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Topological Dynamics of Two-Piece Eventually Expanding Maps

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Topological dynamics of two-piece eventually expanding maps

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Abstract

In this work we show that two-piece eventually expanding maps have the same topological dynamics as two-piece expanding maps. A two-piece eventually expanding map possesses an invariant set that is either a topological attractor or can be perturbed to become one.

Keywords: Expanding map; Attractor; Lorenz attractor; Geometric model; Invariant measure

1. Introduction

The goal of this work is to show that two-piece eventually expanding maps have the same topological dynamics as two-piece expanding maps. We mean by "two-piece map" a map that has one discontinuity and is monotone on either side of it. The original Lorenz map [4,7], which is defined on $[-1, 1]$ to itself and discontinuous at the origin, is an example. While the Lorenz map has derivative greater than $\sqrt{2}$ on $[-1, 0) \cup (0, 1]$, the maps that we consider in this work only have to be monotone off the discontinuity and eventually expanding. We consider only monotone maps because we are interested in generalizing the result of the Lorenz map. Two-piece maps with local extrema are typically studied with the same techniques as are used in the study of multimodal maps, so we leave out such cases here. Once we establish the topological dynamics of maps with one discontinuity, maps with more than one discontinuity can be studied in the same manner.

Recently Morales and Pujals [1] and Choi [2,3] investigated the dynamics of maps with the properties below, which are denoted by $\mathcal{E}_1([a, b], c)$:

(a) $f : [a, b] \longrightarrow [a, b]$.
(b) f has a discontinuity at a c where $a < c < b$ and $f(c) = c$.
(c) $\lim_{x \to c^-} f(x) = r^-$ and $\lim_{x \to c^+} f(x) = r^+$ exist.
(d) f is C^1 on $[a, b] \setminus \{c\}$ and there is a number $\lambda > 1$ such that $|f'(x)| > \lambda$ whenever $f'(x)$ exists. (Or $f(x)$ is expanding off c, allowing corner points.)

Morales and Pujals [1] showed that a map f in $\mathcal{E}_1([a, b], c)$ possesses an invariant set L_f in which the restricted map $f|L_f$ is topologically transitive, and the stable manifold $W^s(L_f)$ of the invariant set is open and dense in $[a, b]$.

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Choi [2] classified the maps in $\mathcal{E}_1([a, b], c)$ and proved that for certain members of $\mathcal{E}_1([a, b], c)$, the invariant set L_f is a topological attractor. Choi later [3] proved that the set of topological attractors is open and dense (in the topology specified in the next section) in the set of invariant sets of the maps in $\mathcal{E}_1([a, b], c)$.

In this work, we replace the derivative condition (d) by (e)–(g) below:

(e) f is piecewise C^1 on $[a, b] \setminus \{c\}$.
(f) f is strictly monotone on $[a, c)$ and $(c, b]$.
(g) There is a number $\lambda > 1$ such that for all x, there is an $n = n(x)$ such that $|\frac{\mathrm{d}}{\mathrm{d}x} f^n(x)| > \lambda > 1$ when the derivative exists.

Condition (e) means there can be finitely many corner points $\{d_1, d_2, \ldots, d_k\}$ such that f is C^1 between corner points. Condition (f) means that f is increasing or decreasing without plateau on each branch of $[a, b] \setminus c$. Condition (g) implies that for a sufficiently small interval U, there is an $n = n(U)$ such that $\ell(f^n(U)) > \lambda \ell(U)$ where $\ell(U)$ denotes the length of the interval U, and hence the map f is *eventually expanding*. Denote by $\mathcal{F}_1([a, b], c)$ the collection of maps satisfying (a)–(c) and (e)–(g). Without loss of generality set $c = 0$. We will need to trace the orbits of points near the discontinuity 0, so we write $f^i(0^+) = f^{i-1}(r^+)$ and $f^i(0^-) = f^{i-1}(r^-)$ for all $i \geq 1$.

In the next section, we show how the invariant set for a map in $\mathcal{F}_1([a, b], c)$ is built and summarize its properties.

2. Adjustment for eventually expanding maps

Consider an open interval U in $[a, b]$ with nonempty interior. The interval U will keep expanding under the iteration of f and eventually hit the discontinuity. Define $S(U)$ to be the number of iterations needed for $f^{S(U)}(U)$ to hit the discontinuity for the first time, $S(U) := \min\{n \in \mathbb{N} : 0 \in f^n(U)\}$. In particular when $U = (-\epsilon, 0)$ and $(0, \epsilon)$, respectively, we write $S(U) = S(-\epsilon)$ and $S(\epsilon)$, respectively. For a subset V of U, it is always true that $S(V) \geq S(U)$.

The following lemma states that for any map in $\mathcal{F}_1([a, b], c)$ there is an interval with a self-covering property. This is a generalization of the idea of Morales and Pujals [1].

Lemma 2.1. *Let $f \in \mathcal{F}_1([a, b], 0)$. There exists a $\delta > 0$ such that for any $0 < \epsilon < \delta$ there are positive integers $n_l = n_l(\epsilon)$, $n_r = n_r(\epsilon)$ such that $(-\delta, \delta) \subset f^{n_l}(-\epsilon, 0)$ and $(-\delta, \delta) \subset f^{n_r}(0, \epsilon)$.*

Proof. We adapt the proof of [1] or [2] for eventually expanding maps. Fix an $\alpha > 1$ and i_γ such that $\lambda^{i_\gamma} > 2\alpha$. Choose a $\gamma > 0$ small so that both $(-\gamma, 0)$ and $(0, \gamma)$ hit the discontinuity 0 after having expanded by a factor λ^{i_γ},

$$\ell(f^{S(-\gamma)}(-\gamma, 0)) > \lambda^{i_\gamma} \ell((-\gamma, 0)) > 2\alpha\gamma \quad \text{and}$$
$$\ell(f^{S(\gamma)}(0, \gamma)) > \lambda^{i_\gamma} \ell((0, \gamma)) > 2\alpha\gamma.$$

Because f is piecewise C^1 on $[a, b] \setminus \{0\}$, any subset of $(0, \gamma)$ has the same amount of minimum expansion under $f^{S(\gamma)}$ for a small enough γ. Therefore for any positive $\eta < \gamma$, we have $S(\eta) \geq S(\gamma)$ and

$$\ell(f^{S(\eta)}(0, \eta)) \geq \ell(f^{S(\gamma)}(0, \eta)) \geq \lambda^{i_\gamma} \eta > 2\alpha\eta.$$

The discontinuity 0 is in $f^{S(\eta)}(0, \eta)$; hence

$$(0, \alpha\eta) \subset f^{S(\eta)}(0, \eta) \quad \text{or} \quad (-\alpha\eta, 0) \subset f^{S(\eta)}(0, \eta).$$

Applying the same argument to $-\eta$, we have $S(-\eta) \geq S(-\gamma)$ and

$$\ell(f^{S(-\eta)}(-\eta, 0)) \geq \lambda^{i_\gamma} \eta > 2\alpha\eta.$$

Consequently

$$(0, \alpha\eta) \subset f^{S(-\eta)}(-\eta, 0) \quad \text{or} \quad (-\alpha\eta, 0) \subset f^{S(-\eta)}(-\eta, 0).$$

Choose $0 < \delta < \gamma$ so that $(-\delta, \delta) \subset f^{S(\gamma)}(0, \gamma) \cap f^{S(-\gamma)}(-\gamma, 0)$. For any $0 < \epsilon < \delta$, let $E_0 = (0, \epsilon)$ and E_1 be the longer component of $f^{S(E_0)}(E_0) \setminus \{0\}$. Then E_1 covers $(-\alpha\epsilon, 0)$ or $(0, \alpha\epsilon)$ because

$$\ell(E_1) > \frac{1}{2} \ell(f^{S(E_0)}(E_0)) > \frac{1}{2} 2\alpha\epsilon = \alpha\epsilon.$$

Inductively, define E_i to be the longer component of $f^{S(E_{i-1})}(E_{i-1}) \setminus \{0\}$ for $i \geq 2$. By the construction and our choice of γ, each E_i satisfies $f^{S(E_i)}(E_i) \supset E_{i+1}$ and covers at least one of the two intervals $(-\alpha^i\epsilon, 0)$ and $(0, \alpha^i\epsilon)$ as long as $\alpha^i\epsilon < \gamma$. Eventually $(0, \gamma) \subset f^{S(E_n)}(E_n)$ for some n. Then

$$(0, \gamma) \subset f^{S(E_n)}(E_n) \subset \cdots \subset f^{S(E_n)+\cdots+S(E_0)}(E_0).$$

By our choice of $(-\delta, \delta)$, we have

$$(-\delta, \delta) \subset f^{S(\gamma)}(0, \gamma) \subset f^{S(\gamma)+S(E_n)+\cdots+S(E_0)}(E_0) = f^{S(\gamma)+S(E_n)+\cdots+S(E_0)}(0, \epsilon).$$

Set $n_r = S(\gamma) + S(E_n) + \cdots + S(E_0)$. We can get n_l in a similar way. □

We can in fact choose intervals $U_l \subset (-\epsilon, 0)$ and $U_r \subset (0, \epsilon)$ that cover $(-\delta, \delta)$ as one piece. In other words $S(U_l) = n_l$ and $S(U_r) = n_r$. This fact is useful for proving the following theorem. We denote by $\mathrm{bd}(X)$, $\mathrm{int}(X)$, and $\mathrm{cl}(X)$, respectively, the boundary, interior, and the closure of X, respectively, and follow the works of [2,3].

Theorem 2.2. *For an $f \in \mathcal{F}_1([a, b], 0)$ define $I_f = \bigcup_{i\geq 0} f^i(-\delta, \delta)$ and $L_f = cl(I_f)$. The set L_f satisfies the following.*

(a) *L_f consists of finitely many closed intervals whose end points are images of 0^- and 0^+.*
(b) *The restricted map $f|L_f$ is topologically transitive.*
(c) *$W^s(L_f) = \{x : f^n(x) \to L_f\}$ is open and dense in $\mathbb{R}$.*
(d) *The set of all periodic points of $f|L_f$, $Per(f|L_f)$, is dense in L_f.*
(e) *For any open set U in L_f there is an integer $K = K(U)$ such that $\bigcup_{i=0}^{K} f^i(U) = I_f = \mathrm{int}(L_f)$.*

Result (e) is a weaker version of *locally eventually onto* defined in [7] which requires just $f^K(U) = I_f$; therefore Choi [2] called this property *weakly locally eventually onto*. The weaker result was caused from the weaker condition on the derivative of f.

Like in the case of expanding maps, L_f is a topological attractor if there is no periodic orbit on its boundary. The end points of the components of L_f are images of 0^- and 0^+, and when one of them is eventually periodic in $\mathrm{bd}(L_f)$ this will create a periodic orbit in $\mathrm{bd}(L_f)$. Let

$$\mathcal{C}_1([a, b], 0) = \left\{ f \in \mathcal{F}_1([a, b], 0) : \bigcup_{i=1}^{\infty} (f^i(0^-) \cup f^i(0^+)) \subset \mathrm{bd}(L_f) \right\}$$

and $\widetilde{\mathcal{F}}_1 = \mathcal{F}_1([a, b], 0) \setminus \mathcal{C}_1([a, b], 0)$. A member f of $\widetilde{\mathcal{F}}_1$ does not have any periodic orbit on L_f. Moreover, we have the following result:

Theorem 2.3. *Let $f \in \widetilde{\mathcal{F}}_1([a, b], 0)$. Then:*

(a) *There is a trapping region $T \subset \mathbb{R}$ such that $L_f = cl(\bigcap_{i\geq 0} f^i(T))$.*
(b) *$\{L_f : f \in \widetilde{\mathcal{F}}_1([a, b], 0)\}$ is open and dense in $\{Ł_f : f \in \mathcal{F}_1([a, b], 0)\}$ in the Hausdorff metric d_H: given $f \in \widetilde{\mathcal{F}}_1([a, b], 0)$ and an $\eta > 0$, there is an open neighborhood $\mathcal{M}$ of f in the C^0-topology such that $\mathcal{M} \subset \widetilde{\mathcal{F}}_1([a, b], 0)$ and for all g in $\mathcal{M}$, $d_H(L_f, L_g) < \eta$; given $f \in \mathcal{F}_1([a, b], 0)$ and an $\eta > 0$, there is a $g \in \widetilde{\mathcal{F}}_1([a, b], 0)$ C^0-close to f such that $d_H(L_f, L_g) < \eta$.*

Note that Lasota and Yorke [5] showed that piecewise C^2 expanding maps[1] have absolutely continuous invariant measures, and according to Li and Yorke [6] those measures are unique and there can be as many of them as, but no more than, the number of discontinuities. This means that when there is only one discontinuity, the unique absolutely continuous invariant measure is ergodic. Therefore if a map $f \in \mathcal{F}_1([a, b], 0)$ is furthermore piecewise C^2, then its invariant set L_f is the support of the ergodic measure discussed in [6].

[1] They mentioned that the result is also true for eventually expanding maps.

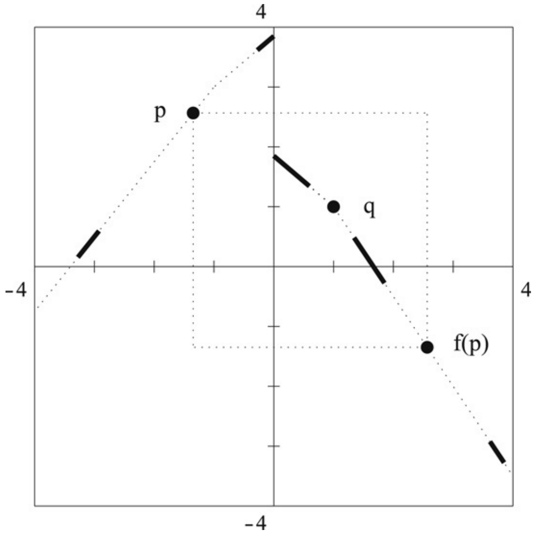

Fig. 1. Periodic points $\{p, f(p)\}$ and a fixed point q in the gaps.

3. Example

Consider $f : [-4, 4] \to [-4, 4]$ such that

$$f(x) = \begin{cases} 1.25x + 4.25 & -4 \le x < -1 \\ 0.85x + 3.85 & -1 \le x < 0 \\ 0 & x = 0 \\ -0.85x + 1.85 & 0 \le x < 1 \\ -1.5x + 2.5 & 1 \le x < 4. \end{cases}$$

The expanding factor $|f'(x)| < 1$ on $(-1, 0)$ and $(0, 1)$, but $|f^{(2)}(x)| = |(0.85) \cdot (-1.5)| = 1.275 > 1$ on $(-1, 0)$ and $|f^{(2)}(x)| = |(-0.85) \cdot (1.25)| = 1.0625 > 1$ on $(0, 1)$. The invariant set consists of four intervals $[f^2(0^-), f^4(0^+)]$, $[f^2(0^+), f^5(0^+)]$, $[f^6(0^+), f(0^+)]$, and $[f^3(0^+), f(0^-)]$ with $0 \in [f^2(0^+), f^5(0^+)]$, and the gaps $[a, b] \setminus L_f$ contain a periodic orbit $\{p, f(p)\}$ and a fixed point q. By taking open neighborhoods of p, $f(p)$, and q, we get a trapping region T; hence L_f is a topological attractor (Fig. 1).

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