How Do Manipulatives Help Students Communicate Their Understanding of Double-Digit Subtraction?

Rabab Abi-Hanna
Montclair State University

Follow this and additional works at: https://digitalcommons.montclair.edu/etd
Part of the Education Commons, and the Mathematics Commons

Recommended Citation
https://digitalcommons.montclair.edu/etd/87

This Dissertation is brought to you for free and open access by Montclair State University Digital Commons. It has been accepted for inclusion in Theses, Dissertations and Culminating Projects by an authorized administrator of Montclair State University Digital Commons. For more information, please contact digitalcommons@montclair.edu.
HOW DO MANIPULATIVES HELP STUDENTS COMMUNICATE THEIR UNDERSTANDING OF DOUBLE-DIGIT SUBTRACTION?

A DISSERTATION

Submitted to the Faculty of Montclair State University in partial fulfillment of the requirements for the degree of Doctor of Education

by

RABAB ABI-HANNA
Montclair State University
Upper Montclair, NJ
2016

Dissertation Chair: Dr. Eileen Fernandez
MONTCLAIR STATE UNIVERSITY

THE GRADUATE SCHOOL

DISSERTATION APPROVAL

We hereby approve the Dissertation

HOW DO MANIPULATIVES HELP STUDENTS COMMUNICATE THEIR
UNDERSTANDING OF DOUBLE-DIGIT SUBTRACTION?

of

Rabab Abi-Hanna

Candidate for the Degree:

Doctor of Education

Dissertation Committee:

Department of Mathematical Sciences

Certified by:

Dr. Joan C. Ficke
Dean of The Graduate School

Date

5/19/16

Dr. Eileen Fernandez
Dissertation Chair

Dr. Steven Greenstein

Dr. Eileen Murray
ABSTRACT

HOW DO MANIPULATIVES HELP STUDENTS COMMUNICATE THEIR UNDERSTANDING OF DOUBLE-DIGIT SUBTRACTION

by Rabab Abi-Hanna

Multi-digit subtraction is difficult for students to learn. The purpose of this study is to explore how second-grade students communicate their understanding of double-digit subtraction through the use of manipulatives/tools. This qualitative study reports on six case studies of second-grade students where clinical interviews were the main source of data. Findings suggest that manipulatives/tools helped reveal cognitive constructs and difficulties that the handwritten algorithms were not conveying. For example, students who exhibited an understanding of the subtraction process had not yet developed an understanding of ten and 10 ones interchangeability. These results highlight the potential role of manipulatives/tools as communication tools that help reveal students’ actual cognitive development. Implications to research and practice in relation to students’ learning trajectories are discussed.

Keywords: manipulatives, cognitive development, mental constructs
ACKNOWLEDGMENTS

This research came to fruition only with the assistance of many individuals. First and foremost, I would like to offer my heartfelt gratitude to my advisor and committee chair, Dr. Eileen Fernandez, for her continuous support, patience, motivation, and immense knowledge. Her guidance facilitated this process; without her continuous encouragement and support, I would not have completed this project. This has been a long journey for us, and I am so happy that you shared the experience with me. The words “thank you” are not adequate to convey my sincere appreciation for all the work you did.

I am grateful to my committee members: Dr. Steven Greenstein for his thoughtful advice and constructive feedback. Thank you for your guidance and pushing me to speak about my findings with authority. I also thank Dr. Eileen Murray for her insightful comments and continued support. Thank you for sharing your expertise and for encouraging me to complete this dissertation.

I would also like to thank Dr. Mika Munakata and Ms. Stacy Pinto for their support. I am very grateful to have had you both in my corner.

I would also like to thank Dr. Nicole Panorkou for her support, and for sharing her expertise with me.

I would like to thank the Mathematics Department at Montclair State University, which has been a major focus of my life for many years. I would especially like to thank Dr. Helen Roberts and Dr. Ken Wolff for their support and encouragement over the
years. I would like to thank the teachers and all the students who agreed to participate in my study without whom this research would not have been possible.

I would like to thank Jacky Dauplaise for being my critical, supportive friend and a great listener.

Above all, I would like to thank my family: Akram for being my best friend and believing in me. You are amazing. Ryan, Yasmine and Dahlia, thank you for taking on responsibilities at such a young age: I hope I make you as proud as I am proud of you. You kept me motivated and smiling. You all are my inspiration. There are no words to express the extent of my love and gratitude to you all. Your love and constant support throughout the years made this day possible. This is definitely our joint accomplishment as a family.

Thank you, Vivian Biron, for your love and daily encouragement; I look forward to hearing your voice every night. To Roger Biron, you will always be in my heart.
DEDICATION

This dissertation is dedicated to my family: Akram, Ryan, Yasmine, Dahlia, and Vivian

and Roger Biron
# TABLE OF CONTENTS

ABSTRACT .......................................................................................................... iv

ACKNOWLEDGEMENT .......................................................................................v

DEDICATION ...................................................................................................... vii

LIST OF FIGURES ............................................................................................... xi

CHAPTER I. INTRODUCTION ............................................................................1
  What makes Subtraction Difficult ...............................................................2
  Tools Used to Help in Understanding Multi-Digit Subtraction ...............5
    Base-ten blocks .......................................................................................6
    Unifix cubes .........................................................................................6
    Hundreds chart ....................................................................................7

CHAPTER II. LITERATURE REVIEW ...............................................................9
  The Child’s Perspective in the Research on Subtraction .........................9
  The Classroom Perspective in the Research on Subtraction .................13
  Summarizing Remarks and Moving Forward .......................................19

CHAPTER III. THE FRAMEWORK..................................................................22
  What We Know About Student Understanding ....................................22
  Relational and Instrumental Understanding ........................................23
  Representations and Tools ....................................................................25
    Representation ....................................................................................25
    The tool ..........................................................................................26
Dual-Representation ..........................................................................................27

CHAPTER IV. METHODOLOGY ..............................................................................30

Design of Study ......................................................................................................30

Sample Selection ...................................................................................................32

Participants .............................................................................................................32

Data Collection ......................................................................................................33

Interview protocol .................................................................................................33

Data Analysis ..........................................................................................................42

Validity ....................................................................................................................44

CHAPTER V. RESULTS ............................................................................................48

Framework Revisited ............................................................................................48

Concept of ten .........................................................................................................48

Connections for understanding .............................................................................49

Tools .......................................................................................................................49

Case Studies ...........................................................................................................50

Deidra ......................................................................................................................50

Mimi ........................................................................................................................56

Billy ..........................................................................................................................63

Troy .........................................................................................................................68

Ivan ..........................................................................................................................76

Sally ..........................................................................................................................82

Cross Analysis ........................................................................................................88
LIST OF FIGURES

Figure 1. Base-ten blocks and Unifix .................................................................7

Figure 2. Hundreds chart .................................................................................7

Figure 3. Connecting base-ten blocks and English word ...............................14

Figure 4. Cardboard with place-value ............................................................15

Figure 5. Subtraction sequence of 25 – 8 with conventional regrouping .......39

Figure 6. Unconventional subtraction sequence of 25 – 8 with Disruptive frame .................................................................39

Figure 7. Subtraction sequence of 25 – 8 with common error .....................40

Figure 8. Subtraction sequence of 33 – 16 with common error .................40

Figure 9. Subtraction sequence of 33 – 16 with conventional regrouping ......41

Figure 10. Unconventional subtraction sequence of 33 – 16 .......................41

Figure 11. Example of case study organization ...........................................42

Figure 12. Mimi’s work .................................................................................56

Figure 13. Mimi’s work (2) ...........................................................................59

Figure 14. Billy’s work ..................................................................................66

Figure 15. Troy’s work ..................................................................................69

Figure 16. Troy’s work (2) ............................................................................72

Figure 17. Troy’s reproduction of 34 and 19 ...............................................73

Figure 18. Ivan’s work ..................................................................................78

Figure 19. Ivan’s work (2) ............................................................................79

Figure 20. Sally’s work ..................................................................................82
Figure 21. Sally’s work (2)..........................................................................................84

Figure 22. Sally’s work (3)..........................................................................................87
Chapter I

INTRODUCTION

Multi-digit subtraction is hard to learn (Baroody, 1984, 1990; Fuson, 1984; Kamii, 2000). For decades, mathematics education researchers have been looking for ways to understand how students learn subtraction. The difficulties inherent in subtraction have led researchers to explore ways to reduce students’ struggles with subtraction. One of the approaches was the use of tools, such as manipulatives, but before researchers could offer any assistance, they needed to identify why students were struggling.

In this study, I focus on exploring how students articulate their understanding of double-digit subtraction. I will explore some of the difficulties children experience in learning subtraction. In the literature review, I will discuss what researchers say about subtraction, how students learn subtraction, and the role of tools, such as manipulatives, in learning subtraction. As a result of these investigations, I will propose the following research questions:

1. What can be inferred about how tools (like manipulatives) enable students to communicate their thinking about double-digit subtraction?

2. Given an interview setting with multi-digit subtraction problems and a selection of tools (like manipulatives), will students opt to use the tools to help them solve the problems? What are students’ perspectives on the use of tools with double-digit subtraction? What do they prefer and why?
What Makes Subtraction Difficult?

In this section, I will discuss what the literature reveals about the difficulty of subtraction. It is important for the reader to gain insight into the experience of a young learner in order to understand these difficulties with subtraction as well as the need to explore additional ways to support students’ learning.

In general, subtraction is more complex than addition (Baroody, 1984, 1990; Fuson, 1984; Kamii, 2000). One reason subtraction is considered to be difficult for students is that it requires students to mentally perform two separate cognitive tasks in opposite directions (Baroody, 1984; Kamii, 2000). “Counting down involves stating the minuend, counting backward a number of times equal to the subtrahend, and announcing as the answer the last number counted” (Baroody, 1984, p. 205). For example, to calculate 27 – 19, the child would have to count down from 27 beginning at 26 until 19 is reached. Keeping track of how many numbers are counted down, the child announces eight.

Another line of research that investigates students’ difficulties considers how different contexts can generate challenges: “subtraction is not just take away but has multiple situational interpretations” (Fuson, 1984, p. 214). For example,

- **Comparison**: Susan has 8 cookies. Her friend Dan has 3 cookies. How many more cookies does Susan have than Dan?
- **Separate or take away**: Mary has 8 cookies. She gives 3 cookies to her friend Scott. How many cookies does she have left?
• **Join missing addend**: Dan has 3 cookies. How many more cookies does he have to get so that he will have 8 cookies?

• **Combine missing addend**: Greg has 3 raisin cookies and some oatmeal cookies. He has 8 cookies. How many oatmeal cookies does he have? (Fuson, 1984, p. 221)

For example, *separate* has one whole, and a part is taken away from it.  

*Comparison* has two wholes, and nothing is being taken away. Thus, comparison is harder than separate because the child has to mentally take the smaller whole and put it against the larger whole to find the relationship. In *combine missing addend*, there are two wholes, and, again, no part is taken away, whereas *join missing addend* begins with a whole and requires increasing to a larger whole, making it easier than comparison or combine missing addend (Kamii, 2000). As Kamii (2000) explains “[c]hildren must first understand the logic of the question before going on to numerical precision” (Kamii, 2000, p. 95). These different contexts and the wording of the questions make subtraction more challenging for students because they have to decipher the language and then figure out the math (Fuson, 1984).

Executing the subtraction comes with its own difficulties. Subtraction involving double-digit and multi-digit numbers presents a new challenge because the focus now becomes place-value (Fuson, 1990). For example, for a problem such as 35 – 14, students are expected to know that the 3 is in the tens place, and it is worth 3 tens, or 30 and the 5 is in the ones place and it is worth 5 ones, or 5. Thus, students now are expected to think 5 ones, take away 4 ones and 30 take away 10. While students may be
able to subtract correctly, their understanding of place-value may be fragile. This becomes apparent in the language they use. For instance, students are taught to align the numbers in a vertical format, with the associated language \(5 - 4\) and \(3 - 1\). When questioned about 3 and 1, students may or may not realize that it is 3 tens and 1 ten, and they may or may not be able to associate it with 30 and 10, respectively.

Another challenge arises when the ones digit in the subtrahend is larger than the ones digit in the minuend. A common strategy taught here is *regrouping* (this is seen in mathematics for elementary school teachers’ textbooks such as Billstein, Libeskind & Lott, 2013, and in elementary textbooks such as Scott Foresman-Addison Wesley mathematics, 2008). For instance, asking students to subtract \(85 - 16\) is a typical question that is intended to reinforce the idea of regrouping. In order to solve this problem with regrouping, base-ten blocks can be used to model these two numbers. With the units blocks, students are shown that 6 ones cannot be taken away from 5 ones. That is, students are taught that they cannot take a bigger number from a smaller number, so they have to take 1 ten from the number in the tens place, break it into 10 ones, and add it to the number in the ones place. In this case, 10 ones and 5 ones are 15 ones. Now, there are enough ones to take away 6 ones.

The concept of regrouping can introduce a host of new problems. Sometimes (using \(85 - 16\)), students subtract 5 from 6 because all they remember is you cannot take a bigger number from a smaller number, and they proceed to state \(6 - 5\) is 1 for the ones place. Then, they continue with \(8 - 1\) is 7, and they end up with 71 instead of 69.
Having zeroes (0s) in one of the place-values presents its own set of impediments. Consider that when a zero occurs in the middle of the numeral, as in 103 – 17. If we regroup, we have to regroup the 1 hundred into 10 tens and regroup one of the tens into 10 ones to get 13 – 7 and 90 – 10. We are left with 86. Some students may make the 3 a 13, but they may skip the zero because they perceive it as nothing and leave the 1 in the hundreds place. They proceed with the subtraction as 13 – 7 and 10 – 1, resulting in 96. Or they may subtract 7 – 3 and 1 – 0, ending up with 114. Each of these different situations can compromise students’ understanding of subtraction.

**Tools Used to Help in Understanding Multi-Digit Subtraction**

Researchers use different approaches to address children’s issues with subtraction (Baroody, 1990; Carpenter, Franke, Jacobs, Fennema & Empson, 1998; Cobb & Wheatly, 1988; Fuson, 1990; Fuson & Briars, 1990; Selter, Prediger, Nuhrenborger, & Hußmann, 2012; Torbeyns, De Smedt, Stassens, Ghesquiere, & Verschaffel, 2009). In this study, I focus on how students use *tools* to communicate their understandings and misunderstandings during the subtraction process. Included among these tools are *concrete manipulatives*.

Concrete or physical manipulatives, as defined by Moyer (2001), are “objects designed to represent explicitly and concretely mathematical ideas that are abstract” (p. 176). Swan and Marshall (2010) extended the definition, specifying “mathematics manipulative material is an object that can be handled by an individual in a sensory manner during which conscious and unconscious mathematical thinking will be fostered”
In multi-digit subtraction, some of the tools used can include base-ten blocks, unifix cubes, and a hundreds chart.

**Base-ten blocks**

Base-ten blocks are intended to support students’ understanding of our base-ten number system. Base-ten blocks (also known as Dienes blocks, see Figure 1) are used to help explain place-value. The little cubes represent the units or ones; the rods represent the tens; the flats represent the hundreds, and the big cubes represent the thousands. Many teachers are taught to use base-ten blocks in the hope that they would make place-value easier for students to understand, as seen in mathematics textbooks for elementary teachers such as Billstein, Libeskind and Lott, 2013 and Long, DeTemple and Millman, 2015.

Consider the number 85. Base-ten blocks allow for multiple representations of 85. For example, students can count 85 ones, or they can use the 10-rods and the ones to represent 85 as 8 tens plus 5 ones. These multiple representations emphasize place-value and allow for re-grouping, which arises during multi-digit subtraction.

**Unifix cubes**

Unifix cubes can also be used to represent ones and tens because they can snap together (see Figure 1). When the 10 unifix cubes are snapped together to make a tower of ten, it becomes easier to show students how one tower of ten can be decomposed into 10 ones. For example, the number 15 comprises one ten and five ones. Using the unifix cubes, the tower of ten can easily be broken into ten ones, yielding 15 ones. Again, these unifix cubes can be used to support regrouping during subtraction.
Figure 1. On the left: Base-ten blocks (http://www.smartfirstgraders.com/base-ten-blocks.html) and on the right: Unifix cubes (http://www.smartfirstgraders.com/unifix-cubes.html)

Hundreds chart

A hundreds chart is a tool to help students subtract. A hundreds chart (as seen in Figure 2) is a 10 x 10 grid where the first row starts at 1 and ends at 10; the second row starts at 11 and ends at 20, and so on. The idea is that every row increases/decreases by ten depending on the direction. Thus, for any number from the grid, the number just above it decreases by ten, and the number just below it increases by ten. Thus, students can subtract on the grid by moving left or up.

Figure 2. Hundreds Chart (http://www.math-aids.com/Hundreds_Chart)
The tools above have played an important role in students’ understanding of mathematical concepts (Cain-Caston, 1996; Flores, 2009, 2010; Fuson, 1990; Fuson, Wearne, Hiebert et al., 1997; Moyer, 2001; Ojose, 2008; Sowell, 1989; Swan & Marshall, 2010). Some proponents of their use highlight the fact that students need concrete experiences for learning to occur. Some argue that manipulatives engage students concretely in the activities or tasks at hand (Cain-Caston, 1996; Hansen-Powell, 2007). However, experts emphasize the need for making a specific connection between the manipulative and the mathematics learned and note that if that connection is not clearly understood, the manipulatives might create misconceptions (Fuson, 1990; Fuson & Briars, 1990; Fuson, Wearne, Hiebert et al., 1997; Moyer, 2001; Swan & Marshall, 2010). That is, the mere presence of manipulatives does not necessarily improve understanding. “[S]tudents must reflect on their actions with the manipulatives to build meaning” (Moyer, 2001, p. 177).
CHAPTER II

Literature Review

In an effort to understand why students experience difficulty with multi-digit subtraction, researchers have approached the problem from a variety of perspectives. In this section, I organize these perspectives under two headings: the child’s perspective in the research on subtraction, and the classroom perspective in the research on subtraction. The research discussed under the former heading comprises studies that focused on how students understand and provided insight into why students have difficulties specifically with subtraction. The latter heading encompassed research that tackled one of the issues that is characteristic of the classroom: performance. This research offered methods of instruction to improve student performance.

The Child’s Perspective in the Research on Subtraction

In order to disentangle students’ difficulties with subtraction, one line of research focused on students’ understanding (Baroody, 1990; Carpenter, Franke, Jacobs, Fennema & Empson, 1998; Cobb, 1988; Cobb & Wheatly, 1988; Fuson, 1990; Hiebert & Carpenter, 1992; Kamii, 2000). For example, in 1988, Cobb and Wheatly criticized instructional materials for the misconceptions they propagate in learning subtraction. They explained that children construct three concepts of ten: ten as a numerical composite, ten as an abstract composite unit, and ten as an “iterable” unit. In the first case, children have “yet to construct ten as a unit of any kind” (Cobb & Wheatly, 1988, p. 5). In other words, they do not recognize that ten is structurally different from any other number. In the second case, children can construct an abstract composite of ten. That is,
they can see ten as a single entity while “simultaneously maintaining [its] tennesss” (p. 5), but counting by ten does not mean ten more or ten less. Rather, it is the last construct of ten as an “iterable” unit that signifies the understanding of ten as a unit that is itself made up of ten ones.

Cobb and Wheatly (1988) posited that although textbooks have pictures of bundles of ten, children are not given the opportunity to construct ten as an “iterable” unit; instead, they treat the ten as a new abstract object, an “abstract singleton” (p. 23). Students do not recognize the fact that one ten is made up of ten ones. The authors argued that the lack of opportunities to build meaning leads to superficial knowledge and rote procedures. They claimed that the textbook approach to teaching place-value emphasizes the value that digits have based on their position (ones, tens), rather than the relation of the positions to each other. This emphasis on the position of digits precedes the introduction of the standard two-digit subtraction procedures. The combination of disassociating one ten and ten ones and the emphasis on the position of the number (ones, tens) results in a misrepresentation of place-value. The authors described this approach as a flawed representation that leads “… teachers to systematically misteach place-value” (Cobb & Wheatly, 1988, p. 3).

Essentially, Cobb and Wheatly’s (1988) article emphasized the need for young learners to construct a mental representation of ten: a ten comprised of ten ones. Until students reach this developmental stage, place-value does not necessarily mean much to them. Rather than blaming students for not understanding, Cobb and Wheatly (1988) referred to the “inadequacy of adult mathematics for understanding children and for
planning instruction” (p. 2). Accordingly, adults impose their own understanding of ten on students and do not take the time to explore students’ understanding of ten.

In his article addressing the goals of mathematics instruction, Cobb (1988) stated that “[a] fundamental goal of mathematics instruction is or should be to help students build structures that are more complex, powerful, and abstract than those that they possess when instruction commences” (p. 89). In other words, students should be given opportunities to make meaning from what they are learning and continually build on the conceptual structures they already have. Cobb (1988) described an episode between a student and her teacher who was teaching his pupils the count backwards method to solve subtraction problems. As the teacher assessed his student, he inferred that she was merely trying to do what she had been told, but that she was not always successful. Faced with a problem the student could not solve, she declared, “Okay, I know it - I just can’t get it in my mind” (p. 94). The teacher then engaged his student in activities that allowed her to construct “a backward counting method that expressed her concepts. First, she did not a use a new method until she was in a situation where her … methods did not work” (p. 95). This interaction between teacher and student highlights the importance of understanding how students are thinking as well as the mental images they are constructing. The process gives the student a voice and allows the student to build on his/her genuine understanding.

Baroody (1987) would agree that mathematics instruction is essentially a process that aims to provide students with opportunities that enable them “to discover relationships and construct meanings” (p. 40). For this reason, Baroody (1987) explained
that children who are actively involved in their learning do not rely on the memorization of steps, but rather look for and use relationships to check their actions. In contrast, children who blindly follow a step-by-step procedure fail “to connect what they do know to what they are doing” (Baroody, 1987, p. 45). For example, when a student is faced with a multi-digit subtraction “such as 206 – 77” (p. 231), he or she may proceed with subtracting 6 from 7 and 0 from 7 and ending up with 271 without considering the fact that the answer he or she came up with (the difference) is greater than the number he or she started with (the minuend).

Similarly, based on the perspective that children can construct ways of adding and subtracting without explicit instruction, Carpenter, Franke, Jacobs, Fennema and Empson (1998) found that students who were given the opportunity to invent strategies to add and subtract multi-digit numbers developed an understanding of base-ten number concepts before the students who only knew the algorithm. Students were able to use their invented strategies “flexibly to transfer their use to new situations” (p. 16).

Accordingly, Kamii (2000) explained that children construct mental relationships from within, and as they put these previously made relationships into new relationships, (that is, new constructs) children construct *logico-mathematical knowledge*. In a study she conducted with Chandler (Chandler & Kamii, 2009), the role that manipulatives played in children’s understanding was addressed. They investigated children’s construction of tens using dimes and pennies. They played a “store” game with students, ranging from kindergarteners through fourth graders, in which the student was the cashier and the interviewer was the customer. Seven different purchases were made; the costs
ranged from two to nine cents, and the payments were made with pennies, a dime, or a combination of pennies and a dime, depending on the task targeted by the interviewer. Their study explored how “children give change when a dime or a dime and a few pennies are offered as payment for a purchase of up to 9 cents” (p. 110). They explained that many students who confidently stated that a dime is worth 10 pennies did not exchange the dime into 10 pennies to give the correct change during the interaction between the student and the interviewer. The failure to make a connection between the dime and the 10 pennies is an indication that, for these students, a dime was something different from 10 pennies.

Chandler and Kamii (2009) explained that children need to be given numerous opportunities to construct mental relationships through their own thinking and actions to build on their logico-mathematical knowledge. This view concurs with Cobb and Wheatly’s (1988) position on the three different constructs of ten that students make and reinforces the learner’s statement about not getting it “in my mind” (Cobb, 1988). Chandler and Kamii (2009) suggested that some students might not have made the mental relationship when they think of tens and ones simultaneously.

The Classroom Perspective in the Research on Subtraction

“Teaching the step-by-step borrowing algorithm for multidigit [sic] subtraction is easier than building up the network of relationships that constitutes knowledge of place value” (Baroody, 1987, p. 43). Baroody is not endorsing the teaching of procedures while ignoring the importance of conceptual structures students develop in mathematical understanding. However, he captures a reality that exists in many public school
classroom settings, in which the emphasis is on demonstrating how well students perform on tests. In this section, I explore a line of research that represents a different perspective on learning. These research studies provide instructional approaches that may enable students to successfully execute multi-digit subtraction problems in the classroom.

For instance, Fuson (1990) attributed elementary students’ low level of competence with place-value and multi-digit subtraction to instructional methods implemented in the United States and to the irregularities of the English language. She argued that the connection between the English word and the written “mark/symbol” of a number is very difficult for young children to grasp. In an effort to alleviate some of the difficulties that subtraction causes, Fuson and Briars (1990) offered a teaching/learning approach that emphasized the connection between the base-ten blocks and the English word. The goal of the initial part of instruction was to explore the relationships between the different blocks and their connection to the English word, including consistent discussion and demonstration of trading one-for-ten and ten-for-one (see Figure 3).

Figure 3. Connecting base-ten blocks and English word in Fuson & Briars, 1990, p. 182
A large piece of cardboard with place-value columns was used to illustrate the problem. Using the example, 3725 – 1647, blocks representing the first number were placed in the first row, blocks representing the second number were placed in the second row, and digits were placed in columns in the corresponding place-values (see Figure 4).

![Place-value chart](image.png)

Figure 4. Cardboard with place-value chart in Fuson & Briars, 1990, p. 186

When students were asked to subtract, they would check the first column on the right to see if the number on the bottom was less than the one on the top; if it was not, they would trade a ten for ten ones so that the number on top was larger than the one on the bottom; then, they were able to proceed. This approach modeled the standard algorithm and was meant to reduce the students’ difficulties in performing the procedure.

Pre- and post-tests measured students’ success; that is, the number of correct answers on the post-test was compared to the pre-test. This type of student assessment is different than that discussed in the section entitled, “The Child’s Perspective in the
Research on Subtraction,” because the score a student gets exemplifies the student’s learning. The emphasis on connecting the blocks with the English word for the number is important because in a typical public school classroom, students are judged based on how well they execute an algorithm. However, we cannot conclude that students have gained a better understanding of place-value. On the other hand, students’ success can be attributed to instrumental understanding (Skemp, 1976); that is, students know the procedure and know how to use it, but they may not understand it. After all, “children can rotely learn procedures with manipulatives as easily as they can with written symbols” (Baroody, 1990, p. 285).

In fact, using the same teaching/learning approach, Fuson and Briars (1990) conducted a second study. However, the authors found that, the “approach did not result in maximal learning in all areas by all children” (p. 202). According to the authors, the approach to the use of the blocks in the first study was more successful in supporting students’ thinking when they were able to self-correct on the post-test or during the interviews. Students in the second study continued to make errors and did not self-correct. This is an interesting observation. Students who are actively involved in their own learning, according to Baroody (1987), are able to rely on the mental relationships they have made. It would be reasonable to infer that these students may be able to self-correct because they rely on their understanding as they engage in solving problems. In contrast, students who memorize and follow procedure steps do not construct such mental relationships (Baroody, 1987), and, therefore, may not be able to self-correct. The results of this study draw attention to the focus of the teaching/learning approach, e.g. what was
measured. The purpose was to learn the standard algorithm of multi-digit subtraction while using manipulatives. The number of correct answers indicated students’ success. Once again, learning was measured by a score on a test.

One reason for the discrepancy in the success of the approaches in the two studies may be attributed to the implementation of the study, the support provided to classroom teacher participants, and the opportunity given to students to develop their understanding of the place-value and the multi-digit subtraction procedure. These conflicting results raise questions about how to characterize the connections students were expected to make. The deciding factor in evaluating students’ success or failure to understand the connections was the post-test score. Despite the fact that successfully executing the multi-digit procedure is an important aspect of the typical public school classroom experience, such success may not necessarily equate to true understanding.

Flores (2009, 2010) demonstrated that students who were experiencing difficulty with subtraction and subtraction with regrouping were successful in meeting the district-wide mandated benchmarks after participating in the Concrete-Representational-Abstract (CRA) instructional sequence. This sequence of instruction comprised three phases. First, the concrete phase required teachers to use concrete/physical manipulatives to support conceptual understanding. Specifically, teachers used base-ten blocks to demonstrate subtraction and subtraction with regrouping, followed by guided and independent practice. Second, the representational phase followed the same steps as the concrete phase, but the manipulatives were not used; they were replaced by drawings and/or pictures. Following this phase, students were provided with a mnemonic
strategy to help them remember the steps involved in a mathematical computation process. Finally, during the abstract phase, when completing a mathematical task, students transitioned from using pictures or drawings to using only numbers. The instruction in this phase focused on computational fluency; it involved “… memorization and continues until the students learn the operation or procedure automatically” (Flores, 2009, p. 145).

The effects of the CRA instructional sequence “… resulted in academic gains” (2009, p. 150). These gains were measured using a pre- and post-test assessment. That is, given subtraction problems that required regrouping, students were able to successfully follow the steps they memorized. However, it was difficult to assess whether students established the relationships (Baroody, 1987) necessary to form meaningful learning of computational fluency merely by successfully performing a procedure.

The study also reported that students were able to retain that knowledge even six weeks after the instruction (Flores, 2010). On one hand, it would seem that the use of base-ten blocks in the first phase of the sequence supported students in successfully computing the subtraction problems. After all, part of the CRA sequence focuses on memorization, which would more likely be related to instrumental understanding (Skemp, 1976). That is, students are proficient in following the steps, but they may not necessarily understand why. On the other hand, it would be difficult to assess whether students will really understand place-value and multi-digit subtraction in the long term.
Students were tested six weeks after the CRA approach, but there was no evidence that they had built a genuine understanding of subtraction after six weeks.

The studies cited in the “The Child’s Perspective in the Research on Subtraction” section focused on students’ understanding. The students’ voices emerged from the message the authors were asserting: students require numerous opportunities to construct their network of mental relationships. In contrast, the studies in this section focused on helping students become adept at computing the step-by-step procedure for multi-digit subtraction. In this case, learning the procedure was a priority, and students’ voices disappeared. Therefore, the focus was on students successfully performing the algorithm.

**Summarizing Remarks and Moving Forward**

Much has been written about the importance of using manipulatives to clarify mathematical concepts and engage students in mathematical thinking to develop their conceptual knowledge (Clements & Sarama, 2005; Cobb & Wheatly, 1988; Flores, 2009, 2010; Fuson & Briars, 1990; Fuson, 1990; Fuson et al., 1997; Kamii, 2000; Chandler & Kamii, 2009; Moyer, 2001; Moyer, Niezgoda & Stanley, 2005; Reimer & Moyer, 2005; Sowell, 1989; Steen, Brooks & Lyon, 2006; Suh & Moyer, 2005; Swan & Marshall, 2010; Uttal et al., 2013). The research consensus is that manipulative use requires teachers to assign careful and thoughtful tasks that are designed to compel students to engage in mathematical ideas and thinking. The mere presence of manipulatives in the classroom does not imply that students are making the intended cognitive connections, which places the responsibility and burden on teachers to select the appropriate tool for each particular student.
Nevertheless, how was success measured? Many of the studies measured success in terms of pre- and post-tests and the number of correct answers. For example, in some studies (Flores, 2009, 2010; Fuson & Briars, 1990), the focus was on the algorithm, whether learning a procedure with or without manipulatives. In contrast, other studies (Baroody, 1987; Cobb, 1988; Kamii, 2000) were more concerned with how students understand the procedures than they were with using a specific method of instruction. Their findings suggested they focused more on the role of manipulatives and assessing where the students were developmentally in learning subtraction. They emphasized issues such as adults imposing their own understanding of multi-digit subtraction and regrouping on students.

Currently, we are experiencing anew the aforementioned issues. The disregard for these issues manifests itself in the expectations set by the Common Core State Standards (CCSS), which reduces evaluating students’ learning to a list of items to be accomplished (Kamii, 2015). For example, the following is a standard objective for second grade students in the domain of “Numbers and Operations in Base Ten:”

Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds (CCSS, http://www.corestandards.org/Math/Content/2/NBT/).
To meet this specific standard, second grade students are expected to add and subtract three-digit numbers using manipulatives. Kamii (2015) maintains that this is unreasonable because meeting the goal depends on whether students have developed a mental construct of tens and hundreds, whereas only a minority of students at this grade level would have built these constructs. It would seem the research that describes how students learn and build their understanding is being ignored, and according to Kamii (2015), “the authors of the CCSS are not aware of the difference between logico-mathematical knowledge and social-conventional knowledge, they urge the direct teaching of logico-mathematical knowledge” (p. 19), which is not teachable.

Consequently, it is now relevant to resuscitate the students' voices because their voices are not being reflected in the policies of the CCSSM. It is time once again to listen to students; therefore, I pose these questions:

1. What can be inferred about how tools (like manipulatives) enable students to communicate their thinking about multi-digit subtraction?

2. Given an interview setting with multi-digit subtraction problems and a selection of manipulatives, will students opt to use the manipulatives to help them solve the problems? What are students’ perspectives on the use of manipulatives with double-digit subtraction? What do they prefer and why?
Chapter III

The Framework

The constructs used to frame the underlying structure of this study originate from different theories. In this chapter, I will discuss each of these theories, and then I will summarize how I will use them.

1. Cognitive theory
2. Relational and instrumental understanding
3. Representations and tools
4. Dual-representation theory

What We Know About Student Understanding

Understanding requires building relationships between mental structures (Hiebert & Carpenter, 1992). Understanding cannot be passively absorbed; it is “actively constructed from within” (Baroody, 1987, p. 10) by connecting new knowledge to existing knowledge. Making these connections can help to strengthen a learner’s understanding because the way he/she thinks about a mathematical idea changes (Baroody, 1987). A good example of this process is the concept of ten as presented by Cobb and Wheatly (1988) earlier in this thesis. When students have an understanding of ten as a singleton, that is, they do not conceptualize it as ten ones; their understanding is limited or weak because they have not yet constructed the relationships between the mental representations of ten and ten ones (Hiebert & Carpenter, 1992). Once they have made that connection and created the relationship between ten and ten ones as the same, or simultaneously, then their understanding of the concept of ten changes. It becomes
more powerful. This new perspective leads to changes in their thinking patterns that are essential to the development of understanding (Baroody, 1987).

Using cognitive theory is crucial for the analysis of my data because it enables me to understand how and what students understand. The purpose of this study is to explore how students articulate their understanding of multi-digit subtraction using manipulatives. In order to recognize the connections of students’ mental representations of mathematical ideas, I must understand what it means for a learner to understand.

**Relational and Instrumental Understanding**

In the typical elementary school, the dominant teacher model of student understanding is a procedural model. It follows that students’ understanding is primarily characterized as procedural or conceptual. In reality, students are assessed on how successful they are at executing a procedure. The relational and instrumental understanding (Skemp, 1976), coupled with a reconceptualization of procedural knowledge (Star, 2005), helped me to recognize what and how students understand multi-digit subtraction. First, I will define the relational and instrumental understanding. Then, I will describe how Star (2005) re-conceptualized procedural understanding.

Relational understanding refers to knowing what to do and why we do it (Skemp, 1976), whereas instrumental understanding is having the rule, using the rule but not knowing why and how the rule works (Skemp, 1976). These two types of understanding are so embedded in a mathematics classroom that using them as a guide to distinguish understanding is only natural. Like a piece of a puzzle, this distinction fits well with the reconceptualization of procedural understanding. Star (2005) disputes the dichotomized
view of procedural and conceptual understanding because it qualifies the types of understanding and puts them on opposite sides (Eisenhart, Borko, Underhill, Brown, Jones & Agard, 1993; Rittle-Johnson & Alibali, 1999; Rittle-Johnson, Siegler & Alibali, 2001). In fact, both play a critical role in students’ learning. Star (2005) introduces the term *deep procedural knowledge* as “knowledge of procedures that is associated with comprehension, flexibility, and critical judgment and that is distinct from (but possibly related to) knowledge of concepts” (Star, 2005, p. 408).

This definition provides a new perspective that challenges the more dominant view of procedural and conceptual understanding that govern classroom practices to this day. The prevailing thought in some of the research studies (Eisenhart et al., 1993; Rittle-Johnson & Alibali, 1999; Rittle-Johnson et al., 2001) and most classrooms is that conceptual knowledge is perceived to be knowledge of concepts, rich with connections of mathematical ideas, which characterizes it as deep understanding. “This knowledge is flexible and not tied to specific problem types and is therefore generalizable” (Rittle-Johnson et al., 2001, p. 347). In contrast, procedural knowledge is described as knowledge of steps in an algorithm with no connections to mathematical ideas, to why the steps are relevant or what they mean. Therefore, procedural knowledge is characterized as superficial understanding (Eisenhart et al., 1993; Rittle-Johnson & Alibali, 1999; Rittle-Johnson et al., 2001). “This type of knowledge is tied to specific problem types and therefore is not widely generalizable” (Rittle-Johnson et al., 2001, p. 346). These definitions favor conceptual knowledge over procedural knowledge.
With his alternative approach, Star (2005, 2007) introduces the possibility that both procedural and conceptual knowledge can be either deep or superficial. In comparison to relational and instrumental understanding, relational refers to deep understanding, whereas instrumental falls under superficial understanding. In any case, this approach to understanding is very important specifically because the way students are assessed in the classroom depends heavily on their success with procedural fluency.

**Representations and Tools**

The use of tools is an important component of this study. Therefore, it is imperative to explain what *representation* is and what role *tools* play in the development of children’s understanding of mathematical concepts.

**Representation**

Kamii (2000) best explained Piaget’s theory of abstraction and representation. Abstraction is a process by which mental representations are created from sensorimotor perceptions. Physical knowledge is knowledge of external aspects of objects, such as color, weight, shape, etc. Social knowledge is knowledge created by individuals, such as language, traditions, etc. Logico-mathematical knowledge refers to the mental relationships created by each individual. These “mental actions on previously constructed mental relationships are known in everyday language as *thinking*” (Chandler & Kamii, 2009, p. 98). Children represent ideas that come from within their mind; it is a mental construct. As children proceed to put these mental constructs into relationships, they construct logico-mathematical knowledge. Base-ten blocks do not themselves inspire an understanding of the base-ten number system if the child has not achieved a
“level of constructive abstraction” (p. 21) that would allow him/her to make such mental relationships. The base-ten blocks can, at best, aid in acquiring social (conventional) knowledge “to show how our system of writing [numbers] works” (Kamii, 2015, p. 12).

**The Tool**

Vygotsky’s (1978) work on the use of tools is also relevant to grounding this study. Tools exist in the child’s social experience. Just as children use physical tools like markers, crayons, computers, or any other object that can be physically manipulated to achieve a goal, they also use cultural tools such as signs, symbols, numbers and, most importantly, language to expand their mental abilities. Furthermore, speech plays a crucial role in the child’s development. Speech starts as a description of the situation, and as a child continues to self-speak, speech turns into analysis and becomes part of planning a solution. Speech is one link between the problem and the solution; this link is referred to as a *stimulus* or *sign*. The convergence of the physical and psychological use of the tool allows the development of higher order functions. According to Vygotsky, the child's behavior influences the tool/object. In other words, it affects the object/activity; thus, the behavior is externally oriented. The sign, on the other hand, does nothing to the object, no change is inflicted on it. However, it represents a "means of internal activity aimed at mastering oneself." (Vygotsky, 1978, p. 54). The combination of tool and sign leads to *higher behavior* or *higher psychological function*, which is learning according to Vygotsky (1978).

The purpose of this study is to examine the role manipulatives play in helping students communicate their understanding of double-digit subtraction. Vygotsky’s theory
and Kamii’s treatment of representation merge the tool (manipulatives), the role of language, and the thinking (the mental activities affecting student understanding).

Representation is what an individual does. As a student uses base-ten blocks, for example, he/she abstracts precepts from handling the blocks. He/she goes through a process of creating mental constructs, also called abstractions. Representations are built through this process. In terms of base-ten blocks, a young learner can look at a long rod and say it is a ten, which only reflects his/her social conventional knowledge. Only after he/she has constructed mental representations that allow him/her to think of ten ones and ten simultaneously would this young learner truly understand that a rod represents a ten.

**Dual-Representation**

The dual-representation theory is related to Kamii’s work on representation. This theory gives me another perspective on how young learners view and relate to the manipulatives they are using. For instance, developmental psychology researchers explain the success and failure of students’ experiences with manipulatives using the dual representation hypothesis (McNeil & Jarvin, 2007; Uttal, Scudder & DeLoache, 1997). The core principle is that a concrete symbol (or model) can be viewed in two different ways: as the object itself with its own properties, or as a tool to model something else.

The suggestion is that a concrete object is intrinsically interesting, and students may be so distracted by its physical properties that the object is no longer a useful tool in facilitating the construction of abstractions. In McNeil, Uttal, Jarvin & Stenberg’s (2009) study they found that using perceptually rich concrete objects to trigger students’ real-world knowledge interfered with students’ learning because it may have created a divide
between formal (school-taught) and informal knowledge. These findings support Kamii’s treatment of representation as a mental construct and maintain that manipulatives in themselves are objects with their own physical traits; manipulatives cannot represent ideas. A representation, as explained by Kamii, is a mental process that occurs when a learner constructs a concept in his/her head. It originates in the learner’s mind. This logico-mathematical construct is not teachable; it is dependent on the learner and where he/she is developmentally. Therefore, unless the student has constructed the knowledge in his/her mind, the physical object is only helping the student achieve social-conventional knowledge (Kamii, 2000).

Each of the theories discussed above will play a role in helping me to understand the student’s thinking process. As I analyze the data, I will be cognizant of how students come to understand double-digit subtraction. Cognitive theory allows me to understand the developmental phase of a student by listening to his/her explanations and by observing his/her behavior while working. Based on these observations, I will be able to infer what kind of mental relationships a student has constructed. Vygotsky’s (1978) research on tools gives me a better understanding of the role tools (physical or mental) can play in learning. As Kamii (2000) explains, these tools support constructive abstractions that lead to mental representations. The dual-representation hypothesis will help me to be aware of a possible separation between the tool as an object that is intrinsically interesting with its own properties, and the tool as an object that facilitates understanding a mathematical idea. Finally, as I observe a student and how he/she proceeds with the task at hand, I will depend on Skemp’s (1976) explanation of relational
and instrumental knowledge as well as Star’s reconceptualization of procedural knowledge to enable me to assess the type of knowledge a student exhibits.

Although I discuss the theories separately, I believe that they are intricately interrelated. For example, if a student exhibits instrumental understanding, then it is reasonable to assume that the student has a superficial understanding of a procedure because the student would know how to use the procedure, but most likely does not understand why the procedure works. Such an observation would lead me to conclude that the student has not yet built the necessary mental networks to promote constructive abstractions that would allow the emergence of mental representations, which demonstrate learning.
Chapter IV
Methodology
Design of study

The purpose of this study is to describe and investigate whether and how tools help students to communicate their understanding of double-digit subtraction. This focus makes the qualitative paradigm an appropriate methodological choice. Qualitative methodology is especially useful because it can provide a rich description and insight into the subjects’ thinking processes. In the present study, the focus is on how students communicated their understanding of double-digit subtraction and, specifically, how they communicated that understanding as they are using the tools. The setting was a naturalistic setting in the sense that the researcher was not manipulating or controlling the student’s activity (Merriam, 2009; Savenye & Robinson, 2005).

Moreover, the approach was phenomenological because the study explored a shared experience (Laverty, 2003): subtraction. For this study, the phenomenological approach attempted to capture a second grader’s experience in solving double-digit subtraction problems with or without the use of tools. Phenomenology was considered to be the most appropriate “to explore the phenomena of pedagogical significance” (Kafle, 2011 p. 183). It attempted to understand the student’s experience and to disclose the meanings of this shared experience. It represents hermeneutic phenomenology because it is informed by the individual participant’s experience (Laverty, 2003). According to Laverty (2003), in Heidegger’s phenomenology, also known as hermeneutic phenomenology, “nothing can be encountered without reference to a person’s
background understanding” (p. 24). As it pertains to this study, a student’s experience has been influenced by the student’s history, for example his/her cognitive development, his/her knowledge of the topic, the tools, and the classroom environment. All of the students shared the experience of solving multi-digit subtraction; however, each student’s approach was influenced by his/her personal background knowledge.

Clinical interviews were used to collect data, because the objective of the research was to encourage students to express their own thinking about their own concepts and methods (Ginsburg, 1997). The clinical interview is “a flexible method of questioning intended to explore the richness of children’s thought, to capture its fundamental activities, and to establish the child’s cognitive competence” (Ginsburg, 1981, p. 4). In this study, I wanted to give the students a voice and to encourage them to verbalize their thinking and understanding of double-digit subtraction. I assumed an interpretive stance in analyzing the data. As discussed in Meriam (2009), interpretive research “…assumes that reality is socially constructed, that is, there is no single, observable reality. Rather, there are multiple realities, or interpretations” (p. 8). The idea here is that researchers make inferences from their interpretation of observed behaviors. These observed behaviors are naturally subjective and vary from one individual to the other. I followed Ginsburg’s (1997) approach to an interpretive stance as it relates to the clinical interview. I was interested in what the students’ behavior revealed about their process of solving double-digit subtraction problems. Thus, I relied on the richness of the interview to “…assign meaning to words or actions on the basis of how they fit into the entire context of the session…” (Ginsburg, 1997, p. 79).
Sample Selection

Purposeful sampling (Merriam, 2009) allows the researcher to learn about a more focused, as opposed to a more general, population. For this study, several factors played a role in selecting the school. I had worked closely with the school’s administration as the math consultant, hired by the district to assist in selecting a new mathematics curriculum and in transitioning fifth-grade students to middle school. In the process, I developed a rapport with the principal who, like me, subscribes to a constructivist approach to learning. In my second year, I worked closely with the teaching faculty helping them implement a new curriculum. As a result, I developed a friendly and comfortable relationship with the teachers, making my presence in the classroom (as a researcher) less intimidating and less intrusive to them.

Due to these pre-existing relationships, selecting this specific school was an easy decision. The focus on the second grade was due to the topic under study--double-digit subtraction. During informal conversations with teachers, they revealed that many of their students struggle with multi-digit subtraction, and they were interested in how to help them. Three second-grade teachers enthusiastically volunteered to participate in this study. They used tools in their mathematics classroom regularly (each in her own way). The teachers were experienced, having between 10 to 38 years of teaching experience at the chosen school.

Participants

The participants in the study were second-grade students, ranging from seven to nine years in age and representing three different classrooms. The majority of them had
attended this school since kindergarten. They recognized me from my having been in their classrooms, in the hallways of the school, but also as one of their schoolmate’s mother. Students in this study were interviewed towards the end of the school year at the beginning of June. The students had all learned one-digit, two-digit and multi-digit subtraction. I selected a total of 17 students. Of these, two students were not interviewed. Of the remaining 15 interviews, six were excluded from this study because they participated in practice interviews that did not follow the same protocol used in this study. Of the nine remaining interviews, I chose two students from each class.

Data Collection

Interviews provide the study’s primary source of data. As the researcher, I conducted and audiotaped the clinical interviews of each participant. The goal was to interview students, who had previously submitted parental consent and assent forms, in their classroom, being careful not to disrupt the classroom’s daily routine and activities. For this reason, the selection of a specific student to interview was left to the teacher’s discretion. For the interviews, teachers suggested the most appropriate time for me to visit the classroom. Upon my arrival, the teacher selected a student to be interviewed, depending on whether the student completed his/her work or whether he/she had a special activity to attend, (for example, speech, resource room, gifted program).

Interview Protocol

Student interviews were conducted after multi-digit subtraction had been taught, towards the end of the school year. Interviews were structured with predetermined questions, but were also semi-structured (Merriam, 2009) in the sense that they allowed
follow-up questions to enable students to elaborate and explain their responses. Student interviews were individual and followed some of Ginsburg’s (1997) suggestions and guidelines. For example, sitting next to the student, or on the floor face-to-face allowed the student to feel that he/she played an equal role in this process, which fostered a mutual respect between the student and me. The student was made to feel that I was not judging his/her responses, but that I was interested in their thinking process.

An area/corner of the classroom (chosen by the teacher) where there was more privacy and less distraction from other students was set up before the student was called. In the interview work area, I officially introduced myself to the student. I explained that my interest was in learning how second-grade students thought about solving subtraction problems. I informed them that there were no wrong answers because they were explaining how they think; I also assured them that they could stop at any time if they did not want to continue with the interview. It was very important for me to have students be open and honest about what they were thinking (as much as is possible for them to verbalize). In order to create such an atmosphere, I believe they had to be as comfortable as possible. I explained to them that they were helping me with “my homework.” I felt that this perspective would alleviate pressure and anxiety (if present) and acknowledged the importance of the role they are playing in completing “my homework.” It also emphasized the fact that I was not judging them and that they were not being tested.

The interview was structured to last approximately 15 to 20 minutes (see interview protocol in Appendix A). The tools employed included (1) the Hundreds chart, (2) unifix (connecting) cubes, (3) base-ten blocks, (4) pencil and paper. These were laid out within
students’ reach and were regularly available in the classroom. To begin, I asked students if they recognized each tool and asked them to do this familiar addition problem using a tool of their choice: \(7 + 5\). This particular fact was one of the addition facts students were expected to know, thereby making it a suitable, that is, a familiar initial question. At the outset of the interview, one goal was to reduce any anxiety they might have had about answering questions related to mathematics. A familiar question gave me a baseline from which to assess their mathematics understanding and opened the door toward their responding as honestly and candidly as possible.

The interview included a total of four types of subtraction items (see Appendix A). It began with an open-ended scenario to elicit their spontaneous thoughts and developed into more closed scenarios, with questions being more tailored to respond to this study’s objectives. I began with “what is \(15 - 8\)?” Whatever the student’s answer was, I followed up with, “How do you know?” While the idea was to ask a question that they knew how to answer, thereby putting them at ease, the expression \(15 - 8\) was a number fact that they were expected and required to know in the second grade. Their descriptions of their solution gave insight, in their own words, into their thought processes. One strategy they had learned was “doubles minus one.” Furthermore, it fell under mental math strategies, in which the work was done “in their head.” For example, since \(16 - 8 = 8\), and 15 is one less than 16, then the difference is 7— one less than 8. On the other hand, they could have thought of the addition fact related to this subtraction, \(8 + \text{ what} = 15\). No matter what strategy they chose to justify their answer, I asked them to show me how this was done using any of the tools available. Their response to this request gave
me insight into their thinking as well as the role that the tools played in this thinking.

At this point in the interview, I expected them to be comfortable with our interaction. I followed up with a double-digit subtraction, 32 – 14, (noting what they did) and asked them to explain or to show me how they were getting the answer. I considered this problem to be more challenging because it was not a previously known fact. Because the ones in the minuend were fewer than the ones in the subtrahend and because this question was written horizontally, it allowed the student the freedom to choose how he/she wanted to solve the problem. The student’s approach to solving this subtraction problem conveyed the constructs of ten that they had. If they did not use any of the tools and reached for a pencil and paper first, I waited for the paper and pencil response, and then I asked them to show me how to solve this using one of the tools of their choice.

The questions discussed above were not typed ahead of time. My intention was to minimize any similarities to test taking and its associated anxieties. Moreover, I felt that if I were to hand them a typed sheet, it would be reminiscent of worksheets or pre-prepared assessments. Writing out the problems on the spot created the feeling of spontaneity as if it were a game, thereby, promoting an atmosphere of ease. I wrote the problems for students in a horizontal form on the paper in front of them. This gave students an opportunity to look at the numerals without any prompts or reminders of using the T-table procedure, in which they aligned the ones and tens in columns and followed through the steps mechanically. Then, I observed their responses. If their initial reaction was to set-up a T-table, then it was their choice to do so.

In the subsequent section of the interview, I provoked students’ thinking by
presenting them with scenarios that contained both correct and incorrect reasoning.

Some of the scenarios were numerical; others were pictorial representations.

I showed them the following and explained to them that some second graders from a different school had worked on these problems.

\[
\begin{align*}
56 - 23 &= 33 \\
40 - 23 &= 17 \\
34 - 19 &= 25
\end{align*}
\]

I asked them if they could tell me which answers were correct or incorrect and to explain their choice. I noted whether they used any tools and asked them if they thought tools helped them to understand better; or if they found one tool more helpful than the other. This gave the student an opportunity to voice their thoughts on using and comparing tools. Because this work was supposed to have been completed by students in their grade from another school, there was less pressure on them since it was less personal. In addition, there was no need to give an answer since the answer had already been supplied. They merely needed to agree or disagree and justify their response. As explained, the horizontal format allowed students the space to tackle the problem in their own natural way, whether they used tools or the procedure spoke to their learning experiences. The first subtraction problem was straightforward and was dependent on whether they knew their number facts. The second and the third subtraction problems gave more insight into the cognitive development the student had achieved. First, the student was prompted to reveal his/her understanding of zero. This understanding was brought about when the minuend had a zero in the ones place, and the subtrahend had a
three in the ones place. Then, having the ones digit in the minuend less than the ones
digit in the subtrahend challenged the student. Again, the solutions to these problems
were indicative of the construct of ten that the student had developed. Did the student
have a construct of ten as an *abstract singleton* (Cobb & Wheatley, 1988) or did the
student see the ten and ten ones simultaneously (Cobb & Wheatley, 1988)?

For the last item, I showed students subtraction models that were similar to what
students see in their textbooks. Each sequence was shown separately, and the student
was asked if the picture matched with the subtraction problem given. This exercise had
multiple goals. First, it gave students’ perspectives on pictures commonly seen in
textbooks. Did these pictorial representations help students get a better understanding of
double-digit subtraction? Second, it provided insight into students’ understanding of the
visual representations. Students had different outlooks on pictorial representations than
adults did. Did they see these representations in the way the adults intended? Third, it
provided insight into how students connected the visual representation to place-value and
the indicated operation. Fourth, and most important, it compelled the student to explain
his/her thinking about subtraction with the visual representation of the tool, specifically
base-ten blocks.

Two subtraction problems were used: 25 – 8 and 33 – 16. Both expressions can
require some form of regrouping, whether using the common procedure or not, and can
result in a variety of solutions.
25 – 8

*Figure 5.* Subtraction sequence of 25 – 8 with conventional regrouping

This sequence (see Figure 5) represents the steps taken using a procedure. Students would have to recognize the regrouping of 25 into one ten and 15 ones and then taking away 8. I consider this type of sequence to be the standard pictorial representation. It was commonly seen in most textbooks in the elementary grades. Although this representation may not be as indicative of a student’s concept of ten, it created an opportunity for the student to demonstrate his/her understanding of the subtraction procedure.

*Figure 6.* Unconventional subtraction sequence of 25 – 8 with Disruptive frame

In this sequence (see Figure 6), the subtraction does not follow the conventional procedure. The subtrahend 8 is taken from the whole 10. This is a mental math strategy that some students may use, who are comfortable with numbers; namely, students who have a construct of ten that is interchangeable with ten ones. The third frame of this
sequence is referred to as the Disruptive frame because it caused a disruption in students’ thinking.

![Figure 7. Subtraction sequence of 25 – 8 with common error](image)

This sequence (see Figure 7) represents one of the most common mistakes students make when subtracting, because it represents subtracting the larger digit in the subtrahend from the smaller digit in the minuend. Rather than regrouping, students sometimes subtract the smaller digit from the larger one.

33 – 16

![Figure 8. Subtraction sequence of 33 – 16 with common error](image)

This sequence (see Figure 8) represents an incorrect procedure in which students subtract three from the six, rather than regroup to make a 13, and continue to subtract one
ten from the 30. Similar to the previous problem, the following sequence reveals whether students recognize a visual representation of the error.

**Figure 9.** Subtraction sequence of 33 – 16 with conventional regrouping

In this sequence, the second frame illustrates the regrouping of 3 tens and 3 ones to 2 tens and 13 ones. In the third frame, 6 ones are crossed out. In the fourth frame, 7 ones are left and a ten is crossed out, leaving the fifth frame with a representation of 17. Again, this is a different representation that sequentially follows the steps of the conventional procedure. It engenders students’ understanding of the concept of ten as itself being made up of ten ones.

**Figure 10.** Unconventional subtraction sequence of 33 – 16

This series (see Figure 10) depicts the deduction taken from the tens, similar to partial sums; this represents partial differences, which is another way of solving the subtraction.
Data Analysis

The data comprised student interviews that were audiotaped and transcribed. As discussed earlier, I used an interpretive stance to analyze the data. The intent was to “make sense of (or interpret) the meanings [students] have” (Creswell, 2003, p. 9) about multi-digit subtraction. The interaction between participant and me required “self-reflexivity, […] actively constructing interpretations of the experience” (Laverty, 2003, p. 30). These interpretations are “varied and multiple, leading me to look for the complexity of views rather than narrowing the meanings into a few categories or ideas” (Creswell, 2003, p. 8).

After I transcribed the interviews, I chose six students, two from each class, and I created a case study for each student (see Appendix B). I organized each case study in a three-column table that identified the speaker, a description of the action and the verbal interaction that took place, and, finally, the last column referred to the line number in the interview transcription as seen below.

![Figure 11. Example of case study organization](image)
In each case study, I included the student’s work and explanation in the student’s own words. In the event that a student’s work was not available, I recreated what the student had done to provide a clearer description of the student’s approach to the task at hand. On several occasions, I found it necessary to revert to the audiotaped interview to ensure that I was reflecting an accurate account of the exchange between us.

I did a preliminary analysis after completing each task. My analysis was based on my interpretation of the student’s observed behavior, his/her work, his/her verbalization of his/her thought process, guided by the framework of this study. My framework has four components; each component influenced my interpretation of the data. These components are intertwined: In order to appreciate the student’s understanding of the concept of ten as described by Cobb and Wheatly (1988), I needed to consider what it means to understand and turned to Baroody (1987) and Kamii (2000), who provided a clear account of how students learn by constructing their own mental constructs. The network of relationships that students build led me to explore Skemp’s (1976) ideas of relational and instrumental understanding. However, the question of what role the manipulatives play in helping students acquire an understanding of double-digit subtraction directed me to explore Vygotsky’s (1978) use of tools as well as Kamii’s (2000) treatment of representations.

In an effort to extract special characteristics of students’ mental constructs related to subtraction, all of these aspects discussed above enabled me to analyze the child’s reflection and the dialogue in which the student engaged.
It is through the same lens, incorporating all four components of the framework, that I studied students’ responses to the pictorial representations. Their reaction to unfamiliar situations, specifically the Disruptive frame (Figure 6), was especially interesting.

I looked over the case studies and examined characteristics of students’ thinking. Initially, I thought of looking for commonalities among the case studies. However, after I examined and re-examined the details of each case study, it became apparent that each case study was unique in its own way. That is, each case study had captured a complexity of the student’s understanding and thought process that required its own recognition. From this observation, I identified two features that characterized students’ thinking processes: the mental constructs and the responses to unfamiliar situations. As a result, these two features are discussed in the cross-analysis section.

**Validity**

The idea of validity in qualitative research is different from that of quantitative research. According to Merriam (1995), “[n]otions of validity and reliability must be addressed from the perspective of the paradigm out of which the study has been conducted” (p. 52). Merriam (1995) also explains that

[q]ualitative research assumes that reality is constructed, multidimensional, and ever-changing; there is no such thing as a single immutable reality waiting to be observed and measured. Thus, there are interpretations of reality; in a sense the researcher offers his or her interpretation of someone else’s interpretation of reality. (p. 54)
In the case of this study, I am trying to understand the phenomenon of double-digit subtraction. Each participant has a unique understanding that has been informed by the student’s learning experience. That is, each student’s reality varies from that of his/her peers.

In this study, I infer students’ understanding from their responses, their approaches to answering the questions and the ways they used the tools, that is, how students used the tools to communicate their understanding of double-digit subtraction. Because I use a clinical interview method, I referred to Ginsburg (1997) to justify validity. Four types of validity are discussed:

1. Content validity measures whether the content of the questions asked are relevant to the subject under examination. That is “to establish content validity, the interviewer needs to determine whether the tasks presented to the child seem to be reasonable and representative” (p. 175) of subtraction problems. The tasks in this study are subtraction problems that are relevant to the second-grade classroom curriculum. The interview questions are designed to give the student the opportunity to convey their mental constructs using the tools displayed in front of them throughout the interview. The interview is deliberately designed to begin with open-ended questions that ultimately become closed to discover the student’s voice.

2. Construct validity relates to “whether the clinical interview provides clear and direct evidence concerning the operation of cognitive processes under investigation” (p. 178). Fundamental to the clinical interview are the observed
behavior, answers to prompts, solutions to problems, verbalized thoughts; all are “key to construct validity” (p.179). According to Ginsburg this is the most important type of validity. In this study, the primary instrument is the student interviews. The tasks are created to elicit student’s articulation of their thought processes. The set-up with tools provided materials that facilitated multiple approaches to the problems, and the structure of the interview proceeding from more open to more closed scenarios first created an option to use tools and, ultimately compelled their use. Some items are designed to provoke students into examining erroneous thinking--sometimes numerically and sometimes pictorially. Students’ treatment of these items and the language they used provided insight into their thought processes and their mental constructs.

3. Domain validity refers to the effectiveness of the clinical interview in a specific domain. Clinical interview “methods are more effective – more valid – in providing information concerning some cognitive processes than others” (p. 179). In this study, I am interested in assessing how students communicate their understanding of double-digit subtraction. Subtraction is the domain. As noted earlier, the set-up with tools providing materials that facilitate multiple approaches to the problems, the structure of the interview from more open to more closed scenarios contributed to providing an atmosphere to elicit student thinking and articulation of their thought process. The items allowed for different ways to approach subtraction. They permitted students to tackle the problems in their own unaffected way. The varied solutions could be in any form exhibited in
the mind of the student, whether it was deferring to the standard algorithm, using numbers, manipulatives or pictures, or revealing their constructs of ten, using numbers, pictures or manipulatives. Double-digit subtraction is an appropriate domain for testing students’ understanding of place-value, positional understanding and number grouping. The interviews allowed me to ask students to explain, or clarify what they were doing or thinking with respect to solving double-digit subtraction problems.

4. Criterion validity questions the “degree to which the [clinical interview] is correlated with some relevant behavior” (p. 176). In other words, does the clinical interview correlate with students’ communicating ideas about multi-digit subtraction with tools? The clinical interview aims to encourage students to verbalize their thinking. It represents the best tool for this study because it allows students the opportunity to express what they think of and how they think about the subtraction tasks presented to them. Other tools may not depict students’ thoughts as accurately. In the case of a written assessment, for example, second-grade students may not yet be able to express in writing what they are thinking.
Chapter V

Results

Framework Revisited

Before I begin this next section, I remind the reader of the theoretical framework that guided my analysis in this chapter (see Chapter II).

Concept of Ten

Recall the concept of ten as explained by Cobb and Wheatly (1988). The authors described the three different constructs of ten that students develop once they have reached the stage of counting by ones. The first is ten as a numerical composite; students at this stage have not yet constructed any kind of mental image of ten of any kind. As the authors explained, “Ten as a numerical composite is structurally no different from the meaning given to other number words by children when they first attain the abstract stage” (p. 4). The second is ten as an abstract composite unit. Students consider ten to be a single entity while preserving its tenness. However, counting by ten does not mean ten more or ten less, but rather “one ten, another ten,” (p. 6). Cobb and Wheatly (1988) pointed out that students at this stage were dependent on “re-presentations” to create an abstract composite of ten. That is, “it is essential that material of some kind (hidden or otherwise) be available that the children can take as abstract composite units of ten” (p. 6). The third concept of ten is ten as an iterable. At this stage, students construct composites of ten without the need for a “re-presentation” (Cobb & Wheatly, 1988, p. 7). That is, they can generate a composite unit of ten, and they understand that counting by ten is an increase or a decrease by ten: Ten is made up of ten ones.
Connections for Understanding

Connections refer to the mental relationships students construct during learning. As discussed by Baroody (1987) and Kamii (2000), students build mental representations that originate from within their mind. As students develop relationships between these representations their thinking structures change; these changes are essential for their understanding to develop. The network of mental constructs allows students to know what to do as well as why they do it. Skemp (1976) refers to this characteristic, knowing what to do and why we do it, as relational understanding. Without this network of relationships students end up following steps without understanding why. Students with instrumental understanding know the rule, follow the rule, but do not understand why or how the rule works (Skemp, 1976).

Tools

According to Vygotsky (1978) tools exist in the student’s experiences. Tools can be physical, such as objects that can be manipulated, or cultural such as language, which affect mental abilities. Self-speech is a psychological tool that allows the student to progress from describing a problem to analyzing it and reaching a solution. Self-speech reflects an internal/psychological activity. The combination of physical and psychological tools leads to higher levels of thinking behavior.

For Kamii (2000), physical tools, such as base-ten blocks, can aid students in gaining social (conventional) knowledge. Only after, students have constructed mental representations that originate from within do they develop logico-mathematical knowledge.
The dual hypothesis theory can be related to Kamii’s (2000) work on tools, which says that if a student considers the tools while merely focusing on their physical aspects, then the tools do not support the student’s learning. However, if the student treats the tools as models to represent an idea, then the tools may be used to support constructive abstractions that lead to mental representations.

Case Studies

The case studies for this research are presented for students who used manipulatives (with or without prompting) and students who struggled to use manipulatives.

Deidra

Deidra was the only student who reached for the base-ten blocks without any prompting from me, the interviewer. She seemed comfortable using base-ten blocks and seemed confident about her manipulation of the blocks. She stated at the beginning of our interview that she did not use manipulatives. Her facility with using the blocks steered me away from assuming that she was using them by rote because of the way she covered 2 cubes instead of breaking up the ten into ten ones, as is typically demonstrated in classrooms. The following excerpt sheds light on this approach:

Interviewer (Int): Can you show me 32 – 14? […] You’re using the base 10 blocks, and what are you doing with it?

Deidra: I'm putting 10s for the 30.

Int: How many 10s do you have?

Deidra: 3 for the 30 and then I'm going to take 2 ones blocks for the two to make 32.
So I’m taking away these 2 which leaves me with 30 —now I take away this [referring to a 10] which left me with 12 now I have 20.

**Int:** You have 20. You took away 2 cubes, the ones, and 1 ten-stick, so if you take away 1 ten and 2 ones how many have you taken away?

**Deidra:** 12

**Int:** 12, and you are supposed to take away 14?

**Deidra:** Which I took 2 more of [with her fingers, she covers the first 2 cubes of a ten-stick] and now this is 8 and this is 10 so it leaves me with 18.

**Int:** I see that you covered the first 2 blocks [referring to the cubes on the ten-stick] with your finger. What were you going to do with those 2 blocks?

**Deidra:** Take them, I mean subtract them.

Deidra’s manipulation of the blocks suggested that she had constructed a concept of ten cognitively that is made up of ten ones. She began the subtraction with the ones. As she kept track of what was left of the 32, she was adding to match the subtrahend while simultaneously deducting from the minuend. She was mentally performing two tasks in opposite directions (Baroody, 1984; Kamii, 2000). She demonstrated an understanding that this unit of 10 has ten ones (little blocks) within it. It could be argued that the presence of the base-ten blocks aided her in constructing ten as an abstract composite unit (Cobb & Wheatly, 1988), and she was able to progress and build an understanding of ten as an iterable. This was evident in her speech as well as her use of the manipulatives. She had developed a mental construct that allowed her to use the model to express her thinking.
The upcoming episodes shed more light on her understanding of a ten being the same as ten ones. I had asked her if she agreed with the equation $56 - 23 = 33$, and if she could justify her answer.

**Deidra:** I’ll check using these [taking the base-ten blocks].

**Int:** Why would we use the base-ten blocks?

**Deidra:** Because they already have the ones stuck together [she was referring to the ten-sticks] I'll take out five tens and now I take six ones, so now I’ll take away two tens cuz 23 has two tens, and now three ones, which leaves me with 33.

Her statement indicated that she had constructed a mental relationship that led her to recognize one ten as ten ones “stuck together.” She had a well-defined definition of what a rod was. It is important to note that she did not perform her subtraction in order. In contrast to her previous approach, she began by taking away from the tens and then the ones, but she was thinking of the composition of the numeral 23. Her ability to decrease 56 by reiterating ten and simultaneously increasing by ten to add up to 23 *without having a representation of 23 in front of her*, suggested that she had an understanding of ten as an iterable. She used the base-ten blocks as models to represent the numbers with which she was working.

When I asked Deidra about her thoughts on $34 - 19 = 25$, she asked to use virtual base-ten blocks. The connections she had constructed may have contributed to how easily she transitioned to using virtual base-ten blocks. She had not used them before, but she knew which represented ones, tens and hundreds. I showed her how the tools worked, that is, how to use the hammer to break apart tens and hundreds, how to use the eraser to
delete or take away ones or tens, and how to use the glue to place ten ones together into a ten.

She chose three tens and four ones to represent 34, but she did not know how to break the 10, so I talked her through the process.

**Int**: What are you going to do now?

**Deidra**: I can erase 19. [She erased the 4 ones] 1, 2, 3, 4. But I only saw a ten [she realized there were only ten-sticks left on the screen. She looked at me for an answer].

**Int**: What do you want to do with the tens?

**Deidra**: Hmmm [thinking].

**Int**: Use the hammer, and you will be able to break it [she did what I had instructed her to do]; and now you have 10 ones, go back to the arrow, click on it, and now put the arrow over one of the cubes, and you can move them around. What do you want to do with them?

**Deidra**: I’m going to erase them [she proceeded to delete them with the eraser tool].

**Int**: Do you know how many you’ve erased already?

**Deidra**: [Continuing the count from the 4 ones she erased earlier] 5, 6, 7, 8, 9, 10, 11, 12, 13, 14. I’m up to 14 but I need to break 5 more [she broke another 10 and erased 5 ones and counted what was left on the screen]…14

**Int**: [She miscounted] 14 are you sure?

**Deidra**: [She recounted] Oh, 15.

**Int**: Are they correct? They said it was 25.

**Deidra**: No they are incorrect. I knew there was something fishy about that.
On the screen, Deidra could not cover portions of the base-ten blocks as she had done previously. She could not cover the blocks with her fingers; yet, she did not seem to have difficulty transitioning from how she subtracted with the physical objects to subtracting on the screen. It is possible that her understanding of ten allowed her to transfer her knowledge of ten as well as her comfort with using the base-ten blocks to virtual base-ten blocks. It is important to note that, initially, the tool created a temporary obstacle because she did not know how to use it. However, once she had learned how the hammer is used to break apart the tens and the hundreds, and how to erase, she did not need any further assistance with the tool. Although she did not verbalize her actions on the screen, her statement “I knew there was something fishy about that” was meaningful because it reflected her thinking about the answer before she began her work on the screen.

Her mental representation may still be a work in progress. Nevertheless, the following scenario pushed her out of her comfort zone and challenged her thinking. The sequence below, revolved around an illustration that represented the sequence in a subtraction that did not follow the conventional algorithm. The problem is 25 – 8:

She counted the cubes, including each cube in the ten-stick.
Deidra: That’s still 25; they crossed out from the tens they [she seemed unsettled about having the 8 crossed out of the ten-stick; then she counted the cubes in the last frame] they have 17 not 25, that is less than 25.

Int: But isn’t that their answer?

Deidra: Because they are trying to take away 8 [she looked back at the third frame and counted the crossed out parts]. I could see that they crossed out 8.

Int: So is this correct or incorrect?

Deidra: Correct.

Deidra tried to understand the reasoning of crossing out 8 from a ten-stick by talking out loud to herself. She used self-speech to describe what occurred in each frame; she was self-explaining why they ended up with “17 not 25, that is less than 25.” This was indicative of her experiencing a disruption in her thinking and her expectations. It might be that she was experiencing a change in her mental activity that would eventually affect her understanding.

Her reaction suggested that even when a student exhibits signs of understanding the concept of ten as being made up of ten ones, that understanding may be fragile. The tool might have brought out that fragility. Deidra showed an understanding of how to use the physical manipulatives. She conveyed an understanding of a ten and ten ones equivalence without having to break the ten into ten ones. The virtual manipulatives created a small obstacle only because she was not familiar with their use. It could be argued that once she was comfortable using the virtual tool, it stimulated a psychological activity that positioned Deidra in the process of building constructive abstractions that
would lead to more complex mental representations of ten.

**Mimi**

Mimi used manipulatives only after I prompted her to do so. Her inclination was to use the conventional algorithm to answer the questions. I asked her to show me how she would solve $15 - 8$. I reminded her that she could use any of the manipulatives or whatever she would normally do.

**Mimi:** I would take $15 - 8$, I would subtract that.

She wrote it vertically (as illustrated below). Then she crossed out the 1 in the tens place, put a 0 above it and crossed out the 5 in the ones place and wrote a 15 above it. She then proceeded with the vertical subtraction; she put a 7 in the ones place and a 0 in the tens place.

![Figure 12. Mimi’s work](image)

It was difficult to interpret whether she was thinking of the problem in terms of tens and ones. Did the 0 above the 1 signify that there were 0 tens and 15 ones? Why place a 0 in front of the 7 in her answer? Was her action a reflex?

I asked her if she had to show me the subtraction using manipulatives, which ones would she use? She chose the hundreds chart.

**Mimi:** I would find 15 and minus 8.
She located 15 and counted eight spaces backward beginning with 14 and landing on 7. She said that the hundreds’ chart is easier to use. Her answer did not give any insight into her understanding of the process. Therefore, it was premature to make any conjectures about her knowledge of the concept of ten or the composition of numerals at this point.

However, when I asked her to show me 32 – 14 she turned to the connecting cubes (also known as unifix cubes) and started snapping them together to create towers of ten.

**Int:** You are using the connecting cubes, why?

**Mimi:** Because when I’m using with a higher number it’s easier to use these. I take 10 of them [she was snapping the cubes together].

**Int:** You’re making towers of 10 with the connecting cubes?

**Mimi:** Uhha [she made 3 towers of 10].

**Int:** How many do you have now?

**Mimi:** 30.

**Int:** But it’s 32.

**Mimi:** Oh, I need 2 more [she took 2 more connecting cubes and started a new tower of 2 cubes], so I would take away 4 of these [as she took the 2 cubes and snapped off 2 more from a tower of 10] and then I would take away a whole ten-stick, and then I have this [referring to the remaining ten-stick] and some ones [she counted how many connecting cubes were left after she had snapped off the 2 from the tower of 10] so I would count them [she counted them] ...8.
Int: You have just 8?

Mimi: 18.

Int: You used the cubes to make towers of 10. What would be different if you used the base-ten blocks? Why are the connecting cubes easier for you to use than the base ten blocks?

Mimi: Because with these, if I had the ones and didn’t have enough ones to take away part of tens, I can’t do that with these [base-ten] because they’re all together.

Mimi’s preference for the connecting cubes over base-ten blocks, when she was solving 32 – 14, suggested that her understanding of the concept of ten did not yet reflect the concept of ten as an iterable; she did not think of one ten as ten ones. From this exchange, it follows that her understanding of ten was in terms of ones, an “abstract composite unit” (Cobb & Wheatly, 1988). It might have been easier for her to snap off the cubes rather than to trade a ten-stick for ten little cubes.

In this next episode, Mimi’s actions contradict the observation made above. When I asked her whether she agreed with the subtraction done by second grade students from another school, she repeatedly checked the answers correctly using the algorithm.

![Figure 13. Mimi’s work (2)](image)

There was no question that she knew how to follow the steps of the conventional
algorithm. However, I requested that she show me how to solve $40 - 23 = 17$ using manipulatives. I was expecting her to choose the connecting cubes (because of what she stated in the previous interaction). Instead, she chose base-ten blocks. She picked 4 ten-sticks to show 40. Then, she put two ten-sticks to the side for the 20; and she traded one of the 2 remaining ten-sticks for ten ones. She took three of the cubes away and she was left with one ten-stick and seven ones.

Her use of the base-ten blocks, and her trading of a ten for ten ones contradicted what she had said earlier. It could be that the problem was influencing which manipulatives she used. The tool she chose to use engendered aspects of her understanding that she did not make clear earlier. Her use of the base-ten blocks indicated that she had an understanding that ten is made up of ten ones. Her actions contradicted her statement made earlier about taking from the ten-stick, “…if I had the ones and didn’t have enough ones to take away part of tens, I can’t do that with these [base-ten] because they’re all together;” it implied that she was not treating the ten-stick as being made up of ten ones. Clearly, she had developed that mental construct and was able to act on it, trading a ten for ten ones; yet her earlier words did not reflect this understanding.

The reader should note that she, like Deidra, began the subtraction out of order. However, by starting with taking away from the tens, the manipulatives enabled her to transform the problem from $40 - 23$ to $20 - 3$. Was this intentional? It is also not clear whether she was representing the 20 from 23 or taking away 20 from 40.
In this next interaction, she recreated the scenarios as they appeared in the sequence of the frames using base-ten blocks. As she looked at the frames and mimicked the illustrations, she described out loud the actions she was replicating. The sequence below followed the conventional algorithm — 25 — 8

Mimi: What they did, they got 2 of these [tens] and 5 of these [ones], then they took away one of the these [pointing to one of the tens in the first frame] and traded it for 10 of these [referring to ones, and examining the second frame], and they took out 8 [looking at the third frame], and what’s left they have is this: they have 7 [counting the 7 ones left in the third frame], I have 6 [referring to the ones in her reproduction of the sequence].

Here, the manipulatives have enabled her to impose her own language from the perspective of a child working with manipulatives to the conventional algorithm. Her actions as well as her words affirmed that she had an understanding of ten being traded for ten ones. Not only did she make the trade, but she also recognized it in the illustration and verbalized it.

Int: Do you think we miscounted?

Mimi: I don’t think we miscounted.

Int: So, you think the steps they used are correct?

Mimi: Yeah, but I don’t think their numbers are correct.

With this statement, Mimi was expressing the fact that she perceived a difference
between the process and the outcome of subtraction. She was convinced that the answer was wrong but was comfortable with the process. This episode conjures up relational and instrumental understanding. Can her understanding be characterized as relational or instrumental? She demonstrated an understanding of trading one ten for ten ones. She was not blindly repeating steps; this fact indicates that she had more than instrumental understanding, but not quite relational.

As she confidently executed the subtraction, I questioned whether she had developed an understanding of ten as an iterable. She was using the base-ten blocks to reproduce each frame; it was not clear whether she depended on the representation she was recreating to help her make sense of the illustrations.

We now observe Mimi’s thinking with the Disruptive Frame. The sequence offers a different solution to the same subtraction problem-- 25 – 8

![Frames](image)

She looked at the frames and disapprovingly pointed to the third frame and said,

**Mimi**: They have this here.

**Int**: You didn’t like that they crossed-out these in the third frame why?

**Mimi**: Because they can’t x-out, that’s a 10, but if they took this [1 ten-stick in the second frame] away and put 10 [referring to 10 ones in the third frame], then they could x-out that [referring to 8].
**Int:** What do you think?

**Mimi:** I think their answer is correct, except for that part [referring to the third frame].

The purpose of the third frame was to cause a disruption in students’ thinking and to observe how students would handle this disruption. Mimi seemed certain that they could not cross out an 8 from a 10, and offered an explanation of how this could have been solved. That is, her process was a little muddled, but she had the correct answer and recognized it. Mimi’s behavior was reflective of her affinity for the conventional algorithm. It is important to note that she found the answer to be correct, but the work to be incorrect. Her behavior in this episode was contrary to that in the section above, when she was comfortable with the process, but claimed that the answer was incorrect. In the previous sequence, she demonstrated an understanding that was more complex than an instrumental one, although not quite relational. This sequence revealed the fragility and the inflexibility of her understanding: She was not ready to make the connection that would allow her to think of ten and ten ones simultaneously.

In summary, Mimi exhibited an understanding of the conventional subtraction algorithm. Through the use of manipulatives, she communicated an understanding of a ten and ten ones equivalence; she made the exchange when she worked with manipulatives, and she was able to identify the regrouping of a ten for ten ones in the illustration. Her use of the tools as well as her self-speech allowed her to express her thinking, and to show that she understands ten as an iterable. Although she demonstrated that she could trade one ten for ten ones, she had not reached a point at which she
identified them simultaneously, that is, interchangeably. It seems that there was a connection missing, and she had not yet constructed that relationship.

Billy

Billy used mental math to answer the first subtraction question that I asked him. He demonstrated his knowledge of decomposing a numeral. He only used the base-ten blocks because I asked him to. As noted in the methodology section, I began the interview with a question that would put the students at ease. I knew $8 + 7 = 15$ was a “fact family” he was familiar with, and he was expected to know. I wrote $15 - 8$ in a horizontal form on the paper in front of him.

Billy: $15 - 8$, subtract 5 from 15 then regroup then minus 3 more is 7.

Int: Okay?

Billy: And also $8 + 7$ is 15.

Int: Can you show me how to do $15 - 8$ using manipulatives?

He chose the hundred’s chart. He started at 15 and counted backwards 5 places

Billy: $15, 1, 2, 3, 4, 5$, regroup, minus $1, 2, 3$, is 7 and also $8 + 7$ [as he located 8 on the hundreds chart and counted 7 spaces] $1, 2, 3, 4, 5, 6, 7$ is 15.

Billy referred to “regroup” in both verbal explanations. What did regroup mean to him? Was he thinking of the procedure or was he thinking of the composition of a numeral? Did he say “regroup” when he reached ten?

I asked him to show me $32 - 14$ and wrote it horizontally on his paper. He did part of the computation in his head, speaking out loud, and in the middle of his calculations, he switched to the hundred’s chart. Using the hundreds chart, he pointed to
32 on the chart and pointed to the number just above it [which was 22] and counted 10, 11, 12, 13, 14 landing on 18. Below I describe his actions and actual words:

**Billy**: Subtract 2, then regroup, then, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 [stopped in the middle of this calculation] also like this [switching to the hundred’s chart]; this is how you can do it [pointed to 22 and counted] 10, 11, 12, 13, 14.

I asked him to show me 32 – 14 using the base-ten blocks. He made two representations: one of 32 (he chose 3 ten-sticks and 2 little cubes), and the other of 14 (1 ten-stick and 4 ones).

**Billy**: That’s 30 [he took 3 ten-sticks, and taking 2 little cubes] 1, 2, that’s 32; minus 1 [taking a ten stick from the 32] that’s 22, that’ll be 10 [referring to the ten from 14]. Then, minus 1, 2 [taking away 2 little cubes]; these are also out [he had covered 2 cubes from the ten-stick with his finger], so 1, 2, 3, 4, 5, 6, 7, 8, 18 [counting what was left of the 32].

Billy referred to “regroup” on three different occasions: first when he was calculating 15 – 8 in his head; then, when he used the hundreds chart to show 15 – 8, and when he showed 32 – 14 using the hundreds chart. However, he did not mention the word “regroup” when he used the base-ten blocks to show 32 – 14. The reader should observe that Billy used “regroup” every time he reached a multiple of ten. What can we conclude about his understanding of ten? It is possible that when he was doing mental math and he was using the hundred’s chart he utilized self-speech to keep in mind the action he needed to take-- regrouping. However, when he used the base-ten blocks he was acting on the object; as a result, he did not need the mental reminder.
Billy’s behavior suggested that he had an understanding of ten that was associated with ten ones. Although he did not break the ten-stick into ten ones, it was clear that he was aware that there were ten ones in a ten-stick when he covered 2 cubes from the ten-stick with his finger to demonstrate what he was taking away.

It is also possible that he was re-enacting his approach when he used the hundred’s chart by subtracting the ten first and keeping track of it in order to make a 14. He was mentally performing multiple tasks. His treatment of the base-ten blocks demonstrated that he was comfortable using them; but when I asked him if he would like to use them for the next problem, he responded “no” and proceeded to perform the algorithm. Here is the exchange:

**Int**: Would you like to use manipulatives for that $[34 - 19 = 25]$?

**Billy**: No [to using the base-ten blocks, and he wrote the minuend and the subtrahend vertically and subtracted] wrong [referring the answer in the given equation].

There was no doubt that Billy had at least an instrumental understanding of the conventional subtraction algorithm. This next interaction confirmed his confidence with using the algorithm.
Billy: 25, they regroup; they took away a 10 [he counted to make sure it’s the same] that’s where they regrouped [referring to the second frame, then in the third frame he counted how many were crossed off, 8, and how many were left] 17 [then he wrote it vertically and did the procedure to check that it was 17] correct.

Billy identified where “they regrouped” in the second frame. His statement and his action of counting the cubes to ensure that there were 10, indicated to his understanding of ten as being made up to ten ones as well as his understanding of regrouping in the conventional sense.

Here again, Billy chose to confirm the answer by using the algorithm. Predictably, he checked himself. It could be argued that these illustrations did not actually support the representations he had constructed. Although Billy demonstrated at least an instrumental understanding of the conventional algorithm, it may be that his confidence in knowing the steps aided him in justifying the work in the illustrations. That is, his mental constructs enabled him to make sense of the visual representations. This next interaction caused instability in his perceived understanding.
25 – 8

He looked at the frames one at a time, but when he got to the third frame,

Billy: This is wrong because they have to regroup; so they did it wrong. They took away that [referring to the 8 in the third frame] this is right [referring to the answer in the last frame].

Int: So, the last frame is right, but what they did is wrong?

He did not answer, but he continued to examine the frames.

Billy: Are these split apart [asking about the ten-stick in the last frame]? [Looking at the last frame and then back to the 3rd frame]. This is correct [referring to the answer in the last frame].

Int: This is correct? So, they took the 8 from a 10, and that’s okay?

Billy: Right [unsure].

The word Billy used, “regroup,” represented that he had at least an instrumental understanding of the algorithm. He relied on that understanding to check and justify his work. The illustrations seemed to be challenging for him: First, he needed the procedure to confirm the process. Then, he found himself faced with an unfamiliar way of subtracting; the subtrahend was taken away from a ten and there was no regrouping. Although he initially categorized it as being wrong, his reaction indicated that his
construct of ten may be evolving, and that perhaps he was ready for the next phase of thinking about ten. He was reluctant, but he still agreed that it was possible to take away from the ten in this way. This agreement was reached after he had spent time examining the frames, which indicated that he had put thought into his answer. It is possible that the combination of the tool, the illustration, and his understanding of ten and of regrouping, that is, the combination of the physical and the psychological tools, enabled him to make connections and exhibit a higher level of thinking behavior. I could only predict that he might be experiencing a change in his mental representations.

Troy

Troy reached for the connecting cubes when I asked him to use manipulatives.

Int: You prefer the connecting cubes?

Troy: Yeah

Int: Why?

Troy: Because it’s easier, because you are allowed to connect them together and they don’t break apart.

Int: Can you show me 15 - 8?

I wrote the expression horizontally on the paper in front of him. He counted 15 cubes, and then took away 8. He treated the connecting cubes as ones.

Int: Is this something you would know quickly? [He nodded yes]. How would you figure it out without the manipulatives?

Troy: I would add the 2 digits, and make them like this; I mean, I would put them in the ones column
**Int**: Can you show me?

He drew a T-table. He made a tens column and a ones column. Then, he drew a vertical line and put a T for tens on top of the left side and an O for ones on the other side and proceeded to perform the algorithm. He recited the rhyme the teacher had taught them.

**Troy**: More on the floor; go next door. Then, I’ll just do if there is more on the floor and do that up here [student’s work is below].

![Figure 15](image)

*Figure 15*. Troy’s work

He proceeded to cross out the 1 in the tens place, write a 0 above it, and to cross out the 5 in the ones place write a 10 above it, and a 5 above that 10, and subtracted. Was this behavior a reflex? Writing the 10 above the crossed out 5 suggested that he was thinking of 15 as 1 ten and 5 ones. But was this thinking imposed on him because of the way he set up the numerals in columns? His behavior with the connecting cubes earlier suggested that he thought of 15 as ones, thereby suggesting that the tool that he used influenced how he thought of 15. What role did the rhyme play in his understanding of the process? It is unclear whether Troy recognized the fact that the ones in the subtrahend were more than the ones in the minuend. Perhaps the rhyme helped him to articulate that observation. The rhyme might be the tool that he was using to initiate basic self-speech, that is, a description of the relationship between the ones in the
minuend and the ones in the subtrahend.

I encouraged him to use the base-ten blocks to show me how he would solve $32 - 14$.

**Int:** Can you show me $32 - 14$?

**Troy:** With these [pointing to manipulatives] or on paper?

**Int:** Could you use base 10 blocks and then do it on paper if you like?

He then took 3 ten-sticks and 2 ones for 32, and then 1 ten-stick and 4 ones for 14. He put them next to each other. Next, he took a ten-stick away from the 30.

**Int:** So what did you take out?

**Troy:** 1 [referring to a ten-stick] and then I have to take out another 1 [he was referring to another the ten-stick]

**Int:** What is this long stick here?

**Troy:** A ten

**Int:** And you took this ten stick away why?

**Troy:** That’s gonna become 20 [he was referring to the 30], but we need to do the ones, to minus the ones, minus 4, so we have to put away another ten, and I think we need to put these away [referring to the ones, the two cubes belonging to 32 and the four cubes belonging to the 14]. I don’t know I forgot.

At this point Troy seemed to be stuck, unsure of how to proceed with the subtraction using the base-ten blocks. He knew he needed to subtract 4 and that he had to take away another ten, “so we have to put away another ten” but he was uncertain about how to continue. He removed the ten-stick, but it was unclear whether he intended to
break it down or to create a 12 in the ones place. His comment “I forgot” suggests that his manipulation of the base-ten blocks was contingent on his memory of something. In his case, a separation existed between the tool and the mathematical idea; therefore, the base-ten blocks revealed a lack of connection in his understanding.

**Int:** That’s okay. So, you put away another 10, and you think we need to put the 2 ones away. That’s a very good strategy. Do you know how to do it on paper? Can you show me how you would do it?

**Troy:** [He smiled] I’m good with this.

I had written the problem horizontally. He drew a T-table and wrote the 32 on top and 14 on the bottom, as seen below,

![Figure 16. Troy’s work (2)](image)

He performed the conventional algorithm, crossed off the 3, put a 2 above it and subtracted 1; then, he moved to the 2 ones crossed that and wrote a 12 above it and subtracted 4; his answer was 18.

**Int:** So, why did you cross off the 3 and make it a 2?
Troy: Because if there is more on the floor you can’t do it, cause then it will be 2 – 4 and that’s not good. I mean 4 – 2, I think [he became confused].

Int: 4 is in the bottom, is that on the floor you’re saying? [He ignored my question and Proceeded to explain what he did].

Troy: I had to go next door and get rid of and put one of the tens on the 2 make it a 12 and turn this into a 2 [pointing to 3 tens] and minusing 1.

Troy seemed to rely on the rhyme to remember the steps of the conventional algorithm. He used language like “put one of the tens on the 2 make it a 12” that suggested that he understood the idea behind regrouping, but it was also possible that he was doing and saying what he had learned by rote. From the previous observation, it is likely that Troy had an understanding of ten as an “abstract singleton.” He had not yet made the connection between one ten and ten ones. On three different occasions, it became clear from his language that he was influenced by the rule “can’t take a bigger number from a smaller number,” because he said, “…2 – 4, and that’s not good. I mean 4 – 2, I think” and “0 – 3 is still 3” and “9 – 4” for 34 – 19 = 25.

Despite indications of misunderstandings, when I asked Troy to tell me whether 34 – 19 = 25 is correct and to justify his answer, he used the relationship between the numbers to validate his answer. He reasoned:

Troy: No, because it’s close to 20, there is a 9 – 4 so it’s gonna go down still … It’s gonna go down to a lower number, lower than 25.

This statement suggests that he had good estimation skills. He recognized that 19 is close to 20 “No, because it’s close to 20,” and the difference will be “lower than 25.”
He was thinking about the relationship between the numerals. For him to be able to reason in the manner he did, he must have developed some understanding of a relationship between numerals. I asked him to show me how to solve $34 - 19 = 25$ using the base-ten blocks. He took 3 ten sticks and 4 ones to show 34 and placed a 1 ten and 9 ones on the side (below is a reproduction of the student’s work).

![Figure 17. Troy’s reproduction of 34 and 19](image)

He began by taking away a ten-stick from the 34.

**Troy:** I’m going through this again. Put away another 10 [now he removed a second ten-stick].

**Int:** Why do you put away another 10?

**Troy:** Because then it’ll turn into 25 [it is possible that he meant 24 because he took a ten from 34] and the other 10 will go away too because of the 9. Yeah I think I need more of these [referring to ones, but he did not say anything about regrouping. He did not know what to do. He was silent and stared at the blocks in front of him].

**Int:** Do you think using these base-ten blocks help you to solve these problems?

**Troy:** No I don’t think these are easier, I think these [pointing to the connecting cubes] are easier.

**Int:** What would you do if you had to use the connecting cubes?

He counted the connecting cubes, made 3 towers of ten, and said
Troy: And then 4, [now he had 34], get 19 [he took one of the towers of ten and counted 9 cubes from another tower of 10] gone. [Counting the left over] 15.

Troy’s preference for using the connecting cubes suggested that he was still counting by ones; it was easier for him to break off the cubes one at a time. His concept of ten seemed to be ten as an abstract singleton and had not yet evolved to ten as an iterable. His treatment of ten was still in a phase in which he did not associate one ten with ten ones. This was evident in this next excerpt from our interview (25 – 8), when, he insisted that he could not take away from a ten.

Int: Can you tell me, 25 – 8, this is how they did this [showing him the sequence]? Can you tell me if what they did is correct?

Troy: [After examining the illustration] No, because minusing, you need to minus that whole 10.

Int: I see, you need to take away a whole 10? You can’t just take it from here [pointing to a ten-stick. He did not look at the final frame to check the final result].

Troy: Nah [meaning you cannot take away from a ten].
His reaction was revealing because it reflected his understanding of what should happen. It could be that the construct of ten that he had developed did not allow him to be open to any solution that deviated from what he was expecting. Troy relied on mnemonics, like the rhyme, to perform the subtraction algorithm. He was successfully representing a number using the base-ten blocks, but the manipulatives were not useful to him when it came to subtracting. His preference was to utilize connecting cubes because he was able to manipulate the cubes one at a time. This preference reflected his fragile constructs of ten; it appears that he might still be thinking and counting by ones, which would explain his success using the connecting cubes versus the base-ten blocks. That is, he was able to take away one cube at a time. The base-ten tools pinpointed the missing connections in his understanding.

In this section, I will share the interactions with students who could not use any manipulatives, and those who tried but were stumped. That is, the manipulatives did not help them carry out their solution. Nevertheless, the extent to which they used the tools revealed the lack of connection that obstructed their ability to proceed with their thought process.

Ivan

I begin with a description of Ivan who seemed comfortable with numbers. He added and subtracted mentally without difficulty. He exhibited an understanding of ten as being made up of ten ones from the beginning of our interview. He traded ten little cubes for a ten. Below is the interaction that occurred.

**Int:** If I were to ask you something like 7 + 5, what is that?
Ivan: 12

Int: 12. Can you show me how to find that answer using the manipulatives?

He chose the base-ten blocks. He got seven cubes and five cubes and counted them, 12, and then he counted ten and traded it for a ten-stick.

Ivan: Then, I trade it [referring to the 10 ones] for a ten-stick.

His action reflected an understanding of a rod being equivalent to ten ones.

His explanation of how he solved 15 – 8 initially indicated that he had some understanding of the composition of a numeral:

Int: What is 15 – 8?

Ivan: 7

Int: How do you know?

Ivan: Because if you have 15, you can just subtract 3 because 5 + 3 = 8.

Int: Okay, so I have 15, I subtract 3, what do I get?

Ivan: 7

I suspected he was subtracting 15 – 5 = 10 and 10 – 3 = 7, but he was having difficulty articulating his thoughts. However, he did not follow through on his thinking process based on what he said; and he did not answer my question “so I have 15, I subtract 3, what do I get?” I questioned whether he had an understanding of a number or whether he had that number fact memorized.

When I asked him to show me how to subtract using manipulatives, he counted 15 little cubes [the ones] and then put eight of them on the side and said,

Ivan: 7
His choice of manipulatives did not convey anything about his understanding other than he was thinking in terms of ones. Next, I asked him to show me 32 – 14 and encouraged him to use manipulatives. He reached for the base-ten blocks and was able to provide a representation of 32 and one of 14. He continued to align two ten-sticks, one from the representation of 32 and one from the representation of 14. He was stuck. He did not know how to proceed. He stared at the representations in an effort to find a solution and then said,

**Ivan:** I think I want to use my pencil.

He proceeded to write the vertical form of the expression and executed the procedure, as seen below,

![Figure 18. Ivan’s work](image)

**Int:** Do you think this is easier, to use the pencil than to use the base ten blocks? You don’t think base ten blocks help you?

**Ivan:** It helps me a little bit

**Int:** A little bit, like when?

**Ivan:** Like...when I add […] yeah it’s harder when you’re subtracting.

His comment aligned with his previous actions in showing 7 + 5 after I asked him to show me how he got his answer using the manipulatives. The action of trading ten
ones for a ten was significant. It indicated his understanding of ten in the context of addition. It also reflected his understanding of ten: he had not yet developed a mental representation that allowed him to see one ten and ten ones simultaneously; otherwise, he would have been able to make the trade when subtracting. In any case, understanding how to trade ten ones for a ten may not be transferring to the concept of subtraction. The difficulty that he had using the base-ten blocks hinted that his representation of the construct of ten did not include the use of base-ten blocks because they interfered with his understanding of how to proceed when subtracting with the blocks. He did not exhibit the same difficulty when using the conventional algorithm for subtraction. The following excerpt highlighted his ability to perform the algorithm not just on paper, but also in his head.

**Int:** How about $34 - 19 = 25$, do you think that’s correct?

**Ivan:** [He shakes his head No] because 3 becomes a 2 and $2 - 1 = 1$; 4 becomes 14 and $14 - 9 = 5$; then, it’s 15 [he did this in his head]. Next, he checked by writing it on the paper and using the procedure.

*Figure 19. Ivan’s work (2)*
Ivan’s words and actions suggested that he had developed an instrumental understanding of the algorithm. However, his understanding of ten had not developed to ten as an iterable unit when he subtracted, as observed in the following interaction.

When I first showed Ivan the sequence of frames it seemed as though he did not understand what the sequence of illustrations represented. I needed to explain to him that these figures illustrate the steps of how the subtraction was done, and prompted him with questions.

25 – 8

**Int**: So, what did they do here [pointing to the second frame]?

**Ivan**: They broke up these to ones [pointing to the ones in the second frame].

**Int**: What are these [repeating his word to clarify what he was referring to]?

**Ivan**: The tens, and then they subtracted 8 [pointing to the third frame], and then they showed what was left. [He looked at the frame and counted] Yes [meaning it is correct].

Once he understood what the sequence represented; that is, each frame shows the step that was taken. He seemed to follow the steps without any difficulty. We moved on to the next figure.

**Int**: So it’s the same problem but done in a different way.
25 – 8

Ivan: 25 [in a whisper, studying the frames]. No [meaning not correct].

Int: Why not?

Ivan: Because they forgot to use this part right here [he was pointing to the ones in the third frame].

Int: What is this part here [repeating after him to clarify what he meant]?

Ivan: It’s the ones. They forgot to take away from the ones and instead of using the ones they used the ten-stick.

Int: Ah, so they can’t just take away from the ten-stick?

Ivan: Mhm [affirmative sound].

Int: They have to use the ones?

Ivan: They use the ones and then they take away the less.

Because his verbal expression was unclear, my interpretation of his comment was that they needed to start taking away from the ones and if they do not have enough ones, then, they can take away from the ten. But they could not start taking away from the ten first. This interpretation would follow from his comfort with performing the algorithm.

Int: Ok, so it’s incorrect?

Ivan: [He studied the final frame] It is correct, but they forgot to do it, to take away the ones.
**Int:** Ok, so they got the right answer but what they did in the third frame is not okay? They can’t just take away from the tens?

**Ivan:** Mmm [affirmative sound].

His disapproval of “taking away from the tens” suggested that although he said “they broke one ten into ten ones,” he still treated the ten and ten ones differently. The tool, as illustrated in the sequence, revealed his strengths as well as his weaknesses. He imposed his knowledge of the algorithm on the illustrated method to make meaning out of the sequence of frames. This behavior substantiated his understanding of the algorithm. However, his disapproval of deducting from a ten indicated that he had not completely connected ten and 10 ones. That is, he had established a ten and ten ones equivalence, but he had not constructed the relationship that makes ten and 10 ones simultaneous for him. He still viewed them as being separate.

His affirmation that the answer is “correct but they forgot to do it, to take away the ones,” suggested that he, like Mimi, differentiated the answer and the process.

**Sally**

Sally was the second student who experienced difficulty using the manipulatives. When I asked her to show me 15 – 8, she chose the hundreds chart and counted backward starting at 15 and landing on 7, as she clarified in her own words.

**Sally:** 7. I counted backwards.

**Int:** Can you show me how to do 32 – 14 using the manipulatives.

I wrote it horizontally on her paper.

**Sally:** I know how to do it another way it’s easier for me.
She rewrote the expression in a vertical form (as illustrated below)

![Figure 20. Sally’s work](image)

Sally: First, you can’t take away 2 from 4 cuz 4 is bigger, so I need to cross out and turn it to 12; so I took one from the 3, and it becomes a 2. It’s 18.

Int: What is the 3?

Sally: Huh?

It seemed that my question confused her. I was expecting her to say that it is 3 tens, but because of her answer, I decided to ask her to show me.

Int: Can you show me how to do this, using base-ten blocks? [She took 3 ten-sticks and 2 little cubes].

Sally: This is 32 — 3 tens usually called rods, and 2 cubes

She was able to show a representation of 32 using the base-ten blocks but when it came to doing the subtraction she did not know how to use them. She just looked at the manipulatives and did not know what else to do. She was stuck. The use of base-ten blocks created a cognitive obstacle for her. As she explained how she was subtracting using the algorithm in a vertical form, the language she used was problematic: she said, “can’t take away 2 from 4;” she did not say, “cannot take 4 from 2.”
I suspected that she was using the algorithm mechanically as steps to follow. Her statements revealed that she neither had an understanding of the meaning of words nor of the mathematics. The following interactions engendered misunderstandings but they also shed light on where she was developmentally.

**Int:** Could you try $40 - 23 = 17$?

**Sally:** [She rewrote the equation vertically on her paper and subtracted using the procedure] I know you can’t take 0 away from 3, so it had to be 3; this is 7 [referring to the ones place in the difference], and 4 take away 2 is 2, so it’s incorrect.

![Figure 21. Sally’s work (2)](image)

It may have been that Sally was influenced by the rule that one *can’t take a larger number from a smaller number*. She evidently did not understand how the rule works, or why, because she ended up subtracting 3 – 0. Her behavior led me to question her understanding of the concept of ten as well as the composition of a number.

Sally did not use the tools available to support her learning. It was apparent that the manipulatives did not lend themselves to support her in building mental connections. The following episodes corroborated this hypothesis.
I needed to explain to her what the sequence of illustrations represented. The problem was 25 – 8:

**Int:** They start off with 25. What do you think happened next [pointing to the second frame]?

**Sally:** It got bigger.

**Int:** How do you think it got bigger?

**Sally:** They added more.

**Int:** How do you think they added more? More of what?

**Sally:** Because they wanted to add to make a bigger number maybe.

**Int:** Where do you think they got the cubes and put them there, or did they get them from somewhere specific?

**Sally:** Specific, like the problem.

**Int:** Like where?

**Sally:** Like the problem.

Sally seemed frustrated. Either she did not understand what the illustrations were reflecting or she was not able to articulate it. I suspected that the illustrations did not support her learning. She was treating them as figures, not as representing the process of
subtraction. In this next scenario, the problem was still 25 – 8, but it did not follow the steps of the conventional algorithm.

Sally: This is correct [she pointed to the 3rd frame] they took away 8.

This third frame did not cause a disruption for her. She recognized that there were eight cubes crossed-out. Sally did not reveal an understanding of ten or of the composition of a number in our previous interactions. It may not be surprising that she was indifferent about the third frame because she only saw ones, and there were 8 ones crossed out. She understood that 8 were to be deducted. Her lack of understanding of the conventional algorithm may have contributed to her indifference to the disruptive frame. It is very likely that Sally had an understanding of ten as a numerical composite; that is, ten has no structural meaning: It is just a name, no different than any other name of any other numeral.

Her answer to the last illustration (25 – 8) confirmed that she did not understand the frames. She looked at the sequence and said, “I don’t know.”
It was possible that she was overwhelmed by the illustrations. Except for being able to represent a numeral, she did not convey any use for the manipulatives. Considering her actions and her responses, there was no evidence of Sally understanding a numeral in terms of tens and ones. There was the possibility that the third frame presented a challenge because she could not tell what was going on: there were five cubes crossed out rather than eight (in the previous sequence, she had recognized the deletion of eight cubes).

The manipulatives revealed her lack of understanding on multiple occasions. The constructs she had seemed to be weak. There was no evidence of a strong network of cognitive relationships that would enable her to perform the tasks asked of her. It appears that she needed opportunities to build on the constructs she already had in order to build a meaningful concept of a number before she could understand double-digit subtraction with the numerals. The following is an example of her lack of connection with numbers and the algorithm.

I had asked her to tell me if $34 - 19 = 25$ was correct and if she could explain her answer. She rewrote it vertically, aligned the numbers, and tried to do the subtraction following the algorithm. She crossed out the 3 tens, made it a 2, crossed out the 4 and put a 3 above it. But she was stuck. She subtracted $2 - 1$, and she did not know what to do with the 4, now a 3. She reached for the hundred’s chart and counted backwards; she mumbles 13 lands on 21.

*Sally:* 21, so that’s wrong.
**Int:** [I reminded her] It’s minus 19, not 13.

**Sally:** I did 19

She did not want to engage in answering any questions related to this problem.

![Figure 22. Sally’s work (3)]

Her unwillingness to engage could be the result of her lack of understanding or of her own realization of not understanding. In both cases, the absence of understanding was affecting her attitude towards her own learning. There was no evidence that she was actively engaged in her learning. Her behavior indicated that she was blindly trying to follow rules and was clearly confusing them. Sally needed to be given opportunities to make mental relationships through her own thinking and actions to build on her logico-mathematical knowledge (Chandler & Kamii, 2009).

**Cross Analysis**

The case studies above suggest that the use of manipulatives can reveal missing mental constructs related to the networks of connections that students have yet to construct. The analysis in this section looks at the students and examines special aspects of their thinking and their responses. The unique differences made students’ actions
meaningful. The analysis that follows sheds light on the different mental constructs students had developed, and on their responses to unfamiliar situations.

**Different Mental Constructs**

**Ten and ten ones equivalence.**

The case studies uncovered signs of differences in the students’ understanding of the concept of ten. Three students demonstrated ten and ten ones equivalence in their own individual ways. However, none of the students exhibited an understanding of ten and ten ones interchangeably. By this I mean that, they demonstrated an understanding that a ten-stick has ten ones within it, but they still treated them distinctly. They did not see them as the same, “simultaneously” (Kamii, 2000).

For example, Deidra’s preference for the base-ten blocks “because they already have the ones stuck together” signified her understanding of what a rod represented. Her ability to reiterate ten to keep track of the composition of 23 when she was deducting from 56 indicated that her concept of ten was iterable. She was taking away a ten from 56, while concurrently building the number 23. The strength of her understanding of ten was portrayed in two ways: First, she did not use the blocks in the way that is usually demonstrated in the classroom. Instead of trading a ten for ten ones, she covered the two cubes on the ten-stick with her fingers. This action illustrated her understanding that this ten-stick is made up of ten ones. Second, she was able to transfer her understanding of ten with concrete base-ten blocks to virtual base-ten blocks. This is important because it points to the strength of her construct of ten. Furthermore, other than learning how to use
the tool, she did not confront obstacles in transferring her understanding to the virtual representations.

Billy, too, established an understanding of ten and ten ones equivalence. His partiality to relying on mental math indicated that he did not need a “re-presentation” (Cobb & Wheatly, 1988) to construct an abstract composite of ten. Billy demonstrated an understanding of ten as an iterable. His explanation of 15 – 8, “subtract 5 from 15, then regroup, then minus 3 more is 7,” showed that he was aware of the composition of a number and that he was mentally conscious of “regrouping.” He was able to determine when regrouping occurs, not only in his speech when referring to a multiple of ten, but also in the sequence of illustrations. This response signified that he repeatedly mentioned regrouping with an understanding of what it meant. He, like Deidra, did not trade ten for ten ones when he was subtracting; instead, he covered the blocks with his fingers. This action suggests that he had an understanding of ten that was clearly associated with ten ones.

Mimi had constructed a concept of ten and ten ones equivalence that was evident in her work. Unlike, Deidra and Billy, Mimi did not use her fingers to cover blocks from the ten-stick. She had an affinity for the conventional algorithm, which may have influenced her actions on the base-ten blocks. When she used the blocks, she communicated her understanding unambiguously by trading ten for ten ones “she traded one of the 2 remaining ten-sticks for ten ones, [and then] she took three of the cubes away.” This construct was fundamental to her understanding of the conventional
algorithm; it characterized her understanding which was more than instrumental but not quite relational.

**Ten, not quite ten ones.**

Most students had constructed some concept of ten, but not all of them had a concept of ten and ten ones equivalence. Troy exhibited an understanding of ten as an “abstract composite unit” (Cobb & Wheatly, 1988). In his explanation of 32 – 14 he said “I had to go next door and get rid of and put one of the tens on the 2 [to] make it a 12,” which suggested that he was aware of a ten being “put” somewhere. His statement signified that he was using the subtraction poem to guide his next move. Moreover, taking his comments about the connecting cubes into consideration “because it’s easier, because you are allowed to connect them together, and they don’t break apart” implied that he was still thinking of ten in terms of ones. His experience with the base-ten blocks indicated that he had some construct of ten, but he still had not yet made a connection between ten and ten ones. He tried to use the base-ten blocks to show 32 – 14; he succeeded in producing a representation of 32 and of 14. He was able to subtract the tens but he found himself stuck not knowing how to proceed with the manipulatives,

That’s gonna become 20 [he was referring to the 30], but we need to do the ones, to minus the ones, minus 4, so we have to put away another ten, and I think we need to put these away [referring to the ones, the two cubes belonging to 32 and the four cubes belonging to the 14]. I don’t know I forgot.
Troy’s understanding of ten depended on his memory of something. His reliance on a tool, whether a poem or connecting cubes, indicated that he did not construct ten and ten ones equivalence.

**Lack of transfer.**

Ivan’s use of the base-ten blocks shed light on his understanding of ten in the context of addition, but that understanding did not appear to transfer to subtraction. To show $7 + 5$, Ivan pulled 7 ones (cubes) and 5 ones individually; he continued to put them together, then counted 10 cubes and traded them for a ten-stick. This behavior indicated that he had a clear understanding of 10 ones being equivalent to one ten in the context of addition. However, Ivan’s understanding was challenged when he tried to show the subtraction $32 – 14$. Like Troy, Ivan successfully represented 32 and 14 using the base-ten blocks. He took 1 ten from the 32 and 1 ten from the 14; he stared at the blocks uncertain as to how to proceed. Ivan did not exhibit an understanding of ten as being equivalent to ten ones, if he had this construct he would have been able to trade 1 ten for ten ones (or cover 4 from the remaining ten rod). His understanding and treatment of the blocks may have been influenced by memorization of the algorithm. After all, he demonstrated the ability to reproduce the algorithm in his head as well as using paper and pencil.

**No tens.**

Sally’s construct of ten was elemental. The way she used, or did not use, the base-ten blocks highlighted the lack of connections she had. Her actions pointed to an understanding of ten as a “numerical composite” (Cobb & Wheatly, 1988). When asked
to show $32 - 14$, her first inclination was to reply, “I know how to do it another way; it’s easier for me.” This implied that the manipulatives were not easy for her, and her constructs did not include the use of manipulatives. The “easier way” she referred to was the algorithm, but the language she used to justify the steps she was taking became problematic. Her explanation, “first you can’t take away 2 from 4 cuz 4 is bigger, so I need to cross out and turn it to 12, so I took one from the 3, and it becomes a 2” evoked memorization of steps. Unlike Ivan and Troy who exhibited some understanding of ten and who successfully performed the algorithm every time, Sally conveyed neither an understanding of the algorithm nor of ten. There was evidence from her approach to $40 - 23 = 17$ that she was influenced by rules. These included rules that she memorized, but did not understand. Statements like, “I know you can’t take 0 away from 3, so it had to be 3,” emphasized that she memorized the rule that one can’t take a larger number from a smaller number, but she did not know when to apply it.

Sally looked at digits in a numeral in isolation. The lack of connection was accentuated in her handling of $34 - 19 = 25$. She crossed out the 3 tens made it a 2 and crossed out the 4 and put a 3 above it. But she was stuck, she subtracted $2 - 1$ and she did not know what to do with the 4 that is now a 3, as seen below.
It was clear that Sally had not constructed mental representations through her own thinking. She needed opportunities to create connections, to make sense of what she was exposed to. The way she dealt with numerals pointed to an understanding of ten as a numerical composite. That is, in her mind, the structure of ten was no different from any other number.

**Responses to Unfamiliar Situations**

Through the use of tools, or the lack thereof, students were able to communicate their mental construct of ten. The strength of this construct influenced their initial reaction to an unfamiliar situation, the Disruptive frame, where the students were pulled out of their comfort zone. Their reaction and the way they handled the frame differentiated their developmental readiness to progress to higher levels of thinking.

Deidra and Billy seemed to have constructed connections that enabled them to be more open to different solutions. Both students initially objected to the deduction from the ten. Deidra reconsidered the third frame when I asked her about the last frame, “isn’t that their answer?” She used self-speech to reach her conclusion. Her action of re-examining the third frame, and counting the crossed out parts convinced her to accept the correctness of the solution.

Billy was very confident that the sequence was wrong because they had to regroup and did not do so in the third frame; however, he recognized that the last frame reflected the correct answer. Without any prompting, he inspected the third frame and the last frame carefully before he gave his final answer. Although he was reluctant, he
changed his initial thought and agreed that the sequence was correct. The vignettes below give a glimpse into his thinking.

**Billy:** Are these split apart [asking about the ten-stick in the last frame]? [Looking at the last frame and then back to the 3rd frame] this is correct [referring to the answer in the last frame].

**Int:** This is correct? So they took the 8 from a 10, and that’s okay?

**Billy:** Right [unsure].

Deidra and Billy put thought into their final decisions. The disruption in their thinking process compelled them to challenge their understanding. This push out of their comfort zone, the introduction to an atypical solution caused them to mentally initiate a connection to their prior knowledge of ten. The strong construct of ten that they had developed allowed them to be open and ready to progress in their thinking patterns.

Mimi, Troy and Ivan, did not experience this challenge to the same extent. Mimi demonstrated her understanding of the conventional algorithm as well as the distinction between the process and the answer. She was able to identify what she perceived to be a mistake and offered a solution, “because they can’t x-out, that’s a 10, but if they took this [1 ten-stick in the second frame] away and put 10 [referring to 10 ones in the third frame], then they could x-out that [referring to 8].” Her reply exemplified the influence of the conventional algorithm on her thinking. She was not relying on her knowledge of ten to justify the illustration, but rather on her memorization of the steps of the algorithm. This reaction pointed to inflexibility in her thinking about the number and the construct of ten. Although she demonstrated an understanding of ten as an iterable, she had not
developed the connection that allowed her to proceed in her thinking of ten and ten ones simultaneously.

Troy and Ivan also had an affinity for the conventional algorithm. They deemed the sequence incorrect because it did not follow the steps with which they were familiar. Like Mimi, they relied on the algorithm to interpret the illustrations. They both exhibited an understanding of ten as an abstract singleton; though they can preserve the “tenness” of ten, they still had not constructed an understanding of ten as ten ones.

Understandably, Sally was the only student who did not experience a disruption when examining the Disruptive frame. Recall that Sally had not constructed a concept of ten, other than a numerical composite. That is, ten was just another name. She did not differentiate the structure of ten from any other number. She thought of numbers in terms of ones. The Disruptive frame had eight cubes crossed out, the subtraction required taking away 8. She did not understand the conventional algorithm so it would not disrupt that either.

For all of these students, the tools played an important role in either supporting them in building a connection or in identifying places where the connections were missing.
Answers to Research Questions

I begin this chapter by answering the research questions that guided this study. Then, I summarize the findings and discuss the implications for research and practice.

Question 1. What can be inferred about how tools (like manipulatives) enable students to communicate their thinking about double-digit subtraction?

Based on the observations in the six case studies, each student’s approach to using manipulatives reflected their cognitive development. Deidra’s use of the base-ten blocks suggested that her mental constructs of ten enabled her to equate a ten with 10 ones. Her understanding allowed her to transition to using virtual manipulatives without difficulty. She physically and virtually exchanged a ten for 10 ones. This act suggested that she thought of them interchangeably. However, the pictorial representation of the process of subtraction from the Disruptive frame indicated that she still treated the ten and 10 ones differently. During this activity, she initially objected to the deduction from the ten, but as she re-examined the frame, she communicated a position that reflected her readiness to move forward with her thinking about the concept of ten. In Deidra’s case, the base-ten blocks brought her mental constructs to light and confirmed her understanding of the conventional algorithm. Nevertheless, the Disruptive frame revealed a missing connection that was not illustrated in her use of the base-ten blocks or in the language she used to explain her thought process. This frame shed light on where she was developmentally with respect to her constructs of ten.
Mimi’s use of the tools revealed how she compartmentalized her knowledge. Her choice of the unifix cubes gave the impression that she was thinking of numbers in terms of ones, when in reality she had a more complex understanding of numbers. Through her use of the base-ten blocks, it became apparent that she had developed an understanding of ten that comprised 10 ones. Although she had established a ten and 10 ones equivalence by physically exchanging a ten for 10 ones as well as by identifying the exchange in the pictorial representation, she did not demonstrate an understanding of the concept of ten and 10 ones interchangeably. Not recognizing this interchangeability became apparent through the activity with the Disruptive frame. Thus, the use of the Disruptive frame shed light on a cognitive connection that she had not yet developed. Mimi had an affinity for the conventional algorithm; her use of the base-ten blocks and the pictorial tool revealed her understanding of the algorithm.

Billy demonstrated knowledge of decomposing a number. He was able to solve the first subtraction problem that I gave him mentally. It was clear that he was aware of the concept of regrouping. In fact, he referred to regrouping when he reached a multiple of ten as he did the mental math and used the hundreds chart. He exhibited an understanding of and reliance on the algorithm throughout the interview. Through his use of the base-ten blocks, he also demonstrated an understanding of ten and 10 ones equivalence. Instead of physically trading a ten for 10 ones, he covered the cubes with his fingers. However, he identified where the regrouping of a ten into 10 ones occurred in the first pictorial representation of the process of subtraction. This response revealed his understanding of ten as being made up of ten ones, and his understanding of
regrouping in the conventional sense. This understanding may have helped him to make sense of the pictorial representations. From his experience with the Disruptive frame, it was evident that he too had not connected a ten and 10 ones interchangeably. However, the way he reconciled this disruption – by re-examining the frames and counting the cubes – pointed to his readiness to progress in his thinking about ten and subtraction.

**Troy** relied on mnemonics to help him facilitate the process of subtracting. His use of the connecting cubes in addition to the language associated with his justifications suggested that he had a construct of ten related to an abstract composite unit. He recognized the “tenness” of ten; however, ten and 10 ones were not necessarily the same for him. Essentially, Troy’s use of the base-ten blocks was limited to modeling a representation of the number. Because he had not yet constructed a ten and 10 ones equivalence, it was difficult for him to proceed further with subtraction using base-ten blocks. Troy’s reaction to the Disruptive frame pointed to his mental constructs. He was not ready to deviate from the conventional and from the steps he had memorized.

**Ivan** exhibited an understanding of 10 ones and a ten equivalence when adding with the base-ten blocks; he exchanged 10 ones for a ten. However, this understanding did not transfer to subtraction when he was using the blocks. He was not able to trade a ten for 10 ones. On the contrary, Ivan relied on his knowledge of the conventional algorithm. He was able to perform the algorithm not just on paper, but also *in his head*. The experience with the pictorial representations supported his understanding of the algorithm, but his response to the Disruptive frame shed light on his mental constructs as well as a missing connection between ten and 10 ones, which characterized his
understanding of the algorithm as being instrumental. The Disruptive frame provoked a reaction from him in relation to what he was expecting the process to be.

For Sally, the use of the tools was difficult from the onset of our interview. First she reached for the hundreds chart, which she used successfully to subtract $15 - 8$. However, she was unable to use the chart correctly for a subsequent problem $34 - 19 = 25$. When asked to use the base-ten blocks, she successfully created a representation of the numbers, but was not able to proceed with the subtraction. Unlike Ivan and Troy, Sally’s understanding of the conventional algorithm was flawed. Her experience with the pictorial representations highlighted the fact that her concept of ten was a numerical composite unit. That is, she did not differentiate the structure of ten from any other number. She did not recognize the conventional algorithm depicted in the sequence of frames. Her response to the Disruptive frame confirmed that she treated the numbers in terms of ones. She did not recognize that the deduction was from a ten. Rather, she only saw a deduction of 8. Her behavior may explain why she used the hundreds chart.

I was able to interpret students’ understanding of the concept of ten by observing the way they used the tools, and from their explanations and responses to the prompts during our interview. Based on my inferences, I identified the struggles in their thinking and the strengths of their mental constructs, whether fundamental or more complex. The depth of their understanding of the concept of ten was a pivotal characteristic when it came to discerning how well they understood double-digit subtraction.
**Question 2.** Given an interview setting with double-digit subtraction problems and a selection of tools (like manipulatives), will students opt to use the tools to help them solve the problems? What are students’ perspectives on the use of tools with double-digit subtraction? What do they prefer and why?

Excluding Deidra who reached for the base-ten blocks on her own, all of the students needed prompting to use any of the tools. The fact that I needed to ask and encourage students to use the tools suggests that they would not, of their own volition, opt to use the tools to help them solve subtraction problems. Based on my observations, students’ choice of manipulatives depended on the mental constructs of ten that they had developed. Mimi was the exception to this behavior since she reached for a different tool each time she began a new problem. I now organize my responses to question 2 according to the tools used.

**Base-ten blocks.**

Students who exhibited an understanding of the ten and 10 ones equivalence were successful in using the base-ten blocks to subtract. For example, Deidra and Billy demonstrated a more advanced understanding of the concept of ten than any of the other students. They reached for and only used the base-ten blocks. Mimi, on the other hand, who had developed a strong sense of the ten and 10 ones equivalence seemed to allow the problem to dictate her choice of manipulatives. She began with the hundreds chart for the first question \(15 - 8\); then she chose the connecting cubes for the second problem \(32 - 14\) until I asked her to use the base-ten blocks to show \(40 - 23 = 17\). Subsequently, she continued to use the base-ten blocks on her own for the pictorial representations.
**Unifix cubes.**

Troy preferred the unifix cubes to the base-ten blocks because it was easier for him to manipulate one block at a time. His concept of ten was an abstract composite unit; accordingly, he had not yet constructed a ten and 10 ones equivalence. His use of the manipulatives was basic and spoke to his comfort level with performing the algorithm. Troy used a subtraction rhyme to remember the steps of the algorithm, which suggests that his understanding of the procedure is more likely to be instrumental than relational.

**Hundreds chart.**

Sally’s first and only choice of tool was the hundreds chart. Her choice reflected her understanding of numbers. She had a fundamental understanding of ten. It was evident that she found it easier to count back spaces on the hundreds chart, although not always successfully, than to use any other tool.

**Discussion of Responses for Questions 1 and 2**

The use of manipulatives revealed missing connections and struggles in ways that students’ handwritten computations were not exposing. A student’s ability, or lack thereof, to perform a task using the base-ten blocks reflected that student’s mental networks. As observed in the case studies, the way students used the tools/manipulatives characterized aspects of their thinking that otherwise would have been difficult to identify based on their pencil and paper work alone. The use of tools gave insight into students’ thinking and their cognitive development. For example, not all of the students were cognitively ready to proceed to the next phase of thinking about the concept of ten, because they had not developed certain cognitive relationships that would make the
connections possible. Students were not always able to articulate what they did and did not understand, but through their use of manipulatives and tools, it was possible to identify some of the challenges students were facing. The tools gave students a voice that acknowledged their strengths and weaknesses and differentiated their mental constructs.

The choice of manipulatives revealed aspects of student thinking that characterized them as students with certain mental constructs. Students who seemed to have a ten and 10 ones equivalence construct, that is, there are 10 ones in a ten, reached for the base-ten blocks, despite the fact that they might not have had a one ten and 10 ones interchangeable construct yet. That is, these students were in the process of thinking about the composition of a number that would eventually lead to a concept of place-value. Students who thought of numbers in terms of ones as well as those who exhibited a construct of ten as an abstract singleton/abstract composite unit were more likely to use the unifix cubes because it was easier for them to connect and disconnect the cubes one at a time. Although they were building towers of 10, they still did not reveal behavior that would suggest they had a construct of ten as an iterable. Moreover, some students who had not yet developed certain mental structures relied on memorized rules to get through the subtraction problems, though not always successfully.

As discussed in the previous paragraph, students had to have reached a certain level of cognitive development to be able to make sense of how to use the manipulatives to help them explain and solve the subtraction problems. Excluding Mimi, the tools students chose and the tools they preferred reflected where they were developmentally in the formation of the concept of ten.
Implications for Research and Practice

Base-ten blocks were designed to embody our decimal number system and to help students understand place-value. This statement may be true for teachers, but not necessarily true for students. The research discussed in Chapter I focused on two distinct objectives. One is motivated by students’ cognitive understanding (Baroody, 1987; Cobb, 1988; Kamii, 2000) and emphasizes the need for students to construct their own cognitive network of relationships before they can begin to make sense of manipulatives. The other objective focuses on students’ performance (Fuson, 1990; Fuson & Briars, 1990; Flores, 2009, 2010) and offers approaches to teaching how to use manipulatives in the hopes that students will grasp the concept.

Tools such as T-tables, hundreds charts, and the subtraction rhyme provide crutches for some students to rely on to accomplish a task. However, their role as temporary aids does not give students an opportunity to build the network of relationships that leads to the development of conceptual structures. Even the base-ten blocks used to model subtraction, as in Fuson and Briars’ (1990) teaching technique, can lead to students’ superficial imitation as opposed to real understanding of the concepts. At best, these aids can often mislead teachers into thinking students know more than they actually do (Ross, 1989). However, as demonstrated in this study, manipulatives have the potential to be a powerful tool in assessing students’ understanding of mathematical constructs. Because my focus was not on students’ successful or unsuccessful use of the manipulatives, it became possible to consider a role for manipulatives different from that typically assigned. As evident in the case studies, the absence of understanding certain
mathematical concepts became obvious through the use of manipulatives. In this study, the tools served a different purpose. They revealed struggles in concept development that the handwritten algorithm did not depict.

Kamii’s (2000) position is that tools are meaningless unless students have built a mental representation that makes sense to them. The current research points to an interaction between students’ mental construct and their use of manipulatives. The Disruptive frame, in particular, is a significant tool that revealed missing connections. Students who demonstrated an understanding of the ten and 10 ones equivalence did not have an understanding of a ten and 10 ones interchangeability when they examined the Disruptive frame. Deidra and Billy both covered the cubes of a ten-stick with their fingers, suggesting that they were aware of the 10 ones comprised in the ten-stick. Billy even pointed out where the regrouping occurs in the pictorial frame; yet, neither student treated them simultaneously. On the other hand, Mimi exchanged a ten for 10 ones and recognized the regrouping in the pictorial representation. Not only did she not treat a ten and 10 ones interchangeably, she refused to accept that a deduction from a ten was even possible. Because the tools helped to reveal these students’ struggles with the concept, they also revealed a fragile component in the students’ mental constructs of double-digit subtraction. Overall, the tools helped to communicate students’ struggles and where they were developmentally in constructing the concept of ten.

To further discuss the potential usefulness of manipulatives for teachers in assessing students’ understanding, I offer an example of a typical scenario witnessed in many second-grade classrooms. A teacher demonstrates how to use the base-ten blocks
to solve a subtraction problem. The student exhibits confusion in not knowing exactly what to do with the blocks. The blocks are tossed aside; next, the student is given instructions on how to follow the steps of the conventional algorithm using one of the temporary aids discussed earlier in this section.

I understand that assessments depend on scores and students’ performance; students need to be able to correctly solve the subtraction problem. However, the teacher could have used this opportunity to evaluate the student’s thought process. The teacher might have been able to assess the mental constructs that the student had developed. Based on this information, the teacher could have designed questions, activities, or lessons to help the student move forward along his/her thinking trajectory. Although the manipulatives seemed useless to the student at that point, by tossing them aside, the teacher loses an opportunity to understand his/her pupil’s cognitive development.

The finding that students’ struggles with the tool can help to reveal aspects of their mental constructs reflects Kamii’s work with children by considering the contrapositive of her results. That is, Kamii states that if students’ mental constructs are intact, they will be able to use manipulatives in their learning. The current study reveals how struggles with manipulative use can help to reveal difficulties with students’ mental construct of double-digit subtraction. Because teachers can observe what students can do with the manipulatives, this implication can enable teachers to better understand their students’ struggles.

Another perspective on the above example is to consider the zone of proximal development (ZPD) or the zone of potential construction (ZPC) (as discussed in Norton &
The child’s zone of proximal development (ZPD) is defined by Vygotsky (1978) as “the distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers” (p. 86). ZPD refers to the relationship between the student’s “learning and development” (Vygotsky, 1978, p. 84). The idea is that teachers facilitate learning by guiding students with questions or probes that enables the student to eventually solve and interpret these types of problems independently.

The zone of potential construction (ZPC), as it is referred to by Norton and D’Ambrosio (2008), is defined by Steffe (1991) as the “zone of potential development of a specific mathematical concept [that] is determined by the modifications of the concept the student might make in, or as a result of, interactive communication in a mathematical learning environment” (p. 193). The focus in this theory is on the student’s cognitive structures; it “obliges the teacher to consider differences between students’ conceptions, or schemes” (Norton & D’Ambrosio, 2008, p. 222). The difference between the two as explained by Norton and D’Ambrosio (2008) is that in Vygotsky’s view “forms of knowledge exist in society first before being internalized by the individual” (p. 222). However, in Steffe’s view, “problems, concepts, knowledge and even society itself have their beginnings in the unique experiences and constructions of the individual” (p. 222). In my study, each student can be viewed as having worked in his/her zone of potential construction. Deidra and Billy both demonstrated a readiness to progress in their

1 Norton and D’Ambrosio (2008) refer to the zone of potential construction, whereas Steffe (1991) calls it the zone of potential development.
thinking about the concept of ten when they interacted with the Disruptive frame. This interaction illuminated their mental structures and triggered a thoughtful examination of the Disruptive frame that may have resulted in a change in their thinking patterns. This process also led to the development of potentially new cognitive constructs—perhaps including a construct of the interchangeability of ten and 10 ones. Although Mimi and the others were more rigid in their thinking about ten, each one of them had developed a different cognitive structure. The manipulatives enabled Mimi to use her own language to express her perspective of the conventional subtraction algorithm. She differentiated between the process and the answer. The Disruptive frame pinpointed her construct of ten, as she affirmed her understanding of the algorithm by providing an alternate solution to the Disruptive frame.

On the other hand, the base-ten blocks did not help Troy in completing the subtraction problems. However, his use of the unifix cubes indicated that his understanding of a ten was an abstract singleton. Troy exhibited good estimation skills when he reasoned that “19 is close to 20,” and, therefore, the difference of 34 – 19 would have to be less than 25.

Ivan demonstrated an understanding of 10 ones and a ten equivalence using the base-ten blocks in the context of addition, but he did not demonstrate this exchange in the context of subtraction using the base-ten blocks. He had an affinity for the conventional subtraction algorithm and demonstrated an instrumental understanding of the process. The Disruptive frame substantiated his understanding of the algorithm and highlighted his cognitive construct of ten.
Sally’s difficulties using the manipulatives helped reveal her struggles with subtraction. She had not yet developed a concept of ten other than as a numerical composite. She presented a passive outlook on learning. In fact, the difficulties that she experienced and her lack of understanding the concept of ten may be affecting her attitude towards learning.

While working in the student’s “zone of potential development” (Steffe, 1991, p. 193), whether it is supported by the teacher (Vygotsky, 1978), or whether it is internalized by the individual student (Steffe, 1991), the student has an opportunity to move forward along his/her learning trajectory. For example, Deidra and Billy worked in their zone of potential construction. That is to say, the “interactive communication in a mathematical learning environment” (Steffe, 1991, p. 193) gave them an opportunity to modify their cognitive constructs. On the contrary, Mimi, Troy, Ivan and Sally demonstrated distinct cognitive constructs. That is, interacting and communicating in a mathematical learning environment, their strengths and weaknesses became apparent. These students may not have exhibited a readiness to modify their understanding of the concept on their own (Steffe, 1991); however, with some guidance, they may be able to develop their cognitive constructs independently (Vygotsky, 1978). Therefore, through the use of speech and tools (Vygotsky, 1978), each one of the students worked in his/her zone of potential construction.

It follows from the above analysis, that manipulatives may not always be useful to students in facilitating a new understanding of mathematical concepts. However, as demonstrated in this study, manipulatives have the potential to be a powerful tool in
conveying students’ struggles, especially when students are not able to articulate and communicate their own understanding and misunderstanding of the concept.

**Limitations**

This study focused on second-grade students’ ability to communicate their understanding of double-digit subtraction. As is the case in any other study, it has its limitations. I acknowledge that students do not necessarily say what they mean or mean what they say; therefore, my inferences are based only on observed behavior and responses communicated verbally. I also recognize that my interpretation of their cognitive constructs is influenced and limited by my own knowledge. That is, “my interpretation could be made only in terms of my own knowledge” (Steffe, 1991, p. 188). The opportunity to re-interview each student and follow-up on questions and responses was not possible. Thus, it is assumed that students’ responses reflect their cognitive development at that moment. Additionally, the fact that this study is based on clinical interviews of second-grade students represents a methodological limitation that restricts the generalizability of the findings to other populations. Finally, my prior relationship with the faculty and students might have affected how I interacted with the participants, students and teachers alike. At times, during the interviews I felt like I wanted to help the students based on my prior relationship with them, but could not because of my research position. In addition, because of my relationship with the teachers, unexpected schedule changes were flexibly accommodated so that I could continue my work with the students. These accommodations may have affected the student’s subsequent dispositions (Were they tired? Was the interview cut short because of resulting time restrictions?).
Recommendations

Additional research on how manipulatives and other educational tools can assist in identifying students’ struggles is imperative if we, as a community of educators -- researchers and teachers – continue to explore students’ cognitive development and their engagement in mathematical ideas. Future research should focus on how to make the identification of students’ cognitive development accessible to classroom teachers.

This study revealed an unconventional use of manipulatives/tools: The manipulatives/tools played an important role in communicating students’ understandings and struggles in ways that the conventional assessments do not convey. The finding in this study was significant because it indicated that traditional problems and assessments could misrepresent students’ actual understanding. As demonstrated in this study, the use of manipulatives/tools as communication tools creates an opportunity for educators to identify stages in students’ cognitive development. As a result, educators have the potential to engage their students in learning at their own cognitive development level, thereby positively affecting their learning experience. By using Vygotsky’s (1978) zone of proximal development, educators can “delineate the child’s immediate future and his dynamic developmental state, allowing not only for what already has been achieved developmentally but also for what is in the course of maturing” (p. 87). Nevertheless, Steffe’s (1991) position is that, a particular modification of scheme cannot be caused by a teacher any more than nutriments can cause plants to grow. . . .Teachers are constrained in specifying what they place in a child’s zone of potential development by what the child makes from
their experiences in particular learning environments. Zones of potential
development are negotiated, then, through the interactive communication that
transpires in learning environments. (p. 190)

Each of these theories of students’ development promotes a way to support students in
progressing along a learning trajectory that is unique to each individual student.

In light of the educational policies and standards in practice today, the
significance of this research is the attention it draws to the schism between students’
cognitive development and the expectations enforced by the Common Core State
Standards (CCSS). Furthermore, current and future classroom teachers could benefit
from programs that would support them in identifying students’ cognitive development,
providing assistance with the use of manipulatives as communication tools as well as
ways to provide an optimal learning environment. These programs could include
designing professional development opportunities to enable teachers to enhance their
creative use of manipulatives in recognizing students’ level of development.

Additionally, for future research, I might be interested in replicating this study over the
course of a year: at the beginning and end of the year to follow students’ development,
and perhaps, at different grade levels.
References


Appendix A

Interview Protocol

The interview will be individual. I will introduce myself to the student and thank him/her for helping me with my homework. I will explain that my interest is learning how second grade students think about solving subtraction problems. I will tell them that there are no wrong answers because they are explaining how they think, I will assure them that they can stop at any time if they do not want to continue with the interview. I will have 100’s chart, connecting cubes and the base-ten blocks laid out within students’ reach. I will ask them if they recognize each manipulative and to demonstrate this addition problem with, $7 + 5$.

Every student will have a pencil and paper on which they can do the work.

1. As a topic opener I will start with “what is $15 - 8$?”
Whatever their answer is I follow up with, “how do you know?”

The idea is to ask them a question they will know the answer to, which will put them at ease if they are nervous. The expression $15 - 8$ is a number fact that they are expected to know. Some of the strategies they have learned so far are doubles minus one, and so, $16 - 8 = 8$ so $15$ is one less than $16$ and one less than $8$ is $7$. Or, they can think of the addition fact related to this subtraction: $8 + \text{what} = 15$.

I will ask them to show me how they know using the manipulatives.

2. I will follow up with these open ended questions (noting what they do) and ask them to explain or show me how they are getting the answer:

\[ 32 - 14 \]
If they don’t use any of the manipulatives, I will ask them to use a manipulative of their choice. If this doesn’t involve base-ten blocks, I will ask them to show me specifically using base-ten.

The open-ended questions discussed above will not be typed ahead of time. I will be writing it for them in a horizontal form on the paper in front of them. Then, I will observe how they proceed.

3. Then I will show them the following and explain to them that children in their grade, from a different school, worked on these problems. I will ask them if they can tell me which is correct or incorrect and then ask them to explain their answer. I will note whether they use manipulatives. The following three problems will be typed in a horizontal form.

- $56 - 23 = 33$
- $40 - 23 = 17$
- $34 - 19 = 25$

4. This last set requires students to identify the correct or incorrect sequence in a subtraction.
These are answers given by second grade students in another school. Can you tell me if they are right or wrong? Explain why?

\[
56 - 23 = 33 \\
40 - 23 = 17 \\
34 - 19 = 25 
\]
25 – 8
25 – 8
25 – 8
33 – 16
33 – 16
33 – 16
Appendix B

Case Studies

CS-Deidra

<table>
<thead>
<tr>
<th>Speaker</th>
<th>Talk or description of action</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>We sat on the floor in a back corner of the classroom, across from each other so that we were face to face. All the manipulatives, base 10 blocks, unifix/connecting cubes, hundred’s chart, and a pencil and paper were all on the floor in front of her and within her reach. She acknowledged that she recognized the manipulatives as she named the hundred’s chart and the cubes (referring to connecting cubes). She knew what base 10 blocks were but she said that she did not use them. I asked her “do you know what 15 – 8 is?” She responded “7”. When I asked her “How do you know?” She replied “I counted it in my head.” I asked, “How do you do that? How do you count in your head?” The following is interaction that occurred</td>
<td></td>
</tr>
<tr>
<td></td>
<td>D: I focus on what the number is?</td>
<td></td>
</tr>
<tr>
<td></td>
<td>R: which number?</td>
<td></td>
</tr>
<tr>
<td></td>
<td>D: 15 – 8 and I took away 5. So I took 5 in my pocket and I was left with 7.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Ln 24</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Ln 27</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Ln 28</td>
<td></td>
</tr>
</tbody>
</table>
R: If you took 5 away from what 15 or 8?
D: 15

R: So 15 take away 5 you’re left with what?
D: 10

R: How did you end up with 7?
She smiled, and was unsure of her answer.
R: You were right. So you took away 5 but you were
supposed to take away 8, right?
So how do you end up with 7?
D: I took away 3 more at the end.
R: three more from what? The 15 or the 10?
[she was not sure how to answer. My question confused her.
Then she answered,
D: from the 15

This report describes the events that occurred when we
discussed subtraction beginning with 32 – 14. At this point the
student seemed at ease with our session together, she even had
a chance to satisfy her curiosity and use virtual base ten blocks
(even though they are not part of this study).
When I asked her to show me how to subtract 32 – 14 (written
in a horizontal form), she reached for the base 10 blocks
| (without prompting) | Deidra | She picked 3 ten-sticks and 2 unit cubes to show 32. [When I asked her what she was doing] she said “I’m putting tens for the 30 […] 3 [ten-sticks] for the 30 and then I’m going to take two ones blocks for the two to make 32” Then to subtract 14, she explained that she took the 2 away and was now left with 30, she then took one ten-stick and explained that she has 12 and now she’s left with 20 | Interviewer | I reminded her that she’s asked to take away 14. | Ln 49 - 51 | Deidra | She continued by covering, with her finger, 2 cubes from the ten-sticks. She then declared that she has “8, and this is 10 [referring to the ten-stick] so it leaves me with 18” | Ln 59-60 | Analysis | So far, the student seemed comfortable using base 10 blocks and seemed confident with her manipulation of the blocks. It was curious for me because she stated at the beginning that she did not use them. Her ease of using the blocks steered me away from assuming that she was using them by rote because of the way she covered 2 cubes instead of breaking up the ten into ten ones, as is usually demonstrated in classrooms. The students’ manipulation of the blocks suggested that cognitively she had constructed a concept of ten that is not simultaneous with ten ones in the sense that she was ready to break the one | Ln 44 |
ten into ten ones, but that she had an understanding that this unit of 10 has ten little blocks within it. The base 10 blocks seemed to aid her with her representation of ten. It was clear that she was using the base 10 blocks to stand for tens and ones.

Interviewer

Out of curiosity I asked the student if she could do the subtraction problem on paper. I wanted to see what that meant to her, I did not specify to use the procedure, I just wanted to see what her interpretation of “do it on paper” was. The expression was written horizontally.

She looked at the paper for a few seconds and did not seem to know what to do. She struggled. I reassured her that she could rewrite it in any way she would like if she wanted to. But she giggled and looked at me. I then, said we did not need to worry about that and we moved on to another question.
<table>
<thead>
<tr>
<th>Analysis</th>
<th>This episode was very interesting to me because I did not expect the student to be uneasy with the question. However, I cannot claim that she was not able to transfer the knowledge she exhibited with the base 10 blocks onto paper, that is, to use the procedure because I do not know what her interpretation of the question was.</th>
</tr>
</thead>
<tbody>
<tr>
<td>For the next set of items, I told the student that second grade students from a different school did these problems and I was interested to see if she thought the problems were done correctly or not and if she can explain to me her reasoning.</td>
<td></td>
</tr>
<tr>
<td>Deidra</td>
<td>She asked me if she could rewrite them and she copied them on her paper in the same way they were written, in horizontal form.</td>
</tr>
<tr>
<td></td>
<td>The first question was $56 - 23 = 33$.</td>
</tr>
</tbody>
</table>
She reached for the base 10 blocks and recreated the problem; she did so for all three problems. She explained every step she made as if she was talking out her reasoning:

“I’ll check using these (taking the base 10 blocks) [I asked her why would we use the base ten blocks?] because they already have the ones together [she was referring to the ten-sticks] I'll take out five 10s and now I take six ones, so now I’ll take away two 10s cuz 23 has two 10s, and now three ones, which leaves me with 33”

She looked at the equation she had written down on the paper and put a check next to it.

Analysis

Again her comfort with using base 10 blocks was very evident and so was her understanding of the composition of a number.

The next equation in this set was 40 – 23 = 17

Deidra

She used the base 10 blocks to represent 40. She now had 4 ten-sticks. Again she spoke out loud the steps she was taking to check the solution “I’ll take away 23”
Then I noticed that she was counting the cubes on the ten-sticks and covering them with her fingers. When I asked her why she was covering them with her hand, she replied “to make sure I don’t lose track of how many I’m counting” And she continued counting “10, 11, …, 20. Now I have 20 [she was referring to two ten-sticks], I need to take three away.” She proceeded to cover with her fingers three cubes from a ten-stick and proceeded to say “that’ll leave me with 17. And that’s correct too.” She proceeded to put a check mark next to the equation she had written on the paper.

<table>
<thead>
<tr>
<th>Analysis</th>
</tr>
</thead>
</table>
| Her understanding did not seem iterative because she was counting the tens as ones: 10, 11, 12, … until she got to 20. This observation was peculiar to me. I was under the impression that she had an understanding of one ten being the same as ten ones; and so, it was surprising to see her count one cube at a time to reach 20. She did not need to do that in the previous problems. Why was she compelled to do that in this problem? Even so, when she covered the three cubes from the
ten-stick she did not need to count to know there were seven left.

<table>
<thead>
<tr>
<th>Deidra</th>
<th>“hmm, they said it was 25?” [her response indicated that she didn’t think it was right] Then, she asked to use the virtual base 10 blocks, but she had never used them before</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interviewer</td>
<td>I explained to her that it is similar to the base 10 blocks she had been using and I only needed to show her how the tools worked, how the hammer is used to break apart a ten-stick, how the glue is used to make a ten-stick out of ten ones, and how to use the eraser to erase cubes or ten-sticks.</td>
</tr>
<tr>
<td>Deidra</td>
<td>She proceeded using the virtual base ten blocks as she did the physical ones. She chose three ten-sticks and four unit cubes. She broke apart the ten-stick and started erasing the cubes one at a time while keeping track of how many she erased. I asked her how many she erased she continued counting from 6,7,…</td>
</tr>
</tbody>
</table>
I’m up to 14 but I need to break 5 more (she broke another 10 and erased 5 ones and counted what was left on the screen)...14. She miscounted and so I asked her if she was sure it was 14? She recounted and replied, “oh, 15” and stated that the answer is not correct and then said, “I knew there was something fishy about that.” She proceeded to put an X next to the equation she had written, scratched the given difference and wrote 15 underneath it.

| Analysis | It was interesting that her first response to this equation was to double-check the answer given and her last comment was “I knew there was something fishy about that”. This behavior suggested that she recognized there is something not right but she only confirmed her “gut feeling” when she used the manipulatives. It was also interesting how the student used the virtual base ten blocks with a manner of ease even though she had just learned how to use them. It is possible that she was able to transfer her knowledge of the physical blocks to the virtual because she had |

| Ln 100 | Ln 106 | Ln 106 |
developed a representation of ten she was comfortable with whether physical or pictorial.

Another observation, on the screen you cannot cover portions of base 10 blocks; and she had just learned how to break them apart, so now when she was counting she could not cover the blocks with her fingers yet she did not seem to have difficulty to shift how she counted on the screen versus how she counted the physical object.

The last item, showed the subtraction problem $25 - 8$ in terms of a sequence of illustrations that broke down how we end up with the answer.

When I explained that this problem was done using pictures and it was done in 3 different ways, I was not expecting her response

| Deidra | “Well there is one I’d say that is correct, that is the one with the x’s but these other 3 they didn’t do them right” | Ln 114 |
**Interviewer**

Her statement made me realize that she was not aware that the frames all belonged together and that they represented a step-by-step process to reach the solution. I explained to her what is happening in these illustrations. First, I began with the expression 25 – 8 so that I ensured she knew what was the process. I explained the first frame; I asked her what is the number in the first frame. When she could not answer, I pointed to the ten-stick and had her count the cubes on each ten-stick until she realized they were a ten, and then I asked her what is this number and she responded 25. Then I asked her about the second frame,

<table>
<thead>
<tr>
<th>Deidra</th>
<th>She responded “they broke it up” [she was referring to the ten-stick]</th>
<th>Ln 126</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interviewer</td>
<td>I asked her if she thought it was still 25, she looked at the frame and confirmed that it was still 25. I asked her again “so</td>
<td>Ln 129</td>
</tr>
</tbody>
</table>
they didn’t change anything?”

<table>
<thead>
<tr>
<th>Deidra</th>
<th>She confirmed again “they just broke it apart”. But then she skipped the third frame and counted the cubes in the last frame and said: “that one is wrong [she looked at the picture and counted the number of cubes left] that one is 17 it doesn’t give me 25 at all, they just crossed out more than 25”. Here she couldn’t answer the question how they crossed out more than 25 and insisted that the answer was incorrect.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analysis</td>
<td>It was curious that she did not seem to grasp what the sequence represented because the illustrations were similar to the illustrations in the books used in school.</td>
</tr>
<tr>
<td>Deidra</td>
<td>She counted the cubes in each of the ten-sticks and the loose cubes and said “that’s still 25”. But she did not approve of the</td>
</tr>
</tbody>
</table>

Ln 130

Ln 132

Ln 133

Ln 142
third frame because they crossed eight out of the ten-stick. She realized that they were left with 17 not 25, “they have 17 not 25, that is less than 25” I reminded her “but isn’t that their answer?” “because they are trying to take away 8. [she looked back at the frame and counted the crossed out parts], then she said “I could see that they crossed out 8”

<table>
<thead>
<tr>
<th>Interviewer</th>
<th>Once again she was reasoning out loud, she was not responding to my question, she was talking to herself and then she said to me “correct”.</th>
<th>Ln 149</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deidra</td>
<td>For this part, she just looked at the third frame and noticed that “they only crossed out 5 not 8, they needed to cross out 3 more” it was incorrect.</td>
<td>Ln 153-155</td>
</tr>
<tr>
<td>Analysis</td>
<td>The interactions in the last item were interesting because, at first she did not connect the illustrations with the subtraction problem. I thought it was peculiar because these illustrations should not have been new; the books used in the classroom and the student work all have similar illustrations. So what did this mean? If the representations did not reflect what the intended purpose was, to facilitate the understanding of the process of subtraction, then I could turn to the dual-representation hypothesis and say that she looked at the illustrations as pictures with their own characteristics, not as representing the process of subtraction. Therefore the student exhibited a disconnect between the tool and its purpose, which made me question the usefulness of these illustrations in helping her understanding of the subtraction process. But then, once she understood the purpose of each frame in a sequence, she was able to follow the steps. Two observations I thought were important and indicative of her cognitive development.</td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• The first is her ability to transfer the comfort and ease of using concrete base 10 blocks to the virtual base 10 blocks, and her understanding of what needs to be taken away.</td>
<td></td>
</tr>
</tbody>
</table>
• The second is her questioning of crossing out 8 from the ten-stick and not starting with the ones. Here it was interesting how she tried to understand the reasoning by talking out loud to herself and she seemed okay with it.

Is it possible that even when a student exhibits signs of understanding the concept of ten as simultaneously made up of ten ones, that understanding could be fragile and the student could go back and forth between ten as an abstract singleton and ten as an iterative? What role do the numerals presented in a problem play in challenging students’ understanding? As we saw with DR, she continually demonstrated her knowledge and comfort with base 10 blocks, but when confronted with 40 – 23, the understanding she exhibited before was not as evident. This may be more related to the concept of zero and having a zero in a place-value than it is related to the concept of ten.
CS – Mimi

<table>
<thead>
<tr>
<th>Speaker</th>
<th>Talk or description of action</th>
<th>Reference Line #</th>
</tr>
</thead>
<tbody>
<tr>
<td>The student and I sat at a table in a corner of the classroom. I made sure she recognized the manipulatives in front of her on the table.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mimi</td>
<td>She said she knew how to use the manipulatives. She referred to the base ten blocks and said “we used those for our hundreds unit”</td>
<td>Ln 6</td>
</tr>
<tr>
<td>Interviewer</td>
<td>I asked her to show me how she would solve 15 – 8. I reminded her that she could use any of the manipulatives “whatever you think you would normally do”</td>
<td>Ln 10</td>
</tr>
<tr>
<td>Mimi</td>
<td>Her first reaction is to do the procedure. She explained “I would take 15 – 8, I would subtract that” She wrote it vertically (as seen below) then she scratched the one in the tens place, put a zero above it and scratched the five in the ones place and wrote a 15 above it. She then proceeded with the vertical subtraction, she put a seven in the ones place and a 0 in the tens place.</td>
<td>Ln 12</td>
</tr>
<tr>
<td>Interviewer</td>
<td>I asked her if she had to show me the subtraction using manipulatives which one would she choose?</td>
<td>Ln 16</td>
</tr>
<tr>
<td>-------------</td>
<td>---------------------------------------------------------------------------------------------</td>
<td>-------</td>
</tr>
<tr>
<td>Mimi</td>
<td>She chose the hundreds chart</td>
<td>Ln 18</td>
</tr>
<tr>
<td></td>
<td>She explained that she “would find 15 and minus 8”</td>
<td>Ln 19</td>
</tr>
<tr>
<td></td>
<td>She counted eight spaces backward beginning with 14 as one.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>She said that the hundreds chart is the easier to use.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>However, when I asked her to show me 32 – 14 she turned to the connecting cubes (also known as unifix cubes).</td>
<td></td>
</tr>
<tr>
<td>Mimi</td>
<td>When I asked her why she chose the connecting cubes she replied, “because when I’m using with a higher number it’s easier to use these. I take 10 of them”</td>
<td>Ln 23</td>
</tr>
<tr>
<td></td>
<td>She took 10 connecting cubes and snapped them together so that she had a tower of 10. She made 3 towers. [I needed to remind her that the question is 32 – 14]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>She then said, “oh, I need 2 more. So I would take away 4 of</td>
<td>Ln 32</td>
</tr>
</tbody>
</table>
these [referring to the cubes] and then I would take away a whole 10 stick, and then I have this [referring to a tower of 10 that she made] and some ones so I would count them….8.”

She said she needed to take away four, she put aside the two cubes she had from forming 32, and she proceeded to snap off two more cubes [totaling four] from one of the towers of ten. She counted how many are left and said 18.

<table>
<thead>
<tr>
<th>Interviewer</th>
<th>I asked her what would be different if she had used the base-ten blocks? And why are the connecting cubes easier to use?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mimi</td>
<td>“Because with these, if I had the ones and didn’t have enough ones to take away part of tens, I can’t do that with these (base10) because they’re all together.”</td>
</tr>
<tr>
<td>Analysis</td>
<td>The student exhibited instrumental understanding of the subtraction algorithm when she placed a zero in the tens place when solving 15 – 8. It was interesting that her first impulse was to use the algorithm and not make connections to what she knew, like 7 + 8 = 15. This was a number fact that students were expected to know. The student was treating the question mechanically and was not relying on her prior knowledge of the relationship between the given numbers.</td>
</tr>
</tbody>
</table>
Her preference to using the connecting cubes over base-ten blocks, when she was solving $32 - 14$, led me to presume that her understanding of the concept of ten was not yet the concept of ten as iterable. Her statement, “if I had the ones and didn’t have enough ones to take away part of tens, I can’t do that with these (base10) because they’re all together” indicated that she did not think of one ten and ten ones simultaneously. It was easier for her to snap off the cubes rather than trade a ten-stick with ten little cubes.

For the next task I explained to her that the problems were done by second graders from another school and I wanted to get her opinion on whether the problems were correct or not and why.

She wrote the problems vertically aligning the ones and tens in columns, and proceeded to perform the algorithm (student’s work is shown below)

For $56 - 23 = 33$, she began with the ones’ place $6 - 3 = 3$ and moved to the tens place $5 - 2 = 3$, and she ended up with 33.

She wrote down the next problem $40 - 23 = 17$, and, in a
similar manner she aligned the ones and the tens in a vertical form scratched out the zero and moved the tens columns scratched the four and placed a three over it and moved to the ones column placed a 10 over the zero and did the subtraction. And said “yeah it’s correct”

She followed the same steps for 34 – 19 = 25 and said “no that’s not correct”

When I asked her why 34 – 19 = 15 is not correct she replied, “they probably did this x-out the 4 and not the 3, so they probably did that”
(She was showing me what she thought the students did)

| Interviewer | I mentioned to the student that I noticed she wrote the problems in vertical form on her paper and solved the problems and that she did not use manipulatives. Then I asked her if she thought it was easy to use manipulatives. |
| Mimi | She said, “sometimes it’s easy, if you write it this way (horizontally) I would use the manipulatives but if I write it this way (vertically) I don’t.” |
| Description | I asked her to show me $40 - 23 = 17$ using the manipulatives. To my surprise (because originally she preferred using the unifix cubes), she chose base-ten blocks. She picked 4 ten-sticks to show 40. Then, she put two ten-sticks on the side for the 20; and she traded one ten-stick for ten ones. |
took three of the cubes and she was left with one ten-stick and seven ones.

| Analysis | At this point, it was evident that the student used the algorithm with comfort and so, when she wrote the equations in vertical form to check whether the answer was correct it was not a surprise. However, it was interesting to see her use the algorithm to show why $34 - 19 = 25$ was incorrect. She showed where they went wrong “they probably did this x-out the 4 and not the 3, so they probably did that” but she did not explain it using language; she did not mention tens, or ones, she did not seem to use her knowledge of the relationships between numbers. I could make two observations:

1. Clearly she understood the algorithm and knew how to execute it.

2. It would seem as if she isolated her prior knowledge as if she compartmentalized her knowledge and only relied on certain aspect depending on the question. For example, when she needed to subtract, she associated it with the procedure and did not connect to other pieces of knowledge she had developed. | Ln 47 |
| **Interviewer** | She challenged my own perceptions when she used the base ten blocks to show $40 - 23 = 17$. From her previous answer, I questioned whether she connected one ten and ten ones because she said, “if I had the ones and didn’t have enough ones to take away part of tens, I can’t do that with these (base 10) because they’re all together” I was expecting her to use connecting cubes. However, her use of the base-ten blocks, and her trading the ten for ten ones contradicted what I was suspecting. This led me to think that she was not connecting all the knowledge she had. |
| **Mimi** | For this last part, I explained to the student that the illustrations showed the steps of a subtraction and I would like her to tell me if it was correct or not. |
|  | She looked at the frames, then, she took out the base-ten blocks (without prompting) and recreated the scenarios as they appeared in the sequence For $25 - 8$ | Ln 36 |
As she looked at the frames and mimicked the illustrations, she described out loud the actions she was replicating.

“What they did they got 2 of these (tens) and 5 of these (ones), then they took away one of the these (tens) and traded it for 10 of these (ones) and they took out 8 and what’s left they have is this they have 7, I have 6”

<table>
<thead>
<tr>
<th>Interviewer</th>
<th>I suggested, “do you think we miscounted?”</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>I purposely chose “we” because I did not want to make her feel uneasy or that she was being judged or tested, this made it sound like it was our project.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mimi</th>
<th>Her response was “I don’t think we miscounted”</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>When I asked her about the steps, she said the steps were correct but she did not think that their numbers were correct.</td>
</tr>
</tbody>
</table>

For part b)

25 - 8
<table>
<thead>
<tr>
<th>Mimi</th>
<th>She looked at the frames and disapproved of the third frame.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>“They have this here.” [I asked her why]</td>
</tr>
<tr>
<td></td>
<td>Her response was “because they can’t x-out, that’s a 10</td>
</tr>
<tr>
<td></td>
<td>but if they took this away and put 10 then they could x-out that.</td>
</tr>
<tr>
<td></td>
<td>[what do you think] I think their answer is correct, except for</td>
</tr>
<tr>
<td></td>
<td>that part. (3rd frame).</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Interviewer</th>
<th>For part c)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>25 – 8</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mimi</th>
<th>As she did with the other two sequences, the student replicated</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>the frames by using base-ten blocks. “I have 2 of these [tens] 5</td>
</tr>
<tr>
<td></td>
<td>of these [ones], they have the same thing here but then they x-</td>
</tr>
<tr>
<td></td>
<td>out these and they have a 3 here [referring to the ones] end up</td>
</tr>
<tr>
<td></td>
<td>with 2 tens, I don’t think they should have this here.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Analysis</th>
<th>The student surprised me again when she used the base-ten</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>blocks to replicate the frames in the sequence of subtraction.</td>
</tr>
<tr>
<td></td>
<td>Her use of the base-ten blocks led me to change my initial</td>
</tr>
</tbody>
</table>
thought, that she did not know how to use them. Her ease with using the base-ten blocks indicated that she knew what to do.

Part a) of the question showed the sequence of the algorithm. She demonstrated her knowledge of the algorithm in the previous sections, and in this section, her actions only confirmed her confidence and knowledge of the procedure.

For part b) her concept of numbers was challenged because the eight was taken from the ten. However, she seemed sure that they could not do that and offered an explanation of how this could have been solved. Once again, she used the steps of the procedure to justify why she thought the step shown in the third frame of part b) is incorrect.

It is evident, again, her concept of ten is still not simultaneous with ten ones. Even though she demonstrated that she can trade one ten with ten ones, I think she does so mechanically without merging her understanding that it is the same.

For part c) she recognized it was incorrect because the illustration showed five ones crossed out and she questioned the steps taken.

She agreed to do one last problem.

33 – 16
<table>
<thead>
<tr>
<th>Mimi</th>
<th>She studied the sequence, and recognized that frame 1 and frame 2 were the same (they both have 33). She then explained why the third frame is incorrect, “if they’re taking out a 10 it will be 23, but then the 3 here they put back … because they have these 3 here and you don’t know where it came from”</th>
<th>Ln 95</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>For 33 - 16 part b)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>She followed the steps and recognized that the second frame showed a ten traded for ten ones. The third frame there are six ones crossed out, the fourth frame has a ten crossed out.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>The sequence in part b) follows the steps of the algorithm, where students start with the ones place and then move on to the tens place.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>For 33 - 16 part c)</td>
<td></td>
</tr>
<tr>
<td>Mimi</td>
<td>She acknowledged that the representation is that of 33 but then looking at the third frame she said “They minus 10 and then minus 3 and they come up with 17 and that’s not correct”</td>
<td>Ln 112</td>
</tr>
<tr>
<td>-----------------------------</td>
<td>-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------</td>
<td>--------</td>
</tr>
<tr>
<td></td>
<td>Her objection was that they deducted three from the ten without showing the ten broken into ten ones.</td>
<td></td>
</tr>
<tr>
<td>Analysis</td>
<td>The student’s reaction in this last interaction supported my interpretations of her knowledge of the subtraction algorithm and her understanding of the concept of ten. She demonstrated numerous times an instrumental understanding of the algorithm. She did not accept any of the subtraction sequences that did not mimic the conventional algorithm. However, she was aware of the incorrect steps and was able to identify what steps should have been taken instead. The student has developed an understanding of the tools provided as representations that stood for an idea, for example, she used the base-ten blocks as ones and tens. Her understanding of the concept of ten seemed to be ten as an</td>
<td></td>
</tr>
</tbody>
</table>
abstract singleton, but I think it’s iterable.
### CS – Billy

<table>
<thead>
<tr>
<th>Speaker</th>
<th>Talk or description of action</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Interviewer</strong></td>
<td>I asked the student “if you were asked to show me $7 + 5$ what would you do?”</td>
<td>Ln 6</td>
</tr>
<tr>
<td><strong>Billy</strong></td>
<td>He said “add”</td>
<td></td>
</tr>
<tr>
<td><strong>Interviewer</strong></td>
<td>What would you do?</td>
<td></td>
</tr>
<tr>
<td><strong>Billy</strong></td>
<td>“$7 + 3$ is ten, so you regroup $+2$ is 12” (he did this in his head)</td>
<td>Ln 9</td>
</tr>
<tr>
<td><strong>Interviewer</strong></td>
<td>I wrote $15 - 8$ in horizontal form on the paper in front of him</td>
<td></td>
</tr>
<tr>
<td><strong>Billy</strong></td>
<td>“$15 - 8$? Subtract 5 from 15 then regroup then minus 3 more is 7 [okay], and also $8 + 7$ is 15”</td>
<td>Ln 16</td>
</tr>
<tr>
<td><strong>Interviewer</strong></td>
<td>“Can you show me how to do $15 - 8$ using manipulatives?”</td>
<td></td>
</tr>
<tr>
<td><strong>Billy</strong></td>
<td>“$15 - 1, 2, 3, 4, 5$, regroup, minus $1, 2, 3$, is 7 and also $8 + 7$, $1, 2, 3, 4, 5, 6, 7$ is 15” (The student chose the hundreds’ chart. He started at 15 and counted backwards 5 places - where 14 was 1- he reached 10 – which was the fifth place)”</td>
<td>Ln 19-20</td>
</tr>
<tr>
<td>Interviewer</td>
<td>“How would you solve 32 – 14 ” I wrote it on his paper in a horizontal form</td>
<td></td>
</tr>
<tr>
<td>Billy</td>
<td>He did it in his head speaking out loud “subtract 2 then regroup then, 1,2,3,4,5,6,7,8,9,10 also like this (using the 100’s chart) this is how you can do it 10, 11,12,13,14”</td>
<td>Ln 24</td>
</tr>
<tr>
<td></td>
<td>Using the 100’s chart he pointed to 32 on the chart and pointed to the number just above it (which was 22) and counted 10, 11, 12, 13, 14 landing on 18.</td>
<td></td>
</tr>
<tr>
<td>Interviewer</td>
<td>I asked him “Do you think you can show me how to do this using the base ten blocks?”</td>
<td>Ln 27</td>
</tr>
<tr>
<td>Billy</td>
<td>“That’s 30” (he took 3 ten-sticks and taking 2 little cubes. He did the same thing to make a representation of 14, he took a ten-stick and 4 little cubes as a reference) “1,2, that’s 32 minus 1 (taking a ten stick from the 32) that’s 22,that’ll be 10. then minus 1,2, these are also out (he had covered 2 cubes from the ten-stick with his finger), so 1,2,3,4,5,6,7,8, 18 (counting what was left of the 32)</td>
<td>Ln</td>
</tr>
<tr>
<td>Interviewer</td>
<td>So when you’re covering 2 of these blocks it’s like breaking it apart? And you can do that with a ten?</td>
<td>Ln 33, Ln 35</td>
</tr>
</tbody>
</table>
Billy | “Yeah” (to both questions) | Ln 34 &36
---|---|---
Analysis | The strategy and the language the student used when he added 7 + 5, he added 7 +3, said out loud regroup, and then added 2, seemed reminiscent of the procedure, he was following the steps in his head. At first I thought this meant that he was comfortable with the relationships between numbers, and was using make a ten strategy, but then he repeated “regroup” when he was subtracting, even when he was using the 100’s chart. Was he thinking about place-value or was he following steps of a procedure he memorized?
---|---|---
Interviewer | For the next set of problems, I rewrote the problems on his paper in horizontal form, I asked him to explain why he thought each problem is correct or not. The first equation was 56 – 23 = 33
---|---|---
Billy | He rewrote it in vertical form and did the procedure.
And then he put a check mark next to the one I wrote down

For the next equation $40 - 23 = 17$

(He rewrote it in vertical form, he scratched the 4 and put a 3 above it and then scratched 0 and put a 10 above it)
<table>
<thead>
<tr>
<th>Interviewer</th>
<th>What did you do here?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Billy</td>
<td>I regrouped</td>
</tr>
<tr>
<td>Interviewer</td>
<td>How?</td>
</tr>
<tr>
<td>Billy</td>
<td>When you regroup, like 1-3 you regroup. you add 10 more</td>
</tr>
<tr>
<td>Interviewer</td>
<td>Why do you have to add 10? Where did the 10 come from?</td>
</tr>
<tr>
<td>Billy</td>
<td>Because if you get a 9 you might have to regroup</td>
</tr>
<tr>
<td>Interviewer</td>
<td>Ok, but where did the 10 come from? You scratched off the 4 and put a 3 and this 0 is scratched and there is a 10 above it. What happened?</td>
</tr>
<tr>
<td>Billy</td>
<td>The 0 becomes this (pointing to the 10 above it)</td>
</tr>
<tr>
<td>Interviewer</td>
<td>But where did you get this?</td>
</tr>
<tr>
<td>Billy</td>
<td>Oh, I took it from the ones column and then the tens column you only have to subtract by 1 so the 3-2 is 1 correct.</td>
</tr>
<tr>
<td>Interviewer</td>
<td>How about 34 – 19 = 25 do you think that’s correct?</td>
</tr>
<tr>
<td>Billy</td>
<td>“The only way is to do it”</td>
</tr>
</tbody>
</table>
Interviewer | “Would you like to use manipulatives for that?” | Ln 63
--- | --- | ---
Billy | “No (he put it vertically and subtracted) wrong” | Ln 64

![Image](image.png)

Interviewer | “I noticed that you’re just using pencil and paper, do you think it’s easier to use the manipulatives or not?” | Ln 65
--- | --- | ---
Billy | “Easier” | Ln 67

Interviewer | “It is?” | Ln 68
--- | --- | ---
Billy | “If you have trouble the manipulatives will help you” | Ln 69

Interviewer | “Yeah? but you don’t need them?” | Ln 71
--- | --- | ---
Billy | “No” | Ln 71

Analysis | The student utilized the procedure to support what he was doing. First he rewrote the equations in vertical form and performed the steps necessary to get the answer. When I pressed him to explain what he did for $40 - 23 = 17$ he knew he had to regroup and he specifically said you have to add 10. | Ln 48 – 59
--- | --- | ---
When he said “I took it from the ones column and then the
tens column you only have to subtract by 1”
He was saying that he added 10 to the ones but that only
needed to subtract one from the ten, that is the digit in the
tens place becomes one less.

| Interviewer | For the next set of questions, I explained to him how the
illustrations show 25 and that each frame shows the step
of what happens before they get to the final answer, which
is the last frame |
| For 25 – 8 | |
| a. | |

| Billy | “25, they regroup, they took away a 10 (he counted to
make sure it’s the same) that’s where they regrouped
(referring to the second frame, then he counted how many
were crossed off, 8, and how many are left) 17 (then he
wrote it vertically and did the procedure to check that it is
17) correct” | Ln 76 |
| For 25 – 8 | |
He looked at the frames one at a time, but when he got to the third frame

<table>
<thead>
<tr>
<th>Billy</th>
<th>“This is wrong because they have to regroup, so they did it wrong. They took away that (referring to 8) this is right.”</th>
<th>Ln 80</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interviewer</td>
<td>“So the last frame is right but what they did is wrong?”</td>
<td>Ln 82</td>
</tr>
<tr>
<td>Student</td>
<td>“Are these split apart? (looking at the last frame and then back to the 3rd frame) this is correct.”</td>
<td>Ln 83</td>
</tr>
<tr>
<td>Interviewer</td>
<td>This is correct? so they took the 8 from a 10, and that’s okay?</td>
<td>Ln 85</td>
</tr>
<tr>
<td>Billy</td>
<td>“Right” (unsure)</td>
<td>Ln 86</td>
</tr>
<tr>
<td>Interviewer</td>
<td>We moved on to the next illustration</td>
<td></td>
</tr>
</tbody>
</table>
Billy looked at the frames and said
“Wrong, because they forgot to regroup and this is still 23”

At this point he was visibly distracted by his classmates and
did not want to continue with the interview.

Analysis
The student seemed to rely on the procedure to justify the work. What kind of mental constructs of ten does the student have? He referred to regrouping on numerous occasions during the interview, which suggested that he knew when to regroup. For instance, he was able to follow the first sequence of frames with no problem because it replicated the procedure.

However, there was evidence to suggest that he still had not developed a construct of ten as ten ones as seen in the second set of illustrations “This is wrong because they have to regroup, so they did it wrong” and yet, he continued examining the frames and said “They took away that
(referring to 8) this is right.”

The student experienced a “disruption”, in the third frame the 8 was crossed out from a ten, he was expecting a regrouping of 10. But he also acknowledged that taking 8 away was correct.

It is interesting to see how the student used the visual representations. It would seem that he was relying on his understanding of the procedure to make sense of the representations.

He was also able to recognize that the last one is incorrect because they did not cross out 8. And so, being able to identify the subtraction problem numerically and visually.
CS – Troy

<table>
<thead>
<tr>
<th>Speaker</th>
<th>Talk or Description of Action</th>
<th>Reference line #</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interviewer</td>
<td>I confirmed that the student had seen and knew how to use all the manipulatives (according to student).</td>
<td></td>
</tr>
<tr>
<td></td>
<td>“Can you show me how you do 7 + 5 using the manipulatives?”</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[He hesitated not sure which manipulatives to use.]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>“You can choose any one you want. Which one would you like to use?”</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[He reaches for the connecting cubes]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>“You prefer the connecting cubes?”</td>
<td></td>
</tr>
<tr>
<td>Troy</td>
<td>“Yeah”</td>
<td>Ln 15</td>
</tr>
<tr>
<td>Interviewer</td>
<td>“Why?”</td>
<td></td>
</tr>
<tr>
<td>Troy</td>
<td>“Because it’s easier, because you are allowed to connect them together and they don’t break apart.”</td>
<td>Ln 17</td>
</tr>
<tr>
<td></td>
<td>He reached for the connecting cubes and made 2 towers</td>
<td></td>
</tr>
<tr>
<td></td>
<td>one consisting of 5 cubes and the other of 7 cubes.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>I asked him to explain to me what he was doing</td>
<td></td>
</tr>
<tr>
<td>Troy</td>
<td>“I add them together.”</td>
<td>Ln 29</td>
</tr>
<tr>
<td></td>
<td>He proceeded to count the total number of cubes.</td>
<td></td>
</tr>
</tbody>
</table>
“Can you show me 15 -8?”
[I wrote the expression horizontally on the paper in front of him].

“On that?” [Pointing to the paper]

“Whatever you want.”

He counted 15 cubes using the connecting cubes, and then took away 8 and said, “7.”

“Is this something you would know quickly? [He nods yes] How would you figure it out without the manipulatives?”

“I would just do it in my mind, I would add the 2 digits, and make them like this, I mean I would put them in the ones column.”

To show me how he would do this problem he made a tens column and a ones column. He drew a vertical line and put a T for tens on top of the left side and an O for ones on the other side and proceeded to perform the algorithm.

He recited part of the rhyme the teacher had taught them, “More on the floor go next door”

The following is the rhyme the student is referring to
The student was checking to see if he needed to regroup.

Troy

“Then I’ll just do if there is more on the floor and do that up here,”

[He proceeded to scratch the 1 in the tens place, write a 0 above it, and scratch the 5 in the ones place and write a 15 above it and subtract]

“Then it will be 8. No, [he looked puzzled and smiled at me] that’s wrong.”
<table>
<thead>
<tr>
<th>Interviewer</th>
<th>“Why?”</th>
<th>Ln 56</th>
</tr>
</thead>
<tbody>
<tr>
<td>Troy</td>
<td>“Because 15 - 8 is not 8. I came up with this [pointing to the cubes he had used earlier, and he counted] 7.”</td>
<td></td>
</tr>
<tr>
<td>Interviewer</td>
<td>“Can you show me 32 – 14?”</td>
<td></td>
</tr>
<tr>
<td>Troy</td>
<td>“With these [pointing to manipulatives] or on paper?”</td>
<td>Ln 64</td>
</tr>
<tr>
<td>Interviewer</td>
<td>“Could you use base 10 blocks and then do it on paper if you like? [He took 3 ten sticks and 2 ones for 32, and then 1 ten stick and 4 ones for 14. He put them next to each other, then he took a 10 stick away and the 2 ones from the 32]. So, what did you take out?”</td>
<td>Ln 65</td>
</tr>
<tr>
<td>Troy</td>
<td>“One and then I have to take out another one” [he was referring to the ten-stick]</td>
<td>Ln 70</td>
</tr>
<tr>
<td>Interviewer</td>
<td>“So what is this long stick here?”</td>
<td></td>
</tr>
<tr>
<td>Troy</td>
<td>“A ten.”</td>
<td>Ln 73</td>
</tr>
<tr>
<td>Interviewer</td>
<td>“And you took this ten stick away why?”</td>
<td></td>
</tr>
<tr>
<td>Troy</td>
<td>“That’s gonna become 20 but we need to do the ones, to minus the ones, minus 4, so we have to put away another ten, and I think we need to put these [the ones] away. I don’t know I forgot.”</td>
<td>Ln 77</td>
</tr>
</tbody>
</table>
Interviewer: “That’s okay. So you put away another 10 and you think we need to put the 2 ones away. That’s a very good strategy. Do you know how to do it on paper? Can you show me how you would do it?”

Troy: [He smiled] “I’m good with this.”

[I had written the problem horizontally. Once again he drew a T-table and wrote the 32 on top and 14 on the bottom. He performed the procedure, scratched off the 3 and put a 2 above it and subtracted 1 from 2, then, he moved to the 2 ones scratch that and wrote a 12 above it and subtracted 4, his answer was 18. When I asked him why he did that]

“Because if there is more on the floor you can’t do it. Cause then it will be 2-4 and that’s not good. I mean 4 -2
<table>
<thead>
<tr>
<th>Role</th>
<th>Response</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interviewer</td>
<td>I asked what he had to do</td>
<td>Ln 95</td>
</tr>
<tr>
<td>Troy</td>
<td>“I had to go next door and get rid of and put one of the 10s on the 2 make it a 12 and turn this into a 2 [pointing to 3 tens] and minusing 1.”</td>
<td>Ln 99</td>
</tr>
<tr>
<td>Analysis</td>
<td>So far the student seemed to rely on the rhyme to remember the steps of the procedure. He used language like “put one of the 10s on the 2 make it a 12” that would suggest that he understands the idea behind regrouping, but it is also possible that he was doing and saying what he had learned by rote. He preferred using connecting cubes when pressed to use a manipulative of his choice which may suggest that he had not constructed a concept of ten as ten ones because he finds it easier to break away one cube at a time that was insinuated when he was not sure what to do with the base ten blocks to take away 2 more to make 14. The student did not convey an instrumental understanding of the procedure, he relied on a rhyme, which he memorized to get through the work without much understanding of the concept of ten or place-value.</td>
<td>Ln 99</td>
</tr>
<tr>
<td>Interviewer</td>
<td>For this set of questions, I explained to the student that second grade students from a different school worked on these problems and I wanted him to explain to me why he thought the equations were correct or not.</td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td></td>
</tr>
</tbody>
</table>
| Troy | For $56 - 23 = 33$
   “This one is correct because $5 - 2 = 3$ and $6 - 3 = 3$. So it’s 33.”
   [He did all the calculations by looking at each equation. He did not use the manipulatives, and he did not write them down.] |
| Interviewer | “What about $40 - 23 = 17$” |
| Troy | “Wrong, because 4, 3, 2 that will be a 2 [referring to the tens place] and 0 - 3 is still 3, so it’ll be 23.” |
| Interviewer | Can you show me using the manipulatives how to solve that? Do you think the manipulatives help you solve these problems?
   He did not want to use the manipulatives to re-do $40 - 23 = 17$. So I asked him to use the manipulatives to show me $34 - 19 = 25$. I asked him if he thought this was correct. |
<table>
<thead>
<tr>
<th>Troy</th>
<th>He said “no” [why?]</th>
<th>Ln 124</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>“Because it’s close to 20, there is a 9 - 4 so it’s gonna go down still.”</td>
<td></td>
</tr>
<tr>
<td>Interviewer</td>
<td>I pressed him to explain what he meant by “it’s gonna go down”</td>
<td></td>
</tr>
<tr>
<td>Troy</td>
<td>“It’s gonna go down to a lower number, lower than 25.”</td>
<td>Ln 126</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Ln 128</td>
</tr>
<tr>
<td>Interviewer</td>
<td>I asked him to show me how using the manipulatives.</td>
<td>Ln 130</td>
</tr>
<tr>
<td></td>
<td>He took 3 ten sticks and 4 ones to show 34 and placed on the side a 1 ten and 9 ones. (Below, is a reproduction of the student’s work)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>He then took away a ten stick from the 34.</td>
<td></td>
</tr>
<tr>
<td>Troy</td>
<td>“I’m going thru this again. Put away another 10.”</td>
<td>Ln 133</td>
</tr>
<tr>
<td>Interviewer</td>
<td>“Why do you put away another 10?”</td>
<td></td>
</tr>
<tr>
<td>Troy</td>
<td>“Because then it’ll turn into 25 and the other 10 will go away too because of the 9.”</td>
<td>Ln 135</td>
</tr>
<tr>
<td></td>
<td>[and so what do you think the answer is going to be?]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>“Ummmm”</td>
<td>Ln 137</td>
</tr>
</tbody>
</table>
[He seemed confused because the 9 ones belonged to the 19 and he knew he could not use them, but there were only 4 ones that belonged to the 34]

“Yeah I think I need more of these” [referring to ones, but he did not say anything about regrouping. He was silent and stared at the blocks in front of him]

<table>
<thead>
<tr>
<th>Interviewer</th>
<th>“Do you think using these base 10 blocks help you to solve these problems?”</th>
<th>Ln 141</th>
</tr>
</thead>
<tbody>
<tr>
<td>Troy</td>
<td>“No I don’t think these are easier, I think these [connecting cubes] are easier.”</td>
<td>Ln 143</td>
</tr>
<tr>
<td>Interviewer</td>
<td>“What would you do if you had to use the connecting cubes?”</td>
<td></td>
</tr>
</tbody>
</table>

He counted the connecting cubes made 3 towers of ten

“Ten and then 4, [now he had 34], get 19 [he took one of the towers of ten and counted 9 cubes from another 10] gone. [Counting the left over] 15.”

<table>
<thead>
<tr>
<th>Troy</th>
<th></th>
<th>Ln 145-147</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analysis</td>
<td>It would appear that the student was using the procedure to help him figure out how to use the manipulatives. He made a representation of 34 and one of 19, and although he did</td>
<td></td>
</tr>
</tbody>
</table>
not combine the two numbers he still was unsure of how to show subtraction using the base-ten blocks. His preference of connecting cubes would suggest that his concept of the ten had not evolved to ten as a iterative number because he would rather break apart one cube at a time. It was interesting, however, that he analyzed why the answer to 34 – 19 = 25 is incorrect.

<table>
<thead>
<tr>
<th>For 25 – 8 part a)</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Diagram" /></td>
</tr>
</tbody>
</table>

**Troy**

“That is wrong because they don’t have this [ten stick] anymore [he counts to make sure he started with 25] no, wait that is correct.”

**Analysis**

He did not recognize at first glance that the ten was broken into 10 ones until he counted 25, this sequence depicts the steps of the algorithm, it was no surprise that he was able to follow what was happening.

<p>| 25 – 8 part b) |</p>
<table>
<thead>
<tr>
<th>Troy</th>
<th>“No because minusing you need to minus that whole 10.”</th>
<th>Ln 163</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>I asked “you need to take away a whole 10, you can’t just take it from here?” pointing to a 10-stick, he did not look at the final frame to check the final result and he did not answer. He moved on to the next illustration</td>
<td></td>
</tr>
<tr>
<td>25 – 8 part c)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Troy</td>
<td>“No, there is too many.”</td>
<td>Ln 168</td>
</tr>
<tr>
<td></td>
<td>[Too many what? I asked]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>“Ones, not enough were taken away.”</td>
<td>Ln 170</td>
</tr>
<tr>
<td>Analysis</td>
<td>The student was trying to reconcile the pictorial representation with his understanding of what steps should be taken. It would appear that he was relying on the procedure to try</td>
<td></td>
</tr>
</tbody>
</table>
and justify the action in the frame. In the first sequence, he double-checked that there were 25 and 8 were crossed out.

In the second frame, he dismissed it as incorrect because the 8 was crossed out of a ten. He would not even explore the idea. And in the last frame, he recognized that not enough ones were crossed out. This is interesting, he was able to catch the mistake when looking at the pictorial representation, however, he made a similar mistake when he looked at $40 - 23 = 17$, and when he was checking $34 - 19 = 25$, he said $9 - 4$.

<table>
<thead>
<tr>
<th>For part a) $33 - 16$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Troy</td>
</tr>
<tr>
<td>Interviewer</td>
</tr>
<tr>
<td>Troy</td>
</tr>
</tbody>
</table>

<p>| For part b) $33 - 16$ |</p>
<table>
<thead>
<tr>
<th>Troy</th>
<th>“Too many ones, they need to subtract more.”</th>
<th>Ln 176</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>He did not recognize that this was the procedure, where a ten is broken into 10 ones, 6 are crossed out and a ten is crossed out.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>For part c) $33 - 16$</td>
<td></td>
</tr>
<tr>
<td>Troy</td>
<td>He looked at the sequence and said “right”</td>
<td>Ln 180</td>
</tr>
<tr>
<td>Interviewer</td>
<td>“How do you know?”</td>
<td></td>
</tr>
<tr>
<td>Troy</td>
<td>“Actually I don’t know if it’s right, let me see, [he counted the little squares] is that a hundred?”</td>
<td>Ln 182</td>
</tr>
<tr>
<td>Interviewer</td>
<td>“Count them and see.”</td>
<td></td>
</tr>
<tr>
<td>Troy</td>
<td>“Yeah ten. This is right because the 6, no this is wrong because they are getting rid of too many. It’s only ten they have to get rid of.”</td>
<td>Ln 185</td>
</tr>
</tbody>
</table>
**Analysis**

In the pictorial representation in part a) he recognized that there were too many ones and that they needed to take away more ones. Once again, the illustration helped him recognize one of the common mistakes students make, and one he made himself but did not realize that he did, and that is to subtract a smaller digit from a larger one.

For part b) it would seem that he looked at 16 as 16 ones not a ten and six ones which would explain his remark about not taking out enough ones. I can infer that his concept of ten had not developed to include one ten and six ones to be the same as 16 ones.

For part c) it was interesting that at first glance he said it was right but then admitted he did not know and needed to check. This led me to think that he was so intent on the procedure as he memorized it that any action that deviated from that he dismissed and would not consider. He was not yet open to a disruption in the process.

In this last interaction he might have been tired and was not as interested in continuing with the interview.
The teacher in this class had given me her desk area to conduct the interviews. The student was sitting in the teacher’s chair at her desk and I was sitting in a chair next to him. The manipulatives were all on her desk in front of him.

I made sure the student recognized the manipulatives in front of him.

He referred to the base ten blocks as powers of ten (this could be because his teacher would refer to them as powers of ten sometimes)

I asked him what $7 + 5$ is. After he immediately responded 12, I asked him to show me how he got his answer using the manipulatives.

He chose the base ten blocks. He got seven cubes and five cubes and counted them, 12, then he counted ten and traded it for a ten-stick
Analysis:
I found it peculiar that he needed to count ten cubes to trade for a ten-stick instead of putting 2 aside.

<table>
<thead>
<tr>
<th>Description</th>
<th>When I asked him to subtract 15 – 8, he came up with the answer rather quickly and said 7. I asked him how he got the answer</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ivan</td>
<td>He tried to explain what he did “because if you have 15 you can just subtract 3 because 5 + 3 = 8” [what he was doing in his head is 15 – 5 = 10 and 10 – 3 = 7, but he had difficulty articulating his thoughts]</td>
<td>Ln 24, Ln 26, Ln 29</td>
</tr>
</tbody>
</table>

Analysis: His reasoning indicated that he had an understanding of the composition of a number, that is the relationship between numbers, but it was evident that it was not an easy task to verbally explain his reasoning.

Interviewer: I asked him to show me 32 – 14 (I wrote it horizontally on his paper) he quickly said 13. But when I asked him how he knew it was 13

He said,
I pointed out to him that he could use the paper and pencil or any of the manipulatives in front of him, however, I did suggest he use manipulatives.

He chose base ten blocks, and he was able to show 32 and 14. He put three ten-sticks and two ones for the 32 in front of him on the left side; and he put one ten-stick and four ones for the 14 on the right side - such as in the reproduction below:

He then took away one ten-stick from the 32 and one ten-stick from the 14 and put them aside. He looked unsure with what he was doing and said “wait, I think I need to start over”. He put back the tens that he took away – one on the left to make the 32 and one on the right to make the 14.
He looked at the representations again, but he seemed to struggle and did not know how to proceed.

He stared at the blocks and looked like he was trying really hard to figure out how and what to do with the blocks he said “I think I want to use my pencil”

He proceeded to write the problem vertically on his paper and did the procedure easily, that is, he scratched the 2 and then scratched the 3, he wrote a 2 above the 3, then he moved to the right and wrote a 12 above the 2. He subtracted 12 – 8 and 2 – 1 came up with 18. Below is the student’s actual work.

<table>
<thead>
<tr>
<th>Interviewer</th>
</tr>
</thead>
<tbody>
<tr>
<td>I asked him if he thought it was easier to do the work on the paper, he did not give me an answer; but when I asked him if he thought the manipulatives help him</td>
</tr>
</tbody>
</table>

Below is the student’s actual work.
<table>
<thead>
<tr>
<th>Ivan</th>
<th>He said “it helps me a little bit” “like … when I add”, “it’s harder when you’re subtracting.”</th>
<th>Ln 53, 55, 57</th>
</tr>
</thead>
</table>

**Analysis**
The student seemed comfortable with numbers, he added and subtracted mentally with no problem. His explanation of how he solved $15 - 8$ indicated that he had an understanding of the composition of a number. This was also apparent when he answered $13$ for $32 - 14$ but he knew that the answer did not seem right because he said he needed to check. His reactions to and treatments of numbers suggested that he had number sense. The difficulty the student had using the base ten blocks suggested that he did not use them regularly, but more importantly, it would seem that the student’s representation of the construct of ten did not include the use of base ten blocks because they hindered his understanding of how to proceed when subtracting with the blocks; and so, the base ten blocks did not stand for an idea, for this student, the base ten blocks were just blocks. Yet, he was able to perform the procedure easily. This event led me to the following observations:

1. The student exhibited relational understanding of the
composition of a number, more so when working with relatively small numbers.

2. When subtracting, the student’s behavior seemed to suggest that his understanding was more instrumental. He showed that he had a good understanding of how the procedure worked.

3. The two separate observations would suggest that the student floats between cognitive developments. The transition between understanding in one area and uncertainty in another is almost visible. I could see the student think and try to make sense of the disruption he felt in his understanding.

Interviewer: For the next task, I explained that second graders from a different school did the work, and I would like him to tell me if they did it correctly or not and to explain to me why he thought so.

The student did not use manipulatives, and I did not push him to use the base ten blocks because it was evident that using the manipulatives obstructed his work.
Ivan | He looked at the first problem $56 - 23 = 33$, turned to his paper and wrote it vertically and performed the procedure, as seen below.

He said it was correct “because $6 - 3 = 3$ and $5 - 2 = 3$”

| Ivan | For $40 - 23 = 17$ he immediately said this was not correct “because it’s 23”

When I repeated “because it’s 23,”

He said “wait, let me check.” He rewrote it on his paper in vertical form and performed the procedure again, and said, “yeah, it’s right”.

| Ln 69 | Ln 71 | Ln 72 |
Ivan

For $34 - 19 = 25$, he shook his head meaning no it’s not correct. When I asked him why he thought it was incorrect, he said “because 3 becomes a 2 and 2-1=1, 4 becomes 14 and 14 - 9 =5, then it’s 15”.

And then he did the procedure to confirm, as shown below

![Image of a calculation]

Analysis

In the previous interaction with the student I was not surprised that he did not turn to manipulatives not even once. From his behavior it was evident that the mental images of numerals he constructed did not include base ten blocks.

Another observation was his understanding of how the procedure works. He was able to perform the procedure mentally and on paper without any difficulty. When he subtracted $40 - 23$. He knew that the 0 became a 10 when he regrouped. He did not make the common mistake of subtracting 3 – 0. However, his reference to “3 becomes a 2,” for example, would suggest that he was treating the numerals at
face value rather than place value.

We were interrupted because of a classmate’s birthday and the class went outside to the playground to have ice cream. And so, this concluded our interview for the day. I did come back the next day and he was willing to finish the interview.

Interviewer

For 25 – 8, I showed him the sequence of frames and needed to prompt him with questions because it seemed as if he did not understand what the sequence of illustrations represented. I asked him what they did in the second frame.

a. 25 - 8

Ivan

He explained that they broke the tens into ones and subtracted 8 (he was referring to the third frame)
<table>
<thead>
<tr>
<th>Interviewer</th>
<th>For part b) I had to explain that it was the same problem but done in a different way. 25 – 8</th>
</tr>
</thead>
</table>
| Description | He studied the frames carefully, and whispered to himself “25,” then said “no”. He did not think it was correct because in the third frame they subtracted the subtrahend from the ten, they did not break the ten into ones. “They forgot to take away from the ones and instead of using the ones they used the 10-stick”. When I asked him “so they can’t just take away from the ten-stick?” He replied:  
“They use the ones and then they take away the less”. (By this statement he meant that they start with the ones and if | Ln 91 | Ln 95 | Ln 97 | Ln 100 |
they don’t have enough, they take away from the ten that was broken into ones)

Interviewer
When I asked him if the sequence is not correct

Ivan
He studied the last frame and said “it is correct but they forgot to do it, to take away the ones”

Interviewer
When I asked him about the next sequence

c. 25 - 8

Ivan
He looked at each frame carefully, and then said it was not correct “because they didn’t take away 8, they only took 5 away so then they took everything else away but then [as he examined the last frame]… wait, the answer is also incorrect; the third frame is wrong because they subtracted 5 not 8”
<table>
<thead>
<tr>
<th>Interviewer</th>
<th>I asked him, if it was still possible to get the correct answer,</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ivan</td>
<td>He replied, “no, because they have 2 ten sticks. they had 2 10s</td>
</tr>
<tr>
<td></td>
<td>here (pointing to the 2nd frame of part c) but then they</td>
</tr>
<tr>
<td></td>
<td>separated all the 10s from the tens (he meant they broke up the</td>
</tr>
<tr>
<td></td>
<td>10 into 10 ones) and they forgot to take away the 10 stick (he</td>
</tr>
<tr>
<td></td>
<td>was referring to the last frame where there was 2 ten-sticks)</td>
</tr>
<tr>
<td></td>
<td>and they also didn’t show what they did over here why there</td>
</tr>
<tr>
<td></td>
<td>are 3 left” [again here, he is referring to the last frame]</td>
</tr>
<tr>
<td>Analysis</td>
<td>The student seemed to follow the sequence of part a) once I</td>
</tr>
<tr>
<td></td>
<td>explained it to him. Part a) was replicating the procedure, and</td>
</tr>
<tr>
<td></td>
<td>so, he seemed comfortable with the actions in the illustration.</td>
</tr>
<tr>
<td></td>
<td>The third frame shows that the deduction of 8 began with the</td>
</tr>
<tr>
<td></td>
<td>ones and then moved to the ones that once belonged to the ten-</td>
</tr>
<tr>
<td></td>
<td>stick. The importance of which “ones” to start with became</td>
</tr>
<tr>
<td></td>
<td>clear when he objected to the sequence of part b). The</td>
</tr>
<tr>
<td></td>
<td>sequence in part b) was not a typical procedure it speaks to how</td>
</tr>
<tr>
<td></td>
<td>well a student understands the make-up of a number and of ten</td>
</tr>
<tr>
<td></td>
<td>in particular. The student’s objection to subtracting from the</td>
</tr>
<tr>
<td></td>
<td>ten indicated his knowledge of the procedure; it also challenged</td>
</tr>
<tr>
<td></td>
<td>his knowledge of numbers and of how to subtract.</td>
</tr>
</tbody>
</table>
For part c) the student exhibited his understanding of subtraction and the procedure.

| Description | I showed him the sequence of the next problem $33 - 16$
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Ivan</td>
<td>He shook his head to mean no, and said that “they only subtract 13 but they didn’t subtract 16.”</td>
</tr>
<tr>
<td></td>
<td>For part b) the sequence replicates the procedure where the ones are subtracted first and then the tens.</td>
</tr>
</tbody>
</table>
|             | b.  $33 - 16$
<p>|             | He said it was correct “because they have 6 ones over here [referring to the third frame] and they crossed out a ten [referring to the fourth frame]. |</p>
<table>
<thead>
<tr>
<th>Interviewer</th>
<th>When I asked him “where did they get 6 ones, they started off with 3 tens and 3 ones”</th>
<th>Ln 129</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ivan</td>
<td>He said, “they deleted one of the tens and turned it into ones and then they subtracted 6 and then they crossed out a ten.”</td>
<td>Ln 130</td>
</tr>
<tr>
<td>Description</td>
<td>This next sequence did not follow the typical steps of the procedure. The third frame shows the subtraction without breaking a ten-stick into ten ones.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>c. 33 - 16</td>
<td></td>
</tr>
<tr>
<td>Ivan</td>
<td>He studied the frames carefully. Then he said it was incorrect. When I asked him why he thought it was not correct he said “because they just took out 6 and they deleted a 10 but they’re</td>
<td>Ln 137</td>
</tr>
<tr>
<td>Analysis</td>
<td>This last interaction with the student led me to observe that the student did not have any difficulty recognizing when the illustration showed a deduction less than or different from the subtrahend. This would suggest that he understood that subtraction was to take away or cross out a certain number, in the context presented to him. Although he seemed to know his numbers as discussed in an earlier analysis, I was inclined to believe that he had not developed a concept of ten that is simultaneous with ten ones. His disapproval of “taking away from the tens” suggested that although he would say they broke one ten into ten ones, he still treated them differently. Comments like “they deleted a 10 but they’re supposed to use the rest” suggest that his understanding of ten is not iterable (as Cobb would explain). His understanding of the procedure process was evident in his explanations and justifications</td>
<td></td>
</tr>
</tbody>
</table>

supposed to use the rest” | Ln 137 |
<table>
<thead>
<tr>
<th>Speaker</th>
<th>Talk or Description of action</th>
<th>Reference line</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interviewer</td>
<td>I asked the student if she recognized the manipulatives in front of her, she responded “No”</td>
<td>Ln 5</td>
</tr>
<tr>
<td></td>
<td>I rephrased my question and named each tool in front of her “you’ve never seen base ten blocks? Hundred’s chart? Connecting cubes?”</td>
<td></td>
</tr>
<tr>
<td>Sally</td>
<td>The student then responded “Oh yeah, I have.” “We only use them when we’re doing math in front of the smart board.”</td>
<td>Ln 7</td>
</tr>
<tr>
<td>Analysis</td>
<td>This interaction was interesting, made me rethink the question. Did she not understand ‘recognize’? Why would she say ‘no’?</td>
<td></td>
</tr>
<tr>
<td>Interviewer</td>
<td>I asked her to show me how to solve 15 – 8</td>
<td></td>
</tr>
<tr>
<td>Sally</td>
<td>She used the hundred’s chart located 15 and counted backwards eight spaces beginning with 14, and said “7.”</td>
<td></td>
</tr>
<tr>
<td>Interviewer</td>
<td>I asked “can you show me how to do 32 – 14 using manipulatives?” I wrote the expression on her paper in a horizontal format.</td>
<td>Ln 15</td>
</tr>
</tbody>
</table>
| Sally          | She replied “I know how to do it another way, it’s easier for me.”  
|               | She rewrote the expression in vertical form. She explained “first you can’t take away 2 from 4, so I need to make it a 12 so I took one from the 3 and it becomes a 2, it’s 18.”  
|               | Student’s work is below  
|               | ![Image of student's work](image)  
| Interviewer   | I asked her to show me how to subtract 32 – 14 using the base-ten blocks.  
| Sally         | She said “this is 32, 3 tens usually called rods, and 2 cubes.”  
<p>|               | She was able to show a representation of 32 using the base ten blocks but when it came to doing the subtraction she did not know how to use them. She just looked at the manipulatives and did not know what else to do. She was stuck |</p>
<table>
<thead>
<tr>
<th>Analysis</th>
<th>I suspected the student was not comfortable using the base-ten blocks. Her initial response to the question of showing me how to use base-ten blocks to solve 32 - 4 was “I know how to do it another way, it’s easier for me” her statement suggested that base-ten blocks cause a difficulty for her. As she explained how she was subtracting (using the algorithm in a vertical form) she said “can’t take away 2 from 4,” I noticed the language she used; she did not say ‘cannot take 4 from 2’ I suspected that she was using the algorithm in a mechanical way, steps to follow. Her statements did not indicate that she had an understanding of place-value or that she was aware of what she was saying.</th>
<th>Ln 17 Ln 20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interviewer</td>
<td>We moved on to the next set of questions. I asked To show me 56 – 23 = 33. I wrote on her paper in horizontal form.</td>
<td></td>
</tr>
<tr>
<td>Sally</td>
<td>She rewrote it vertically, aligned the numbers, and did the subtraction following the procedure. She whispered 6 take away 3 is 3 and 5 take away 2 is 3</td>
<td>Ln 30</td>
</tr>
</tbody>
</table>
and said, “it’s correct”

Sally

When I asked her about $40 - 23 = 17$, she did the same thing as before rewrote the equation in vertical form and subtracted, as seen below and said “it’s incorrect."

She explained “I know you can’t take 0 away from 3 so it had to be 3, this is 7, and 4 take away 2 is 2 so it’s incorrect.”

Sally

She followed the same steps with the last equation in
this set, but she crossed out the 3 made it 2 and crossed out the 4 and put a 3 above it.

But she was stuck, she subtracted $2 - 1$ and she did not know what to do with the 4 that is now a 3. She reached for the 100’s chart and counted backwards, she mumbles 13 lands on 21) “21, so that’s wrong.” I reminded her “it’s minus 19 not 13.”

S: “I did 19.”

She did not want to engage in answering questions related to this problem. We moved on to the next set.

<table>
<thead>
<tr>
<th>Analysis</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>The student did not exhibit an understanding of place-value. It appeared she was executing a rule, although she was incorrectly using it; for instance, in $40 - 23 = 17$, she said “I know you can’t take 0 away from 3” I suspect she meant to say ‘we cannot take 3</td>
<td></td>
</tr>
</tbody>
</table>
away from 0,’ I presume she was incorrectly using the rule ‘can’t take a bigger number from a smaller number.’ She clearly did not understand how the rule works and why because she ended up subtracting 3 – 0.

Her behavior made me question her understanding of the concept of ten.

Her treatment of $34 - 19 = 25$ seemed to support the idea that she did not understand how the rule of regrouping works or why.

<table>
<thead>
<tr>
<th>Interviewer</th>
<th>For the next set of questions I had to explain to her that the frames are a sequence that explains the steps, $25 - 8$ a.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sally</td>
<td>“It got bigger.”</td>
</tr>
</tbody>
</table>
| Interviewer       | “How do you think it got bigger?”                                                                                      | Ln 49
<table>
<thead>
<tr>
<th>Interviewer</th>
<th>“How do you think they added more? More of what?”</th>
<th>Ln 54</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sally</td>
<td>“Because they wanted to add to make a bigger number maybe.”</td>
<td>Ln 55</td>
</tr>
<tr>
<td>Interviewer</td>
<td>“Where do you think they got the cubes and put them there or did they get them from somewhere specific?”</td>
<td></td>
</tr>
<tr>
<td>Sally</td>
<td>“Specific. Like the problem.”</td>
<td>Ln 58</td>
</tr>
<tr>
<td></td>
<td>[when I asked her if she thinks what they did is correct] she said “yes”</td>
<td>Ln 64</td>
</tr>
</tbody>
</table>

For part b) 25 – 8

<p>| Sally       | “This is correct (she pointed to the 3rd frame) they” | Ln 68 |</p>
<table>
<thead>
<tr>
<th>Sally</th>
<th>She looked at the sequence and said “I don’t know”</th>
<th>Ln 75</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analysis</td>
<td>It was surprising that the student did not seem to understand what the frames represent. In part a) she said, “the number got bigger” she was not able to articulate a clear explanation of the second frame. She could not even give a reason why. For part b) she realized that they took away 8.</td>
<td></td>
</tr>
</tbody>
</table>