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# Candy Sharing and Chip Firing Games on Graphs

Joseph DeGaetani  
*Montclair State University*

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# ABSTRACT

The Candy Game begins with a finite number of players sitting in a circle, each with an initial amount of candy. At each time step, each player passes half of their pile to the player on their left (with odd sized stacks receiving an extra piece of candy). The original question was whether every initial distribution of candy results in every player holding the same number of pieces after a finite number of turns. For arbitrary initial distributions, we prove asymptotically tight bounds on the final amount of candy. The diffusion chip firing game assigns integral chip amounts to each vertex of a graph. At each time step, a vertex sends a chip to each neighbor who has less chips than itself. We show that this game on the infinite path, with bounded chip labels remains bounded for all time.

MONTCLAIR STATE UNIVERSITY

Candy Sharing and Chip Firing Games

on Graphs

by

Joseph DeGaetani

A Master's Thesis Submitted to the Faculty of

Montclair State University

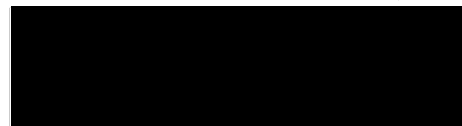
In Partial Fulfillment of the Requirements

For the Degree of

Master's in Mathematics

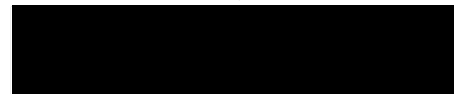
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College of Science and Mathematics



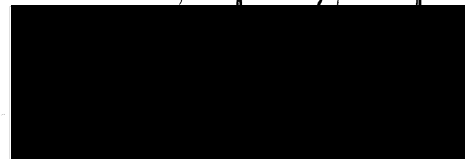
Department of Mathematics

Dr. Deepak Bal, Thesis Advisor



Thesis Committee:

Dr. Aihua Li, Committee Member



Dr. Jonathan Cutler, Committee Member

# CANDY SHARING AND CHIP FIRING GAMES ON GRAPHS

A THESIS

Submitted in partial fulfillment of the requirements  
for the degree of Master's in Mathematics

by

Joseph DeGaetani

Montclair State University

Montclair, NJ

May 2018

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# 1 Introduction

The 1962 Beijing Mathematical Olympiad posed the following question: A number of students sit in a circle while their teacher gives them candy. Each student initially has an even number of candies. When the teacher blows a whistle, each student simultaneously gives half of his or her own candy to the neighbor on the left. Any student who ends up with an odd number of pieces of candy gets one more piece from the teacher. Show that no matter how many pieces of candy each student has at the beginning, after a finite number of iterations of this transformation all students have the same number of pieces of candy. Initially posed as a challenge problem in the mathematical community, this problem has now transitioned into an entertaining game to play with students of all ages to investigate math in a way that doesn't feel like math.

Despite the fact that the game is known to terminate for any initial distribution, it is still open as to the final amount of candy held by each student when the game ends (or equivalently, the number of pieces drawn in a game), as well as the length of the game. The first part of this thesis will address these questions. In Section 2, we give some preliminary definitions and lemmas and in Section 3 we closely analyze the 3 player game. Intuitively, in a game initialized with  $n$  total pieces of candy and  $k$  players, the amount that each player ends the game with should be approximately  $\frac{n}{k}$  plus some amount negligible in comparison to  $n$ . Numerical experiments confirm this intuition and indeed, in Section 4, we prove this to be the case using tools from the theory of Markov Chains.

An adjacent field of study is the topic of chip firing games. Since their introduction in the 1980s, chip firing games on graphs have received much attention. In the general setting, an initial amount of chips are placed on the vertices of a graph. A firing rule determines how the chips are distributed moving to the next round. Typically the rule is if the label of a vertex matches or exceeds its degree, that vertex passes a chip to each of its neighbors. In 2015 Duffy et al. [1] analyzed this game under the rule that every vertex will pass a chip to each neighbor which

currently has a lesser amount. This team conjectured that this variation of the game, termed the diffusion game, played on finite graphs, will always result in a stable or periodic configuration. Long and Narayanan [3] proved a strong version of this conjecture, namely that any initial configuration will result in a constant labeling or an oscillation of period 2. In their paper, titled “Diffusion on Graphs is Eventually Periodic”, Long and Narayanan questioned whether anything could be said of the diffusion game played on infinite graphs of bounded degree. In particular, they conjectured that if the initial labeling is bounded, then the vertex labeling will remain bounded throughout the game. In Section 5, we will address this conjecture in the case of maximum degree 2.

## 2 Initial Analysis

As a warm-up, we first prove the result of the original Olympiad problem, namely the candy sharing game terminates in a finite number of steps.

Given that the game is played with a finite number of players (and thus a finite number of different candy amounts), there exists at least one player each who has the maximum amount of candy ( $M$ ) and the minimum ( $m$ ). Multiple players may have these amounts, but we know that at least one will. Each turn, either the number of players holding  $M$  pieces will remain unchanged (which would happen if every one of these players is seated to the left of a player who also holds  $M$  or  $M-2$  pieces), or there will be less players holding the maximum amount. Similarly, every turn the number of players holding  $m$  pieces *will* decrease, as at least one player with  $m$  pieces is sitting to the left of a player with more if all players don't have equal amounts. Within  $k$  turns we are then guaranteed that the minimum number of pieces a player holds has increased. Since  $M$  either never increases, or decreases and the minimum will perpetually increase, in a finite number of steps the minimum will match the maximum leading to the end of the game.

Notice this proof implies naively that the number of turns is bounded by  $k(M-$



$m$ ), a poor bound for large  $M - m$ .

Let  $\mathbf{d} = [d_1, d_2, \dots, d_k] \in (2\mathbb{Z})^k$  represent an initial distribution of candy among the  $k$  players in the candy sharing game, i.e. player  $i$  is given  $d_i$  pieces initially. We say that a game *stabilizes* when every player has the same amount of candy,  $s$ . In this case, we say the game stabilizes at  $s$ . Furthermore, we refer to a piece of candy being inserted into the game as a “draw”.

**Definition 1.** Let  $D(\mathbf{d})$  be the number of draws in a candy game with initial distribution  $\mathbf{d}$ . Let  $D(n, k) = \max \left\{ D(\mathbf{d}) : \mathbf{d} \in (2\mathbb{Z})^k \text{ and } \sum_{i=1}^k d_i = n \right\}$ .

**Definition 2.** Let  $T(\mathbf{d})$  be the number of turns before stabilization in a candy game with initial distribution  $\mathbf{d}$ . Let  $T(n, k) = \max \left\{ T(\mathbf{d}) : \mathbf{d} \in (2\mathbb{Z})^k \text{ and } \sum_{i=1}^k d_i = n \right\}$ .

**Lemma 1.** Let  $\mathbf{d} = [d_1, d_2, \dots, d_k]$  be the initial distribution of candy, and  $a \in 2\mathbb{Z}$  and let the candy sharing game stabilize at  $s$ . Then, the candy sharing game played with  $\tilde{\mathbf{d}} = [d_1 + a, d_2 + a, \dots, d_k + a]$  stabilizes at  $s + a$ .

In the context of the game, this lemma asserts that giving each player the same additional amount of candy (or removing the same amount from every player) leaves the structure of the game unchanged. We proceed with the convention that a player having a negative candy amount is interpreted as them being in a “candy-debt”.

*Proof of Lemma 1.* From the definition of stabilization, a game in which every player starts with the same amount of candy is already stabilized, and that amount will never change. Suppose we are given initial distribution  $\mathbf{d} = [d_1, d_2, \dots, d_k]$  which stabilizes to amount  $s$ , and an additional amount  $a \in 2\mathbb{Z}$ . Instead of visualizing the  $i^{\text{th}}$  player’s candy pile as a stack of size  $d_i + a$ , imagine that each player divides his or her pile into two: an inner pile of size  $a$  and an outer pile of size  $d_i$ . Each turn the players will exchange the inner piles according to the rules of the game, and the outer piles. Since the inner pile is the same amount for every player, that game has already stabilized and the common amount will never change. All that remains is the outer piles of size  $d_i$ , which is simply the

original candy game. This stabilizes to  $s$  by assumption, leading every player to hold exactly  $s + a$  pieces.  $\square$

Often games played with  $n$  players will be modeled on the  $n$ -cycle graphs. Turns are indicated by arrows while removing (or adding) candy, termed “stripping” is denoted with  $\equiv$  and  $+$ . The figures below affirm that  $D([2, 4, 6]) = 6$  and  $T([2, 4, 6]) = 3$  since 6 pieces of candy are drawn and the game lasts 3 turns.

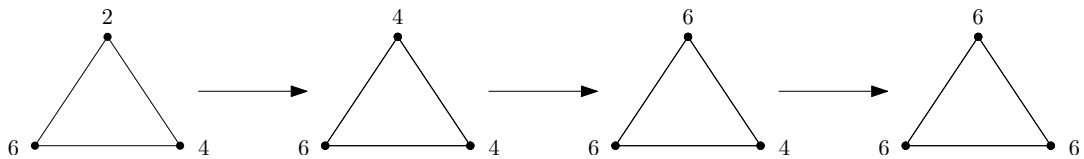


Figure 1: A candy game showing turns passing.

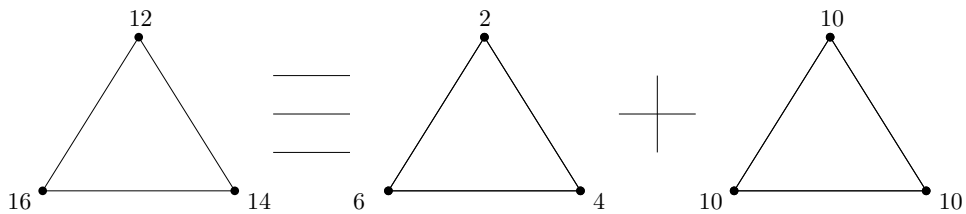


Figure 2: A candy game that shows the stripping equivalence.

### 3 The Three-Player Game

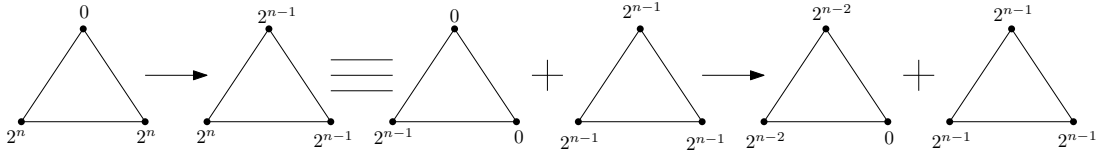
A first natural problem is to determine an exact formula for  $D(n, k)$  and  $T(n, k)$  depending on the initial candy distribution. To that end, we first considered the simplest cases of the game. A game played on two players ends in one turn, and  $s$  is just the average of the players’ pieces plus or minus one piece depending on the parity of the average. The first non-trivial game we consider is the three player game.

Note that in analysis of any  $n$  player game, at most  $n - 1$  players are required to receive candy, as we could simply strip away the minimum value to reduce the game to one in which the players who held the minimum amount now have zero pieces.

**Theorem 1.** Consider the 3-person game with initial distribution  $\mathbf{d} = [0, 2^n, 2^n]$  for any  $n \geq 1$ . Then the stabilization amount is  $s = 1 + (-1)^n \left(\frac{1}{3}\right) + \frac{2^{n+1}}{3}$  and the game will stabilize in  $n + \frac{1}{2} + (-1)^n \left(\frac{1}{2}\right)$  turns.

Note that this theorem confirms the intuition that the game only draws a negligible amount compared to the initial total, at least in a specific class of instances.

*Proof of Theorem 1.* Let  $P_n$  be the stabilization amount for the game with initial distribution  $\mathbf{d} = [0, 2^n, 2^n]$ , and  $T_n$  the length of the game. For completeness, define  $P_0 = 2$  and  $T_0 = 1$ . Inspection of the game in question shows that  $P_1 = 2$  and  $T_1 = 1$ . The following sequence of turns and stripping can then be established:



Taking the first and last game states, we can establish the following two recurrence relations (since two turns have passed and we stripped away  $2^{n-1}$  pieces):

$$P_n = P_{n-2} + 2^{n-1}$$

$$T_n = T_{n-2} + 2$$

Here we used a fact that will be proved in Section 4: that rotating candy amounts doesn't change the structure of a candy sharing game. To extract exact formulas, we use the method of generating functions. Let  $P(x) = \sum_{n=0}^{\infty} P_n x^n$  and  $T(x) = \sum_{n=0}^{\infty} T_n x^n$ . Then, multiplying through by  $x^n$ :

$$P_n x^n = P_{n-2} x^n + 2^{n-1} x^n$$

$$T_n x^n = T_{n-2} x^n + 2x^n$$

Now summing over all values of  $n$  we get:

$$\begin{aligned}\sum_{n=2}^{\infty} P_n x^n &= \sum_{n=2}^{\infty} P_{n-2} x^n + \sum_{n=2}^{\infty} 2^{n-1} x^n \\ \sum_{n=2}^{\infty} T_n x^n &= \sum_{n=2}^{\infty} T_{n-2} x^n + \sum_{n=2}^{\infty} 2x^n\end{aligned}$$

$$\begin{aligned}\sum_{n=2}^{\infty} P_n x^n &= x^2 \sum_{n=2}^{\infty} P_{n-2} x^{n-2} + x \sum_{n=2}^{\infty} (2x)^{n-1} \\ \sum_{n=2}^{\infty} T_n x^n &= x^2 \sum_{n=2}^{\infty} T_{n-2} x^{n-2} + 2 \sum_{n=2}^{\infty} x^n\end{aligned}$$

So, using the definitions of  $P(x)$  and  $T(x)$  as well as the formula for the sum of a geometric series:

$$\begin{aligned}P(x) - P_1 x - P_0 &= x^2 P(x) + x \left( \frac{2x}{1-2x} \right) \\ T(x) - T_1 x - T_0 &= x^2 T(x) + 2 \left( \frac{x^2}{1-x} \right)\end{aligned}$$

Solving for  $P(x)$  and  $T(x)$ , followed by implementing partial fraction decomposition:

$$\begin{aligned}P(x) &= \frac{2+2x}{1-x^2} + \frac{2x^2}{(1-2x)(1-x^2)} = \frac{1}{1-x} + \frac{1/3}{1+x} + \frac{2/3}{1-2x} \\ T(x) &= \frac{1+x}{1-x^2} + \frac{2x^2}{(1-x)(1-x^2)} = \frac{1}{(1-x)^2} - \frac{1/2}{1-x} + \frac{1/2}{1+x}\end{aligned}$$

Re-expanding the series and collecting the like terms together will allow us to extract the equations for  $P_n$  and  $T_n$ .

$$\begin{aligned}P(x) &= \sum_{n \geq 0} \left( 1 + \frac{1}{3}(-1)^n + \frac{2}{3}(2^n) \right) x^n = \sum_{n \geq 0} P_n x^n \\ T(x) &= \sum_{n \geq 0} \left( n + \frac{1}{2} + (-1)^n \left( \frac{1}{2} \right) \right) x^n = \sum_{n \geq 0} T_n x^n\end{aligned}$$

Therefore:

$$P_n = 1 + \frac{1}{3}(-1)^n + \frac{2}{3}(2^n)$$

$$T_n = n + \frac{1}{2} + (-1)^n \left(\frac{1}{2}\right)$$

□

*Alternate Proof of Theorem 1.* Claim: The stabilization amount of the game with initial distribution  $\mathbf{d} = [0, 2^n, 2^n]$  is given by  $P_n = 1 + \frac{(-1)^n}{3} + \frac{2^{n+1}}{3}$ , and it will take  $T_n = n + \frac{1}{2} + \frac{(-1)^n}{2}$  turns,  $n \geq 1$ .

For  $n = 1$  inspection shows the game with initial distribution  $[0, 2, 2]$  stabilizes at 2 pieces, and lasts 1 turn.

$$P_1 = 1 - \frac{1}{3} + \frac{4}{3} = 2$$

$$T_1 = 1 + \frac{1}{2} - \frac{1}{2} = 1$$

Assume that the formula holds true for  $n \leq k$ , that is,  $P_k = 1 + \frac{(-1)^k}{3} + \frac{2^{k+1}}{3}$  and  $T_k = k + \frac{1}{2} + \frac{(-1)^k}{2}$ . Next consider the game played with distribution  $\mathbf{d} = [0, 2^{k+1}, 2^{k+1}]$ .

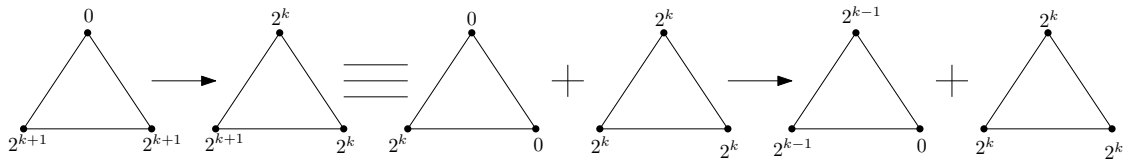


Figure 3: Labels following the  $[0, 2^{k+1}, 2^{k+1}]$  game

Therefore:

$$P_{k+1} = P_{k-1} + 2^k = 1 + \frac{(-1)^{k-1}}{3} + \frac{2^k}{3} + 2^k$$

$$= 1 + \frac{(-1)^2(-1)^{k-1}}{3} + \frac{4 \cdot 2^k}{3}$$

$$= 1 + \frac{(-1)^{k+1}}{3} + \frac{2^{k+2}}{3},$$

and

$$\begin{aligned}
T_{k+1} &= T_{k-1} + 2 = (k-1) + \frac{1}{2} + \frac{(-1)^{k-1}}{2} + 2 \\
&= (k+1) + \frac{1}{2} + \frac{(-1)^2(-1)^{k-1}}{3} \\
&= (k+1) + \frac{1}{2} + \frac{(-1)^{k+1}}{3}
\end{aligned}$$

thus the proof is completed by induction.  $\square$

The next logical step in establishing an explicit formula is to consider two players receiving different powers of 2 as their initial amounts of candy (initial distribution  $\mathbf{d} = [0, 2^n, 2^m]$ ). Though recurrences can be found for both the stabilization amount  $P_{n,m}$  and game length  $T_{n,m}$  extracting an exact formula proves wildly difficult. The recurrences are provided for completeness:

$$P_{n,m} = P_{n-2,m-2} + 2^{n-2} + 2^{m-2}$$

$$T_{n,m} = T_{n-2,m-2} + 2.$$

Moving away from searching for an explicit formula begins the analysis of a more general three person game, beginning with an analysis of games with initial distribution of the form  $\mathbf{d} = [n, 0, 0]$ .

**Definition 3.** Let  $\tilde{D}(n) = D([n, 0, 0])$  and  $\tilde{T}(n) = T([n, 0, 0])$

**Lemma 2.** Let  $k \in \mathbb{Z}^+$ . Then the following eight equations hold:

- |  |  |
|--|--|
| 1. $\tilde{D}(8k) = \tilde{D}(2k)$         | 5. $\tilde{T}(8k) = 2 + \tilde{T}(2k)$       |
| 2. $\tilde{D}(8k+2) = 4 + \tilde{D}(2k)$   | 6. $\tilde{T}(8k+2) = 2 + \tilde{T}(2k)$     |
| 3. $\tilde{D}(8k+4) = 2 + \tilde{D}(2k)$   | 7. $\tilde{T}(8k+4) = 2 + \tilde{T}(2k)$     |
| 4. $\tilde{D}(8k+6) = 2 + \tilde{D}(2k+2)$ | 8. $\tilde{T}(8k+6) = 2 + \tilde{T}(2k+2)$ . |

*Proof.* These recurrences are established in the same way the previous relations were, simply by tracking the games and recording how many pieces were drawn

in addition to the number of turns taken. In the following figure the players who have drawn a piece are marked with a  $\times$  symbol.

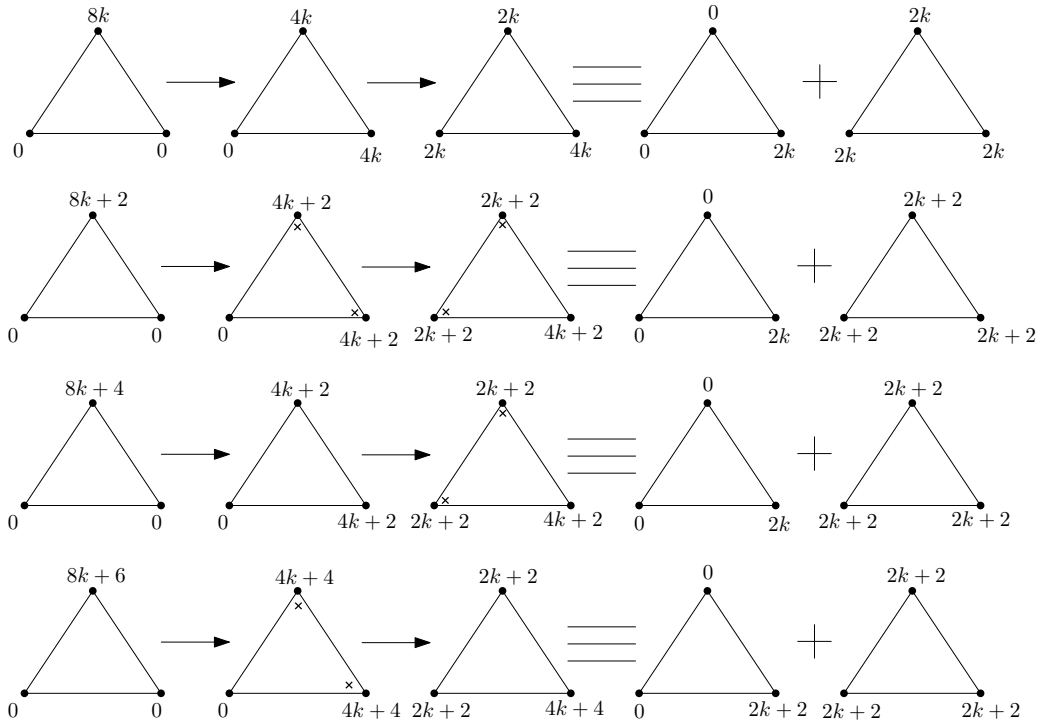


Figure 4

□

Two natural questions are for which values of  $n$  does the game require the most draws, and which games take the longest.

**Theorem 2.** *Let  $\ell \in 2\mathbb{N}$ ,  $\ell \geq 4$ , and let  $r_\ell$  be the smallest integer such that  $\tilde{D}(r_\ell) \geq \ell$ . Then the  $r_\ell$  satisfy the recurrence  $r_\ell = 4r_{\ell-4} + 2$  with  $r_4 = 2$  and  $r_6 = 6$ .*

The  $r_\ell$  in the previous theorem are referred to as the *record breakers* of the  $[n, 0, 0]$  game. The  $\bar{r}_\ell$  in the following theorem are the *turn record breakers* of the  $[n, 0, 0]$  game. In practice, this recurrence is reindexed to the labels  $\{0, 1, 2, \dots\}$  using the substitution  $n = \frac{\ell-4}{2}$ . This allows us to think of the now  $\hat{r}_n$  as the  $n^{\text{th}}$  record breaker, while  $r_\ell$  is the record breaker for  $\ell$ . We will instead show  $\hat{r}_n = 4\hat{r}_{n-2} + 2$  with  $\hat{r}_0 = 2$  and  $\hat{r}_1 = 6$ .

*Proof of Theorem 2.* Let  $R \geq 4 \in 2\mathbb{Z}$  be fixed. We will proceed with induction on  $R$ . Let  $n, m$  be the smallest integers such that  $\tilde{D}(n) \geq R$  and  $\tilde{D}(m) \geq R - 2$ .

Furthermore, suppose that  $n^*$  is the smallest integer such that  $\tilde{D}(n^*) \geq R + 2$ . By hypothesis  $m < n$  (if this were not the case, then this contradicts  $m$  being the record breaker for  $R - 2$ ). We make the claim that  $n^* = 4m + 2$ .

Since  $m \in 2\mathbb{Z}$ ,  $\tilde{D}(4m + 2)$  follows the second recurrence from Lemma 2, with  $k = m/2$ .

$$\tilde{D}(4m + 2) = 4 + \tilde{D}\left(2\binom{m}{2}\right) = 4 + \tilde{D}(m) \geq R + 2.$$

Therefore,  $n^* \leq 4m + 2$ . It will suffice to show  $n^* \geq 4m + 2$ . Assume otherwise, that  $n^* < 4m + 2$ . Depending on the remainder of  $n^*$  modulo 8, we have four cases:

$$R + 2 \leq \tilde{D}(n^*) = \begin{cases} \tilde{D}\left(\frac{n^*}{4}\right) & \text{if } 8|n^* \\ 2 + \tilde{D}\left(\frac{n^*-2}{4}\right) & \text{if } 8|n^* - 2 \\ 4 + \tilde{D}\left(\frac{n^*-4}{4}\right) & \text{if } 8|n^* - 4 \\ 2 + \tilde{D}\left(\frac{n^*-2}{4}\right) & \text{if } 8|n^* - 6. \end{cases}$$

In the first case, we have that  $\tilde{D}\left(\frac{n^*}{4}\right) \geq R + 2$ . But  $\frac{n^*}{4} < \frac{4m+2}{4} = m + \frac{1}{2} < n$ . This is a contradiction, since  $n$  is supposed to be the smallest integer such that  $\tilde{D}(n) \geq R$ .

Secondly, the next inequality can be rearranged to produce  $R \leq \tilde{D}\left(\frac{n^*-2}{4}\right)$ . This leads to the chain  $\frac{n^*-2}{4} < \frac{4m+2-2}{4} = m < n$ , eliciting another contradiction as in case 1.

The third inequality is rearranged to form  $R - 2 \leq \tilde{D}\left(\frac{n^*-4}{4}\right)$ . Then  $\frac{n^*-4}{4} < \frac{4m+2-4}{4} = m - \frac{1}{2}$ . This contradicts the fact that  $m$  is the smallest integer such that  $\tilde{D}(m) \geq R - 2$ .

Finally, the fourth inequality asserts that  $R \leq \tilde{D}\left(\frac{n^*-2}{4}\right)$  and  $\frac{n^*-2}{4} < \frac{4m+2-2}{4} < m$ , contradicting both assumptions that  $n$  and  $m$  were the smallest integers to



draw more than  $R$  and  $R - 2$  pieces of candy respectively.

Each case resulted in a contradiction, and thus our assumption that  $n^* < 4m + 2$  must be false. We then conclude that  $n^* = 4m + 2$ . Since  $\hat{r}_n$  denotes the  $n^{\text{th}}$  record breaker,  $\hat{r}_n = 4\hat{r}_{n-2} + 2$ .

□

**Corollary 1.** Let  $\ell \in \mathbb{N}$ ,  $\ell \geq 2$ , and let  $r_{2\ell}$  be the smallest integer such that  $\tilde{D}(n^*) \geq 2\ell$ . Then  $r_{2\ell}$  has binary representation of the form

$$r_{2\ell} = \begin{cases} \underbrace{10 \dots 10}_{\ell/2} & \ell \in 2\mathbb{Z} \\ 1 \underbrace{10 \dots 10}_{(\ell-1)/2} & \ell \notin 2\mathbb{Z}. \end{cases}$$

*Proof.* In binary,  $r_0 = (10)_2$  and  $r_1 = (110)_2$ . The recursion that the record breakers follow can be expressed, in binary, as appending a “10” to the end of the string. Since the recursion only depends on the term two indices back, a “10” will be continually appended to  $(10)_2$  and  $(110)_2$  □

**Theorem 3.** Let  $\ell \in 2\mathbb{N}$ ,  $\ell \geq 2$ , and let  $\bar{r}_\ell$  be the smallest integer such that  $\tilde{T}(\bar{r}_\ell) \geq \ell$ . Then these  $\bar{r}_\ell$  satisfy the recursion  $\bar{r}_\ell = 4\bar{r}_{\ell-2} - 2$  with  $\bar{r}_2 = 2$ .

This is proved in the exact manner as Theorem 2, which shouldn’t be surprising as the recurrences and theorems are stated almost identically. Also similarly, we can reindex the recurrence with the substitution  $n = \frac{\ell-2}{2}$  so that  $\tilde{r}_n$  refers to the  $n^{\text{th}}$  turn record breaker. We now show that  $\tilde{r}_n = 4\tilde{r}_{n-1} - 2$  with  $\tilde{r}_0 = 2$ .

*Proof.* Let  $R \geq 4 \in 2\mathbb{Z}$  be fixed. Let  $n$  be the smallest integer such that  $\tilde{T}(n) \geq R$ . Furthermore, suppose that  $n^*$  is the smallest integer such that  $\tilde{T}(n^*) \geq R + 2$ . We claim that  $n^* = 4n - 2$ .

Since  $n^* \in 2\mathbb{Z}$ ,  $\tilde{T}(n^*)$  follows the  $\tilde{T}(8k + 6)$  recurrence, with  $k = \frac{m}{2} - 1$ . Then:

$$\tilde{T}(4m - 2) = 2 + \tilde{T}\left(2\left(\frac{m}{2} - 1\right) + 2\right) = 2 + \tilde{T}(m) \geq R + 2$$

As in the proof of Theorem 2, we've established that  $n^* \leq 4m - 2$ , and it will suffice to show that  $n^* \geq 4m - 2$ . For the sake of contradiction, assume that  $n^* < 4m - 2$ . Again, we have four cases based on the divisibility of  $n^*$  with respect to 8:

$$R + 2 \leq \tilde{T}(m^*) = \begin{cases} 2 + \tilde{T}\left(\frac{n^*}{4}\right) & \text{if } 8|n^* \\ 2 + \tilde{T}\left(\frac{n^*-2}{4}\right) & \text{if } 8|n^* - 2 \\ 2 + \tilde{T}\left(\frac{n^*-4}{4}\right) & \text{if } 8|n^* - 4 \\ 2 + \tilde{T}\left(\frac{n^*-2}{4}\right) & \text{if } 8|n^* - 6. \end{cases}$$

In every case, we can eliminate the 2's and just consider the inequalities with  $R$  and  $\tilde{T}$ .

Specific to case 1, we see that  $R \leq \tilde{T}\left(\frac{n^*}{4}\right)$ . But  $\frac{n^*}{4} < \frac{4n-2}{4} < n$ , a contradiction since  $n$  is supposedly the smallest integer such that  $\tilde{T}(n) \geq R$ .

For cases 2 and 4 we have  $\frac{n^*-2}{4} < \frac{4n-2-2}{4} < n$ , another contradiction.

Finally, for case 3, we arrive at the fact that  $\frac{n^*-4}{4} < \frac{n^*-2-4}{4} < n$ , the same contradiction.

Therefore, our assumption is false and we conclude that  $n^* = 4n - 2$ . Since  $n$  is the previous turn record breaker, we arrive at the desired recurrence  $\tilde{r}_n = 4\tilde{r}_{n-1} - 2$  with  $\tilde{r}_0 = 2$ . □

**Corollary 2.** Let  $\ell \in \mathbb{N}$ ,  $\ell \geq 2$ , and let  $\bar{r}_{2\ell}$  be the smallest integer such that  $\tilde{T}(m^*) \geq 2\ell$ . Then  $\bar{r}_{2\ell}$  has binary representation of the form

$$m^* = \underbrace{10 \dots 10}_{\ell-2} 110.$$

This corollary is proved in exactly the same manner as the corollary to Theo-

rem 2, simply apply the recursive operation to the elements of  $\bar{r}_n$  in binary.

To help motivate further analysis, we may view  $\tilde{D}$  as a function,  $\tilde{D} : 2\mathbb{N} \mapsto \mathbb{N}$  and consider its (asymptotic) behavior. As a function,  $\tilde{D}$  is graphed in figure 5.

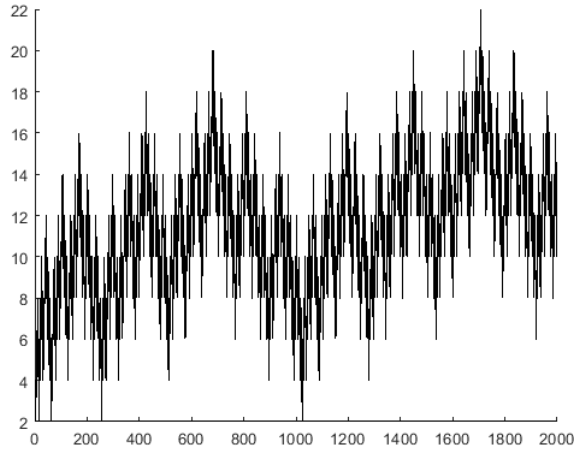


Figure 5: The graph of  $\tilde{D}(n)$ ,  $n \in \{2, 4, \dots, 2000\}$

**Proposition 1.** *The minimum of  $\tilde{D}(n)$  is 2, and occurs only when  $n = 2^{2^j}$ ,  $j \in \mathbb{N}$*

*Proof.* By examining the  $[4, 0, 0]$  game it is easily seen that  $\tilde{D}(2^{2^1}) = 2$ . Next, assume that  $\tilde{D}(2^{2^k}) = 2$  for all  $j \leq k$ . For  $k > 1 \in \mathbb{N}$ ,  $2^{2^k}$  is a power of 8. Thus,  $\tilde{D}(2^{2^k}) = \tilde{D}(2^{2^{k-2}}) = 2$  by Lemma 2. Thus, the proof is complete through induction on  $j$ .  $\square$

This establishes a lower bound. Inspection of the record breaker recursion will allow us to algebraically establish an upper bound for  $\tilde{D}(n)$ .

**Proposition 2.**  $\tilde{D}(n) = \mathcal{O}(\log(n))$

*Proof.* The peaks of  $\tilde{D}(n)$  occur when  $n$  is one of the record breakers. The first step is to solve the recurrence for the record breakers for an explicit formula. Using the method of generating functions, let  $R(x) = \sum_{n=0}^{\infty} \tilde{r}_n x^n$ . Then:

$$\tilde{r}_n = 4\tilde{r}_{n-2} + 2.$$

Multiplying by  $x^n$ :

$$\tilde{r}_n x^n = 4\tilde{r}_{n-2} x^n + 2x^n.$$

Summing over all values of  $n$ :

$$\begin{aligned} \sum_{n=2}^{\infty} \tilde{r}_n x^n &= \sum_{n=2}^{\infty} 4\tilde{r}_{n-2} x^n + \sum_{n=2}^{\infty} 2x^n \\ \sum_{n=2}^{\infty} \tilde{r}_n x^n &= 4x^2 \sum_{n=2}^{\infty} \tilde{r}_{n-2} x^{n-2} + 2 \sum_{n=2}^{\infty} x^n. \end{aligned}$$

Using the definitions of  $R(x)$  and the formula for the sum of a geometric series:

$$R(x) - \tilde{r}_0 - \tilde{r}_1 x = 4x^2 R(x) + 2 \left( \frac{x^2}{1-x} \right).$$

Solving for  $R(x)$  and using partial fraction decomposition:

$$\begin{aligned} (1 - 4x^2)R(x) &= 2 + 6x + \frac{2x^2}{1-x} \\ R(x) &= \frac{2 + 6x}{1 - 4x^2} + \frac{2x^2}{(1 - 4x^2)(1 - x)} = -\frac{2/3}{1-x} + \frac{3}{1-2x} - \frac{1/3}{1+2x}. \end{aligned}$$

Finally, re-expanding the series and collecting like terms will let us extract the formula we need.

$$\begin{aligned} R(x) &= \sum_{n=0}^{\infty} \left( -\frac{2}{3} + 3(2^n) - \frac{1}{3}(-2)^n \right) x^n = \sum_{n=0}^{\infty} \tilde{r}_n x^n \\ \therefore \tilde{r}_n &= -\frac{2}{3} + 3(2^n) - \frac{1}{3}(-2)^n. \end{aligned}$$

This equation gives the record breaker (amount of candy) in terms of which number record breaker you want (i.e.  $r_{200}$  is the 200<sup>th</sup> record breaker). Undoing the substitution, letting  $n = \frac{\ell-4}{2}$  transforms our formula into one which gives the record breaker in terms of the record broken:  $r(\ell) = -\frac{2}{3} + 3(2^{\frac{\ell-4}{2}}) - \frac{1}{3}(-2)^{\frac{\ell-4}{2}}$  (e.g.  $r(8)=10$  since 10 is the first time the  $[n, 0, 0]$  game draws 8 pieces of candy). Inverting this would give an equation,  $\ell(r)$ , that returns the peaks of the  $\tilde{D}$  function

in terms of the record breakers (values of  $n$ ).

If  $n$  is even

the base of the last term is always positive

$$\tilde{r}_n = -\frac{2}{3} + 3(2^n) - \frac{1}{3}(2^n).$$

Rearranging the equality and factoring

$$\tilde{r}_n + \frac{2}{3} = 2^n \left( \frac{8}{3} \right).$$

Isolating the exponential and taking the  $\log_2$  of both sides

$$\frac{3}{8} \left( \tilde{r}_n + \frac{2}{3} \right) = 2^n$$

$$n = \log_2 \left( \frac{3}{8} \left( \tilde{r}_n + \frac{2}{3} \right) \right).$$

Undoing the substitution that was made for  $\ell$

$$\frac{\ell - 4}{2} = \log_2 \left( \frac{3}{8} \left( \tilde{r}_n + \frac{2}{3} \right) \right)$$

$$\ell = 2 \log_2 \left( \frac{3}{8} \left( \tilde{r}_n + \frac{2}{3} \right) \right) + 4.$$

If  $n$  is odd

the exact process above is repeated with the base of the last term strictly negative.

$$\tilde{r}_n = -\frac{2}{3} + 3(2^n) + \frac{1}{3}(2^n)$$

$$\ell = 2 \log_2 \left( \frac{3}{10} \left( \tilde{r}_n + \frac{2}{3} \right) \right) + 4.$$

□

Treating  $\tilde{r}_n$  as our variable lets us plot either of these functions as a suitable

upper bound.

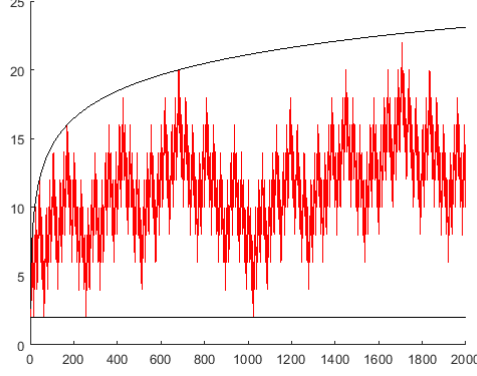


Figure 6: The graph of  $\tilde{D}(n)$  and its bounds

Following this bound, we move on to searching for a similar result for the general three person game with initial distribution  $\mathbf{d} = [n, m, 0]$ .

**Definition 4.** Let  $\hat{D}(n, m) = D([n, m, 0])$  and  $\hat{T}(n, m) = T([n, m, 0])$

**Lemma 3.** Let  $\ell, k \in \mathbb{Z}^+$ . Without loss of generality, let  $\ell \geq k$ . Then the following eight equations hold:

- |   |   |
|---|---|
| 1. $\hat{D}(4\ell, 4k) = \hat{D}(2\ell, 2(\ell - k))$                 | 5. $\hat{T}(4\ell, 4k) = \hat{T}(2\ell, 2(\ell - k))$                 |
| 2. $\hat{D}(4\ell + 2, 4k) = 2 + \hat{D}(2\ell + 2, 2(\ell - k) + 2)$ | 6. $\hat{T}(4\ell + 2, 4k) = 2 + \hat{T}(2\ell + 2, 2(\ell - k) + 2)$ |
| 3. $\hat{D}(4\ell, 4k + 2) = 2 + \hat{D}(2\ell, 2(\ell - k) - 2)$     | 7. $\hat{T}(4\ell, 4k + 2) = 2 + \hat{T}(2\ell, 2(\ell - k) - 2)$     |
| 4. $\hat{D}(4\ell + 2, 4k + 2) = 2 + \hat{D}(2\ell, 2(\ell - k))$     | 8. $\hat{T}(4\ell + 2, 4k + 2) = 2 + \hat{T}(2\ell, 2(\ell - k))$ .   |

These equations and the following figure are included for completeness as efforts are swiftly refocused towards the general candy sharing game.

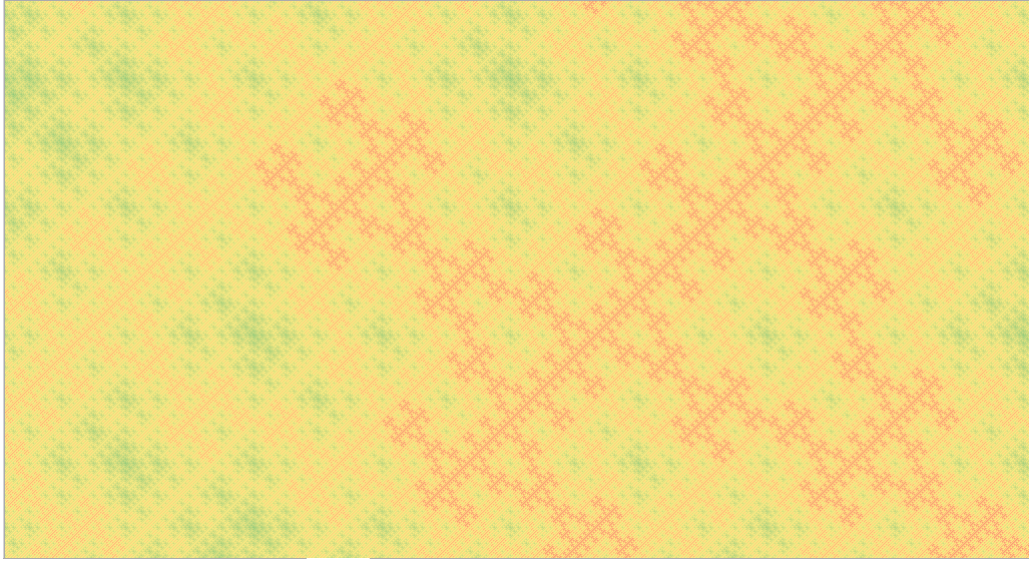
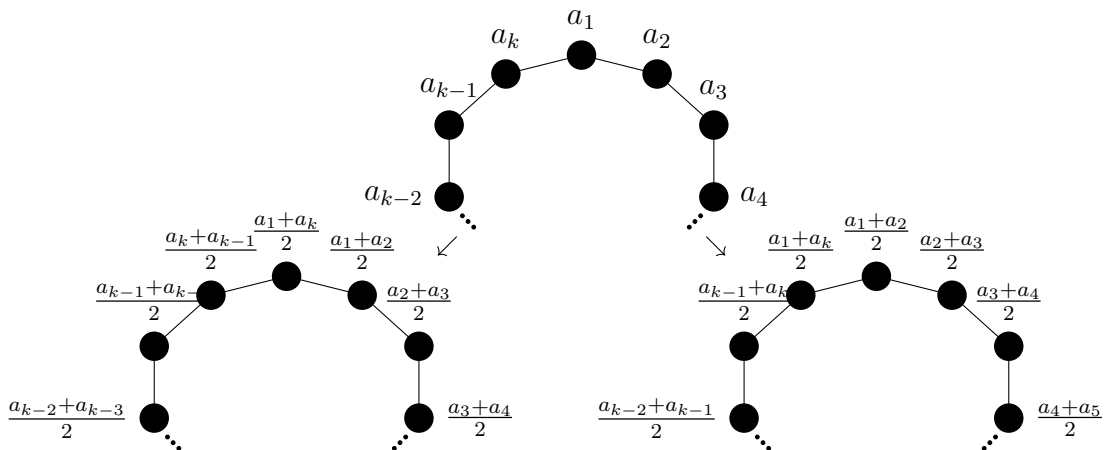


Figure 7: A table of values of  $\hat{D}(n, m)$  shaded according to value.

## 4 The General Candy Sharing Game

**Proposition 3.** *Any element of the dihedral group  $D_k$  acting on any game state in a  $k$ -player game doesn't affect the stabilization amount nor the game length.*

*Proof.* Let  $[a_1, a_2, \dots, a_k]$  represent the amounts held by players 1 through  $k$  at some arbitrary turn of the game. A rotation by  $\frac{360^\circ}{k}$  moves every candy amount over by one person, giving distribution  $[a_2, a_3, \dots, a_k, a_1]$ . We can simply rename the players to arrive back at the original game. Note that a flip in this game amounts to passing the candy in the opposite direction. Examining the amounts present for an arbitrary turn of the game passing left and right:



We can see that these two game states are just rotations of each other. Therefore, any arbitrary turn plays the same passing right or left. Since the dihedral groups are generated from some number of rotations of  $\frac{360^\circ}{k}$  and a flip, we have that any dihedral action leaves the structure of the game unchanged.  $\square$

This has a further reaching consequence than simply passing left or right throughout the entire game. Since we considered any arbitrary turn, any candy sharing game could be played with the teacher arbitrarily deciding each turn whether the students will pass right or left and the game will end at the same amount as the game where the students passed right or left consistently throughout.

In their paper titled “Candy Sharing” Iba and Tanton [4] rephrase analysis of this game in an attempt to examine it in a broader context. These more general candy sharing games in [4] allow players to share different portions of their candy, not just half their stack. The authors determine criteria for arbitrary candy sharing games to have an ending state on general graphs. To accomplish this, they shifted focus to the movements of an individual piece of candy as opposed to the piles as a whole. This piece of candy can be in one of a finite number of “states”: in possession of one of the players (i.e. a single piece of candy is shared with probability  $\frac{1}{2}$ ). The punch line of this is that we can now view any candy sharing game as a Markov chain, about which there is a great deal of theory. The text *Markov Chains and Mixing Times* by David Levin, Yuval Peres, and Elizabeth Wilmer [2] is used as reference for the required pieces of Markov Theory.

“A finite Markov Chain is a process which moves among the elements of a finite set  $\Omega$  in the following manner: when at  $x \in \Omega$ , the next position is chosen according to a fixed probability distribution  $P(x, \cdot)$ ” [2]. In practice, and for our purposes, we can encode this probabilistic information into a  $|\Omega| \times |\Omega|$  matrix, where the  $(i, j)$  entry is the probability of transitioning from state  $i$  to state  $j$ .



The  $(i, j)$  entry of  $P^n$  is the probability of transitioning from state  $i$  to state  $j$  in  $n$  steps. If  $\mathbf{c}$  is a  $1 \times |\Omega|$  vector describing the state of the system at time  $t$ , then  $\mathbf{c}P$  is the state of the system at time  $t + 1$ .

Now consider a version of the game where players no longer draw a piece to ensure divisibility by 2. Instead, at each time step, for each piece of candy in their possession the players flip a fair coin. They keep that piece if the coin is heads.

We call this version of the game the *continuous game*, while the traditional candy sharing game with rounding is called the *discrete game*. Let  $\mathbf{c}_t = [a_1, a_2, \dots, a_k]$  be the vector such that  $a_i$  is the expected amount of candy held by player  $i$  at turn  $t$  of the continuous game. Similarly, let  $\mathbf{d}_t = [a_1, a_2, \dots, a_k]$  be the vector such that  $a_i$  is the amount of candy held by player  $i$  at turn  $t$ .

The Markov Chain modeling the continuous game has *transition matrix*

$$P = \left[ \begin{array}{cccccc} 1/2 & 1/2 & 0 & 0 & \dots & 0 \\ 0 & 1/2 & 1/2 & 0 & \dots & 0 \\ 0 & 0 & 1/2 & 1/2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/2 & 0 & 0 & 0 & \dots & 1/2 \end{array} \right] \Bigg\}^k \quad (1)$$

Suppose a piece of candy is initially in possession of player  $i$ . This state can be modeled by the  $1 \times k$  vector  $\delta_i^k$  where  $\delta^k(j) = 1$  if  $j = i$  and 0 otherwise. Then the  $j^{\text{th}}$  entry of  $\delta_i^k P^t$  represents the probability that the piece of candy is in possession of player  $j$  at time  $t$ . Thus, if we start with initial distribution  $\mathbf{c}_0 = [a_1, a_2, \dots, a_k]$  then  $\mathbf{c}_t = \mathbf{c}_0 P^t$ .

Two important properties of Markov chains (which are heavily used throughout this section) are irreducibility and aperiodicity [2]. A chain is called *irreducible* if for any two states  $x, y \in \Omega$  there exists an integer  $t$  (possibly depending on  $x$  and  $y$ ) such that the  $(x, y)$  entry in  $P^t > 0$ . Furthermore, a chain is *aperiodic* if 1 is the greatest common divisor of all positive integers  $t$  such that the  $(i, i)$  entry

of  $P^t > 0$  for all  $i$ .

**Proposition 4.** *The Markov chain with transition matrix  $P$  from (1) is irreducible.*

*Proof.* We claim that this integer is  $k$  for all pairs of states. To demonstrate this, we need to show that the  $(i, j)$  entry of  $P^k$  is non-zero for all pairs of  $i$  and  $j$ , i.e. in  $k$  turns there is the possibility that a piece of candy transitions from player  $i$ 's possession to player  $j$ 's. In  $k$  turns, a piece of candy chosen to move at every turn would cycle back to the player it started with. So, one possibility to transition from player  $i$  to  $j$  is to move around the circle until reaching player  $j$ , and then being chosen to remain for the remainder of the turns. Therefore, every entry of  $P^k$  is nonzero.  $\square$

**Proposition 5.** *The Markov chain modeling our candy sharing game is aperiodic.*

*Proof.* The  $(i, i)$  entry of  $P$  is the probability that the piece of candy will stay in the current player's possession. As such, the  $(i, i)$  entry of  $P^t$  is at least  $\frac{1}{2^t}$ . Therefore, the set of  $t$  for each  $(i, i)$  is just  $\{1, 2, 3, \dots\}$ .  $\gcd(1, 2, 3, \dots) = 1$  and the proof is complete.  $\square$

It is a classical, well known result (see for example Proposition 1.14 in [2]) that for any irreducible chain, there exists a unique stationary distribution (that is, a vector  $\boldsymbol{\pi}$  such that  $\boldsymbol{\pi}P = \boldsymbol{\pi}$ ). Furthermore, irreducible and aperiodic chains converge to their stationary distribution regardless of the initial state. Since our chain is irreducible and aperiodic, a stationary distribution  $\boldsymbol{\pi}$  exists and each starting state will converge to  $\boldsymbol{\pi}$ . Before moving forward, we introduce some new notation that will be needed in the main result.

Note that  $\mathbf{c}_t = \mathbf{c}_0 P^t$ . Let  $\boldsymbol{\pi}$  denote the stationary distribution of the continuous candy sharing game.

**Proposition 6.** *Let  $\mathbf{c}_0 = [a_1, a_2, \dots, a_k]$ ,  $a_i \in \mathbb{R}$  be the initial distribution in*

the continuous candy sharing game. Furthermore, let  $s = \frac{1}{k} \sum_{i=1}^k a_i$ . Then  $\boldsymbol{\pi} = [s, s, \dots, s]$  is the stationary distribution.

*Proof.* We first note that the transition matrix  $P$  was built on the probability that an individual piece of candy makes a transition from player  $i$ 's pile to player  $j$ 's. We first claim that the stationary distribution for this process is  $\vec{v} = [\frac{1}{k}, \frac{1}{k}, \dots, \frac{1}{k}]$  where  $\text{length}(\vec{v}) = k$ .

$$\begin{aligned} \vec{v} * P &= \left[ \frac{1}{k}, \frac{1}{k}, \dots, \frac{1}{k} \right] \cdot \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & \dots & 0 \\ 0 & 1/2 & 1/2 & 0 & \dots & 0 \\ 0 & 0 & 1/2 & 1/2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/2 & 0 & 0 & 0 & \dots & 1/2 \end{bmatrix} \\ &= \left[ \frac{2}{2k}, \frac{2}{2k}, \dots, \frac{2}{2k} \right] \\ &= \left[ \frac{1}{k}, \frac{1}{k}, \dots, \frac{1}{k} \right] \\ &= \vec{v}. \end{aligned}$$

So  $\vec{v}$  is the unique stationary distribution for the motions of a single piece of candy. Any starting point (the piece of candy being held by player  $i$ ) is modeled by the  $1 \times k$  vector  $\delta_i^k$  where  $\delta_i^k(j) = 1$  if  $j = i$  and 0 otherwise, and each starting point will tend to  $\vec{v}$ . With this, we return to  $\mathbf{c}_0$ . Note we have

$$\mathbf{c}_0 = [a_1, a_2, \dots, a_k] = a_1 \delta_1^k + a_2 \delta_2^k + \dots + a_k \delta_k^k.$$

And so

$$\begin{aligned}
\lim_{t \rightarrow \infty} \mathbf{c}_t &= \lim_{t \rightarrow \infty} \mathbf{c}_0 P^t = \lim_{t \rightarrow \infty} (a_1 \delta_1^k + a_2 \delta_2^k + \dots + a_k \delta_k^k) P^t \\
&= \lim_{t \rightarrow \infty} a_1 \delta_1^k P^t + a_2 \delta_2^k P^t + \dots + a_k \delta_k^k P^t \\
&= a_1 \lim_{t \rightarrow \infty} \delta_1^k P^t + a_2 \lim_{t \rightarrow \infty} \delta_2^k P^t + \dots + a_k \lim_{t \rightarrow \infty} \delta_k^k P^t \\
&= a_1 \vec{v} + a_2 \vec{v} + \dots + a_k \vec{v} \\
&= \left[ \frac{1}{k} \sum_{i=1}^k a_i, \frac{1}{k} \sum_{i=1}^k a_i, \dots, \frac{1}{k} \sum_{i=1}^k a_i \right] \\
&= \boldsymbol{\pi}.
\end{aligned}$$

□

This gives strong evidence towards our intuition that the ending amount of the candy sharing game should be approximately the average, since the continuous game tends to exactly the average.

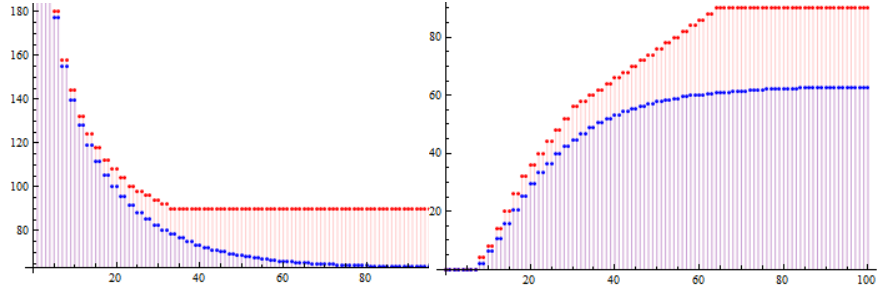
The final pieces needed before proving an asymptotic bound on the number of pieces drawn are connections between the discrete and continuous game, and a strong theorem from Markov Theory.

**Definition 5.** Let  $\Delta_t$  be the number of pieces drawn in the discrete game up to and including turn  $t$ .

**Definition 6.** Let  $\max(\mathbf{d}_t) = \max \{a_i \in \mathbf{d}_t \mid 1 \leq i \leq k\}$ ,  
 $\max(\mathbf{c}_t) = \max \{a_i \in \mathbf{c}_t \mid 1 \leq i \leq k\}$

**Definition 7.** Let  $\min(\mathbf{d}_t) = \min \{a_i \in \mathbf{d}_t \mid 1 \leq i \leq k\}$ ,  
 $\min(\mathbf{c}_t) = \min \{a_i \in \mathbf{c}_t \mid 1 \leq i \leq k\}$ .

The amounts between the discrete and continuous games are very close. In particular, the minimums will only differ by the fact that the discrete game continually draws pieces. Therefore  $\min(\mathbf{d}_t) \geq \min(\mathbf{c}_t)$ . Likewise,  $\max(\mathbf{d}_t) \leq \max \mathbf{c}_t + \Delta_t$ .



(a) The max of  $\mathbf{d}_t$  in red vs the max of  $\mathbf{c}_t$  in blue

(b) The min of  $\mathbf{d}_t$  in red vs the min of  $\mathbf{c}_t$  in blue

**Definition 8.** The *total variation distance* between two probability distributions  $\mu$  and  $\nu$  is given by  $\|\mu - \nu\|_{TV} = \max_{A \subset \Omega} |\mu(A) - \nu(A)|$ .

This definition is the maximum difference between probabilities assigned to a single event by these two distributions. It turns out there is a much more convenient way to calculate the total variation distance between two probability distributions, given by the following lemma.

**Lemma 4** (Proposition 4.2 in [2]). Let  $\mu$  and  $\nu$  be two probability distributions on space  $\Omega$ . Then  $\|\mu - \nu\|_{TV} = \frac{1}{2} \sum_{x \in \Omega} |\mu(x) - \nu(x)|$ .

The main tool which we will use is the following.

**Theorem 4** (The Convergence Theorem, Theorem 4.9 in [2]). *If  $P$  is an irreducible and aperiodic chain, with stationary distribution  $\pi$ , then for any initial probability distribution  $x$  on  $\Omega$  there exists constants  $\alpha \in (0, 1)$  and  $C > 0$  such that for all  $t \geq 0$   $\|xP^t - \pi\|_{TV} \leq C\alpha^t$ .*

With the preliminaries out of the way we are now ready to state and prove our main theorem.

**Theorem 5.** *For any initial distribution  $\mathbf{d}_0 = [a_1, \dots, a_k]$  such that  $\sum_{i=1}^k a_i = n$  the discrete candy sharing game ends in  $\mathcal{O}(\log(n))$  turns, and every player will be holding  $\frac{n}{k} + \mathcal{O}(\log(n))$  pieces of candy.*

*Proof.* Let  $\mathbf{c}_0 = \mathbf{d}_0$ ,  $\tilde{\mathbf{c}}_t = \frac{\mathbf{c}_t}{n}$  be the normalized game state of the continuous candy game, and  $\boldsymbol{\pi} = [\frac{1}{k}, \frac{1}{k}, \dots, \frac{1}{k}]$ . Dividing by  $n$  transforms  $\mathbf{c}_t$  into a probability

distribution, since  $\sum_{i=1}^k a_i = n$  and in the continuous game the total amount of candy remains unchanged. Let  $\alpha$  and  $C$  be given by the Convergence Theorem for matrix  $P$  and initial distribution  $\mathbf{c}_0$ . Then after  $t$  steps we have  $\|\tilde{\mathbf{c}}_t - \boldsymbol{\pi}\|_{TV} < C\alpha^t$ . Let  $t_0 = \frac{\ln(2Cn)}{\ln(\frac{1}{\alpha})}$ . Then for  $t > t_0$  we have

$$\|\tilde{\mathbf{c}}_{t_0} - \boldsymbol{\pi}\|_{TV} < \frac{1}{2n}. \quad (2)$$

Using our above inequalities:

$$\begin{aligned} |\max(\mathbf{d}_t) - \min(\mathbf{d}_t)| &\leq |\max(\mathbf{c}_t) - \min(\mathbf{c}_t)| + \Delta_t \\ &\leq n |\max(\tilde{\mathbf{c}}_t) - \min(\tilde{\mathbf{c}}_t)| + \Delta_t \\ &\leq n \left( \left| \max(\tilde{\mathbf{c}}_t) - \frac{1}{k} \right| + \left| \min(\tilde{\mathbf{c}}_t) - \frac{1}{k} \right| \right) + \Delta_t. \end{aligned}$$

The sample space we consider is size  $k$ . In the sum above we include 2 terms used in the total variation norm's sum formula. Including the remaining terms increases the value and leads to the next inequality.

$$\begin{aligned} n \left( \left| \max(\tilde{\mathbf{c}}_t) - \frac{1}{k} \right| + \left| \min(\tilde{\mathbf{c}}_t) - \frac{1}{k} \right| \right) + \Delta_t &\leq n \sum_{i=1}^k \left| \tilde{\mathbf{c}}_t(i) - \frac{1}{k} \right| + \Delta_t \\ &\leq 2n \|\tilde{\mathbf{c}}_t - \boldsymbol{\pi}\|_{TV} + \Delta_t \end{aligned}$$

where in the second inequality we've used Lemma 4. Now note that  $\Delta_t$  is bounded above by  $kt$ , since at most every player will have to draw every turn. Therefore, after  $t_0$  turns we have that  $|\max(\mathbf{d}_{t_0}) - \min(\mathbf{d}_{t_0})|$  is bounded above by

$$\begin{aligned} 2n \|\tilde{\mathbf{c}}_{t_0} - \boldsymbol{\pi}\|_{TV} + \Delta_{t_0} &\leq 2n \left( \frac{1}{2n} \right) + kt_0 \\ &< 1 + k \cdot \frac{\ln(2Cn)}{\ln(\frac{1}{\alpha})} \end{aligned}$$

where in the first inequality we used (2) and the bound on  $\Delta_{t_0}$ . Thus we have that there exists a constant  $C^*$  such that after  $t_0$  turns,  $|\max(\mathbf{d}_{t_0}) - \min(\mathbf{d}_{t_0})| < C^* \log(n)$ .

We know that the minimum in the discrete game is guaranteed to increase every  $k$  turns. Therefore, after at most  $kC^* \log(n)$  more turns, the difference between the maximum and minimum will be less than 1, and thus the game will have ended. From this we know the total number of turns the game took is at most  $t_0 + kC^* \log(n) = \frac{\ln(2Cn)}{\ln(\frac{1}{\alpha})} + kC^* \log(n) = C^{**} \log(n)$  for some constant  $C^{**}$ . At worst, each player draws a piece of candy every turn. So the total amount of candy at the end of the game is at most  $n + kC^{**} \log(n)$  implying that each player has  $\frac{n}{k}$  plus at most  $C^{**} \log(n)$  pieces.  $\square$

As a last remark on the main result, the proof can be extended to candy sharing games played on more general graphs and the result is unchanged providing the graphs and game fit the following criteria. The graphs must be strongly connected, to ensure that the Markov Chain remains irreducible. The greatest common divisor of all the cycle lengths must be 1, which ensures the chain remains aperiodic. Finally, the in and out degree of every vertex must be equal (say  $\delta$ ), forcing the Chain to have a uniform steady state. If these conditions on the underlying graph are met, then the candy sharing game in which each player shares a  $\frac{1}{\delta}$  portion of their candy to each neighbor along an out-edge followed by drawing up to the next multiple of  $\delta$  ends in logarithmic time at approximately the average, plus a negligible amount.

## 5 Chip Firing Games

As stated above, the final part of this thesis deals with a result in the field of chip firing games. Recall that a chip firing game assigns an integral number of chips to each vertex and is advanced in time according to some firing rule which determines how the chips will move. The most significant result proven in this field is on the topic of the diffusion game. In the diffusion game, each vertex “fires off” one chip to each of its neighbors with less chips than itself. Duffy et al. [1] proved that any initial chip configuration on a finite graph would end in a constant game state or

oscillate between two game states in a finite amount of time. His team end their paper with the idea to analyze the diffusion game on infinite graphs of bounded degree and bounded vertex label (chip amounts). We answer this in the case of the infinite path.

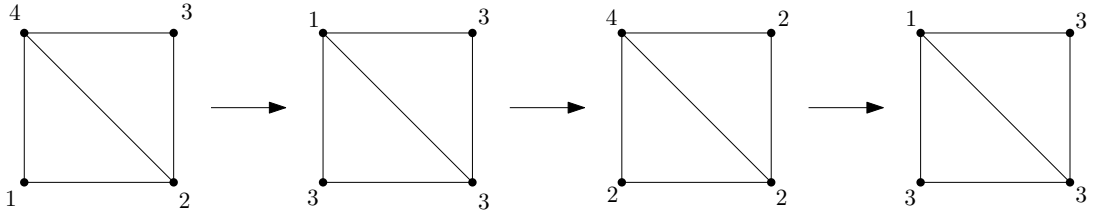


Figure 9: A chip firing game that ends in oscillation.

**Theorem 6.** *If the diffusion chip firing game is played on the infinite path with initial vertex labeling from  $\{0, 1, 2, \dots, k\}$  then the highest possible vertex label is  $k + 1$  and the lowest possible vertex label is  $-1$ .*

*Proof.* To prove this, we will show an equivalent property that directly implies the theorem. We will show that in this diffusion game, the neighbors of a vertex with label  $k + 1$  must be less than or equal to  $k - 1$ , and the neighbors of a vertex with label  $-1$  must be greater than or equal to  $1$ .

Let the initial chip labeling on the infinite path be labels from  $\{0, 1, 2, \dots, k\}$ . At turn  $t = 0$ , the conclusions are trivially satisfied. Assume that at turn  $t = k$ , the label of every neighbor of a vertex with label  $k + 1$  is less than or equal to  $k - 1$  and every label of a neighbor of a vertex with label  $-1$  has label greater than or equal to  $1$ .

At turn  $t = k + 1$ , if a  $k + 1$  label has been created, it must have come from one of the following configurations present at turn  $t = k$ . Following each configuration one step forward we can glean information about the possibilities for the neighbors of the label  $k + 1$  (up to symmetry).

Configuration 1: A string of the form  $[\dots, (\leq k), k, (k - 1), k, (\leq k), \dots]$ . Tracking this through to the next turn, we lose information about the outermost players, but this isn't required as our induction hypothesis guarantees they won't



draw to a  $k + 1$ . The next turn is then represented as  $[\dots, (k - 1)/(k - 2), (k + 1), (k - 1)/(k - 2)]$  depending on whether the players holding  $k$  chips had a neighbor holding  $k$  chips or  $< k$  many chips.

Configuration 2: A string of the form  $[\dots, k, (k - 1), (k + 1), (\leq k - 1), \dots]$ . Following this through in a similar manner, we arrive at the string  $[\dots, (k - 1)/(k - 2), (k + 1), (\leq k - 2)]$  again depending on the neighbors of the player holding  $k + 1$  chips.

Configuration 3: A string of the form  $[\dots, (\leq k - 1), (k + 1), (k - 1), (k + 1), (\leq k - 1), \dots]$ , leading to the next game state of  $[\dots, (\leq k - 2), k + 1, (\leq k - 2), \dots]$ .

At turn  $t = k + 1$ , if a label of  $-1$  has been created, it must have come from the following configuration present at turn  $t = k$ :  $[\dots, \geq 0, 0, 1, 0, \geq 0]$ . Again, we lose information about the outermost players, but that information isn't necessary. At turn  $t = k + 1$ , this configuration becomes  $[\dots, 1/2, -1, 1/2, \dots]$ , depending on whether the players holding  $\geq 0$  chips held  $0$  or  $> 0$  many chips.

Each possible configuration that generates a  $k + 1$  label results in its neighbors being less than or equal to  $k - 1$ , and the configuration that leads to a  $-1$  label results in its neighbors being greater than or equal to  $1$ . Our proof is now complete by induction. □

## 6 Conclusions and Future Work

### 6.1 Candy Sharing

In this paper, we analyze the candy sharing game when the number of players (length of cycle) is fixed, and the initial amount of candy is variable and tending to infinity. One interesting aspect of the candy sharing game is to consider what happens when the initial amount of candy is fixed and the number of players tends to infinity. The most natural instance of this game would be the following.

Consider the game played on  $\mathbb{N} \cup \{0\}$  with  $n$  pieces of candy starting at 0. At each time step half the pile of candy at position  $i$  remains in place while half moves to  $i + 1$ . The rounding rule remains the same. Playing on the natural numbers simulates a very large cycle. A run through of this game with  $n = 10$  yields

$$(10) \rightarrow (6, 6) \rightarrow (4, 6, 4) \rightarrow (2, 6, 6, 4) \rightarrow (2, 4, 6, 4, 2) \rightarrow ((2, 4, 6, 6, 4, 2) \rightarrow (2, 4, 6, 6, 6, 4, 2) \dots$$

where the ellipses represents a continued propagation of the number 6. On all simulations of the game, we have observed that the game with initial amount  $n$  eventually reaches a state

$$(2, 4, 6, \dots, \alpha(n) - 2, \alpha(n), \alpha(n), \alpha(n), \dots, \alpha(n), \alpha(n) - 2, \dots, 6, 4, 2)$$

where the  $\alpha(n)$  propagates. We would like to understand the asymptotics of this function  $\alpha(n)$ . Numerical experiments seem to suggest that it grows approximately on the order of  $n^{.6968\dots}$ .

Another direction to go would be to prove an analogue of Theorem 5 for more general candy sharing games. That is, instead of each player passing half to their neighbor at each step, each player passes a rational proportion ( $p/q$ ). At each step each player rounds up to the nearest multiple of  $q$ . This more general game was analyzed by Iba and Tanton in [4] where they prove that such games stabilize after finitely many steps. However, they do not prove any bounds on the length of the game or the amount of candy drawn. It would certainly be interesting to study whether the techniques used in this thesis transfer to the more general setting.

Moving along these lines, it would also be interesting to prove analogous results to Theorem 5 for candy sharing games played on general graphs (not just the cycle). In their paper [4], Iba and Tanton analyzed a game similar to the one described in the previous paragraph with the change that a player(vertex) shares some proportion of their candy to each neighbor, and always rounds up to the

nearest multiple of their vertex's degree. Iba and Tanton determine criteria for such games to reach a steady state, but no bounds on length nor amount of candy drawn.

## 6.2 Diffusion Game

The main open problem concerning the diffusion game is still the problem stated by Long and Narayanan [3]. The number of possible configurations grows very quickly with the maximum degree, and a brute force argument used in Theorem 6 will not suffice. We conjecture that the diffusion game played on an infinite graph  $G$  with maximum degree  $\Delta(G)$ , using vertex labels  $\{0, 1, 2, \dots, k\}$  will have its vertex labels bounded above by  $k - 1 + \Delta(G)$  and below by  $1 - \Delta(G)$  for all time.

Long and Narayanan proved that any initial chip configuration on a finite graph will always exhibit oscillatory or stable behavior in a finite number of steps. This team didn't address the length of the games. Further analysis of this game focuses on the expected number of turns needed for a game to settle in its final stage(s). Simulations on cycles of increasing length  $n$  using random labels of 0 and 1 give evidence that this expected number of turns is sub-linear, despite  $n$  of the possible configurations needing a linear amount of time to end.

## References

- [1] C. Duffy, T.F. Lidbetter, M.E. Messinger, R.J. Nowakowski, *A Variation on Chip-Firing: the diffusion game*, 2017. <https://arxiv.org/abs/1609.05792>
- [2] D. A. Levin, Y. Peres, E.L. Wilmer, *Markov chains and mixing times. With a chapter by James G. Propp and David B. Wilson*. American Mathematical Society, Providence, RI, 2009. xviii+371 pp.
- [3] J. Long, and B. Narayanan. *Diffusion on Graphs is Eventually Periodic*, 2017. <https://arxiv.org/abs/1704.04295>
- [4] G. Iba, J. Tanton, *Candy sharing*. Amer. Math. Monthly 110 (2003), no. 1, 25 - 35. <https://www.jstor.org/stable/3072341>