i(t), K_i(t)$, and $D_i(t)$ are processes with continuous sample paths, and Eqs. (30)–(32) are discrete observations of them.

For each agent i, define its *state* at time t as the triplet

$$X_i(t) = (L_i(t), K_i(t), D_i(t)) \in \mathbb{R}^3, \quad X = (X_1, X_2, \ldots, X_n) \tag{34}$$

It is further assumed that:

- $L_i(t) \geqslant 0$: any negative liquidity (shortage of money) is immediately converted to a debt increase .
- $|\widetilde{\Delta} K_i(t+1)| \leqslant \kappa_i(t)K_i(t)$: liquidation of $K_i(t)$ is limited to a fraction $\kappa_i(t)$.
- $D_i(t) \leqslant D_{i\max}(t)$: each agent i has a maximum level of debt $D_{i\max}(t)$ which depends on its wealth $w_i(t)$ and credit market condition.

The internal return $\gamma_i(t)K_i(t)$ of the invested asset $K_i(t)$ can be interpreted as a result of "self-investment," hence replaced by

$$F_{ii}(t) = \gamma_i(t)K_i(t) \tag{35}$$

From Eq. 27, (31)–(33), and 35

$$w_i(t+1) = w_i(t) + \sum_{j=1}^{n} F_{ij}(t) - \sum_{k \neq i}^{n} F_{ki}(t) \tag{36}$$

The flow of funds $F_{ji}(t)$ from i to j at t can be considered as an *investment* by agent i. We assume that it affects only the counterpart j and induces a stream of returns $F_{ij}(s)$ at dates $s > t$, and each agent "invests" to maximize the utility (value, benefit) of receiving $F_{ij}(s)$ for all $s > t$ subject to liquidity and solvency constraints. This optimization is formulated by Eqs. (6)–(9). The optimal solution $F_{ji}^*(t)$ of the NLP leads to a random dynamical system f in $\mathbb{R}^{3n}$:

$$X^*(t+1) = f(X^*(t)) \tag{37}$$

where the components of $X^*(t+1)$ are given by Eqs. (30)–(32) and (36) with optimal $F_{ji}^*(t)$, then optimal debt and wealth allocation, under the constraints (7)–(9). From this random f, we can derive a deterministic dynamical system $\bar{f}$[25] that acts on rescaled state $\overline{X}(t) = \alpha(t)^{-1}X(t)$, and from the Jacobian $d\bar{f}$ of $\bar{f}$, a reduced Jacobian $B(t)$ which is in fact the Jacobian matrix of dynamical system of *rescaled* wealth $\bar{\mathbf{w}}(t)$, i.e.

$$B(t) = d\phi(t), \ \ \phi(\bar{\mathbf{w}}(t)) = \bar{\mathbf{w}}(t+1). \tag{38}$$

The construction of $\bar{f}$ and $B(t)$ above are described in detail in Appendix of (Choi and Douady, 2012). In Castellacci and Choi, 2013 a deterministic wealth dynamical system f[26] was constructed by observing historical flow of funds, and both discrete and continuous cases were covered.

For notational simplicity, we remove "bar" from $\bar{f}$ and $\bar{w}(t)$ and write f and $w(t)$ in the rest of the Appendices as well as in the main body the paper. We also dropped the * symbol of the optimal solution of the NLP to write $X(t+1) = f(X(t))$ etc.

[22] Note that $w_i(t)/S(w(t))$ represent the relative wealth or "weight" of the i-th agent with respect to total wealth.

[23] This can be thought of as a deflator.

[24] The notation $\widetilde{\Delta}$denotes a contribution that is different from the ordinary time-increments, e.g., $\Delta D_i(t+1) = D_i(t+1) - D_i(t)$.

[25] In general we obtain a predictable process, yet when it is further assumed that there is no exogenous random contribution, the predictable process becomes deterministic.

[26] The article used notation f for the map that corresponds to ϕ in Eq. (38).

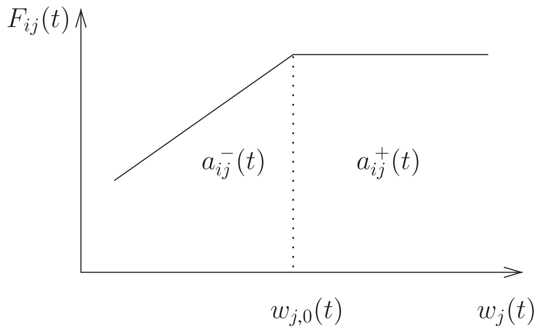

Fig. 5. When the wealth $w_j(t)$ falls below the target wealth $w_{j,0}(t)$, the flow of funds $F_{ij}(t)$ decreases, but it stays flat when $w_j(t) > w_{j,0}(t)$.

A.2. Elasticity coefficient

The *elasticity coefficient* between two agents i and j is defined as the change rate of outgoing cash flow with respect to the wealth of the payer,

$$\Delta F_{ij}(t) = a_{ij}(t)\Delta w_j(t). \tag{39}$$

Since Castellacci and Choi, 2013 already showed how the dynamical system of wealth could be constructed to be differentiable modulo isolated points, we use the partial derivative notation throughout the paper. [27] However care should be taken before writing

$$a_{ij} = \frac{\partial F_{ij}}{\partial w_j}, \tag{40}$$

for this definition assumes the existence of the partial derivative for both signs of the wealth shock δw_j. In reality, we can observe discontinuities of elasticity coefficients. For example, when major US banks were bailed out during the 2008 US financial crisis, the lendings to consumers and firms did not increase as much as the policy makers expected, for the banks hoarded cash instead of lending. Fig. 5 explains such a phenomenon.

To resolve this problem, we refine the definition of elasticity coefficient as follows:

$$\begin{cases} a_{ij}^+(t) = \lim_{\Delta w_j \to 0^+} \frac{F_{ij}(t+\Delta t) - F_{ij}(t)}{\Delta w_j} \\ a_{ij}^-(t) = \lim_{\Delta w_j \to 0^-} \frac{F_{ij}(t+\Delta t) - F_{ij}(t)}{\Delta w_j} \end{cases} \tag{41}$$

where $w(t + \Delta t) = w_j(t) + \Delta w_j$.

In case the subeconomies involved in the cash flows need to be specified, we use the upper indices to denote them, so the cash flow from agent j to i in subeconomy k is

$$F_{N(k)+i,N(k)+j}(t) = F_{ij}^k(t) \tag{42}$$

We do the same thing for the Jacobian matrix and elasticity matrix:

$$b_{N(k)+i,N(k)+j}(t) = b_{ij}^k(t) \tag{43}$$

and

$$a_{N(k)+i,N(k)+j}(t) = a_{ij}^k(t) \tag{44}$$

where $B^{(k)}(t) = \left(b_{ij}^k(t)\right)$ and $A^{(k)}(t) = \left(a_{ij}^k(t)\right)$ are the Jacobian matrix and the elasticity matrix of subeconomy k, respectively. In case two subeconomies k and l are involved, we denote the cash flow from agent j in subeconomy l to agent i in subeconomy k by

$$F_{ij}^{kl}(t) = F_{N(k)+i,N(l)+j}(t). \tag{45}$$

[27] When the differentiability is not applicable, we modify the derivative notation to difference quotient and write $a_{ij} = \frac{\Delta F_{ij}}{\Delta w_j}$.

Therefore

$$\begin{cases} a_{ij}^{kl+}(t) = \lim_{\Delta w_j^l \to 0^+} \frac{F_{ij}^{kl}(t+\Delta t) - F_{ij}^{kl}(t)}{\Delta w_j^l} \\ a_{ij}^{kl-}(t) = \lim_{\Delta w_j^l \to 0^-} \frac{F_{ij}^{kl}(t+\Delta t) - F_{ij}^{kl}(t)}{\Delta w_j^l} \end{cases} \tag{46}$$

where $w_j^l(t)$ is the wealth of the j-th agent in the l-th economy at time t and $w_j^l(t + \Delta t) = w_j^l(t) + \Delta w_j^l$.

More generally, it may be economically meaningful to consider mild discontinuities along arbitrary directions. We can model this phenomenon by introducing directional Jacobians as follows. Recall that if $\mathbf{u} \in \mathbb{R}^n$ is a unit vector and $g : \mathbb{R}^n \to \mathbb{R}$, then we can define the left and right directional derivatives at $\mathbf{x}_0$ along $\mathbf{u}$ as

$$D_{\mathbf{u},\mathbf{x}_0}^{\pm}(g) := \lim_{\epsilon \downarrow 0} \frac{g(\mathbf{x}_0 \pm \epsilon \mathbf{u}) - g(\mathbf{x}_0)}{\epsilon}. \tag{47}$$

We need to generalize the notion of gradient, too. The gist of the idea is that given n linearly independent direction, we can pick a side (left or right) for each of them and consider the corresponding lateral directional derivative. Consider the set of n-ary multi-indexes with values ± 1, that is the space of functions

$$\mathcal{S} = \{s : \mathbf{n} \to \{-1, 1\}\} \cong 2^{\mathbf{n}} \tag{48}$$

where $\mathbf{n} := \{1, \ldots, n\}$. Given a base of directions $U = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$, we define the (U, s)-lateral gradient at $\mathbf{x}_0$ as the vector

$$D_{U,\mathbf{x}_0}^s(g) := \left(D_{\mathbf{u}_1,\mathbf{x}_0}^{s(1)}(g), D_{\mathbf{u}_2,\mathbf{x}_0}^{s(2)}(g), \ldots, D_{\mathbf{u}_n,\mathbf{x}_0}^{s(n)}(g)\right), \tag{49}$$

where we are identifying $D_{\mathbf{u},\mathbf{x}_0}^+(g) = D_{\mathbf{u},\mathbf{x}_0}^1(g)$ and $D_{\mathbf{u},\mathbf{x}_0}^-(g) = D_{\mathbf{u},\mathbf{x}_0}^{-1}(g)$. Now, for any vector field $\mathbf{G} : D \to \mathbb{R}^m$, where $D \subset \mathbb{R}^n$ is open, we define the (U, s)-lateral differential or Jacobian matrix at $\mathbf{x}_0$ as the matrix whose columns are the gradients of the components of $\mathbf{G}$, namely

$$D_{U,\mathbf{x}_0}^s(\mathbf{G}) := \left(D_{U,\mathbf{x}_0}^s(G_1) \quad D_{U,\mathbf{x}_0}^s(G_2) \quad \cdots \quad D_{U,\mathbf{x}_0}^s(G_m) \right), \tag{50}$$

Notice that this generalizes the customary notion of differential or Jacobian matrix of a vector field. Indeed, when the field admits first partial derivatives at $\mathbf{x}_0$, and $U = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, then

$$D_{U,\mathbf{x}_0}^s(\mathbf{G}) = D_{\mathbf{x}_0}(\mathbf{G}), \tag{51}$$

where $D_{\mathbf{x}_0}$ denotes the Jacobian at $\mathbf{x}_0$. For, in this case the lateral partial derivatives of $\mathbf{G}$ coincides with the partial derivatives.[28] We can apply this new notion to our wealth dynamical system at time t by taking $f^t = \mathbf{G}$. Then we can relax the assumption that the dynamical system admits partial derivatives everywhere to the condition that it admits directional Jacobians everywhere. Consistently with Castellacci and Choi, 2013, we assume that the directional Jacobians may differ from the classical ones only at a set of isolated points.

We assume that if agent j experiences a change δw_j in its wealth at time t, then the cash outflow from j to i is changed by $a_{ij}(t)\delta w_j(t)$, while those from other agents are not affected. To reflect the internal change of wealth, "self-elasticity" a_{ii} of agent i is defined to be

$$a_{ii}(t) = \frac{\partial(F_{ii}(t))}{\partial w_i(t)} = \frac{\partial(\gamma_i(t)K_i(t))}{\partial w_i(t)} \tag{52}$$

This implies that $a_{ii}\delta w_i = \delta F_{ii}$, hence we have $a_{ij}\delta w_j = \delta F_{ij}$ for all $1 \leqslant i, j \leqslant n$, including $j = i$.

Let $A = (a_{ij})_{1 \leqslant i,j \leqslant n}$ be the $n \times n$ matrix of elasticities, with entries a_{ij}, and let $A^\sharp$ be the diagonal matrix with entries $a_i^\sharp = \sum_{k \neq i}^n a_{ki}$ for $1 \leqslant i \leqslant n$,

[28] If, furthermore $n = m$, then $\mathbf{G}$ is locally invertible iff $D_{U,\mathbf{x}_0}^s(\mathbf{G})$ is.

$$A^{\sharp} = \mathrm{diag}\left(\sum_{k \neq 1}^{n} a_{k1}, \ldots, \sum_{k \neq i}^{n} a_{ki}, \ldots, \sum_{k \neq n}^{n} a_{kn}\right).$$

It was shown in (Choi and Douady, 2012 and Castellacci and Choi, 2013) that matrices A and B, regardless of the number of the subeconomies forming the global economy, are related by the equation:

$$B = I + A - A^{\sharp}, \tag{53}$$

Which means:

$$b_{ii} = 1 + a_{ii} - \sum_{k \neq i}^{n} a_{ki} \quad \text{and} \tag{54}$$

$$b_{ij} = a_{ij} \qquad \text{for } i \neq j \tag{55}$$

As a result, both the local elasticity matrices $A^{(k)}(t)$ and the local Jacobian matrices $B^{(k)}(t)$ can be canonically embedded into the global counterparts, hence at given time t,

$$A(t) = \begin{pmatrix} A^{(1)}(t) & A^{(12)}(t) & \ldots & A^{(1s)}(t) \\ A^{(21)}(t) & A^{(2)}(t) & & \\ \vdots & & \ddots & \\ A^{(s1)}(t) & \ldots & & A^{(s)}(t) \end{pmatrix} \tag{56}$$

where

$$A^{(kl)}(t) = \left(a_{ij}^{kl}(t)\right)_{\substack{1 \leqslant i \leqslant n_k, \\ 1 \leqslant j \leqslant n_l}} \tag{57}$$

and

$$B(t) = \begin{pmatrix} B^{(1)}(t) & A^{(12)}(t) & \ldots & A^{(1s)}(t) \\ A^{(21)}(t) & B^{(2)}(t) & & \\ \vdots & & \ddots & \\ A^{(s1)}(t) & \ldots & & B^{(s)}(t) \end{pmatrix}. \tag{58}$$

A.3. Market instability indicator

The "market instability indicator" ρ is defined as the spectral radius of the reduced Jacobian matrix $B(t)$.

$$I(t) = \rho(B(t)) \tag{59}$$

The higher the indicator, the more unstable the market. In stable market conditions, an equilibrium point $\widetilde{X}$ is an attractor, and the eigenvalues of $\widetilde{B} = B(\widetilde{X})$ have modulus less than 1. when the market is close enough to the equilibrium and, as a consequence, in its basin of attraction, the instability indicator $I(t)$ is also below the critical value 1.

When $I(t) < 1$ then perturbations of the system tend to be absorbed and disappear. On the contrary, when $I(t) > 1$ then most of the perturbations contain a component that will increasingly propagate within the system, either as a propagation of contraction of payments, or simply as an increase of leverage making liquidity constraints tighter and tighter and reactions to variations of income stronger and stronger.

References

BBC Timeline June 13, 2012. The unfolding eurozone crisis. http://www.bbc.co.uk/news/business-13856580.
Castellacci, G., Choi, Y., 2013. Financial instability contagion: a dynamical systems approach, to appear in quantitative finance, available at Taylor & Francis Online: http://www.tandfonline.com/doi/full/10.1080/14697688.2014.890737.
Choi, Y., Douady, R., 2012. Financial crisis dynamics: attempt to define a market instability indicator. Quantitative Finance 12 (9), 1351–1365.
Choi, Y., Douady, R., 2013. Financial Crisis and Contagion: A Dynamical Systems Approach. *Handbook on Systemic Risk*, Cambridge University Press.
Friedman, Milton., Savage, L.J., 1948. The utility analysis of choices involving risk. Journal of Political Economics 56 (4), 279–304.
Kahneman, Daniel, Tversky, Amos, 1979. Prospect theory: an analysis of decision under risk. Econometrica XLVII, 263–291.
Karolyi, G., 2003. Does international financial contagion really exist? International Finance 6 (2), 179199.
MarketWatch October 26, 2011. EU agrees debt deal; 50% Greek debt haircut. http://www.marketwatch.com/story/greek-bondholders-to-take-50- haircut-2011-10-26.
Von Neumann, John, Morgenstern, Oskar, 1944. Theory of Games and Economic Behavior. Princeton University Press.
Reinhart, Carmen M., Rogoff, Kenneth S., This Time is Different: A Panoramic View of Eight Centuries of Financial Crises. Working paper, NBER, April 2008. http://www.nber.org/wbuiter/cr1.pdf.
Rieger, Marc Oliver, Wang, Mei, 2006. Cumulative prospect theory and the St. Petersburg paradox. Economics Theory 28, 665–679.
Riskdata April, 2012. Greek debt default: investors' and risk managers' perspective. http://www.riskdata.com/resources/greek-debt.html.
Straffin, Philip D., 1993. Game Theory and Strategy. The Mathematical Association of America.
The Financial Times November 22, 2011. ECB lending to Eurozone banks hits high. http://www.ft.com/intl/cms/s/0/c35c37f2-1527-11e1-855a-00144feabdc0.html#axzz1esJ3oXoK
The Financial Times June 6, 2012. Europe weighs up limited Spanish rescue. http://www.ft.com/intl/cms/s/0/81e1c8ec-afe5-11e1-ad0b-00144feabdc0.html#axzz1x3iXThn9.
The Financial Times June 7, 2012. Fitch downgrades Spain's credit rating. http://www.ft.com/intl/cms/s/0/3175685e-b081-11e1-8b36-00144feabdc0.html#axzz1x3iXThn9.
Trading Economics. http://www.tradingeconomics.com/ (accessed 13.07.12).
Tversky, Amos, Kahneman, Daniel, 1992. Advances in prospect theory: cumulative representations of uncertainty. Journal of Risk and Uncertainty 5, 297–323.
Wall Street Journal, September 16, 2011. Central Banks pour dollars into Europe. http://online.wsj.com/article/SB10001424053111904060604576572442555810356.html.
Weber, Elke U., 1994. From subjective probabilities to decision weights: the effect of asymmetric loss functions on the evaluation of uncertain outcomes and events. Psychological Bulletin 115 (2), 228–242.
Wikipedia. Greek government-debt crisis—Wikipedia, the free encyclopedia, 2004. Available at: <http://en.wikipedia.org/wiki/Greek_government-debt_crisis> (29 April 2010).
Yahoo! Finance. http://finance.yahoo.com/.