Gravitational-Wave Memory from Black Hole and Neutron Star Mergers

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Abstract

The detection of gravitational waves from binary black hole and binary neutron star mergers has ushered in a new age of observational astronomy. Anticipation of detection from these coalescing compact binaries has led to the development of models for comparison using analytical and numerical techniques. Typically, these methods model gravitational-wave signals as small oscillations that grow over time, reach some maximum value, and eventually decay to zero. However, these models are incomplete: compact binaries can emit gravitational waves that decay to a non-zero value. This phenomenon is known as the gravitational-wave memory. In particular, the signal from compact binaries displays a nonlinear memory effect, which arises from gravitational waves produced by the previously emitted gravitational-wave energy. Using a semi-analytic approach we generate nonlinear memory signals for a range of binary black hole parameters, extending previous work. We also, for the first time, compute the nonlinear memory for binary neutron star mergers. Additionally, we perform the first comparison between our semi-analytic approach and full numerical relativity simulations of the nonlinear memory. These waveforms will be useful in future searches of the nonlinear memory in ground and space-based detectors.
Gravitational-Wave Memory from Black Hole and Neutron Star Mergers

by

Matthew Karlson

A Master's Thesis Submitted to the Faculty of
Montclair State University
In Partial Fulfillment of the Requirements
For the Degree of
Master of Science
May 2018
GRAVITATIONAL-WAVE MEMORY FROM BLACK HOLE AND NEUTRON STAR MERGERS

A THESIS

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Montclair State University

Montclair, NJ

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Chapter 1

Introduction

1.1 Gravitational Waves, Compact Binaries, and Detection

Over a hundred years ago in 1916, Albert Einstein predicted the existence of gravitational waves in his theory of general relativity. Gravitational waves (GW) are ripples in spacetime curvature that propagate at the speed of light. They are sourced from rapidly changing mass motions. The strongest sources of GWs are compact binaries such as binary black hole (BBH) and binary neutron star (BNS) systems. As the masses in these systems orbit each other they emit gravitational radiation and their orbit slowly shrinks, marking a phase known as the inspiral. Over time, the amount of gravitational radiation emitted grows, moving the masses closer together until they collide and coalesce into one. The collision is known as the merger phase, which leaves a single mass in a highly unstable state, giving off enormous amounts of gravitational radiation. After the merger, the mass enters the ringdown phase and continues to give off GWs until it settles into a stable state. It is during the late inspiral to merger when prospects of GW detection from compact binaries are greatest. Nevertheless, the amplitudes $h$ (the strength of the signal) are on the order of $10^{-19}$ m, roughly 1/10000 the width of a proton. GW detectors at the Laser Interferometer Gravitational-Wave Observatory (LIGO) in the United States [1], Virgo in Italy [2], and GEO 600 in Germany [3] use laser interferometry to measure such small distances with high precision. The detectors are modeled after the Michelson interferometer, which is a device for measuring light interference patterns. LIGO detectors, for instance, consist of two perpendicular arms each formed by two mirrors separated by a distance $L_x = L_y = L = 4$ km. A laser beam is emitted at the junction of the arms and passes through a beam splitter, which sends light down both arms; the light then reflects off the mirror at the end of each arm and recombines at a photodetector generating an interference pattern. When GWs pass through the detector, the length between the mirrors in both arms contracts and expands during the disturbance, effectively altering the arm lengths by amounts $\delta L_x$ and $\delta L_y$. These changes in arm lengths produce an interference pattern that forms when the light rays recombine. The passage of GWs alter the arm lengths in such a
way that the amplitudes measured from the interference pattern are \( h = \frac{\Delta L}{L} \), where \( \Delta L = \delta L_x - \delta L_y \) is the differential length difference. This means that the detectors are sensitive enough to measure \( \Delta L = 10^{-16} \) m.

Since the early 2000’s, this global network of GW detectors has been collecting data on compact binary systems. BBH and BNS mergers are the primary sources due to their higher rate of occurrence (than e.g., supernovae) and they can emit GWs within the detectable range of 10 Hz - 1 kHz [4]. History was made on September 14, 2015 when LIGO detected GWs from a BBH merger for the first time [5]. Since then, LIGO has detected GWs from four more BBH mergers, including the more recent event on August 14, 2017 detected by both LIGO and Virgo [6-9]. On August 17, 2017, only three days after the last BBH detection, LIGO and Virgo made history once more by detecting GWs from a BNS merger for the first time [10]. These detection events are undoubtedly historic achievements. But another important aspect apart from detection is the data analysis done during searches for GWs. LIGO and Virgo use a matched filtering technique that compares observational data with templates of the GW signal. These templates are models generated using analytical techniques and numerical relativity simulations depending on the stage of binary coalescence. The post-Newtonian approximation method is used to model GW emission during the inspiral; it involves solving Einstein’s equations analytically in series expansions of \( v^2/c^2 \), where \( v \) is the orbital velocity and \( c \) is the speed of light. For the merger and ringdown, full numerical relativity simulations of Einstein’s equations are used to model GW emission. There are, essentially, two ways of accomplishing this, either by an extrapolation method or by using Cauchy-Characteristic Extraction (CCE). In the extrapolation method, the GW signal is calculated on the surface of spheres far from the source and a polynomial is fit to the data. The polynomial is then extrapolated out to a region where the signal resembles what would be measured by a GW detector on Earth. CCE, on the other hand, evolves Einstein’s equations in a way that allows the GW signal to be calculated directly and free of coordinate effects.

### 1.2 Gravitational-Wave Memory

Gravitational waves are typically modeled as small oscillations that grow in amplitude over time, reach some maximum value, and then eventually decay down to zero. This model, unfortunately, is not entirely accurate: some sources can emit GWs that decay to a non-zero value. This phenomenon is known as the gravitational-wave memory. The memory refers to a permanent distortion of an idealized detector made up of a ring of freely-falling test masses [11-13]. A GW signal without memory will cause the ring to contract and expand during the disturbance, but then return the ring to its initial state. A GW signal with memory, however, will cause the ring to assume a different state after its passage. The ring, thus, retains “memory” of the GW.

There are two kinds of memory: linear and nonlinear. The linear memory has been known since the 1970’s and arises due to non-periodic motions of binaries on an unbounded (hyperbolic) orbit or from sources that emit matter (e.g. supernovae
and gamma-ray bursts) [11-13]. On the other hand, systems with bounded orbits such as BBHs and BNSs are sources of nonlinear memory. The nonlinear memory is a non-oscillatory effect that arises when the previously emitted gravitational radiation produces GWs; it was discovered independently in the 1990’s by Blanchet and Damour [14], and Christodoulou [15]. During the evolution of the binary system, the nonlinear memory\textsuperscript{1} starts out small during the inspiral and builds up over time as the GW energy is lost, eventually saturating to some final value during the merger and ringdown. This build-up of memory results in an overall vertical shift in the GW signal (see Figure 1.1 below).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Plot of the GW $h_+$ signal with (blue) and without (red) memory for a 1:1 mass ratio BBH. The dashed black curve shows the memory’s growth up to the merger.}
\end{figure}

1.3 Motivation

Over the past several decades, the post-Newtonian (PN) approximation method and numerical relativity (NR) have been used extensively to model GW emission from coalescing binaries. However, these models are incomplete as they do not account for the memory. Currently, the memory can be detected by LIGO, but the sensitivity of the detectors to the oscillatory parts of the GW signal makes this challenging [11-13]. Furthermore, although NR simulations account for the full nonlinear effects of Einstein’s equations, they too have difficulty capturing the memory [16-26]. Despite these challenges, recent work has shown that the detectability of the memory significantly improves as more compact binary mergers are detected [27]. Radio astronomers have determined that the memory could be detected using Pulsar Timing Arrays (PTAs), which can measure GWs over periods of several months to years [28, 29]. As a result, two independent groups have completed searches for memory using PTAs [30, 31]. The detectability and active searches of the memory emphasize a need for accurate models to compare future observational data. The purpose of this

\textsuperscript{1} In the rest of this document, we use the terms ‘nonlinear memory’ and ‘memory’ interchangeably.
thesis is to develop accurate models of the GW memory. The PN approximation method provides accurate analytical formulas of the memory for the inspiral [11,32,33]; while in the merger, the memory can be calculated using NR extracted oscillatory waveforms as input [19,24-26]. Calculations of the memory during the merger can then be matched with the inspiral portion using PN formulas and added to the total GW signal. This PN/NR hybrid approach can benefit both observational and NR data by providing templates for comparison. Ultimately, this work can give new insights on the evolution of BBH and BNS mergers, and serve as another test of general relativity. It could also provide a means of studying the nonlinear effects of general relativity and the neutron star equation of state.

1.4 Summary

Our intention is to build on previous work [11-13,34] in developing the PN/NR hybrid method for computing the memory from non-spinning BBH mergers. In light of GW170817 [10], we extended calculations of the memory with this semi-analytic approach to non-spinning BNS mergers for the first time. We also for the first time compare the memory extracted from NR simulations with semi-analytic calculations. The simulation names and relevant parameters are listed below in Tables 1.1 and 1.2 for reference.

This work is organized as follows. Chapter 2 describes the semi-analytic procedure beginning with the memory formalism in Section 2.1, while Sections 2.2 - 2.3 summarize the numerical schemes and matching technique used in the calculations. In Chapter 3, we discuss the memory calculations in the black hole case. Sections 3.1 - 3.7 give an overview of the oscillatory modes; examine the dependence of the memory on choice of summation indices $l' = l'' = \ell_{\text{max}}$ and multipolar order $l$; compare an approximation of the memory in terms of the change in GW energy $\Delta E$; examine the dependence of the memory on the reduced mass ratio $\eta$ and source angle $\Theta$; and compare the $h_+$ signal with and without memory. In Chapter 4, we discuss the memory calculations in the neutron star case. In Sections 4.1 - 4.7, we do similar analyses as in the black hole case. The only difference is in Section 4.5, where we examine the dependence of the memory on the neutron star equation of state. In Chapter 5, we compare the memory extracted from NR simulations with our semi-analytic calculations. In Sections 5.1 - 5.2, we compare the CCE and extrapolated $\dot{h}_{22}$ modes, and the CCE and semi-analytic CCE memory modes. Lastly, Chapter 6 summarizes conclusions drawn from our results and discusses future work.

Some key results for the black hole case are Figures 3.3 - 3.6, 3.13 - 3.16, and 3.18 - 3.22. Figures 3.3 - 3.6 show that summing past $\ell_{\text{max}} = 4$ and $\ell_{\text{max}} = 5$ in (2.1.5) for the (2,0) and (4,0) memory modes, respectively, adds small corrections. In Figures 3.13 - 3.16, the memory amplitude decreases as $\eta$ gets smaller. We include the new 7:1, 9:1, and 10:1 mass ratio cases for completeness (cf. Ref. [34]). Figures 3.13 - 3.16 show that the memory significantly alters the total GW $h_+$ signal during the late

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2 The neutron star equation of state is a thermodynamic equation relating the pressure and density of nuclear matter in the core of the star.
inspiral to merger. For the neutron star case, some key results are Figures 4.2 - 4.3, 4.10 - 4.11, and 4.13 - 4.16. Figures 4.2 - 4.3 show that summing up to $\ell_{\text{max}} = 4$ in (2.1.5) adds significant corrections to the memory modes beyond $\ell_{\text{max}} = 2$. Figures 4.10 - 4.11 show that the amplitude of the memory depends on the equation of state. In these plots, there are notable differences in the memory amplitudes despite the binaries being close to or at equal mass. Figures 4.13 - 4.16 show, as in the black hole case, the memory significantly alters the total GW $h_+$ signal during the late inspiral to merger. In Chapter 5, the main results are Figures 5.4- 5.19. These figures show good agreement between the CCE and semi-analytic CCE memory.

<table>
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<tr>
<th>Name</th>
<th>$q$</th>
<th>$\eta$</th>
<th>$e$</th>
<th>$\max{|\chi|}$</th>
<th>$N$</th>
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Table 1.1: Parameters for the BBH simulations. The columns indicate the simulation name, mass ratio $q = m_1/m_2$, reduced mass ratio $\eta = m_1m_2/M^2$ where $M = m_1 + m_2$, initial eccentricity $e$, spin component with the maximum magnitude $\max\{\|\chi\|\}$, and number of orbits prior to the merger $N$.

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Table 1.2: Parameters for the BNS simulations. The columns indicate the equation of state EOS, simulation name, mass $M_A$ of neutron star $A$ (in solar masses), mass $M_B$ of neutron star $B$ (in solar masses), mass ratio $q$, reduced mass ratio $\eta$, and number of orbits prior to the merger $N$. 
Chapter 2

PN/NR Hybrid Method

2.1 Memory Formalism

Once a source emits GWs, they propagate throughout all three dimensional space in combinations of two polarizations: (+) and (×), which differ by a 45° rotation and have amplitudes denoted by \( h_+ \) and \( h_\times \). Given this scenario, it is more convenient to use spherical coordinates \((R, \Theta, \Phi)\) to describe GWs emitted from quasi-circular compact binaries. Here \( R \) is the distance to the source and the angles \((\Theta, \Phi)\) indicate the direction from source to observer specified by a unit vector \( \vec{N} \). We restrict the discussion to orbits lying in the \( x-y \) plane. This means the angular momentum \( \vec{L} \) of the binary system points along the \( z \) axis and \((\Theta, \Phi)\) are relative to \( \vec{L} \) (see Figure 2.1 below). In this configuration, we follow the formalism in Ref. [11] to describe the memory. We begin by decomposing the GW polarizations onto an infinite sum of spin-weighted spherical harmonic basis functions

\[
h_+ - i h_\times = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} h_{lm} - 2Y_{lm}(\Theta, \Phi),
\]

(2.1.1)

where \( h_{lm} \) are the gravitational waveform modes and \(-2Y_{lm}(\Theta, \Phi)\) are the spin-weighted spherical harmonic functions expressed in terms of the Wigner \( d \) functions by

\[
-sY_{lm}(\Theta, \Phi) = (-1)^s \sqrt{\frac{2l+1}{4\pi}} d_{ms}^l(\Theta)e^{im\Phi}.
\]

(2.1.2)

In (2.1.2)

\[
d_{ms}^l(\Theta) = \sqrt{(l + m)!(l - m)!(l + s)!(l - s)!} \times \sum_{k=k_i}^{k_f} (-1)^k (\sin \frac{\Theta}{2})^{2k-m+s} (\cos \frac{\Theta}{2})^{2l-2k+m-s} \frac{k!(k - m + s)!(l + m - k)!(l - k - s)!}{k!(l + m + s)!(l + m - k)!},
\]

(2.1.3)

where \( s \) is the spin weight, \( k_i = \max(0, m - s) \), and \( k_f = \min(l + m, l - s) \).
Figure 2.1: Configuration of the binary system consisting of masses $m_1$ and $m_2$ orbiting in the $x$-$y$ plane.

The memory contribution to (2.1.1) is

$$h_{\pm}^{\text{(mem)}} - i h_{\times}^{\text{(mem)}} = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} h_{lm}^{\text{(mem)}} Y_{lm}(\Theta, \Phi), \quad (2.1.4)$$

where $h_{lm}^{\text{(mem)}}$ are the memory waveform modes. We can express $h_{lm}^{\text{(mem)}}$ in terms of the GW energy flux $\frac{dE_{gw}}{dt d\Omega}$ by

$$h_{lm}^{\text{(mem)}} = \frac{16\pi}{R} \sqrt{\frac{(l-2)!}{(l+2)!}} \int_{-\infty}^{T_R} dt \int d\Omega \frac{dE_{gw}}{dt d\Omega}(\Omega) Y_{lm}^*(\Omega), \quad (2.1.5)$$

where $T_R = T - R$ is the retarded time, $d\Omega = \sin \theta d\theta d\phi$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, and $*$ means to take the complex conjugate. Note that $(\theta, \phi)$ are different from $(\Theta, \Phi)$.

The GW energy flux is given by

$$\frac{dE_{gw}}{dt d\Omega} = R^2 \frac{G^{2-20}}{16\pi} \sum_{l'=2}^{\infty} \sum_{l''=2}^{\infty} \sum_{m'=l'} \sum_{m''=-l''}^{l''} \hat{h}_{l'm'} \hat{h}_{l''m''}^{*} Y_{l'm'}^{*}(\theta, \phi) Y_{l''m''}(\theta, \phi). \quad (2.1.6)$$

To calculate the memory during the late inspiral to merger, (2.1.5) is the primary formula to be evaluated. The calculation is done beginning with the angular integral first. The angular integral can be simplified by substituting (2.1.6) into (2.1.5). The result is the first time derivative of the memory waveform mode for a given $(l, m)$

$$\hat{h}_{lm}^{\text{(mem)}} = R \sqrt{\frac{(l-2)!}{(l+2)!}} \sum_{l'=2}^{\infty} \sum_{l''=2}^{\infty} \sum_{m'=l'} \sum_{m''=-l''}^{l''} (-1)^{m+m''} \hat{h}_{l'm'} \hat{h}_{l''m''}^{*} G_{l'l''lm'm''}, \quad (2.1.7)$$
where

$$G_{l_1l_2l_3m_1m_2m_3}^{s_1s_2s_3} \equiv \int -s_1 Y_{l_1m_1}(\Theta, \Phi) -s_2 Y_{l_2m_2}(\Theta, \Phi) -s_3 Y_{l_3m_3}(\Theta, \Phi) d\Omega \quad (2.1.8)$$

can be expressed in terms of Gamma functions in Appendix A of Ref. [11]. For instance, if \((l, m) = (2, 0)\) the right hand side of (2.1.7) expanded up to \(l' = l'' = \ell_{\text{max}} = 3\) is

$$\dot{h}_{20}^{\text{(mem)}} = \frac{R}{84} \sqrt{\frac{30}{\pi}} \left\{ |h_{22}|^2 + |h_{22-2}|^2 - |\dot{h}_{20}|^2 - \frac{1}{2} \left( |\dot{h}_{21}|^2 + |\dot{h}_{2-1}|^2 \right) \right\}$$

$$+ \frac{\sqrt{14}}{4} \left( h_{31} \dot{h}_{21}^* + h_{21} \dot{h}_{31}^* - h_{3-1} \dot{h}_{2-1}^* - h_{2-1} \dot{h}_{3-1}^* \right)$$

$$+ \frac{\sqrt{35}}{4} \left( h_{22} \dot{h}_{32}^* - h_{2-2} \dot{h}_{3-2}^* + h_{32} \dot{h}_{22}^* - h_{3-2} \dot{h}_{2-2}^* \right) + \cdots \right\}. $$

The sum in (2.1.7) relates \(\dot{h}_{lm}^{\text{(mem)}}\) to the first time derivatives of the oscillatory waveform modes that appear in the product \(\hat{h}_{l'm'} \hat{h}_{l''m''}^*\). We use oscillatory modes extracted from BBH and BNS simulations [19, 24-26] that are publicly available at [35-37]. These waveform modes are numerically differentiated with respect to time to compute \(\dot{h}_{lm}^{\text{(mem)}}\) in (2.1.7). Although the sum in (2.1.7) is infinite\(^3\) the SXS Gravitational Waveform Catalog [35] provides waveform data up to \(l' = l'' = \ell_{\text{max}} = 8\), while Zenodo [36, 37] provides data up to \(\ell_{\text{max}} = 4\). Therefore, we calculate (2.1.7) up to \(\ell_{\text{max}} = 8\) in the black hole case and up to \(\ell_{\text{max}} = 4\) in the neutron star case.

### 2.2 Numerical Differentiation and Integration of \(h_{lm}\)

In order to calculate \(\dot{h}_{lm}^{\text{(mem)}}\), we must numerically differentiate the oscillatory waveform modes \(h_{lm}\) for each given \((l', m')\) and \((l'', m'')\) prior to computing the product \(\hat{h}_{l'm'} \hat{h}_{l''m''}^*\) that appears in (2.1.7). The data for each waveform mode are output in columns containing the time parameter, the real part \(\text{Re}(h_{lm})\), and the imaginary part \(\text{Im}(h_{lm})\). We use a third order scheme for a set of points \((t_{i-2}, t_{i-1}, t_i, t_{i+1})\) to differentiate the real and imaginary parts of the oscillatory modes in (2.1.7). Note that the derivative is evaluated at each point \(t_i\). The formula is derived by expanding (4.2) for \(n = 3\) in Ref. [38]

\(^3\) Note that while the time derivatives of the memory \((m = 0)\) modes show up in the sum in (2.1.7), upon examining the PN scaling (in \(c = 1\) units), the oscillatory product in (2.1.7) scales like \(h_{l'm'} h_{l''m''}^* \sim \frac{\nu^2}{c^2} v^{10} + O(2)\), where \(\nu\) is the orbital velocity of the binary, while the memory product scales like \(h_{l'0} h_{l''0}^* \sim \frac{\nu^4}{c^2} v^{20} + O(6)\) [11]. This implies that the memory product adds only a 5PN order correction and consequently, we ignored its contribution in the calculations.
\[f'(t_i) = \frac{f(t_{i-2})(t_i - t_{i-1})(t_i - t_{i+1})}{(t_i - t_{i-1})(t_i - t_{i-2})(t_i - t_{i+1})} + \frac{f(t_{i-1})(t_i - t_{i-2})(t_i - t_{i+1})}{(t_i - t_{i-1})(t_i - t_{i-2})(t_i - t_{i+1})} + \frac{f(t_i)(2t_i - t_{i-1} - t_{i+1}) + (t_i - t_{i-1})(t_i - t_{i+1})}{(t_i - t_{i-2})(t_i - t_{i-1})(t_i - t_{i+1})} + \frac{f(t_{i+1})(t_i - t_{i-2})(t_i - t_{i-1})}{(t_i + t_{i-2})(t_{i+1} - t_{i-1})(t_{i+1} - t_i)} + O(h^3), \tag{2.2.1}\]

where \( f \) is either \( \text{Re}(h_{lm}) \) or \( \text{Im}(h_{lm}) \) and \( h = \max(||t_i - t_{i-2}||, ||t_i - t_{i-1}||, ||t_i - t_{i+1}||) \).

The scheme permits the use of a non-uniform step-size, which arises in the data of the NR simulations.

After computing the derivatives of the real and imaginary parts of the oscillatory mode for a given \((l', m')\), the total time derivative of the \( h_{lm} \) waveform is obtained by

\[\dot{h}_{lm} = \text{Re}\left(\hat{h}_{lm}\right) + i \text{Im}\left(\hat{h}_{lm}\right). \tag{2.2.2}\]

For the \((l'', m'')\) modes on the right hand side of (2.1.7), we apply the same differentiation scheme only we take the complex conjugate of (2.2.2) at the end. This procedure is repeated for all the oscillatory modes in (2.1.7) to obtain the total time derivative of the memory waveform \( \dot{h}_{lm}^{(\text{mem})} \).

Once \( \dot{h}_{lm}^{(\text{mem})} \) is calculated, the time derivative is numerically integrated to give the memory waveform during the merger

\[h_{lm}^{(\text{mem}), \text{NR}} = \int_{t_0}^{t} \dot{h}_{lm}^{(\text{mem})} \, dt + C_{lm}, \tag{2.2.3}\]

where the constant of integration \( C_{lm} \) will be determined by a matching technique in Section 2.3. The integral in (2.2.3) is calculated using Simpson’s rule adapted for non-uniform step-size. The scheme is fourth order for a set of points \((t_{i-2}, t_{i-1}, t_i)\).

Note the integral is evaluated at each point \( t_i \). The formula is derived by integrating (3.1) for \( n = 2 \) in Ref. [38]

\[
\int_{t_{i-2}}^{t_i} \dot{h}_{lm}^{(\text{mem})} \, dt = \frac{(t_{i-2} - t_i)^2[\dot{h}_{lm}^{(\text{mem})}(t_{i-2}) - \dot{h}_{lm}^{(\text{mem})}(t_{i-1})]}{6(t_{i-2} - t_{i-1})} - \frac{(t_{i-2} - t_i)[\dot{h}_{lm}^{(\text{mem})}(t_{i-2}) + \dot{h}_{lm}^{(\text{mem})}(t_i)]}{2} - \frac{(t_{i-2} - t_i)^2[\dot{h}_{lm}^{(\text{mem})}(t_{i-1}) - \dot{h}_{lm}^{(\text{mem})}(t_i)]}{6(t_{i-1} - t_i)} + O(h^4), \tag{2.2.4}\]

where \( h = \max(||t_i - t_{i-2}||, ||t_{i-1} - t_i||) \).
2.3 Matching Technique

After computing $h_{\text{ln}}^{(\text{mem})}$ in the merger using (2.2.4), the integration constant $C_{\text{ln}}$ is determined by matching the result with PN calculations of the memory during the inspiral. To find the match point, it is more convenient to use $\omega$, the orbital frequency, rather than the time parameter since PN formulas of the memory depend on $\omega$. An expression for $\omega$ can be derived by noting that during the inspiral the oscillatory modes have the form

$$h_{\text{ln}} = A_{\text{ln}} e^{-i m \varphi},$$  \hspace{1cm} (2.3.1)

where $A_{\text{ln}}$ is the amplitude of the mode and $\varphi$ is a phase that modulates the waveform. Setting $(l, m) = (2, 2)$ in (2.3.1), $\omega$ can be obtained indirectly by differentiating (2.3.1) with respect to time and using the fact that $\omega = \dot{\varphi}$ to give

$$\omega = -\frac{1}{2} \text{Im} \left( \frac{\dot{h}_{22}}{h_{22}} \right).$$  \hspace{1cm} (2.3.2)

After computing $\omega$, the result is used to calculate the PN parameter $x = \left( M \omega \right)^{2/3}$, where $M = m_1 + m_2$ is the total mass of the binary, which is then substituted into PN expressions for the first time derivative of the memory modes. The formulas for the (2,0) and (4,0) modes up to 3PN order are

$$\dot{h}_{20}^{(\text{mem})} = \frac{256}{21} \sqrt{\frac{3\pi}{10R}} \eta^2 x^5 \left\{ 1 + x \left( -\frac{1219}{288} + \frac{1}{24} \eta \right) + 4\pi x^{3/2} \right. \right.$$

$$+ x^2 \left( -\frac{793}{1782} - \frac{14023}{6336} \eta - \frac{4201}{1584} \eta^2 \right) + \pi x^{5/2} \left( -\frac{2435}{144} - \frac{23}{12} \eta \right)$$

$$\left. + x^3 \left\{ \frac{174213949439}{1816214400} + \frac{16}{3} \pi^2 - \frac{856}{105} (2\gamma_E + \ln 16x) + \left( -\frac{126714689}{4447872} + \frac{41}{48} \pi^2 \right) \eta \right. \right.$$  \hspace{2cm} (2.3.3a)

$$\left. + \frac{4168379}{123552} \eta^2 + \frac{142471}{46332} \eta^3 \right) + O(7) \right\}$$

$$\dot{h}_{40}^{(\text{mem})} = \frac{64}{315} \sqrt{\frac{\pi}{10R}} \eta^2 x^5 \left\{ 1 + x \left( -\frac{10133}{704} + \frac{25775}{528} \eta \right) + 4\pi x^{3/2} \right. \right.$$

$$+ x^2 \left( \frac{322533}{4576} - \frac{721593}{2288} \eta - \frac{237865}{5148} \eta^2 \right) + \pi x^{5/2} \left( -\frac{1028}{11} + \frac{11114}{33} \eta \right)$$

$$\left. + x^3 \left\{ \frac{32585924257}{403603200} + \frac{16}{3} \pi^2 - \frac{856}{105} (2\gamma_E + \ln 16x) + \left( \frac{4669843}{164736} + \frac{41}{48} \pi^2 \right) \eta \right. \right.$$  \hspace{2cm} (2.3.3b)

$$\left. + \frac{16531}{52} \eta^2 - \frac{1145725}{92664} \eta^3 \right) + O(7) \right\},$$

where $\eta = m_1 m_2 / M^2$ is the reduced mass ratio and $\gamma_E$ is the Euler-Mascheroni constant.
The match point is found by plotting the PN and NR forms of $h_{lm}^{\text{mem}}$ against $\omega$ on the same graph and finding a point where the two curves overlap, or are close together (within a $10^{-5}$ tolerance) at a small value of $\omega$. This value of $\omega$ gives the corresponding points $(x_{\text{match}}, t_{\text{match}})$, which are used as initial conditions to integrate

$$\frac{dx}{dt} = \frac{64}{5} \eta M x^5 \left\{ 1 + x \left( -\frac{473}{336} - \frac{11}{4} \eta \right) + 4\pi x^{3/2} + x^2 \left( \frac{34103}{18144} + \frac{13661}{2016} \eta + \frac{59}{18} \eta^2 \right) + \eta \right\}$$

$$+ \pi x^{5/2} \left( -\frac{4159}{672} - \frac{189}{8} \eta \right) + x^3 \left[ \frac{16447322263}{139708800} + \frac{16}{3} \eta^2 - \frac{856}{105} (2\gamma_E + \ln 16) \right]$$

$$+ \left( -\frac{56198689}{217728} + \frac{451}{48} \eta^2 \right) + x^4 \left[ 1 + x \left( -\frac{743}{336} - \frac{11}{4} \eta \right) + 4\pi x^{3/2} + x^2 \left( \frac{34103}{18144} + \frac{13661}{2016} \eta + \frac{59}{18} \eta^2 \right) + \eta \right\}$$

backwards in time, generating $x(t)$ and the time parameter from an arbitrary starting time to the matching time. These $x(t)$ values are substituted into PN formulas for the memory

$$h_{lm}^{\text{mem}} = 8 \sqrt{\frac{\pi}{5}} \eta M x \hat{H}_{lm},$$

where $\hat{H}_{lm}$ are the GW polarization modes. For instance, the (2,0) and (4,0) polarization modes up to 3PN order are

$$\hat{H}_{20} = \frac{5}{14\sqrt{6}} \left\{ 1 + x \left( -\frac{4075}{4032} + \frac{67}{48} \eta \right) + x^2 \left( -\frac{151877213}{76060224} - \frac{123815}{44352} \eta + \frac{205}{352} \eta^2 \right) + \pi x^{5/2} \left( -\frac{253}{336} + \frac{253}{84} \eta \right) + x^3 \left[ \frac{4397711103307}{532580106240} + \left( \frac{700464542023}{13948526592} - \frac{205}{96} \eta \right) \right] + O(7) \right\},$$

$$\hat{H}_{40} = \frac{1}{504\sqrt{2}} \left\{ 1 + x \left( -\frac{180101}{29568} + \frac{27227}{1056} \eta \right) + x^2 \left( \frac{2201411267}{158505984} - \frac{34829479}{432432} \eta + \frac{84951}{27456} \eta^2 \right) + \pi x^{5/2} \left( -\frac{13565}{1232} + \frac{13565}{308} \eta \right) + x^3 \left[ \frac{15240463356751}{781117489152} + \left( -\frac{1029744557245}{27897053184} - \frac{205}{96} \eta \right) \right] - \frac{4174614175}{36900864} \eta^2 + \frac{221405645}{11860992} \eta^3 \right\} + O(7).$$

After substituting $x(t)$ into (2.3.5) and (2.3.6), we obtain the memory modes as a function of time during the inspiral $h_{lm}^{\text{mem}}, \text{PN}(t)$. 

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The integration constant $C_{lm}$ is found by taking the difference between the inspiral and merger memory waveforms at the match point, i.e.

$$C_{lm} = h_{lm}^{\text{(mem)}, \text{PN}}(t_{\text{match}}) - h_{lm}^{\text{(mem)}, \text{NR}}(t_{\text{match}}),$$

where $h_{lm}^{\text{(mem)}, \text{PN}}$ is given by (2.3.5) and $h_{lm}^{\text{(mem)}, \text{NR}}$ is given by (2.2.3). The total memory waveform $h_{lm}^{\text{(mem)}}$ is then constructed by combining (2.2.3), (2.3.5), and (2.3.7) to give

$$h_{lm}^{\text{(mem)}}(t) = \begin{cases} h_{lm}^{\text{(mem)}, \text{PN}}(t) & \text{for } t < t_{\text{match}} \\ h_{lm}^{\text{(mem)}, \text{NR}}(t) + C_{lm} & \text{for } t \geq t_{\text{match}} \end{cases}.$$ 

Finally, now that we have the full memory waveform, the GW polarization signals in (2.1.1) can be computed.

Note that in the preceding formalism, the memory comes out real despite being a function of complex numbers (cf. (2.1.7)). The reason for this is that the spherical harmonics $-2Y_{lm}(\Theta, \Phi)$ in (2.1.2) are real. The spherical harmonics come out real in part due to restricting the memory formalism to binary orbits in the $x$-$y$ plane (see Section III B in Ref. [11] for further discussion). For this particular orbital configuration, only the $m = 0$ modes have memory, which implies that the exponential term $e^{im\Phi}$ is 1 in (2.1.2). This means that the memory does not depend on the source angle $\Phi$. Moreover, the $h_+$ memory vanishes. Therefore, the memory only affects the $h_\times$ polarization signal. However, due to machine precision the memory calculations in this study resulted in complex numbers with imaginary parts on the level of numerical noise (i.e. $<10^{-20}$). As a consequence, for all plots of the memory modes in Chapters 3 - 5, the labels indicate the real part $\text{Re}(h_{l0}^{\text{(mem)}})$. 


Chapter 3

Memory from Binary Black Holes

In the black hole case, we calculated the memory for ten non-spinning BBH simulations with mass ratios from 1:1 to 10:1 in integer increments. We included the 7:1, 9:1, and 10:1 cases, which were not previously considered (cf. Ref. [34]). Section 3.1 begins with an overview of the oscillatory modes from the SXS Catalog [35]. Since these waveforms are used as input in our procedure, this investigation gives insights into which oscillatory modes influence the memory. In Section 3.2, we analyze the dependence of the memory on choice of summation index $l' = l'' = \ell_{\text{max}}$ in (2.1.7). The purpose is to see if the higher modes add significant corrections to the memory. Note that the SXS group [35] provides waveforms up to $l = 8$ only, so we calculated (2.1.7) up to $\ell_{\text{max}} = 8$. Section 3.3 examines the amplitudes of the memory modes. This analysis will determine which modes are the most influential in (2.1.4). Section 3.4 then compares the GW energy flux formula (2.1.5) with an approximation in terms of the change in GW energy $\Delta E$. This comparison serves as a test of the approximation formulas (3.4.10). In Section 3.5, we analyze the memory’s dependence on the reduced mass ratio $\eta$. This analysis shows how the amplitude varies for different mass ratio configurations. Section 3.6 examines the dependence of the memory on the source angle $\Theta$ to see how the amplitude varies as the direction of the source changes. Section 3.7 compares the $h_+$ signal with and without memory to show visually how the signal is altered.

As a reminder, the main results are Figures 3.3 - 3.6, 3.13 - 3.16, and 3.18 - 3.22. These plots show that summing past $\ell_{\text{max}} = 4$ in the $(2,0)$ mode and $\ell_{\text{max}} = 5$ in the $(4,0)$ mode result in small corrections; that the memory’s amplitude decreases as $\eta$ gets smaller; and that the memory notably alters the $h_+$ polarization signal near the merger. Other important results are in Figures 3.9 - 3.12 and 3.17. These plots show two main things: that the GW energy flux formula (2.1.5) adds small, but significant corrections to the $\Delta E$ approximations in (3.4.2) near the merger; and that the memory has the largest amplitude when the source angle is at $\Theta = 90^\circ$, i.e. when the unit vector $\vec{N}$ lies in the $x$-$y$ plane in Figure 2.1. This is equivalent to observing the binary edge on. Another noteworthy result is in Figures 3.7 and 3.8. These plots show that the $(2,0)$ and $(4,0)$ modes are the most influential in the $h_+^{(\text{mem})}$ waveform, while the higher even modes make minimal contributions and the odd modes are zero.
3.1 Overview of the Oscillatory Modes $h_{lm}$

We first start with an overview of the range of amplitudes of the oscillatory modes. In the equal mass case, Figure 3.1 shows a wide range of amplitudes of the oscillatory modes. The $h_{22}$ mode has the largest amplitude, while for the higher $(l, m)$ modes the amplitude is much smaller. For the 4:1 case, Figure 3.2 shows that the $h_{22}$ mode dominates again; however, the higher $(l, m)$ modes have a larger amplitude than in the equal mass case. Although not shown here, this behavior is similar in the other unequal mass ratio configurations considered (see Table 1 in Section 1.4). Also, the amplitude of the $(2, 2)$ mode maintains a similar range for the other unequal mass ratio cases. These plots show that the $(2, 2)$ mode has the largest amplitude and, due to the structure of (2.1.7), is the most influential in the memory modes.

![Figure 3.1](image1.png)  
Figure 3.1: Log plot of $|h_{lm}|$ for a 1:1 mass ratio ($\eta = 0.25$) BBH showing how the amplitudes vary among the oscillatory $(l, m)$ modes.

![Figure 3.2](image2.png)  
Figure 3.2: Log plot of $|h_{lm}|$ for a 4:1 mass ratio ($\eta = 0.16$) BBH showing how the amplitudes vary among the oscillatory $(l, m)$ modes.
3.2 Dependence of $h_{l0}^{(\text{mem})}$ on Choice of $\ell_{\text{max}}$

The choice of summation limit $\ell_{\text{max}}$ can change the amplitude of the memory modes. We examine the corrections added to the memory modes when $\ell_{\text{max}}$ is varied. Figures 3.3 and 3.4 show the 1:1 and 9:1 mass ratio configurations. In both plots, the $\ell_{\text{max}} = 2$ waveform clearly reaches a smaller value than the curves for $\ell_{\text{max}} \in [3, 8]$. If the sum were truncated at $\ell_{\text{max}} = 2$, this would result in roughly a 10.2% error for the 1:1 case in Figure 3.3 and a 5.3% error for the 9:1 case in Figure 3.4. Upon further investigation, we found that as $\eta$ gets smaller, the error between the $\ell_{\text{max}} = 2$ and higher $\ell_{\text{max}}$ curves decreases, though still remains significant ($\geq 3\%$) in the other mass ratio configurations. For instance, in the 7:1 and 10:1 cases (not shown here), the error would be about 7.7% and 5.2%, respectively. In all cases considered, summing past $\ell_{\text{max}} = 4$ results in small corrections ($< 1\%$) to the memory modes.

![Figure 3.3](image1.png)

Figure 3.3: Plot of $\text{Re}(h_{20}^{(\text{mem})})$ without PN matching and with $\ell_{\text{max}} \in [2, 8]$ for a 1:1 mass ratio ($\eta = 0.25$) BBH.

![Figure 3.4](image2.png)

Figure 3.4: Plot of $\text{Re}(h_{20}^{(\text{mem})})$ without PN matching and with $\ell_{\text{max}} \in [2, 8]$ for a 9:1 mass ratio ($\eta = 0.09$) BBH.
For the (4, 0) mode, summing up to $\ell_{\text{max}} = 2$ would result in an error of about 51% for the 1:1 case in Figure 3.5 and 162% for the 9:1 case in Figure 3.6. In particular, in Figure 3.6 not only is the $\ell_{\text{max}} = 2$ waveform different, but now the $\ell_{\text{max}} \in [3, 8]$ curves behave differently near the end of the simulation. This qualitative feature of the (4, 0) mode will be discussed further in Section 3.6. For the remaining mass ratio configurations (not shown here), the error goes down to 11% in the 2:1 case, jumps up to 600% in the 3:1 case, and then decreases as $\eta$ gets smaller thereafter. For the 7:1 and 10:1 cases, the error is around 175% and 160%, respectively. In contrast to the (2, 0) mode, summing past $\ell_{\text{max}} = 5$ results in small corrections ($\leq 2\%$) in all cases considered. However, note that the (4, 0) mode is typically a factor of 10 smaller than the (2, 0) mode.

Figure 3.5: Plot of Re($h_{40}^{(\text{mem})}$) without PN matching and with $\ell_{\text{max}} \in [2, 8]$ for a 1:1 mass ratio ($\eta = 0.25$) BBH.

Figure 3.6: Plot of Re($h_{40}^{(\text{mem})}$) without PN matching and with $\ell_{\text{max}} \in [2, 8]$ for a 9:1 mass ratio ($\eta = 0.09$) BBH.
3.3 Dependence of $h_{l0}^{(\text{mem})}$ on $l$

We now analyze how the memory’s amplitude depends on the multipolar order $l$. In both Figures 3.7 and 3.8, the $(2, 0)$ mode clearly has the highest amplitude, followed by the $(4, 0)$ mode. The higher even modes ($(6, 0)$ and $(8, 0)$) contribute minimally to the memory signal with amplitudes ranging from $10^{-9}$ to $10^{-5}$. As for the odd modes ($(3, 0)$, $(5, 0)$, $(7, 0)$), they vanish [11,13]; but due to machine precision, they are near the level of numerical noise in both plots. These features were observed in the other binary configurations considered. The $(2, 0)$ and $(4, 0)$ modes contribute the most to the total memory waveform $h_+$ in (2.1.4).

Figure 3.7: Log plot of $|h_{l0}^{(\text{mem})}|$ with $l \in [2, 8]$ without PN matching for a 1:1 mass ratio ($\eta = 0.25$) BBH.

Figure 3.8: Log plot of $|h_{l0}^{(\text{mem})}|$ with $l \in [2, 8]$ without PN matching for a 9:1 mass ratio ($\eta = 0.09$) BBH.
3.4 Comparison of $h_{l0}^{(\text{mem})}$ in Terms of $\Delta E$

The memory is related analytically to the GW energy flux in (2.1.5). Using (2.1.6) and (2.1.8), the memory can be approximated in terms of $\Delta E$, the change in GW energy. This approximation is derived as follows. Since the change in GW energy is

$$\Delta E = \int \frac{dE}{dt} dt,$$

(3.4.1)

where $dE/dt$ is the rate of change of the GW energy, it follows from the GW energy flux formula (2.1.6) that

$$\frac{dE}{dt} = \int d\Omega \frac{dE}{d\Omega dt}.$$

(3.4.2)

Plugging (2.1.6) into (3.4.2) gives

$$\frac{dE}{dt} = \frac{R^2}{16\pi} \sum_{l'=2}^{\infty} \sum_{m'=l'}^{m'} \sum_{m''} \hat{h}_{l'm'} \hat{h}^{*}_{l'm''} \int d\Omega_{-2} Y_{l'm'}(\theta, \phi)_{-2} Y_{l''m''}(\theta, \phi).$$

(3.4.3)

Now we could proceed by expressing the integral in (3.4.3) in terms of (2.1.8), but an easier way is to note that the spherical harmonics satisfy the orthogonality condition

$$\int d\Omega_{s_1} Y_{l1}^{m1}(\theta, \phi)_{s_2} Y_{l2}^{m2}(\theta, \phi)^* = \delta_{l_1} \delta_{l_2} \delta_{m_1} \delta_{m_2}$$

$$= \begin{cases} 1, & \text{if } l_1 = l_2 \text{ and } m_1 = m_2 \\ 0, & \text{otherwise}. \end{cases}$$

(3.4.4)

The orthogonality condition (3.4.4) implies $l' = l''$ and $m' = m''$ in (3.4.3). After applying this condition, the rate of change of GW energy is

$$\frac{dE}{dt} = \frac{R^2}{16\pi} \sum_{l'=2}^{\infty} \sum_{m'=l'}^{l'} |\hat{h}_{l'm'}|^2.$$  (3.4.5)

The sum on the right hand side of (3.4.5) up to $l = 3$ is

$$\frac{dE}{dt} = \frac{R^2}{16\pi} \left\{ |\hat{h}_{22}|^2 + |\hat{h}_{2-2}|^2 + |\hat{h}_{20}|^2 + |\hat{h}_{21}|^2 + |\hat{h}_{2-1}|^2 + |\hat{h}_{33}|^2 + |\hat{h}_{3-3}|^2 + |\hat{h}_{32}|^2 + |\hat{h}_{3-2}|^2 + |\hat{h}_{31}|^2 + |\hat{h}_{3-1}|^2 + \cdots \right\}.  \quad (3.4.6)$$

Next we expand (2.1.7) for the $(2,0)$ and $(4,0)$ modes

$$h_{20}^{(\text{mem})} = \frac{R}{84} \sqrt{30} \left\{ |\hat{h}_{22}|^2 + |\hat{h}_{2-2}|^2 - |\hat{h}_{20}|^2 - \frac{1}{2} (|\hat{h}_{21}|^2 + |\hat{h}_{2-1}|^2) \\
+ \frac{\sqrt{14}}{4} (\hat{h}_{31} \hat{h}_{21}^* + \hat{h}_{21} \hat{h}_{31}^* - \hat{h}_{3-1} \hat{h}_{2-1}^* - \hat{h}_{2-1} \hat{h}_{3-1}^*) \\
+ \frac{\sqrt{35}}{4} (\hat{h}_{22} \hat{h}_{32}^* - \hat{h}_{2-2} \hat{h}_{3-2}^* + \hat{h}_{32} \hat{h}_{22}^* - \hat{h}_{3-2} \hat{h}_{2-2}^*) + \cdots \right\} \quad (3.4.7a)$$
\[
h_{20}^{(\text{mem})} \approx \frac{R}{5040} \sqrt{\frac{10}{\pi}} \left\{ |\dot{h}_{22}|^2 + |h_{2-2}|^2 + 6|\dot{h}_{20}|^2 - 4 \left( |h_{21}|^2 + |\dot{h}_{2-1}|^2 \right) + 2\sqrt{35} \left( \dot{h}_{32} h_{22}^* - \dot{h}_{3-2} h_{2-2}^* + \dot{h}_{22} h_{32}^* - \dot{h}_{2-2} h_{3-2}^* \right) - 5\sqrt{14} \left( \dot{h}_{31} h_{21}^* - \dot{h}_{3-1} h_{2-1}^* + \dot{h}_{21} h_{31}^* - \dot{h}_{2-1} h_{3-1}^* \right) - \frac{49}{11} \left[ |\dot{h}_{31}|^2 + |\dot{h}_{3-1}|^2 \right] - 7 \left( |\dot{h}_{32}|^2 + |\dot{h}_{3-2}|^2 \right) + 3 \left( |\dot{h}_{33}|^2 + |\dot{h}_{3-3}|^2 \right) + 6|\dot{h}_{30}|^2 \right\} + \cdots \text{ (3.4.7b)}
\]

If \( \eta = 0.25 \), then \( h_{21} = h_{2-1} = 0 \), and so from (3.4.5)
\[
\frac{16\pi}{R^2} \frac{dE}{dt} \approx |\dot{h}_{22}|^2 + |\dot{h}_{2-2}|^2. \text{ (3.4.8)}
\]

After plugging (3.4.8) into (3.4.7) and simplifying, we have
\[
\dot{h}_{20}^{(\text{mem})} \approx \frac{4\sqrt{30\pi}}{21} \frac{1}{R} \frac{dE}{dt} \text{ (3.4.9a)}
\]
\[
\dot{h}_{40}^{(\text{mem})} \approx \frac{\sqrt{10\pi}}{315} \frac{1}{R} \frac{dE}{dt}. \text{ (3.4.9b)}
\]

Then after integrating the left and right hand sides of (3.4.9), the approximations for the (2, 0) and (4, 0) memory modes become
\[
h_{20}^{(\text{mem})} \approx \frac{4\sqrt{30\pi}}{21} \frac{1}{R} \Delta E \text{ (3.4.10a)}
\]
\[
h_{40}^{(\text{mem})} \approx \frac{\sqrt{10\pi}}{315} \frac{1}{R} \Delta E. \text{ (3.4.10b)}
\]

Note that for \( \eta \neq 0.25 \), generally the \( h_{21} \) and \( h_{2-1} \) modes are not zero. However, these modes are a factor \( v/c \) smaller than the \( h_{22} \) and \( h_{2-2} \) modes and so the approximations (3.4.8), (3.4.9), and (3.4.10) still hold.

As a test of (3.4.10), we calculated the approximated \( h_{20}^{(\text{mem})} \) and \( h_{40}^{(\text{mem})} \) modes for \( \ell_{\text{max}} = 2 \) and \( \ell_{\text{max}} = 8 \). We then compared these approximations to the same modes computed using the GW energy flux formula (2.1.5). The purpose of this comparison is to see how well the formulas (3.4.10) approximate (2.1.5). To evaluate (3.4.10), we first calculated the time derivative \( dE/dt \) in (3.4.8) by differentiating the oscillatory modes using (2.2.1), summing, and then integrating the result using (2.2.4). We also calculated the PN waveforms for the (2, 0) and (4, 0) modes by using (2.3.2) to compute the PN parameter \( x = (M\omega)^{2/3} \) and plugging the result into (2.3.5). For the comparison, we shifted\(^4\) the (2.1.5) and (3.4.10) curves to match the PN (2, 0) and (4, 0) modes in (2.3.5) at the starting time. We included the PN waveforms in the plots for the black hole case.

\(^4\) We shifted the curves by a constant \( \Delta h = h_{0}^{\text{PN}}(t_0) - h_{0}^{\text{NR}}(t_0) \), where \( t_0 \) is the starting time.
For the $(2,0)$ mode, there are some differences between (2.1.5) and the approximations (3.4.10). In the equal mass case, Figure 3.9 shows that both (3.4.10) waveforms are quite different from (2.1.5) with $\ell_{\text{max}} = 8$. The error between them is about 10%. The error between (2.1.5) with $\ell_{\text{max}} = 2$ and both (3.4.10) waveforms is less than 1%. In the 5:1 case, Figure 3.10 shows better agreement between (2.1.5) with $\ell_{\text{max}} = 8$ and (3.4.10) with $\ell_{\text{max}} = 2$. The error is about 3.3%, in contrast to 10% for (3.4.10) with $\ell_{\text{max}} = 8$. The better agreement is likely due to a smaller error between (2.1.5) with $\ell_{\text{max}} = 2$ and $\ell_{\text{max}} = 8$ in the smaller $\eta$ simulations. The PN curve in both plots agrees well with the other curves up to the merger, but then starts to diverge thereafter.

![Figure 3.9: Plot of Re($h_{20}^{\text{(mem)}}$) for a 1:1 mass ratio ($\eta = 0.25$) BBH. The waveforms are (2.1.5) with $\ell_{\text{max}} = 2$ (solid blue), (2.1.5) with $\ell_{\text{max}} = 8$ (solid red), (3.4.10) with $\ell_{\text{max}} = 2$ (solid green), (3.4.10) with $\ell_{\text{max}} = 8$ (solid black), and (2.3.5) (dashed blue).](image)

![Figure 3.10: Plot of Re($h_{20}^{\text{(mem)}}$) for a 5:1 mass ratio ($\eta = 0.1389$) BBH. The waveforms are (2.1.5) with $\ell_{\text{max}} = 2$ (solid blue), (2.1.5) with $\ell_{\text{max}} = 8$ (solid red), (3.4.10) with $\ell_{\text{max}} = 2$ (solid green), (3.4.10) with $\ell_{\text{max}} = 8$ (solid black), and (2.3.5) (dashed blue).](image)
There is a larger difference between (2.1.5) and the approximations (3.4.10) for the (4, 0) mode. In the equal mass case, Figure 3.11 shows a more notable difference between (2.1.5) with $\ell_{\text{max}} = 8$ and (3.4.10) (both curves) than in the (2, 0) mode. The error is around 51%. Both (3.4.10) waveforms lie on top of (2.1.5) with $\ell_{\text{max}} = 2$; the error is less than 1%. For the 5:1 case, in Figure 3.12 the difference between (2.1.5) with $\ell_{\text{max}} = 8$ and both (3.4.10) curves is much larger than in the equal mass case. This is due to the sign change in (2.1.5) with $\ell_{\text{max}} = 8$. The error is much greater than 100%. There is also a larger difference between (2.1.5) with $\ell_{\text{max}} = 2$ and (3.4.10) (both curves). The error is around 4% and 12% for (3.4.10) with $\ell_{\text{max}} = 2$ and $\ell_{\text{max}} = 8$, respectively. The PN curve in both plots agrees well with the other curves prior to the merger, but then begins to the diverge thereafter.

Figure 3.11: Plot of $\text{Re}(h_{40}^{(\text{mem})})$ for a 1:1 mass ratio ($\eta = 0.25$) BBH. The waveforms are (2.1.5) with $\ell_{\text{max}} = 2$ (solid blue), (2.1.5) with $\ell_{\text{max}} = 8$ (solid red), (3.4.10) with $\ell_{\text{max}} = 2$ (solid green), (3.4.10) with $\ell_{\text{max}} = 8$ (solid black), and (2.3.5) (dashed blue).

Figure 3.12: Plot of $\text{Re}(h_{40}^{(\text{mem})})$ for a 5:1 mass ratio ($\eta = 0.1389$) BBH. The waveforms are (2.1.5) with $\ell_{\text{max}} = 2$ (solid blue), (2.1.5) with $\ell_{\text{max}} = 8$ (solid red), (3.4.10) with $\ell_{\text{max}} = 2$ (solid green), (3.4.10) with $\ell_{\text{max}} = 8$ (solid black), and (2.3.5) (dashed blue).
3.5 Dependence of $h_{l0}^{(mem)}$ on the Reduced Mass Ratio $\eta$

The memory can be parameterized by the reduced mass ratio $\eta = m_1 m_2 / M^2$ in (2.1.5) and (2.3.5). In this analysis, we vary $\eta$ and examine the amplitude of the memory modes. For the $(2,0)$ mode, Figures 3.13 and 3.14 show that as $\eta$ decreases the amplitude decreases. However, the decrease itself reduces as $\eta$ gets smaller. Dividing the waveforms by $\eta$ lines up the curves better prior to the merger, but the decreasing trend in the peak amplitude is still maintained. The $(2,0)$ mode has the largest amplitude when $\eta = 0.25$.

Figure 3.13: Plot of $\text{Re}(h_{20}^{(mem)})$ with PN matching for ten BBHs with mass ratios from 1:1 to 10:1 in integer increments, corresponding to different values of $\eta$. The waveforms were time shifted to the peak of $|h_{22}|$ for each simulation.

Figure 3.14: Plot of $\text{Re}(h_{20}^{(mem)})/\eta$ with PN matching for ten BBHs with mass ratios from 1:1 to 10:1 in integer increments, corresponding to different values of $\eta$. The waveforms were time shifted to the peak of $|h_{22}|$ for each simulation.
For the (4, 0) mode, the memory also decreases in amplitude for decreasing $\eta$ in Figures 3.15 and 3.16. The (4, 0) mode also behaves differently near the merger starting at $\eta = 0.1875$. This is due to negative terms in (2.1.5) for the (4, 0) mode having larger magnitudes than the positive terms. As in the (2, 0) mode, dividing by $\eta$ in Figure 3.16 lines up the curves better prior to the merger, but the decreasing trend in the peak amplitude is still maintained. The (4, 0) mode also has the largest amplitude when $\eta = 0.25$. These results are consistent with previous work [34].

Figure 3.15: Plot of $\text{Re}(h^{(\text{mem})}_{40})$ with PN matching for ten BBHs with mass ratios from 1:1 to 10:1 in integer increments, corresponding to different values of $\eta$. The waveforms were time shifted to the peak of $|h_{22}|$ for each simulation.

Figure 3.16: Plot of $\text{Re}(h^{(\text{mem})}_{40})/\eta$ with PN matching for ten BBHs with mass ratios from 1:1 to 10:1 in integer increments, corresponding to different values of $\eta$. The waveforms were time shifted to the peak of $|h_{22}|$ for each simulation.
3.6 Dependence of $h_+^{(\text{mem})}$ on the Source Angle $\Theta$

The $h_+$ polarization signal in (2.1.1) is a function of the spherical angles $(\Theta, \Phi)$. These angles indicate the direction to the detector in the source frame (cf. Figure 2.1). However, because $m = 0$ in the memory modes, we found at the end of Chapter 2 that the memory piece $h_+^{(\text{mem})}$ does not depend on $\Phi$. As a consequence, here we examine the dependence of $h_+^{(\text{mem})}$ on the source angle $\Theta$ only. In Figure 3.17, it is clear that in all cases the memory has the highest amplitude when $\Theta = 90^\circ$. Table 3.1 below shows some errors calculated between the solid and dotted curves for each $\eta$.

Figure 3.17: Plot of the final value of $h_+^{(\text{mem})}$ for ten BBHs with mass ratios from 1:1 to 10:1 in integer increments, corresponding to different values of $\eta$. The dotted curves correspond to the solid counterparts and were calculated using only the $(2,0)$ mode in (2.1.4).

<table>
<thead>
<tr>
<th>$q$</th>
<th>$\eta$</th>
<th>$E_{\text{avg}}$</th>
<th>$E_{90^\circ}$</th>
<th>$E_{\text{max}}$</th>
<th>$\Theta_{\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.2500</td>
<td>3.8%</td>
<td>1.5%</td>
<td>8.0%</td>
<td>0.5$^\circ$</td>
</tr>
<tr>
<td>2</td>
<td>0.2222</td>
<td>2.3%</td>
<td>0.84%</td>
<td>4.8%</td>
<td>0.5$^\circ$</td>
</tr>
<tr>
<td>3</td>
<td>0.1875</td>
<td>0.74%</td>
<td>0.27%</td>
<td>1.6%</td>
<td>179.5$^\circ$</td>
</tr>
<tr>
<td>4</td>
<td>0.1600</td>
<td>0.18%</td>
<td>0.064%</td>
<td>0.39%</td>
<td>0.5$^\circ$</td>
</tr>
<tr>
<td>5</td>
<td>0.1389</td>
<td>0.70%</td>
<td>0.25%</td>
<td>1.5%</td>
<td>179.5$^\circ$</td>
</tr>
<tr>
<td>6</td>
<td>0.1224</td>
<td>0.98%</td>
<td>0.34%</td>
<td>2.1%</td>
<td>179.5$^\circ$</td>
</tr>
<tr>
<td>7</td>
<td>0.1094</td>
<td>1.1%</td>
<td>0.38%</td>
<td>2.3%</td>
<td>179.5$^\circ$</td>
</tr>
<tr>
<td>8</td>
<td>0.0988</td>
<td>1.1%</td>
<td>0.40%</td>
<td>2.5%</td>
<td>179.5$^\circ$</td>
</tr>
<tr>
<td>9</td>
<td>0.0900</td>
<td>1.2%</td>
<td>0.41%</td>
<td>2.5%</td>
<td>0.5$^\circ$</td>
</tr>
<tr>
<td>10</td>
<td>0.0826</td>
<td>1.2%</td>
<td>0.41%</td>
<td>2.5%</td>
<td>179.5$^\circ$</td>
</tr>
</tbody>
</table>

Table 3.1: Errors between the solid and dotted curves in Figure 3.17 for BBHs with different $\eta$. The columns indicate the average error $E_{\text{avg}}$, error at $\Theta = 90^\circ$ $E_{90^\circ}$, maximum error $E_{\text{max}}$, and angle at maximum error $\Theta_{\text{max}}$. 

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3.7 $h_+$ Polarization with and without Memory

After calculating the memory modes $h^{(\text{mem})}_{l0}$, we see that only the (2, 0) and (4, 0) modes make significant contributions to $h^{(\text{mem})}_+$ in (2.1.4). We can now calculate the total memory waveform $h^{(\text{mem})}_+$ and finally, the $h_+$ polarization signal in (2.1.1) for non-spinning coalescing binaries with any mass ratio. In Figures 3.18 - 3.22, it is clearly seen that the $h_+$ waveform with memory is vertically shifted and decays to a positive value. Furthermore, from Section 3.5, the $h_+$ waveform has the highest amplitude at $\Theta = 90^\circ$. In all figures, we include a plot of $h^{(\text{mem})}_+$ to show the memory’s growth near the merger.

Figure 3.18: Plot of $h_+$ with (solid blue) and without (solid red) memory, and $h^{(\text{mem})}_+$ (dashed black) near the merger for a 1:1 mass ratio ($\eta = 0.25$) BBH.

Figure 3.19: Plot of $h_+$ with (solid blue) and without (solid red) memory, and $h^{(\text{mem})}_+$ (dashed black) near the merger for a 5:1 mass ratio ($\eta = 0.1389$) BBH.
Figure 3.20: Plot of $h_+$ with (solid blue) and without (solid red) memory, and $h_+^{\text{(mem)}}$ (dashed black) near the merger for a 7:1 mass ratio ($\eta = 0.1094$) BBH.

Figure 3.21: Plot of $h_+$ with (solid blue) and without (solid red) memory, and $h_+^{\text{(mem)}}$ (dashed black) near the merger for a 9:1 mass ratio ($\eta = 0.90$) BBH.

Figure 3.22: Plot of $h_+$ with (solid blue) and without (solid red) memory, and $h_+^{\text{(mem)}}$ (dashed black) near the merger for a 10:1 mass ratio ($\eta = 0.0826$) BBH.
Chapter 4

Memory from Binary Neutron Stars

In the neutron star case, we calculated the memory for fourteen non-spinning BNS simulations with five neutron star equations of state (EOS). The EOS is a thermodynamic equation that relates the pressure and density of nuclear matter in the core of the star. There are four different mass ratio simulations each for EOS DD2, LS220, and SFHo [24], and one equal mass simulation each for EOS MS1b and SLy [25, 26]. We do similar analyses as in the black hole case. Section 4.1 begins with an overview of the oscillatory modes used in our procedure. Then in Section 4.2, we examine the dependence of the memory on choice of summation index \( l' = l'' = \ell_{\text{max}} \) in (2.1.7). Note that Zenodo [36, 37] provides waveforms up to \( l = 4 \) only, so we calculated (2.1.7) up to \( \ell_{\text{max}} = 4 \). Section 4.3 examines the amplitudes of the memory modes, while Section 4.4 compares the GW energy flux formula (2.1.5) with the approximations (3.4.10) in terms of the change in GW energy \( \Delta E \). Section 4.5 analyzes the dependence of the memory on the neutron star EOS. Section 4.6 then examines the dependence of the memory on the source angle \( \Theta \), and Section 4.7 compares the \( h_+ \) signal with and without memory.

As a reminder, the main results are Figures 4.2, 4.3, 4.10, and 4.11; and 4.13 - 4.16. These figures show that summing up to \( \ell_{\text{max}} = 4 \) adds significant corrections to the memory modes; that small changes in the mass ratio of the binary can result in large fluctuations in the memory amplitude depending on the EOS of the system; and that the memory notably alters the \( h_+ \) polarization signal near the merger. Other important results are in Figures 4.6 - 4.9, and 4.12. As in the black hole case, these plots show two main things: that the GW energy flux formula (2.1.5) adds small, but significant corrections to the \( \Delta E \) approximations (3.4.10) near the merger; and that the memory has the largest amplitude when the source angle is at \( \Theta = 90^\circ \). Again, this is equivalent to observing the binary edge on. Another result is in Figures 4.4 and 4.5. These plots show that the \((2, 0)\) and \((4, 0)\) modes are the most influential in the \( h_+^{(\text{mem})} \) waveform, while the higher even modes make minimal contributions and the odd modes are zero.
4.1 Overview of the Oscillatory Modes \( h_{lm} \)

We begin with an overview of the oscillatory modes used in our calculations. As in the black hole case, the \( h_{22} \) mode has the largest amplitude. This was confirmed upon further investigation, but for clarity we omit plots of the higher modes. Instead, Figure 4.1 plots the real part of the \( h_{22} \) mode \( \text{Re}(h_{22}) \) against time. For each BNS simulation considered the \( h_{22} \) mode is roughly 100 times larger than the higher order modes examined (i.e. the modes \((l, m)\) with \( l \in [2, 4] \) and \( m \in [-l, l] \)). While the time range varies in each panel, there is little change in the amplitudes of \( h_{22} \) for each EOS and mass ratio (note that all the BNS simulations are close to equal mass). In some cases the amplitude decays down to zero, while in others the signal is still in the process of decaying when the simulation ends.

Figure 4.1: Plot of \( \text{Re}(h_{22}) \) near the merger for equal and unequal mass ratio BNSs with different EOS. In the first twelve panels, each column and color represents a given EOS: DD2 (blue), LS220 (red), and SHFo (green), while each row of panels corresponds to a given mass ratio, i.e. the 1:1 case (row 1, \( \eta = 0.25 \), labeled M135135), 1.036:1 case (row 2, \( \eta = 0.2499 \), labeled M144139), 1.092:1 case (row 3, \( \eta = 0.2495 \), labeled M1365125), and 1.167:1 case (row 4, \( \eta = 0.2485 \), labeled M140120). The remaining panels are for equal mass ratio (\( \eta = 0.25 \)) BNSs with EOS MS1b (row 5, purple) and SLy (row 6, magenta). For these cases, the left panels show the entire waveform, while the right panels show a close-up of the merger.
4.2 Dependence of $h_{l0}^{(\text{mem})}$ on Choice of $\ell_{\max}$

We now analyze the dependency of the memory modes on the summation limit $\ell_{\max}$. For the (2, 0) mode, in Figure 4.2 there is a notable difference between the $\ell_{\max} = 2$ and $\ell_{\max} = 4$ waveforms in all cases. Summing up to $\ell_{\max} = 2$ would produce an error in the 1:1, 1:039:1, 1.092:1, and 1.167:1 cases of around 5.7%, 2.9%, 1.9%, and 0.71% for DD2; 4.2%, 4.9%, 4%, and 2.5% for LS220; 5.1%, 4.6%, 4.8%, and 4.7% for SFHo; 2.5% for MS1b (1:1 case), and 6% for SLy (1:1 case). The error between the $\ell_{\max} = 3$ and $\ell_{\max} = 4$ waveforms is less than 1% in all cases except for the DD2 1:1 (2%) and 1.167:1 (1.1%) cases.

For the (4, 0) mode, in Figure 4.3 there is a notable difference between the $\ell_{\max} = 2$, $\ell_{\max} = 3$, and $\ell_{\max} = 4$ waveforms. Summing up to $\ell_{\max} = 2$ would result in a large range of errors in the 1:1, 1:039:1, 1.092:1, and 1.167:1 cases: 24%, 20%, 10.1%, and 6.5% for DD2; 26%, 29%, 25%, and 10.2% for LS220; 34%, 26.2%, 27%, and 21.5% for SFHo; 24% for MS1b (1:1 case), and 65% for SLy (1:1 case). The error between the $\ell_{\max} = 3$ and $\ell_{\max} = 4$ waveforms in the 1:1, 1.039:1, 1.092:1 and 1.167:1 is around 5.4%, 1.9%, 0.40%, and 0.76% for DD2; 1.9%, 3.3%, 5.0%, and 2.1% for LS220; 2.5%, 2.5%, 4.1%, and 3.2% for SFHo; 3.7% for MS1b (1:1 case), and 9.0% for SLy (1:1 case). Although the errors are larger than in the (2, 0) mode, note that the (4, 0) is at least ten times smaller, as in the black hole case.
Figure 4.3: Plot of Re($h_{40}^{\text{mem}}$) with PN matching and $\ell_{\text{max}} \in [2, 4]$ for equal and unequal mass ratio BNSs. Each panel corresponds to a given mass ratio: 1:1 (top left), 1.036:1 (top right), 1.092:1 (bottom left), and 1.167 (bottom right). Each color represents an EOS: DD2 (blue), LS220 (red), SFHo (green), MS1b (purple), SLy (magenta). Lastly, each line style represents a value for $\ell_{\text{max}}$: 2 (dash-dotted), 3 (dashed), and 4 (solid).

4.3 Dependence of $h_{l0}^{\text{mem}}$ on $l$

Now we examine how the amplitude of the BNS memory modes changes when the multipolar order $l$ increases. In Figures 4.4 and 4.5, the $(2, 0)$ mode has the largest amplitude followed by the $(4, 0)$ mode. Again, the $(6, 0)$ and $(8, 0)$ modes make a minimal contribution to the memory with amplitudes ranging from $10^{-10}$ to $10^{-5}$. The odd modes are near the level of numerical noise. As in the black hole case, the results confirm that the $(2, 0)$ and $(4, 0)$ modes are the most influential.

Figure 4.4: Log plots of $|h_{l0}^{\text{mem}}|$ without PN matching for four 1:1 mass ratio ($\eta = 0.25$) BNSs with $l \in [2, 8]$ and EOS DD2 (top left), LS220 (top right), SFHo (bottom left), and MS1b (bottom right).
Figure 4.5: Log plots of $|h_{l0}^{(\text{mem})}|$ without PN matching for four 1.167:1 mass ratio ($\eta = 0.2485$) BNSs with $l \in [2, 8]$ and EOS DD2 (top left), LS220 (top right), and SFHo (bottom).

4.4 Comparison of $h_{l0}^{(\text{mem})}$ in Terms of $\Delta E$

The approximations (3.4.10) of the memory in terms of the change in GW energy $\Delta E$ derived in Section 3.4 are valid in the neutron star case as well. We restate the formulas here for easier reference:

$$h_{20}^{(\text{mem})} \approx \frac{4\sqrt{30\pi}}{21} \frac{1}{R} \Delta E$$  \hspace{1cm} (3.4.10a)

$$h_{40}^{(\text{mem})} \approx \frac{\sqrt{10\pi}}{315} \frac{1}{R} \Delta E.$$  \hspace{1cm} (3.4.10b)

where

$$\Delta E = \int \frac{dE}{dt} dt$$  \hspace{1cm} (3.4.1)

and

$$\frac{dE}{dt} = \frac{R^2}{16\pi} \sum_{\ell'=2}^{\infty} \sum_{m'=-\ell'}^{\ell'} |\hat{h}_{\ell'm'}|^2.$$  \hspace{1cm} (3.4.6)

As another test of these approximations, we compare the $h_{20}^{(\text{mem})}$ and $h_{40}^{(\text{mem})}$ memory modes from BNS simulations calculated using (3.4.10) with the same modes computed from the GW energy flux formula (2.1.5). This comparison was done for the $\ell_{\text{max}} = 2$ and $\ell_{\text{max}} = 4$ waveforms. As in the black hole case, we calculated the PN (2, 0) and (4, 0) modes using (2.3.5) and shifted$^5$ the (2.1.5) and (3.4.10) curves to match the PN waveforms at the starting time. We omit the PN curves in the plots for clarity.

$^5$ As a reminder, we shifted the curves by a constant $\Delta h = h_{l0}^{\text{PN}}(t_0) - h_{l0}^{\text{NR}}(t_0)$, where $t_0$ is the starting time.
Beginning with the \((2,0)\) mode, there are some differences between \((2.1.5)\) and the approximations \((3.4.10)\). In the equal mass case, the error between \((2.1.5)\) with \(\ell_{\text{max}} = 4\) and \((3.4.10)\) (both curves) is around 2.7-3\%, 3.8-4\%, 5-5.4\%, and 2.8-3.2\% for DD2, LS220, SFHo, and MS1b in Figure 4.6. For the 1.167:1 case, this error is about 1-1.8\%, 1.5-2.4\%, and 3.2-4.4\% for DD2, LS220, and SFHo in Figure 4.7. In both cases, the error between \((2.1.5)\) with \(\ell_{\text{max}} = 2\) and both \((3.4.10)\) waveforms is <1-1.6\% for all EOS. We find similar errors in the other mass ratio configurations. While the error between the \((2.1.5)\) and \((3.4.10)\) waveforms decreases with increasing mass ratio, this is partly due to summing up to \(\ell_{\text{max}} = 4\) rather than \(\ell_{\text{max}} = 8\) in the black hole case.

![Figure 4.6: Plot of Re(\(h_{20}^{\text{mem}}\)) for four 1:1 mass ratio (\(\eta = 0.25\)) BNSs with EOS DD2 (top left), LS220 (top right), SFHo (bottom left), and MS1b (bottom right). The waveforms are (2.1.5) with \(\ell_{\text{max}} = 2\) (solid blue), (2.1.5) with \(\ell_{\text{max}} = 4\) (solid red), (3.4.10) with \(\ell_{\text{max}} = 2\) (solid green), and (3.4.10) with \(\ell_{\text{max}} = 4\) (solid black).](image1)

![Figure 4.7: Plot of Re(\(h_{20}^{\text{mem}}\)) for three 1.167:1 mass ratio (\(\eta = 0.2485\)) BNSs with EOS DD2 (top left), LS220 (top right), and SFHo (bottom). The waveforms are (2.1.5) with \(\ell_{\text{max}} = 2\) (solid blue), (2.1.5) with \(\ell_{\text{max}} = 4\) (solid red), (3.4.10) with \(\ell_{\text{max}} = 2\) (solid green), and (3.4.10) with \(\ell_{\text{max}} = 4\) (solid black).](image2)
There are larger differences in the (4,0) mode. In the equal mass case, the error between (2.1.5) with $\ell_{\text{max}} = 4$ and both (3.4.10) curves is about 23%, 25-26%, 34%, and 25% for DD2, LS220, SFHo, and MS1b in Figure 4.8. For the 1.167:1 case, Figure 4.9 shows smaller differences than in the equal mass case between (2.1.5) with $\ell_{\text{max}} = 4$ and both (3.4.10) waveforms. The error is around 6%, 9%, and 20% for DD2, LS220, and SFHo. The error between (2.1.5) with $\ell_{\text{max}} = 2$ and both (3.4.10) curves is <1-1% in the 1:1 case and 1-2.5% in the 1.167:1 case for all EOS. We also find similar errors in the other mass ratio configurations, as in the (2,0) mode.

Figure 4.8: Plot of $\text{Re}(h_{40}^{\text{mem}})$ for four 1:1 mass ratio BNSs with EOS DD2 (top left), LS220 (top right), SFHo (bottom left), and MS1b (bottom right). The waveforms are (2.1.5) with $\ell_{\text{max}} = 2$ (solid blue), (2.1.5) with $\ell_{\text{max}} = 4$ (solid red), (3.4.10) with $\ell_{\text{max}} = 2$ (solid green), and (3.4.10) with $\ell_{\text{max}} = 4$ (solid black).

Figure 4.9: Plot of $\text{Re}(h_{40}^{\text{mem}})$ for three 1.167:1 mass ratio BNSs with EOS DD2 (top left), LS220 (top right), and SFHo (bottom). The waveforms are (2.1.5) with $\ell_{\text{max}} = 2$ (solid blue), (2.1.5) with $\ell_{\text{max}} = 4$ (solid red), (2.3.8) with $\ell_{\text{max}} = 2$ (solid green), and (3.4.1) with $\ell_{\text{max}} = 4$ (solid black), and (2.3.5).
4.5 Dependence of $h_{l0}^{\text{mem}}$ on the Neutron Star Equation of State

The neutron star EOS must be determined in order to simulate BNS mergers. Here we examine the dependence of the memory modes on the EOS of the system. In Figures 4.10 and 4.11 the memory amplitude varies significantly as mass ratio and EOS are varied. For instance, in the (2, 0) mode the difference between the SFHo equal mass and other BNS equal mass waveforms is 32%, 5.3%, 61%, and 20% for DD2, LS220, MS1b, and SLy, respectively; while in the (4, 0) mode it is 41%, 14.4%, 78%, and 65%. As another example, the sets of waveforms with decreasing amplitudes for decreasing $\eta$ are for EOS DD2 (except the $\eta = 0.25$ waveform) and LS220. An interesting feature arises in the sets for DD2 and SFHo: the 1:1 and 1.036:1 waveforms have very different amplitudes despite their reduced mass ratios being close. For DD2, the difference is about 11% in the (2, 0) and (4, 0) modes, while for SFHo it is 58% in the (2, 0) mode and 66% in the (4, 0) mode. The SFHo equal mass waveform has the highest amplitude out of all the BNS simulations. The difference between this waveform and the equal mass black hole curve is around 18% in the (2, 0) mode and 43% in the (4, 0) mode.

Figure 4.10: Plot of $\text{Re}(h_{20}^{\text{mem}})$ with PN matching for fourteen BNSs with different EOS. There are four simulations each for EOS DD2 (blue), LS220 (red), and SFHo (green) with mass ratios 1:1 ($\eta = 0.25$, solid), 1.036:1 ($\eta = 0.2499$, dashed), 1.092:1 ($\eta = 0.2495$, dash-dotted), and 1.167:1 ($\eta = 0.2485$, dotted). For EOS MS1b (purple) and SLy (magenta), each has one equal mass ($\eta = 0.25$) simulation. The black curve is for an equal mass BBH. These plots were time shifted to the peak of $|h_{22}|$ in each simulation.
Figure 4.11: Plot of Re($h_{40}^{\text{mem}}$) with PN matching for fourteen BNSs with different EOS. There are four simulations each for EOS DD2 (blue), LS220 (red), and SFHo (green) with mass ratios 1:1 ($\eta = 0.25$, solid), 1.036:1 ($\eta = 0.2499$, dashed), 1.092:1 ($\eta = 0.2495$, dash-dotted), and 1.167:1 ($\eta = 0.2485$, dotted). For EOS MS1b (purple) and SLy (magenta), each has one equal mass ($\eta = 0.25$) simulation. The black curve is for an equal mass BBH. These plots were time shifted to the peak of $|h_{22}|$ in each simulation.

4.6 Dependence of $h_+^{\text{mem}}$ on the Source Angle $\Theta$

As in the black hole case, the memory piece $h_+^{\text{mem}}$ for BNS mergers does not depend on $\Phi$. Figure 4.12 shows that the memory has the highest amplitude when $\Theta = 90^\circ$. Table 4.1 below lists different errors between the solid and dashed curves for each $\eta$.

Figure 4.12: Plot of the final value of $h_+^{\text{mem}}$ for BNSs with different $\eta$. Each panel is for an EOS: DD2 (top left), LS220 (top right), and SFHo (bottom). The dashed curves are computed from (2.1.4) with the (2,0) mode only and correspond to the same $\eta$ as its solid counterpart.
<table>
<thead>
<tr>
<th>EOS</th>
<th>Name</th>
<th>q</th>
<th>$\eta$</th>
<th>$E_{\text{avg}}$</th>
<th>$E_{90^\circ}$</th>
<th>$E_{\text{max}}$</th>
<th>$\Theta_{\text{max}}$</th>
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<td>1.1%</td>
<td>6%</td>
<td>0.5°</td>
</tr>
<tr>
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<td>0.2499</td>
<td>2.8%</td>
<td>1.1%</td>
<td>5.9%</td>
<td>179.5°</td>
</tr>
<tr>
<td>DD2</td>
<td>M1365125</td>
<td>1.092</td>
<td>0.2495</td>
<td>2.5%</td>
<td>1%</td>
<td>5.3%</td>
<td>179.5°</td>
</tr>
<tr>
<td>DD2</td>
<td>M140120</td>
<td>1.167</td>
<td>0.2485</td>
<td>2.5%</td>
<td>0.9%</td>
<td>5.2%</td>
<td>0.5°</td>
</tr>
<tr>
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<td>M135135</td>
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<td>3%</td>
<td>1.1%</td>
<td>6.2%</td>
<td>179.5°</td>
</tr>
<tr>
<td>LS220</td>
<td>M144139</td>
<td>1.036</td>
<td>0.2499</td>
<td>3.1%</td>
<td>1.2%</td>
<td>6.5%</td>
<td>179.5°</td>
</tr>
<tr>
<td>LS220</td>
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<td>1.092</td>
<td>0.2495</td>
<td>2.9%</td>
<td>1.1%</td>
<td>6.1%</td>
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</tr>
<tr>
<td>LS220</td>
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<td>2.5%</td>
<td>0.94%</td>
<td>5.3%</td>
<td>0.5°</td>
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<tr>
<td>SFHo</td>
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<td>0.2500</td>
<td>3.3%</td>
<td>1.2%</td>
<td>6.8%</td>
<td>0.5°</td>
</tr>
<tr>
<td>SFHo</td>
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<td>0.2499</td>
<td>3%</td>
<td>1.1%</td>
<td>6.3%</td>
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<tr>
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<td>1.092</td>
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<td>3%</td>
<td>1.1%</td>
<td>6.2%</td>
<td>179.5°</td>
</tr>
<tr>
<td>SFHo</td>
<td>M140120</td>
<td>1.167</td>
<td>0.2485</td>
<td>2.8%</td>
<td>1%</td>
<td>5.8%</td>
<td>179.5°</td>
</tr>
</tbody>
</table>

Table 4.1: Errors between the solid and dashed curves in Figure 4.12. The columns indicate the EOS, name, mass ratio $q = m_1/m_2$, average error $E_{\text{avg}}$, error at $\Theta = 90^\circ$ $E_{90^\circ}$, maximum error $E_{\text{max}}$, and angle at maximum error $\Theta_{\text{max}}$.

4.7 $h_+$ Polarization with and without Memory

In the neutron star case, the memory only affects the $h_+$ polarization signal. As in the black hole case, the (2, 0) and (4, 0) modes are the most influential. In Figures 4.13 - 4.16, the $h_+$ waveform with memory is vertically shifted in all cases; however, in all the DD2 and the LS220 1:1, 1.092:1, and 1.167:1 cases, the signal is still in the process of decaying. In all figures, we include a plot of $h_+^{(\text{mem})}$ to show the memory’s growth near the merger.

Figure 4.13: Plot of $h_+$ with (solid blue) and without (solid red) memory, and $h_+^{(\text{mem})}$ (dashed black) near the merger for three 1:1 mass ratio ($\eta = 0.25$) BNSs.
Figure 4.14: Plot of $h_+$ with (solid blue) and without (solid red) memory, and $h_+^{\text{mem}}$ (dashed black) near the merger for three 1.036:1 mass ratio ($\eta = 0.2499$) BNSs.

Figure 4.15: Plot of $h_+$ with (solid blue) and without (solid red) memory, and $h_+^{\text{mem}}$ (dashed black) near the merger for three 1.092:1 mass ratio ($\eta = 0.2495$) BNSs.

Figure 4.16: Plot of $h_+$ with (solid blue) and without (solid red) memory, and $h_+^{\text{mem}}$ (dashed black) near the merger for three 1.167:1 mass ratio ($\eta = 0.2485$) BNSs.
Chapter 5

Memory from NR simulations and Semi-Analytic Calculations

Numerical relativity simulations of compact binary mergers use different methods of gravitational waveform extraction. A popular approach is to calculate the Weyl curvature scalar $\Psi_4$ on spheres of finite radius far from the source and extrapolate the data to future null infinity $\mathcal{I}^+$. On each sphere, $\Psi_4$ is decomposed onto a sum over the spherical harmonic functions in (2.1.2) by

$$\Psi_4 = \ddot{h}_+ - i \ddot{h}_x = \sum_{l=2}^{\infty} \sum_{m=\pm l} \psi_{lm}(t, R) Y_{lm}(\Theta, \Phi), \quad (5.1)$$

where $\psi_{lm}$ are the curvature modes. Comparing with (2.1.4), the gravitational waveform modes $h_{lm}$ are related to $\psi_{lm}$ by

$$\ddot{h}_{lm} = \psi_{lm}. \quad (5.2)$$

The coefficients $\psi_{lm}$ in (5.1) are measured at a set of coordinate times $\{t_i\}$ on a set of spheres of radius $\{R_j\}$ [20]. A polynomial in $1/R$, where $R$ is the distance from the source, is fit to the data, which is then extrapolated to $\mathcal{I}^+$. The waveform modes $\dot{h}_{lm}$ can then be determined by using the polynomial and integrating (5.2) twice.

A newer method of gravitational waveform extraction is Cauchy-Characteristic Extraction (CCE). The Cauchy$^7$ code evolves Einstein’s equations in the strong field regime near the source on successive time slices of spatial hypersurfaces (see Figure 1 in Refs. [20, 22]). The characteristic$^8$ code then takes Cauchy data on a worldtube $R_{\Gamma}$ defined on an inner boundary and evolves Einstein’s equations along null hypersurfaces$^9$ out to $\mathcal{I}^+$. The worldtube is constructed so that its intersection with each

$^6$ $\mathcal{I}^+$ is a region of spacetime where the GW signal resembles what would be measured by a detector on Earth [20].

$^7$ The term ‘Cauchy’ refers to a particular definition of spacetime near the binary system that separates space and time in the computational grid (see (2.1) in Ref. [23]).

$^8$ The term ‘characteristic’ refers to a definition of spacetime that enables gravitational waveforms free of coordinate effects to be extracted at $\mathcal{I}^+$ (see (10) in Ref. [20] and (2.2) in Ref. [23]).

$^9$ A null hypersurface defines a surface along which light rays propagate (see Figure 1 in Refs. [20, 22]).
\( t = t_0 \) Cauchy slice forms a sphere of radius \( R_{\Gamma} \) [20-23]. Using the characteristic data, a gravitational radiation function is computed at \( I^+ \) and then transformed from simulation coordinates to inertial coordinates. The result is the time derivative of the GW signal \( \dot{h}(t) \), which is decomposed onto the spherical harmonic functions in (2.1.2) for extraction, i.e.

\[
\dot{h}_+ - i \dot{h}_x = \sum_{l=-2}^{\infty} \sum_{m=-l}^{l} \dot{h}_{lm} Y^{lm}(\Theta, \Phi) .
\] (5.3)

Unlike the extrapolation method, the waveforms extracted using CCE are free from coordinate effects. CCE can also extract the memory modes, which cannot be extracted by the extrapolation method. For instance, the plot of the \((2,0)\) mode in Figure 5.1 shows no memory, only the \((2,0)\) quasi-normal mode.

![Figure 5.1: Plot of Re(\(h_{20}\)) extracted from an equal mass (\(\eta = 0.25\)) BBH simulation using extrapolation.](image)

Recently, the SXS group [35] implemented CCE in their code [22]. Using CCE, the memory can now be extracted from all previous BBH simulations in their catalog. In Section 5.1, we begin by comparing the CCE and extrapolated \( \dot{h}_{22} \) mode in the equal mass case. We differentiated the extrapolated \( \dot{h}_{22} \) mode using (2.2.1). Section 5.2 then compares the CCE memory modes with semi-analytic calculations for the first time. We considered two BBH simulations with mass ratio configurations 1:1 and 2:1. This comparison was done for the \( \dot{h}_{20}^{(\text{mem})} \) and \( \dot{h}_{40}^{(\text{mem})} \) memory modes and their time derivatives \( \dot{h}_{20}^{(\text{mem})} \) and \( \dot{h}_{40}^{(\text{mem})} \). Note that we calculated (2.1.7) up to \( \ell_{\text{max}} = 8 \) for the semi-analytic waveforms. The errors in Sections 5.1 and 5.2 are defined as the absolute difference \(|A - B|\), where \(A\) and \(B\) refer to either the CCE and extrapolated \( \dot{h}_{22} \) modes, or the CCE and semi-analytic memory modes. As a reminder, the main results are in Figures 5.4 - 5.19. These plots show good agreement between the CCE and semi-analytic memory modes. The error is on the order of the error between the CCE and extrapolated \( \dot{h}_{22} \) modes. Other results are in Figures 5.2 and 5.3, which show that the CCE and extrapolated \( \dot{h}_{22} \) modes agree well.
5.1 Comparison of CCE and Extrapolated $\dot{h}_{22}$

We begin with a comparison of the CCE and extrapolated $\dot{h}_{22}$ modes. In Figure 5.2, we see that the CCE waveform lines up almost equally close with the extrapolated waveform. In Figure 5.3, the error ranges between $10^{-9}$ to $10^{-4}$ up to $t/M \approx -200$, then jumps up to $10^{-2}$ near the merger. However, the error between the peaks of $\text{Re}(\dot{h}_{22})$ at the merger (around $t/M \approx 0$) is less than 1%.

Figure 5.2: Plot of $\text{Re}(\dot{h}_{22})$ near the merger from CCE (dashed black) and extrapolated (solid red) equal mass ($\eta = 0.25$) BBH simulations. The curves were time shifted to the peak of $|\dot{h}_{22}|$ from their respective files.

Figure 5.3: Log plot of the error between the CCE and extrapolated $\text{Re}(\dot{h}_{22})$ in Figure 5.2 for the entire simulation time. To compute the error, we interpolated the CCE waveform to the extrapolated waveform. The curves were time shifted to the peak of $|\dot{h}_{22}|$ from their respective files prior to interpolating and computing the error.
5.2 Comparison of $\dot{h}_l^{(\text{mem})}$ and $h_l^{(\text{mem})}$ with CCE and Semi-Analytic Calculations

We now compare the CCE and semi-analytic CCE $\dot{h}_l^{(\text{mem})}$ and $h_l^{(\text{mem})}$ modes. Figure 5.4 shows that the CCE $\dot{h}_l^{(\text{mem})}$ mode lines up well with the semi-analytic CCE version up to the merger ($t/M \approx 17880$). At the merger, the CCE curve deviates from the CCE semi-analytic curve, then oscillates about it briefly. The oscillations dampen out and then the CCE curve follows along semi-analytic CCE curve thereafter. In Figure 5.5 for the $h_l^{(\text{mem})}$ mode, again the CCE waveform lines up with the semi-analytic version, but they are still somewhat different near the merger. The two curves are most notably offset where the curves are rapidly increasing.

Figure 5.4: Plot of Re($\dot{h}_l^{(\text{mem})}$) without PN matching for an equal mass ($\eta = 0.25$) BBH simulation. The solid red curve is the CCE waveform and the dashed black curve is the semi-analytic CCE version. The CCE waveform was just plotted from the simulation file.

Figure 5.5: Plot of Re($h_l^{(\text{mem})}$) without PN matching for an equal mass ($\eta = 0.25$) BBH simulation. The solid red curve is the CCE waveform and the dashed black curve is the semi-analytic CCE version. The CCE waveform was integrated using (2.2.4).
In Figure 5.6, there is good agreement between the CCE and semi-analytic CCE $\dot{h}_20^{(\text{mem})}$ modes for the inspiral. The error ranges from $10^{-12}$ to $10^{-6}$, increases to nearly $10^{-3}$ at the merger ($t/M \approx 17880$), and drops to around $10^{-7}$ thereafter. The error also displays small oscillations, although there are two regions where the oscillations grow larger. In Figure 5.7, there is also good agreement between the CCE and semi-analytic CCE $h_20^{(\text{mem})}$ modes. The error ranges from about $10^{-7}$ to $10^{-3}$ in the inspiral, increases to nearly $10^{-2}$ near the merger and drops to around $10^{-3}$ thereafter. The larger errors in the $\dot{h}_20^{(\text{mem})}$ and $h_20^{(\text{mem})}$ modes near the merger correspond to where the $(2,0)$ quasi-normal mode is excited. The errors in both the $\dot{h}_20^{(\text{mem})}$ and $h_20^{(\text{mem})}$ modes are about as large as the error between the CCE and extrapolated $\dot{h}_{22}$ modes.

Figure 5.6: Log plot of the error between the CCE and semi-analytic CCE $\text{Re}(\dot{h}_20^{(\text{mem})})$ from the equal mass BBH in Figure 5.3 for the entire simulation time.

Figure 5.7: Log plot of the error between the CCE and semi-analytic CCE $\text{Re}(h_20^{(\text{mem})})$ from the equal mass BBH in Figure 5.6 for the entire simulation time.
In Figure 5.8 showing the $\hat{h}^{(\text{mem})}_{40}$ mode, the curves line up well during the inspiral, but then the CCE waveform starts to deviate from the semi-analytic CCE version near the merger ($t/M \approx 17780$). As in the $\hat{h}^{(\text{mem})}_{20}$ mode, the CCE curve oscillates briefly about the semi-analytic CCE curve, then follows along the semi-analytic CCE curve after the oscillations dampen out. For the $h^{(\text{mem})}_{40}$ mode, again the CCE and semi-analytic CCE curves line up well in Figure 5.9, however the difference becomes larger near the merger. The curves are most notably different in the region where they flatten out towards the end of the simulation. Although the scale is larger in Figure 5.9, the difference in the $h^{(\text{mem})}_{40}$ modes near the merger is about as large as the difference between the $\hat{h}^{(\text{mem})}_{40}$ modes.

![Figure 5.8](image1.png)

Figure 5.8: Plot of Re($\hat{h}^{(\text{mem})}_{40}$) without PN matching for an equal mass ($\eta = 0.25$) BBH simulation. The solid red curve is the CCE waveform and the dashed black curve is the semi-analytic CCE version. The CCE waveform was just plotted from the simulation file.

![Figure 5.9](image2.png)

Figure 5.9: Plot of Re($h^{(\text{mem})}_{40}$) without PN matching for an equal mass ($\eta = 0.25$) BBH simulation. The solid red curve is the CCE waveform and the dashed black curve is the semi-analytic CCE version. The CCE waveform was integrated using (2.2.4).
In the error plots, Figure 5.10 shows good agreement between the CCE and semi-analytic CCE $\dot{h}_{40}^{\text{(mem)}}$ modes, despite the more pronounced oscillatory feature. The error ranges from about $10^{-13}$ to $10^{-7}$ during the inspiral, increases to nearly $10^{-4}$ at the merger ($t/M \approx 17880$), and drops to around $10^{-9}$ afterwards. In Figure 5.11, there is also good agreement between the CCE and semi-analytic $h_{40}^{\text{(mem)}}$ modes. The error ranges from about $10^{-10}$ to $10^{-5}$ during the inspiral, increases to nearly $10^{-4}$ at the merger, and drops to around $10^{-5}$ thereafter. The errors are smaller than in the $\dot{h}_{20}^{\text{(mem)}}$ and $h_{20}^{\text{(mem)}}$ modes. The larger errors in the $\dot{h}_{40}^{\text{(mem)}}$ and $h_{40}^{\text{(mem)}}$ modes near the merger correspond to where the (4,0) quasi-normal mode is excited. The errors in both the $\dot{h}_{40}^{\text{(mem)}}$ and $h_{40}^{\text{(mem)}}$ modes are smaller than the error between the CCE and extrapolated $h_{22}$ modes.

Figure 5.10: Log plot of the error between the CCE and semi-analytic CCE Re($\dot{h}_{40}^{\text{(mem)}}$) from the equal mass BBH in Figure 5.8 for the entire simulation time.

Figure 5.11: Log plot of the error between the CCE and semi-analytic CCE Re($h_{40}^{\text{(mem)}}$) from the equal mass BBH in Figure 5.9 for the entire simulation time.
For the 2:1 mass ratio case, Figure 5.12 shows that the CCE and semi-analytic CCE $\dot{h}_{20}^{(\text{mem})}$ modes line up well during the inspiral. The CCE waveform then deviates from the semi-analytic CCE curve near the merger ($t/M \approx 16880$). As in the equal mass $h_{20}^{(\text{mem})}$ mode, the CCE curve oscillates briefly about the semi-analytic CCE curve near the merger. The CCE curve then follows along the semi-analytic CCE curve after the oscillations dampen out. In Figure 5.13, the CCE and semi-analytic CCE $h_{20}^{(\text{mem})}$ waveforms again line up close during the inspiral, but then the difference becomes larger near the merger. In particular, the two curves are most notably offset where they are rapidly increasing, in a similar way as the equal mass $h_{20}^{(\text{mem})}$ mode.

![Figure 5.12](image1.jpg)

Figure 5.12: Plot of Re($\dot{h}_{20}^{(\text{mem})}$) without PN matching for a 2:1 mass ratio ($\eta = 0.22$) BBH simulation. The solid red curve is the CCE waveform and the dashed black curve is the semi-analytic CCE version. The CCE waveform was just plotted from the simulation file.

![Figure 5.13](image2.jpg)

Figure 5.13: Plot of Re($h_{20}^{(\text{mem})}$) without PN matching for a 2:1 mass ratio ($\eta = 0.22$) BBH simulation. The solid red curve is the CCE waveform and the dashed black curve is the semi-analytic CCE version. The CCE waveform was integrated using (2.2.4).
In the error plots for the \((2, 0)\) mode, Figure 5.14 shows good agreement between the CCE and semi-analytic CCE \(\dot{h}_{20}^{\text{(mem)}}\) modes. The error between the curves ranges from about \(10^{-10}\) to \(10^{-6}\) during the inspiral, increases to nearly \(10^{-4}\) at the merger \((t/M \approx 16880)\), and drops to around \(10^{-9}\) afterwards. The error also displays an oscillatory feature similarly to the equal mass \(\dot{h}_{20}^{\text{(mem)}}\) and \(\dot{h}_{40}^{\text{(mem)}}\) modes. In Figure 5.15, there is also good agreement between the CCE and semi-analytic CCE \(h_{20}^{\text{(mem)}}\) modes. The error ranges from about \(10^{-8}\) to \(10^{-3}\) in the inspiral, increases to nearly \(10^{-2}\) at the merger, and drops to around \(10^{-3}\) thereafter. Again, the larger errors in the \(h_{20}^{\text{(mem)}}\) and \(h_{20}^{\text{(mem)}}\) modes near the merger correspond to where the \((2, 0)\) quasi-normal mode is excited. The errors in both the \(\dot{h}_{20}^{\text{(mem)}}\) and \(h_{20}^{\text{(mem)}}\) modes are also about as large as the error between the CCE and extrapolated \(h_{22}\) modes, as in the equal mass \(\dot{h}_{20}^{\text{(mem)}}\) and \(h_{20}^{\text{(mem)}}\) modes.

![Figure 5.14](image1.png)

Figure 5.14: Log plot of the error between the CCE and semi-analytic CCE \(\text{Re}(\dot{h}_{20}^{\text{(mem)}})\) from the 2:1 mass ratio BBH in Figure 5.12 for the entire simulation time.

![Figure 5.15](image2.png)

Figure 5.15: Log plot of the error between the CCE and semi-analytic CCE \(\text{Re}(h_{20}^{\text{(mem)}})\) from the 2:1 mass ratio BBH in Figure 5.13 for the entire simulation time.
Figure 5.16 shows that for the $\dot{h}_{40}^{\text{(mem)}}$ mode, the CCE and semi-analytic CCE waveforms line up closely during the inspiral. Near the merger ($t/M \approx 16880$), the CCE curve oscillates briefly about the semi-analytic CCE curve and then lines up with the semi-analytic CCE curve after the oscillations dampen out. For the $\dot{h}_{40}^{\text{(mem)}}$ mode, Figure 5.17 shows some difference between the CCE and semi-analytic CCE waveforms near the merger as well. The curves are most notably different in the region where they flatten out towards the end of the simulation, as in the equal mass $\dot{h}_{40}^{\text{(mem)}}$ mode. Although the scale in Figure 5.17 is larger, the difference in the $\dot{h}_{40}^{\text{(mem)}}$ mode near the merger is about as large as the difference in the $\dot{h}_{40}^{\text{(mem)}}$ mode. This difference is also smaller than in the $h_{20}^{\text{(mem)}}$ mode, but the $h_{40}^{\text{(mem)}}$ mode is at least ten times smaller than the $h_{20}^{\text{(mem)}}$ mode.

Figure 5.16: Plot of $\text{Re}(\dot{h}_{40}^{\text{(mem)}})$ without PN matching for a 2:1 mass ratio ($\eta = 0.22$) BBH simulation. The solid red curve is the CCE waveform and the dashed black curve is the semi-analytic CCE version. The CCE waveform was just plotted from the simulation file.

Figure 5.17: Plot of $\text{Re}(h_{40}^{\text{(mem)}})$ without PN matching for a 2:1 mass ratio ($\eta = 0.22$) BBH simulation. The solid red curve is the CCE waveform and the dashed black curve is the semi-analytic CCE version. The CCE waveform was integrated using (2.2.4).
In the error plots, Figure 5.18 shows good agreement between the CCE and semi-analytic CCE \( \hat{h}_{40}^{(\text{mem})} \) modes. The error ranges from about \( 10^{-14} \) to \( 10^{-7} \) during the inspiral, increases to nearly \( 10^{-4} \) at the merger, and drops to around \( 10^{-10} \) afterwards. As in the equal mass \( \hat{h}_{20}^{(\text{mem})} \) and \( \hat{h}_{40}^{(\text{mem})} \) modes, and the 2:1 mass ratio \( \hat{h}_{20}^{(\text{mem})} \) mode, the error displays an oscillatory feature. In Figure 5.19, there is also good agreement between the CCE and semi-analytic CCE \( h_{40}^{(\text{mem})} \) modes. The error ranges from about \( 10^{-10} \) to \( 10^{-5} \) in the inspiral, increases to nearly \( 10^{-4} \) at the merger, and drops to around \( 10^{-5} \) thereafter. As in the same modes for the equal mass case, the larger errors in the \( \hat{h}_{40}^{(\text{mem})} \) and \( h_{40}^{(\text{mem})} \) modes near the merger correspond to where the \((4,0)\) quasi-normal mode is excited. Also as in the same modes for the equal mass case, the errors in both the \( \hat{h}_{40}^{(\text{mem})} \) and \( h_{40}^{(\text{mem})} \) modes are smaller than the error between the CCE and extrapolated \( h_{22} \) modes.

\[ \text{Figure 5.18: Log plot of the error between the CCE and semi-analytic CCE Re}(\hat{h}_{40}^{(\text{mem})}) \text{ from the 2:1 mass ratio BBH in Figure 5.16 for the entire simulation time.} \]

\[ \text{Figure 5.19: Log plot of the error between the CCE and semi-analytic Re}(h_{40}^{(\text{mem})}) \text{ from the 2:1 mass ratio BBH in Figure 5.17 for the entire simulation time.} \]
Chapter 6

Conclusions

The nonlinear memory is an interesting prediction of general relativity that has a unique, visible feature. Because the nonlinear memory arises from previously emitted gravitational radiation, gravitational waves are, effectively, sources of gravitational waves. The goal of this work was to produce highly accurate nonlinear memory waveforms so they can be used in future searches by LIGO and Pulsar Timing Arrays, and eventually the space-based detector LISA (Laser Interferometer Space Antenna). Previous work [11-13, 34] focused on non-spinning, binary black hole simulations with mass ratios 1:1, 2:1, 3:1, 4:1, 5:1, 6:1, and 8:1. The current work includes the 7:1, 9:1, and 10:1 cases, completing the list of mass ratios from 1:1 to 10:1 in integer increments. For the first time, we applied the semi-analytic approach to binary neutron star simulations with five different equations of state and mass ratios 1:1, 1.036:1, 1.092:1, and 1.167:1. The results presented in Chapters 3 and 4 indicate that the memory adds significant corrections to the gravitational-wave signal from coalescing compact binaries. In the black hole case, there is significant variation of the nonlinear memory as the mass ratio is varied. In the neutron star case, the nonlinear memory varies as the mass ratio and equation of state vary. Our procedure can be used as a means of studying the equation of state.

Detection of the nonlinear memory can serve as another test of general relativity. While ground-based detectors such as LIGO could potentially detect this effect, their poor low frequency sensitivity makes this challenging. LISA, on the other hand, has potentially better prospects of detecting the nonlinear memory, but this depends on the mass of the black holes in the binary system. For supermassive binary black holes (e.g. a $10^5 M_\odot/10^5 M_\odot$ system [12], where $M_\odot$ is one solar mass), the frequency of the memory falls within LISA’s range of 0.1 mHz - 1 Hz [39]. The waveforms calculated here could serve as templates for comparison with future observational data from LISA once it becomes operational.

Numerical relativity simulations have led to greater understanding of gravitational waveforms. Now that Cauchy-Characteristic Extraction has been implemented in the SXS group’s code [22], we were able to compare numerical relativity extracted nonlinear memory with semi-analytic calculations for the first time. The results indicate good agreement between both approaches. The waveforms computed in this study could be used to compare the nonlinear memory extracted from future simulations.
However, numerical relativity simulations extract the quasi-normal modes along with the nonlinear memory. As another comparison between numerical relativity extracted nonlinear memory and semi-analytic calculations, we will investigate ways of separating the quasi-normal to compare only the nonlinear memory portion. Ultimately, this will lead to better understanding of the nonlinear memory.

For other future work, the nonlinear memory from coalescing binaries with spins is in need of further investigation. Only the equal mass binary black hole case with aligned and anti-aligned spins has been examined [40]. It would be worthwhile to examine the unequal mass ratio cases, and also binaries with misaligned spins. The resulting waveforms would be a significant contribution considering they would model more closely the physics of compact binary systems. For the neutron star case, it would be interesting to examine the nonlinear memory from simulations with equations of state MS1b and SLy for unequal mass ratios. We also have not examined the detectability of the nonlinear memory from binary neutron star mergers. Lastly, the nonlinear memory from neutron star-black hole mergers has not been examined. This would also be a significant contribution considering neutron star-black hole mergers are another source for LIGO and Virgo.
Bibliography


