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Pre-service teachers' figurative and operative graphing actions

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ABSTRACT

We report on semi-structured clinical interviews to describe U.S. pre-service secondary mathematics teachers' graphing meanings. Our primary goal is to draw on Piagetian notions of figurative and operative thought to identify marked differences in the students' meanings. Namely, we illustrate students' meanings dominated by fragments of sensorimotor experience and compare those with students' meanings dominated by the coordination of mental actions in the form of covarying quantities. Our findings suggest students' meanings that foreground operative aspects of thought are more generative with respect to graphing. Our findings also indicate that students can encounter perturbations due to potential incompatibilities between figurative and operative aspects of thought.

1. Introduction

Representational activity is critical to the study of mathematics. Coordinate systems and associated graphs form important representations because they provide perceptual material that permits conceiving of and operating on quantities. Specifically, coordinate systems and graphs enable constructing quantities, obtaining measures via partitioning those quantities based on a unit magnitude, and representing relationships between quantities' values in the generated coordinate system. For instance, the Cartesian coordinate system consists of oriented segment magnitudes and a coordinate point that unites and enables representing these magnitudes and corresponding measures simultaneously. Compare the Cartesian coordinate system with tables of values or formulas that provide symbolic inscriptions or glyphs; the latter representations can signify quantities and their relationships, but they do not typically provide the perceptual material on which an individual can operate quantitatively.

Due in part to the affordances of coordinate systems, researchers have used graphing contexts to investigate students' meanings for topics rooted in relationships between quantities such as function and modeling (e.g., Carlson, 1998; diSessa, Hammer, Sherin, & Kolpakowski, 1991; Leinhardt, Zaslavsky, & Stein, 1990; Stalvey & Vidakovic, 2015). Relatedly, scholars have identified covariational reasoning—reasoning about how quantities vary in tandem—as productive for students' and teachers' mathematical development (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Castillo-Garsow, 2012; Ellis, Özgür, Kulow, Williams, & Amidon, 2015; Johnson, 2012; , 2015a; Moore, 2014; P. W. Thompson & Carlson, 2017; P. W. Thompson, Hatfield, Joshua, Yoon, & Byerley, 2017; Trigueros & Jacobs, 2008). These same scholars have simultaneously echoed Saldanha and Thompson's sentiments from two decades ago—"[U]nderstanding graphs as representing a continuum of states of covarying quantities is nontrivial and should not be taken for granted" (Saldanha & Thompson, 1998, p. 303)—in order to motivate more detailed investigations into the extent individuals conceive graphs

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as entailing covarying quantities.

In this paper, we respond to the need to better understand individuals' covariational reasoning by investigating the graphing activity of undergraduate students enrolled in a pre-service secondary teacher program. We accomplish three related goals in this paper. First, we adopt Piagetian constructs to characterize important differences in students' graphing meanings. We provide examples in which students held meanings dominated by *figurative thought* (Piaget, 1976, 2001; P. W. Thompson, 1985)—thought that foregrounds carrying out (sensorimotor) actions. In contrast, we provide examples in which students held meanings dominated by *operative thought* (Piaget, 1976, 2001; P. W. Thompson, 1985)—thought that foregrounds the coordination of internalized mental actions so that figurative aspects of thought are subordinate to this coordination. Second, we clarify an operative meaning for graphing that underscores the productive role of reasoning about magnitudes (cf. numbers or values) when graphing covarying quantities. Third, by adopting the aforementioned constructs, we discuss the implications of different graphing meanings including potential incompatibilities between them that result in students experiencing perturbations.

2. Background

Our primary goal is to draw distinctions between students' graphing meanings, and thus we first define our use of *figurative* and *operative* with respect to characterizing aspects of thought including the extent such aspects are internally compatible. We then explain our approach to covariational reasoning as emphasizing quantities' *magnitudes*. Against this backdrop, we extend Moore and Thompson's (Moore & Thompson, 2015; Paoletti & Moore, 2017) construct of emergent shape thinking. Our primary purpose of this extension is to illustrate an operative meaning for graphing rooted in reasoning about quantities' magnitudes that is productive for representing covariational relationships.

2.1. *Figurative and operative aspects of thought*

In characterizing the nature of schemes constituting students' meanings for graphing, we draw on Piaget's (1976, 2001 distinctions between thought based in and constrained to figurative material including perceptual objects and sensorimotor actions and thought in which figurative material is subordinate to logico-mathematical operations. Important to our work here, meanings that foreground figurative aspects of thought are dominated by re-presentations of specific perceptual material or sensorimotor experience (von Glasersfeld, 1995). Meanings that foreground operative aspects of thought are instead dominated by the coordination and transformation of mental actions (Norton, 2014; von Glasersfeld, 1995), and thus not constrained to specific perceptual material or sensorimotor actions.

An illustrative example of these distinctions exists in Steffe and colleagues' (see Steffe, 1991; Steffe & Olive, 2010) use of Piaget's constructs to characterize children's construction of number. They explained that a figurative counting scheme (or meaning) is a scheme in which a child is able to count without perceptual material in her sensory field, yet she requires re-presenting particular perceptual material or sensorimotor actions when counting. Examples of such material include those that are visual, such as an array of dots, or those that are sensory, such as finger taps. Steffe and Olive (2010) called such re-presentational objects figural unit items due to their basis in figurative material generated through thought or sensorimotor actions.

Defining an operative counting scheme, Steffe and Olive (2010) articulated a scheme in which a child has unitized records of counting so that they are not constrained to re-presenting particular perceptual material or sensorimotor actions. Steffe (1991) explained, "[the unitizing] operation strips the figurative unitary items of their sensory-motor quality and creates a sequence of abstract unit items that contain the records of the sensory-motor material" (p. 35). An important marker of a child with an operative counting scheme is that he or she is able to take a unitized record of counting as input on which to operate independent of specific material or carried out actions.

Reflecting his interest in working with children as they progress at any level of mathematical development, P. W. Thompson (1985) loosened the distinction between figurative and operative thought in order to characterize activity that necessarily entails both. He explained:

> When a person's actions of thought remain predominantly within schemata associated with a given level (of control), his or her thinking can be said to be figurative in relation to that level. When the actions of thought move to the level of controlling schemata, then the thinking can be said to be operative in relation to the level of the figurative schemata. That is to say, the relationship between figurative and operative thought is one of figure to ground. Any set of schemata can be characterized as figurative or operative, depending upon whether one is portraying it as background for its controlling schemata or as foreground for the schemata that it controls. (p. 195)[1]

Adopting Thompson's framing, a researcher drawing distinctions between figurative and operative thought is thus an issue of characterizing whether an individual's meanings are tied to carrying out particular actions and their results or if the individual can call forth and control a scheme and its results. Steffe and Olive's notion of a figurative counting scheme is an example of the former; a constraint of the meaning is that it is necessary for the child to re-present particular perceptual material or sensorimotor actions. Thompson broadens the definition of figurative thought to include those cases in which the action in question, which need not be

[1] We note that we can interpret Thompson's use of schemata to be consistent with the term meaning. "The scheme is the meaning" (P. W. Thompson et al., 2014, p. 12).

sensorimotor, has not been abstracted in a way that it can be transformed to accommodate novel figurative material. In contrast, Thompson emphasizes that an operative meaning can entail figurative fragments, but the individual's meaning is such that they can transform their actions (both mental and physical) to account for novel figurative material and experiences.

Thompson's framing of figurative and operative thought is productive when investigating students' meanings in the moment of graphing quantitative relationships because it emphasizes the need to be sensitive to both aspects of thought simultaneously. A student's graphing actions necessarily involve sensorimotor activity, producing perceptual material, or imbuing perceptual material with meaning. For the purposes of this paper, we also take it as a given that an undergraduate student's meanings entail aspects of operative thought with respect to concepts including number, length, and quantitative relationships including additive and multiplicative comparisons. But, because a student holds such meanings does not imply that her or his graphing actions will foreground these meanings including the coordination of quantities (P. W. Thompson et al., 2017). Furthermore, it does not imply that her or his graphing actions will entail aspects of figurative and operative thought in the form of an internally compatible complex conceptual structure (von Glasersfeld, 1995).

As we illustrate in the results section, giving attention to both figurative and operative aspects of thought enables us to discuss the extent these aspects are compatible in the context of a student's graphing activity, an important feature of operational intelligence (Montangero & Maurice-Naville, 1997). We use the term compatible to mean that a student's meanings are such that he or she does not experience a sustained *perturbation* in activity. A perturbation is the occurrence of an unexpected result when assimilating an experience to a meaning or meanings (Hackenberg, 2010; Harel, 2008; von Glasersfeld, 1995), and we consider the perturbation to be sustained if the student does not reconcile it (e.g., achieve equilibration) to their satisfaction.

With respect to the present work, an unexpected result, or perturbation, can take two forms. First, it could take the form of a person being unable to achieve their perceived goal, including perceiving that their available meanings do not yield expected results. Second, and potentially related to the first form, it could take the form of a person identifying an incompatibility in her or his actions and associated results that stems from two disparate meanings; a person can hold two meanings that yield contradictory results or solutions to what they perceive the same situation. For example, a person could hold two meanings for graphs that result in producing or anticipating different drawn graphs for the same prompt, resulting in the person experiencing a perturbation due to differing graphs. Although the present work is not a study of learning, we note that either form of perturbation is an important catalyst for learning as each form can contribute to a need for adaptation and searching for characteristics possibly disregarded in assimilation (Harel, 2008; Steffe, 1990; von Glasersfeld, 1995). Yet, we emphasize that a perturbation is a product of the meanings a person has available. Perturbations are a property of thinking based in assimilation and the interaction of meanings, and are not thinking in and of itself (Becker, 2004; von Glasersfeld, 1995).

2.2. Covariational reasoning, magnitudes, and an operative meaning for graphs

Our research agenda focuses on students' covariational reasoning—coordinating how quantities vary in tandem (Carlson et al., 2002; Saldanha & Thompson, 1998)—in conceiving graphs and quantitative situations (e.g., a person taking a road trip). Some researchers have described covariational reasoning in terms of patterns in successive, corresponding *numerical values* of two sets (see Confrey & Smith, 1994, 1995). Our use of covariational reasoning is rooted in Thompson's (2011) more extensive description of *quantity*.

Thompson (2011) defined quantity as an attribute with a measurable *magnitude*. Magnitude refers to the size or amount-ness of a quantity that remains invariant with respect to changes in the unit used to measure the quantity (P. W. Thompson, Carlson, Byerley, & Hatfield, 2014).[2] For instance, a person's height is the same size or amount regardless if conveyed in inches, feet, or meters. We distinguish between a quantity's magnitude and a quantity's value or measure because it enables us to describe a person's covariational reasoning in terms of their operating on figurative material associated with magnitudes in flux as opposed to their using numerical values as proxies for quantities (Castillo-Garsow, Johnson, & Moore, 2013).[3] For instance, a person who conceives a situation as entailing two segment magnitudes can covary lengths in flux while *anticipating* that these lengths have specific measures in an associated unit at any instantiation of their covariation (Saldanha & Thompson, 1998; P. W. Thompson, 2011). It is not necessary for the person to have specified measures to conceive, coordinate, and compare lengths.

Although we focus on the covariation of magnitudes in this work, we highlight the importance of students' reasoning about specified measures and patterns in these measures. Such reasoning is important for students constructing various function classes, quantifying rate of change and accumulation, reasoning about limits, and using representations like formulas and tables (see Ayalon, Watson, & Lerman, 2015; Confrey & Smith, 1995; Ellis, 2007; Ellis et al., 2015; Johnson, 2015a, 2015b; Oehrtman, 2008). Approaching covariation in terms of coordinating magnitudes provides a complementary focus, the explanatory affordances of which have been recognized by several researchers (i.e., Carlson et al., 2002; Johnson, 2012; Moore, 2014; Paoletti & Moore, 2017; P. W. Thompson, 2011; P. W. Thompson et al., 2017). Such a focus has enabled these scholars to describe how students might construct productive images of covariation and reason about relationships between quantities, and we build on this work in the area of graphing.

[2] Concepts of magnitudes can and do differ from person to person, particularly in development. A discussion of these differences is beyond the scope of this manuscript. We point the reader to Steffe and Olive (2010) and P. W. Thompson et al. (2014) for more detail.

[3] Our use of flux is consistent with Newton's image of dynamic quantities in the process of changing, such as one's image of a segment in the process of changing length (P. W. Thompson, 1994).

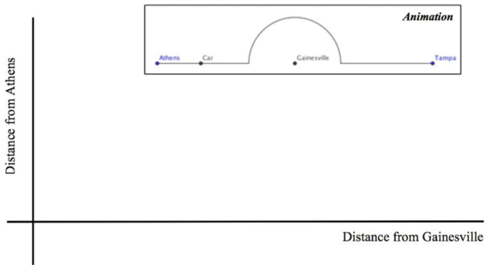

Fig. 1. The Going Around Gainesville (GAG) task.

Moore and Thompson's (Moore & Thompson, 2015; Paoletti & Moore, 2017) construct of *emergent shape thinking* informs our approach to students' covariational reasoning and graphing meanings. Moore and Thompson (2015) explained that emergent shape thinking involves conceiving of a graph as a locus or in-progress trace that results from the simultaneous coordination of two covarying quantities. As the authors identified, one cannot veridically illustrate emergent shape thinking in print because it relies upon conveying quantities in the progress of covarying. We thus provide instantiations of covariation in Fig. 2 that are associated with *Part II* of the task in Fig. 1. The reader should conceive the quantities' magnitudes, represented by segments along the axes, in flux from instantiation to instantiation.[4]

A person engaging in emergent shape thinking to represent the relationship between the quantities identified in *Part II* of the *GAG* task[5] could proceed as follows:

(1) The person conceives that she starts at a magnitude of zero from Athens ($||Y|| = 0$) and a non-zero magnitude from Gainesville ($||X|| > 0$). The person represents this relationship by plotting a point on the horizontal axis understanding that this point simultaneously represents both magnitudes (Fig. 2a).
(2) The person conceives that over the first portion of the trip, for any particular magnitude increase in the distance from Athens, the distance from Gainesville simultaneously decreases by that same magnitude (i.e., $|\Delta||X||| = |\Delta||Y|||$). The person conceives the distance from Athens increases at a constant rate with respect to decreases in the distance from Gainesville.
(3) The person represents the relationship she constructed in (2) by constructing two magnitudes covarying along the axes in a way that maintains (2), with a point moving correspondingly to represent the *uniting* both magnitudes (Fig. 2b–d) as a single construct (P. W. Thompson et al., 2017).
(4) The person conceives the point leaving a trace representing all instantiated pairs of covarying magnitudes (Fig. 2e).
(5) And so on (Fig. 2f).[6]

We use this example to illustrate a form of emergent shape thinking that entails pairing and coordinating two *magnitudes* to produce a graph in a way that maintains a relationship equivalent to that in the situation. Adopting the notation Thompson (2011) used to represent covariation as uniting and coordinating two attributes simultaneously, we can model a person's emergent shape thinking as described above with $(||X||, ||Y||)$. Here, $(||X||, ||Y||)$ represents the uniting of two quantities' magnitudes, $||X||$ and $||Y||$, such that these magnitudes are understood as covarying (Fig. 3a). Upon choosing unit magnitudes for each quantity, a person can determine specified values that represent relative comparisons between the respective unit magnitudes and each quantity's magnitude at each instantiation of covariation. The result of comparisons between magnitudes and unit magnitudes enables the person to represent the attributes of $(||X||, ||Y||)$ in terms of corresponding values, x and y (with associated unit magnitudes), thus representing the uniting and coordination of these values as (x, y) (Fig. 3b, with a unit of the mile).

We have provided an elaboration of Moore and Thompson's (Moore & Thompson, 2015; Paoletti & Moore, 2017) emergent shape thinking for three related reasons. First, we underscore that such a meaning could involve conceiving a relationship between two *magnitudes* and constructing a graph to represent a relationship between magnitudes (Fig. 3a) as opposed to the more typical graph conveying quantities' values (Fig. 3b). Second, the aforementioned description of emergent shape thinking provides the foundation for our clinical interview design and analysis (see Sections 3–5).

Third, emergent shape thinking is an apropos example of an operative meaning for graphing that necessarily entails figurative fragments. Constructing a graph (whether engaging in emergent shape thinking or not) involves the sensorimotor action of drawing in different physical directions (e.g., up, down, left, or right). Also, a drawn graph has properties that could be interpreted figuratively including its direction (e.g., vertical or horizontal) and shape (e.g., curved or not curved). Emergent shape thinking additionally entails the mental coordination of covarying quantities so that the aforementioned figurative aspects are subordinate to this coordination. Specifically, emergent shape thinking entails mental actions associated with quantitative reasoning (P. W. Thompson, 2011) including partitioning, iterating, disembedding, units coordination, and related counting actions (Steffe & Olive, 2010). These mental actions are coordinated with images of quantities' variation in order to make additive and multiplicative comparisons that

[4] This task is a modification of the task Saldanha and Thompson (1998) presented.
[5] We suggest the reader work the problem in detail before we describe a partial solution.
[6] A person's actions are not likely to proceed in such a linear progression.

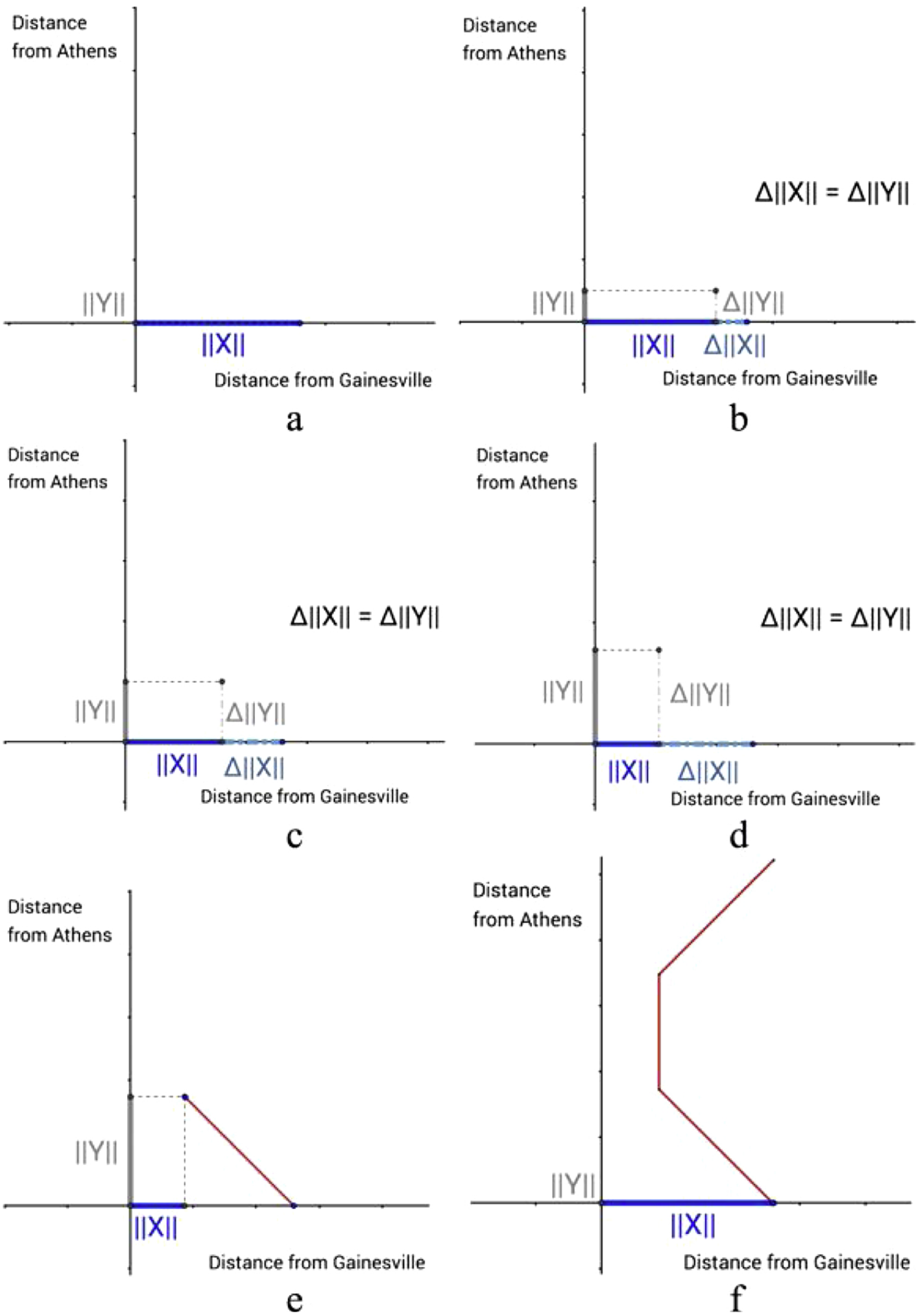

Fig. 2. (a–f) A graph as a coordination of two magnitudes for the trip *there and back*. We adopt the Euclidean distance notation of ||*|| in order to differentiate a quantity's magnitude from a quantity's value (signed or absolute).

describe how quantities covary (Carlson et al., 2002; Johnson, 2012; P. W. Thompson & Carlson, 2017). For example, a student may construct and compare increases or decreases in one quantity for equal variations in a second quantity within a dynamic situation (e.g., a car trip). In order to represent such a relationship graphically, the student then disembeds the results of those quantitative operations from the situation and inserts them into a coordinate system orientation to represent an equivalent covariational relationship (Lee, 2017; Lee, Hardison, & Paoletti, 2018; Lee, Moore, & Tasova, 2018, online; Paoletti, Lee, & Hardison, 2018). This latter action is a hallmark of operative thought due to the individual transforming figurative entailments associated with their mental actions to maintain quantitative equivalence in a context (e.g., a coordinate system) that differs figuratively from that in which the quantitative relationship was initially constructed (e.g., a dynamic situation).

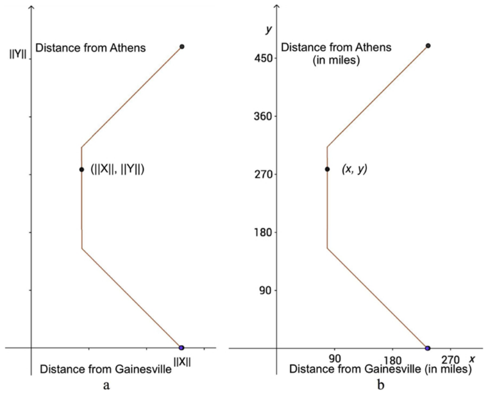

Fig. 3. (a) A magnitude graph and (b) a values graph, both understood as representing the same covariational relationship.

Although the reader might deem such a meaning trivial, our and other researchers' results with students and teachers suggest that such a meaning is anything but trivial (Johnson, 2015b; Moore, Paoletti, & Musgrave, 2014; P. W. Thompson & Carlson, 2017). Hadjidemetriou and Williams (2002) identified that UK students show a propensity for drawing a graph through the origin regardless of context. Janvier (1998) briefly alluded to students' propensity to sketch a graph from left-to-right regardless of relationship. Recently targeting teachers' capacity to reason emergently, P. W. Thompson et al. (2017) illustrated that 67.8% of 121 U.S. secondary mathematics teachers' responses did not indicate their assimilating covarying magnitudes oriented orthogonally to an emergent meaning for graphs. This result is a significant concern and we infer there is a need to provide more detailed analyses of students' and teachers' activity with the following question in mind: To what extent do individuals' meanings for graphs foreground figurative and operative aspects of thought?

3. Participants, setting, and methods

We collected data with six prospective secondary mathematics teachers (hereafter referred to as students) enrolled in an undergraduate secondary mathematics education program in the southeastern U.S. The students ranged from juniors to seniors in credits taken and had completed at least one mathematics course beyond an undergraduate calculus sequence. We chose the students on a volunteer basis and selected students from the volunteer pool based on availability. The students received financial compensation for their time. Within the results section, we report five data excerpts from four students—Annika, Betsy, Lydia, and Patty. We chose the excerpts as representative illustrations of important differences in how students' meanings can foreground figurative or operative thought.

We conducted semi-structured clinical interviews (Ginsburg, 1997; Goldin, 2000) with each of the students. Each interview lasted approximately 75 minutes. The lead author and another member of the research team (i.e., members of the author team and additional mathematics educators) acted as co-researchers at each interview. Consistent with semi-structured clinical interview principles described elsewhere, we began each interview by asking the student to talk-aloud as much as possible (Carlson & Bloom, 2005) and then asked open-ended questions during her progress (Ginsburg, 1997; Goldin, 2000). For instance, because we do not take the production of a drawn graph to be indicative of covariational reasoning, we first asked each student to explain how she determined to draw her graph in order to not direct the student's attention to a specific aspect of the graph. Also consistent with semi-structured clinical interview principles, we made modifications to the interview protocols both within and between interview sessions in order to be sensitive to an individual student's thinking and ongoing findings. Allowing modifications to interview protocols based on ongoing findings enabled us to gain deeper insights into each student's thinking when compared to using a protocol fixed with respect to tasks and questions. Each interview included tasks designed using the principles we describe in Section 4, and we provide a collection of tasks in Appendix A for reference.

All interviews were video- and audio-recorded using video cameras to capture student work, physical actions (e.g., drawing and pointing), and researcher-participant interactions. For tasks including dynamic animations, we also captured the computer-displayed animations and students' interactions with the display. We digitally captured all written work after each interview, which enabled us to compare video data to written records. The lead author and co-researcher(s) also recorded and digitized observation notes after each interview.

We analyzed the data following selective open and axial methods (Corbin & Strauss, 2008) in combination with *conceptual analyses*—attempts to build models of students' mental actions that explain their observable activity and interactions (P. W. Thompson, 2008; von Glasersfeld, 1995). Members of the research team first identified instances that provided insights into each student's thinking. The research team then viewed these selected instances in order to build models of the student's thinking with attention to the student's quantitative and covariational reasoning, which we also compared to the observation notes captured after each interview. As the research team developed these models, we continually returned to previously identified instances across all students to compare, revise, or provide alternative models based on interpretations of latter instances. This iterative process generated themes among characterizations of students' meanings, several of which we report in this paper.

One area in which we narrowed our focus during analysis (both between interviews and retrospectively) was the instances during which students experienced perturbations (e.g., their obtaining an unexpected outcome including searching for a solution or identifying a contradiction in their actions and results) as this represented an emergent theme that ultimately informed protocol modifications. We did so due to the prevalence of students' perturbations in combination with several students being unable to reconcile their perturbations to their satisfaction. In our analysis, we hypothesized that one explanation for a student's perturbation was her holding two meanings that were incompatible at that moment as indicated by the student questioning the efficacy of her actions; the students experienced perturbations due to different meanings entailing results that they perceived to contradict each other (see Section 5.3). We further inferred that these incompatibilities stemmed from tensions between the sensorimotor activity of drawing a graph and the goal of constructing a graph that represents a covariational relationship. We thus adopted the figurative and operative distinctions in subsequent analysis and protocol design hoping to generate more viable explanations of students' graphing meanings including the aspects of thought that produced perturbations. Namely, we became more attentive to considering whether students' graphing actions implicitly or explicitly privileged characteristics of the physical movements of drawing a graph including how the resulting graph 'looks', or if they privileged quantitative attributes such as two magnitudes along axes and properties of their covariation.

4. Task design

We designed a series of tasks in which we either asked the students to construct a graph representing a relationship defined by an analytic rule, interpret a given graph, or construct a graph representing some relationship in a dynamic situation. A subset of tasks included aspects we perceived to be unconventional, which we often presented as the work of a hypothetical student. In such cases, we asked the participants to comment on the thinking that might generate the posed graph. Some tasks included a second part that clarified the unconventional aspects of the hypothetical student's work. Figs. 4a, b and 5 illustrate these principles with respect to hypothetical students graphing $y = 3x$ under unconventional axes orientations. With respect to Fig. 4, we provided Fig. 4b only after participants addressed how a hypothetical student may be thinking if he constructed the graph in Fig. 4a as representing $y = 3x$. With respect to Fig. 5, we left the axes unlabeled to gain insights into the orientations the participants imposed.

With respect to constructing a graph representing some relationship in a dynamic situation, each task: (a) provided a dynamic, albeit often simplified, situation through video; (b) did not include numerical values for attributes of the situation; (c) prompted the student to graph a relationship between two quantities; and (a majority of which) (d) prompted the student to create a second graph, either between similar quantities or the same quantities but under different axes orientations. Like the tasks associated with Fig. 4, we only prompted the students to create a second graph after they completed the first part of the task to their satisfaction. *Going Around Gainesville* (*GAG*) (Fig. 1), which involves a video depicting a car starting in Athens and traveling back and forth from Tampa,

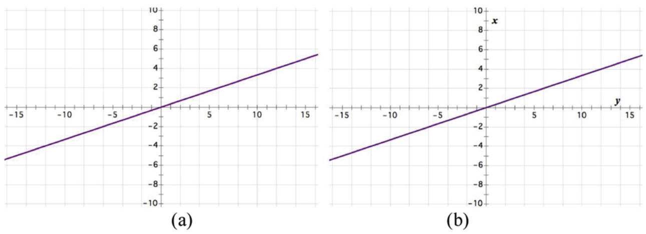

Fig. 4. (a and b) A hypothetical student's work when graphing $y = 3x$.

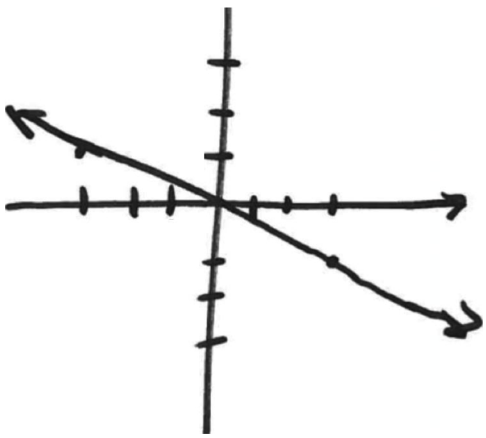

Fig. 5. A hypothetical student's graph of $y = 3x$.

illustrates each of these guiding principles.[7]

Regardless of task type, we intended that the students construct and interpret graphs of familiar relationships (e.g., piecewise linear, quadratic, and trigonometric) but that likely differed figuratively from those they had previously experienced. We operated under the assumption that incorporating tasks that provide a variety of sensorimotor experiences was necessary to gain insights into the extent a student's meanings foregrounded figurative or operative aspects of thought. As the interviews progress, we thus designed the tasks so that the students would engage in a multitude of sensorimotor actions if constructing normative solution graphs (i.e., a graph that an educator would likely judge as correct). We attempted to engage students in familiar and unfamiliar actions including drawing curves left-to-right, drawing curves right-to-left, drawing vertical segments, drawing horizontal segments, retracing a curve, 'starting' curves at different locations in the Cartesian coordinate system, perceiving curves that might not be familiar to them in shape, etc.

5. Results

We structure the results section around examples that illustrate students' meanings that foreground either figurative or operative aspects of thought. We then draw attention to examples in which students experienced perturbations stemming from incompatibilities between figurative and operative aspects of thought. We choose this structure due to our goal of differentiating students' meanings based on the extent they foreground figurative or operative aspects of thought.

We hasten to point out that when imputing a particular meaning to a student, we are providing one hypothesized model that explains our observations of the student's *in-the-moment* activity (P. W. Thompson et al., 2014). The meanings we infer from these students' in-the-moment activity allow us to illustrate the ways in which they may engage in figurative or operative thought; we do not use the following excerpts to make holistic claims or generalizations about the meanings a particular student may or may not hold and enact.

5.1. Foregrounding figurative aspects of thought

We first draw on Lydia's response, which is analyzed in detail elsewhere (Lee, Moore, et al., 2018, online), to Fig. 4b in order to illustrate a students' actions foregrounding figurative aspects of thought. After Lydia rotated the paper counterclockwise 90-degrees, apparently to orient the x-axis horizontally, she conceived the "slope" as "rising negative three." Lydia clarified, "If I were to rise here…I'm rising this three [*referring to the denoted vertical line segment*, Fig. 6]…and then I'm running negative one [*referring to the denoted horizontal line segment*, Fig. 6], which would then [be] three over negative one x $\left[\textit{writes } `\frac{3}{-1}x'\right]$ still equals negative three x [*writes* '$= -3x = y$' *after* $\frac{3}{-1}x$]." In order to ensure that Lydia was aware of the positive and negative orientations of the rotated axes, the interviewer asked her to determine the coordinates of the point identified in Fig. 6. Lydia did so without difficulty.

As Lydia continued, she maintained the "slope" of the given graph (Fig. 4b) as positive and the "slope" of the rotated graph (Fig. 6) as negative due to "rising" and "running" attributes. Yet, she was perturbed because the equation $y = 3x$ defined the coordinate points for both the given and rotated graphs. In light of Lydia's perturbation, a researcher rotated the given graph (Fig. 4b) 90-degrees *clockwise* and asked her to consider that orientation (Excerpt 1).

Excerpt 1. Lydia considering an alternate rotation of the given graph.

Int.2:	What about like the slope this way? How would we compute like a slope?
Lydia:	The slope is negative again [*pausing and then making sounds implying her thinking*].
Int.2:	Because we do it the same way, or how would we even compute it?
Lydia:	Because I, to get from, um [*pause*]. This would- this way would correspond with that $\left[\textit{pointing to } \frac{3}{-1}x\right]$, in that we're, okay, so we're rising 3 [*tracing upward along the vertical axis, see* Fig. 7] and going the negative direction for 1 [*tracing to the graph from right-to-left, see* Fig. 7] and so that's still a negative slope of 3 [*pointing to* -3 *in* $y = -3x$].
Int.1:	So then that's the rule for that one [*pointing to the graph*]?
Lydia:	Mm-hm. Yes.
Int.1:	That's kind of interesting, right? Because what, we do 5 and 15, like 5 gives us 15 [*re-identifying circled point and corresponding coordinate values on axes in* Fig. 7], but if I plug in 5 there I get negative 15 [*pointing to* $y = -3x$].
Lydia:	Mh-hm. [*tapping pen repeatedly, laughs*] This is [*speaking emphatically*] so annoying.

Consistent with her activity when the given graph was rotated counterclockwise 90-degrees, Lydia associated the sensorimotor actions of moving increments right or left and up or down with negative or positive "rising" and "running" values. For instance, a negative run necessarily involved a sensorimotor movement to the left and a positive rise necessarily involved a sensorimotor movement upward; her conception of how a quantity's value varied was figurative in that its increase or decrease was constrained to a direction movement (Lee, Moore, et al., 2018, online). Furthermore, we interpret her actions to be a contraindication of a meaning

[7] An evolving collection of tasks is located at goo.gl/mS68fj.

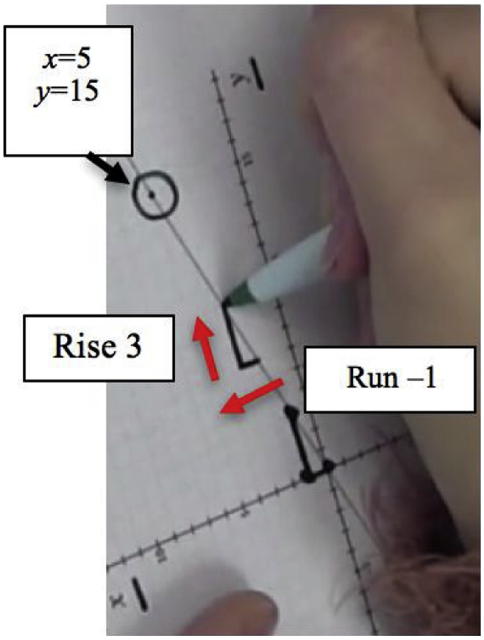

Fig. 6. Lydia's annotated work on the given graph rotated *counterclockwise* 90-degrees. (Lee, Moore, et al., 2018, online).

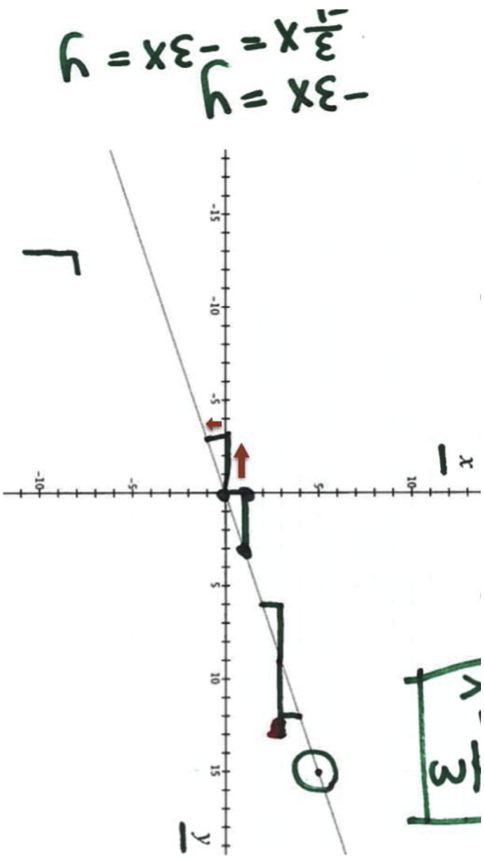

Fig. 7. Lydia's annotated work on the given graph rotated clockwise 90-degrees.

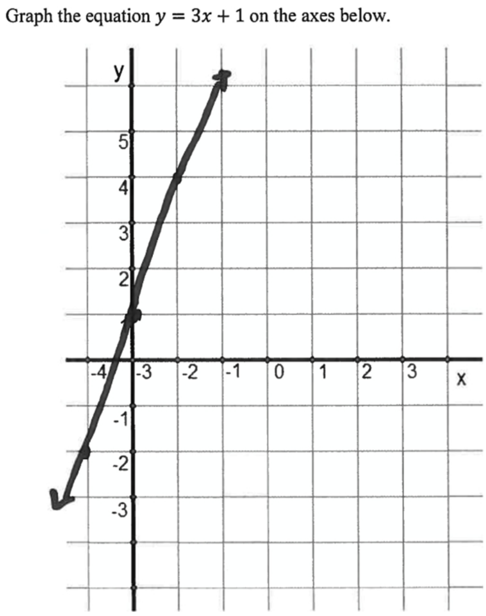

Fig. 8. Betsy's work graphing the line $y = 3x + 1$ on the given axes.

that foregrounds operative aspects of thought because her associations with movement persisted when her attention was drawn to specific coordinate values. Lydia maintained $y = -3x$ as an appropriate equation based on her “slope” despite understanding that her equation was not consistent with the coordinate pair values. She ultimately ended the problem unsatisfied due to such inconsistencies, further suggesting that her meaning for graphing and slope had not been abstracted beyond particular sensorimotor activity.

As a second example of a student's graphing meanings foregrounding figurative thought, we present Betsy's response when asked to graph $y = 3x + 1$ on coordinate axes that are denoted such that they intersect at the point $(-3, 0)$. Betsy initially did not notice our unconventional location of the intersection of the axes. She graphed the line by moving up one from the intersection of the axes and plotting the point often referred to as the y-intercept. Then, based on the value of 3 in the given equation, Betsy moved up three units and right one unit from her first point, plotted a point, and then repeated this process to plot another point. She then moved down three units and left one unit from her first point to draw four total points, which she connected with a line (see Fig. 8 for her work). Later in the interview, and based on the conjecture that Betsy did not explicitly identify the coordinates of the axes intersection as implied, the interviewer drew her attention to the coordinates (Excerpt 2).

Excerpt 2. Betsy reconsidering how to graph y = 3x + 1.

Int:	Look specifically at the axes below.
Betsy:	Uh! Gosh. I need to pay more attention. Come on Betsy. Okay, yeah yeah yeah, that's funny, okay so. Why, how, why? I don’t know [*puts hand on head*]. Why!? [*laughs*] Why is that necessary [*motioning over axes*]? I don’t know. [*4-second pause*] I don’t know why, like. [*5-second pause*] So then. [*14-second pause*] I don’t know but, I don’t know this is so confusing. I don’t know.
Int:	[*laugh*] So how would you, if they asked you to graph this [*pointing to the equation y = 3x + 1*] on that [*pointing to axes*]?
Betsy:	That's not right [*motioning over her graph*], obviously. [*interviewer asked her to use a different color marker if she wants to draw a new graph.*] Yeah, I don’t, wait. [*7-second pause*] Okay wait, so, this [*indicating the equation*] is saying. Okay, you’re y-intercept is one, your slope is three. So [*3-second pause*].
Int:	So could this be right then [*pointing to her graph]*?
Betsy:	I mean you still go up three and over one [*motioning from her first drawn point up then to the right*].
Int:	Okay.
Betsy:	So that's, that's the right slope and the right y-intercept. You just moved your axes negative, I don’t know [*laughs*], over [*motioning as if moving vertical axes to the left*]. So. [*6-second pause*]. But with this being. I don’t know why, I don’t.
Int:	So would you just, if you were doing it with these axes now, would you just, same line [*motioning over line*]?
Betsy:	I guess so cause I mean that's, I don’t. I wouldn’t say, I don’t think that that's wrong. You know?
Int:	Okay.
Betsy:	Technically, your slope is three. You know, that is three [*motioning up then over from*

her first drawn point to her second drawn point and tracing over the line] and you're *y*-intercept is one [*pointing to her first drawn point*], so that's what that gives you [*motioning over the line*].

As indicated by both her words ("Why, how, why? I don't know") and her frequent pausing, Betsy experienced a minor perturbation upon perceiving an unconventional location of axes intersection. Although Betsy initially conjectured her original graphed line was incorrect ("That's not right… obviously"), as she considered her constructed line in relation to her meanings for graphing linear functions of the form $y = mx + b$, she concluded that her line accurately represented a *y*-intercept of 1 and a slope of 3. We characterize Betsy's meaning as figurative in that we infer it was constrained to the action of moving *b* units up the *y*-axis from the intersection of the axes, then using the slope value to move up (or down) and over a certain number of units. Furthermore, we interpret Betsy forgoing the quantitative implications of her initial point's location with respect to the *x*-axis to be a contraindication of a meaning that foregrounds operative thought; despite being aware of the unconventional axes location, her meaning did not entail coordinating the quantitative implications of this with her meaning for graphing linear functions.

5.2. Foregrounding operative aspects of thought

We use Annika's responses to two tasks to illustrate a student constructing or interpreting a graph in a way that implies the foregrounding of operative thought in the form of emergent shape thinking with respect to magnitudes or values. We first present Annika's activity addressing *GAG Part II*. Excerpt 3 occurred after Annika had constructed a normative solution to *GAG Part I* by engaging in emergent shape thinking. Recall that we anticipated students' solutions to *Part II* of the task could result in their making comparisons with *Part I* that identified similarities or differences in sensorimotor actions or the represented covariational relationships.

Excerpt 3. Annika and emergent shape thinking on GAG Part II.

[*The interviewer has provided Annika with GAG Part II*]

Annika: [*Drawing an image of the road on the paper, seen in* Fig. 9] Okay. This is Athens, and this is Gainesville, right, cool [*denoting both Athens and Gainesville on her drawn image*]. Alright, so we're starting in Athens, so our distance from Athens better be zero, but we're obviously not in Gainesville, and we're the furthest from Gainesville that we're ever going to be in our trip [*tracing a magnitude on her drawn image to indicate the distance from Gainesville*]. So [*tracing an equivalent magnitude on the horizontal axis of her graph*] we're like here [*plotting a point on the horizontal axis, seen in red in* Fig. 9].

Alright. As we travel [*traces along the first road segment on her drawn image*], black's gonna be distance from Athens [*drawing a black segment below the road to represent an accumulated distance from Athens*], and red will be distance from Gainesville [*drawing a red segment above the road to represent an accumulated distance from Gainesville seen at the bottom of* Fig. 9]…

So for equal changes in distance from Athens as we go from here to here [*denoting equal changes in the accumulated distance from Athens on her drawn image*], we see that our distance from Gainesville is also changing by the same amount [*denoting equal changes in the accumulated distance from Gainesville on her drawn image*], but it's decreasing…And we're saying that it's changing by the same amount [*referring to corresponding equal changes on each axis*]. We're going to pretend I scaled this better, [*drawing a segment for her graph*] but we're going to pretend that corresponds to equal changes in distance and the same equal change from Gainesville, the same equal change from Athens…

[*See* Fig. 9 *for completed graph and annotated situation Annika continues her graph to represent the car's distance from each city as it travels around the semicircle by drawing a vertical segment*]. I know I want my distance from Gainesville to stay the same. But my distance from Athens is increasing.

Suggestive of emergent shape thinking, Annika maintained a persistent focus on coordinating two magnitudes and how they vary. She did this first within the situation, specifically through partitioning segments and constructing additive comparisons of those partitions to infer how the two distances covaried. Disembedding these quantities and the result of their covariation from the situation, Annika then coordinated two magnitudes in order to graphically represent the relationship (||Distance from Gainesville||, ||Distance from Athens||) she constructed within the situation. Also indicative of emergent shape thinking, Annika maintained an explicit focus on axes through the process of graphically representing the relationship. For instance, Annika engaged in partitioning acts to indicate corresponding changes in orthogonal magnitudes before constructing her graph, which suggests she disembeded the quantities from the situation and inserted them into a coordinate system. Importantly, the mental (quantitative) operations she used to construct her graph (and likely her graph to *GAG Part* I) were not constrained to an axes orientation or the situation, which is a contraindication of a meaning that foregrounds figurative thought.

We also use Annika's activity addressing the task associated with Fig. 5 to highlight a student's meaning dominated by operative thought. Annika initially identified several axes orientations by which the graph is a viable representation of $y = 3x$. This included identifying positive *x*- and *y*-values oriented up and left of the origin, respectively. She also identified positive *x*- and *y*-values oriented down and right of the origin, respectively. With respect to the latter orientation, the interviewer asked her to discuss a

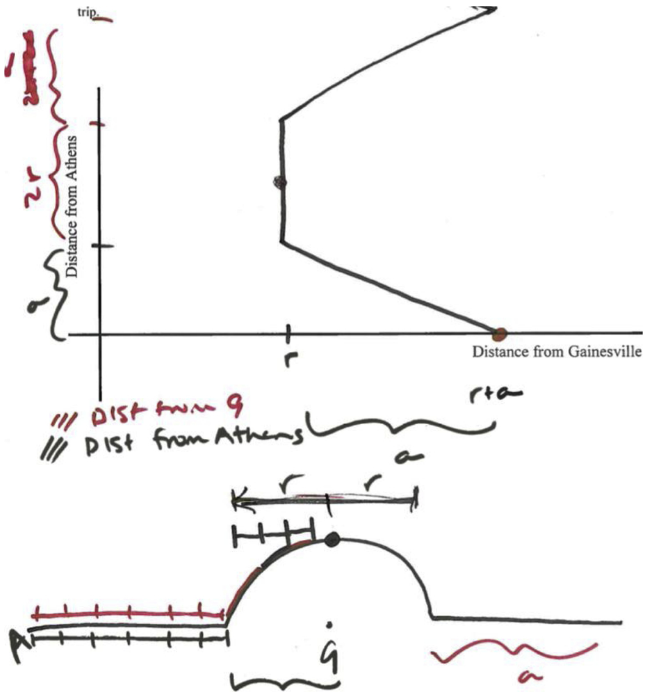

Fig. 9. Annika's completed work for *GAG Part II*.

hypothetical student claiming that the line has “a negative slope” and cannot be a graph of $y = 3x$ (Excerpt 4).

Excerpt 4. Annika coordinating two quantities’ values (see Fig. 10 for Annika’s annotated figure).

Annika:	You’d have to notice that even though it looks like a negative slope [*making a hand motion down and to the right*] because we call it slope because it's visual and it's easy to visualize a negative and positive slope [*making hand motions to indicate different slopes*]. But that's only visual on our conventions of how we set it up. Um, but like [*pointing to the graph*] if slope is rate of change we can still see that for like equal increases of x [*making hand motions to indicate equal magnitude increases*] we have an equal increase of y [*making hand motions to indicate equal magnitude increases*] of three. And so for equal positive increase of one [*sweeping fingers vertically downward to indicate an increase of one*], we have an equal positive increase of three [*sweeping fingers horizontally rightward to indicate an increase of three*]. And so it is a positive slope.

Annika identified associations between figurative aspects like direction of a line and properties of “slope.” She also immediately described “slope” as “rate of change,” and her subsequent actions involved partitioning two quantities and additively comparing those partitions and associated values under the quantitative constraints of the coordinate system in which their relationship is graphed (i.e., emergent shape thinking). As a contraindication of a meaning that foregrounds figurative aspects of thought, Annika anticipated other graphs that differed perceptually but represented the same coordination of quantities, including those under “our conventions”. Similar to her actions with magnitudes on *GAG Part II*, these actions indicate foregrounding operative thought in the form of coordinating covarying quantities’ values (x, y) so that figurative aspects of activity are a consequence of this coordination, and thus change based on different coordinate orientations.

5.3. The compatibility of figurative and operative aspects of thought

Returning to Annika (Excerpt 4), her fluidity suggests that figurative and operative aspects of her meanings were internally compatible. Namely, we infer that she understood figurative aspects of her actions were a consequence of coordinating quantities’ magnitudes or values under the quantitative constraints of a coordinate system. In contrast, Lydia (Excerpt 1) experienced a perturbation because figurative and operative aspects of her meanings were internally incompatible. In the moment of her solving the task associated with Fig. 4, the implications of her conceiving a graph as composed of coordinate points contradicted her meaning for “slope” that foregrounded directions of sensorimotor movement. The incompatibility of these two meanings, in that moment, led Lydia to end the task perturbed and unsatisfied, claiming “this is so annoying.”

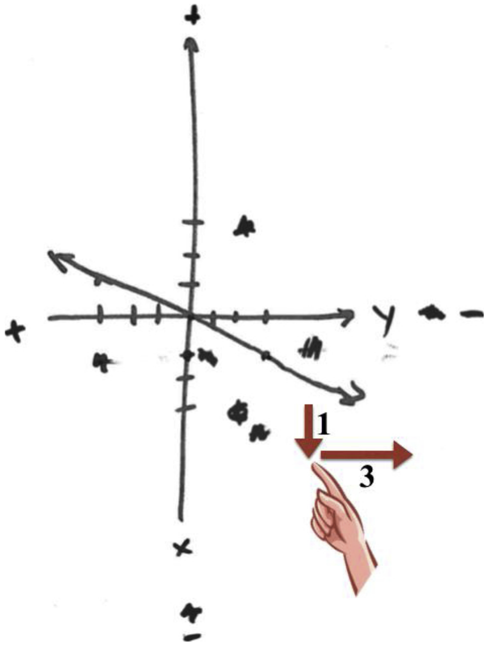

Fig. 10. Annika's annotated Fig. 5, each tick mark based on a unit change of 1. In Excerpt 4 she considers positive *x*- and *y*-values oriented down and to the right of the origin, respectively.

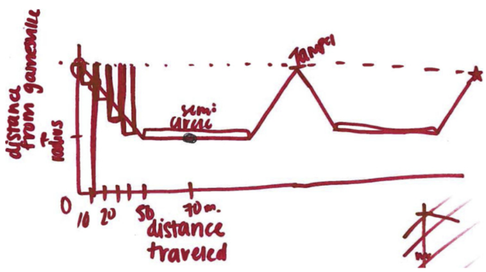

Fig. 11. Patty's work on *GAG Part I*.

We experienced several instances in which students held meanings such that figurative and operative aspects of thought were internally incompatible, resulting in them experiencing perturbations when those incompatibilities were raised due to engaging in a variety of figurative activity. As another illustration, consider Patty's activity on *GAG Part I-II* (Fig. 1). Patty constructed a normative solution to *Part I* by identifying different intervals over which the distances changed by equivalent magnitudes or a distance remained constant. She then constructed segments and appropriate partitions to graphically represent that relationship (Fig. 11).

In response to *GAG Part II*, Patty immediately marked a point on the vertical axis and anticipated drawing a curve emanating from that point (Excerpt 5). Recall that a normative solution to *GAG Part II* includes no points representing a zero distance from Gainesville (Fig. 2).

Excerpt 5. Patty 'starting' graphs along the vertical axis.

Patty:	Your distance from Athens starts at zero [*plots point at origin*] because you're in Athens. Um, so as you get. Mmm, no, you're gonna start up here [*plots point on vertical axis but not at origin to represent a non-zero distance from Gainesville*]. Ignore that [*covering origin*]. 'Cause, oh wait, no, stop [*crosses out second plotted point*]. No, you're here [*points to origin*].

Patty plotted an initial point by focusing on one quantity's initial magnitude and identifying a point on the vertical axis based on that magnitude. Patty alternated which quantity she considered, maintaining her point along the vertical axis regardless of the quantity she was considering.

Patty continued, eventually determining an alternative 'starting' point that was not on the vertical axis (Excerpt 6). During this time, she was perturbed by that 'starting' point and how to draw a graph emanating from that point. We provide a condensed account of her activity due to the extended time over which it occurred.

Excerpt 6. Patty continues *GAG Part II*.

Patty: [*Patty has determined an initial point that is not at the origin but is along the vertical axis. She motions as if drawing a segment sloping downward left to right from this point that she later crosses out—see the crossed out point on vertical axis in* Fig. 12] I wanted to start here because I wanted to show that the distance was decreasing [*motioning down and to the right from the point plotted on the vertical axis*], but that means your distance from Athens is decreasing [*tracing vertical axis from the initial point to the origin*]…But your distance from Athens is growing. But your distance from Gainesville is decreasing. So, if that's growing and that's decreasing, so [*draws an arrow pointing downward beside horizontal axis label and then an arrow pointing upwards beside the vertical axis label*]

[*Patty then works for six additional minutes maintaining her 'starting' point on the vertical axis, without making progress, and explaining "this is so hard". She repeatedly identifies the distance from Gainesville as decreasing and the distance from Athens as increasing, including drawing a graph in an alternative axes orientation (i.e., Distance from Athens ("dA") being on the horizontal axis, see the bottom right of* Fig. 12*). She eventually has an insight*.]

Ohhhh, what if I started it like here [*plots point on the right end of the horizontal axis*]. Okay…but I don't want to start like, like

I don't like starting graphs. You know

I don't know work backwards that's weird…[*in the next minute and a half Patty draws in a normative initial segment of the graph, as seen in* Fig. 12, *hesitating throughout while explaining how the distances covary*] But it's backwards so I don't like it…My graph is from right-to-left, which is probably not right…[*describes the covariational relationship between the two distances*] I guess I just don't like this.

Int.: And why don't you like it?

Patty: Because it's backwards.

Int.: And by backwards we mean?

Patty: Backwards is traveling from right-to-left. But I think my graph is just, I think I'm just not clicking. I think I'm missing something.

Patty's perturbation is noteworthy for a few reasons. First, she had constructed a developed image of how the two relevant quantities' magnitudes covaried with respect to the situation; her perturbation did not stem from an undeveloped image of how the magnitudes covaried in the situation. Second, Patty had constructed a graph that she understood to represent that relationship in a different axes orientation. Third, after nearly ten minutes of activity (which is also notable due to her quickly graphing this relationship in an alternative orientation; see lower right of Fig. 12), she constructed a normative partial graph via emergent shape thinking (Fig. 12). But, Patty became perturbed during and post-construction of her partial graph due to figurative aspects, including the physical location of "starting" the graph to the right of the vertical axis, "work[ing] backward," and "traveling right-to-left."

Collectively, we infer that Patty held particular figurative-dominated graphing meanings that were incompatible with the figurative activity occurring as a consequence of her emergent shape thinking, leading her to claim, "I'm missing something." Important to note, her reasoning entailed operative aspects with respect to the situation *and* constructing a normative graph to *Part I*; however, with respect to *Part II*, Patty's reasoning was dominated by figurative activity in that her graphing meanings foregrounded sensorimotor actions that restricted her in representing an equivalent relationship with efficacy in an alternative axes orientation.

6. Discussion

We inferred that the study's participants held meanings constrained to several, sometimes intertwined, figurative aspects: drawing a graph left-to-right, drawing a graph without a vertical segment, 'starting' a graph on the vertical axis, and drawing a graph without

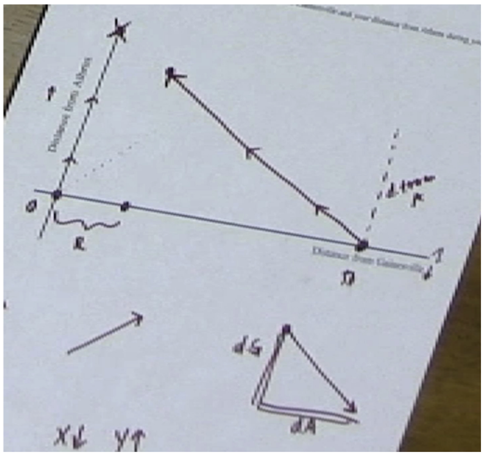

Fig. 12. Patty's graph for the first portion of the trip for *GAG Part II*.

retracing part of the graph. Additionally, we inferred some students to hold meanings for graphing and related topics that were constrained to carrying out a sequence of actions that were not sensitive to the quantitative organization of unconventional coordinate systems. For example, some students associated slope or graphing linear relationships with a prescriptive sequence of sensorimotor movements and point plotting that was not sensitive to the quantitative organization of the coordinate systems as designed. On the other hand, we also inferred that the students held meanings that foregrounded operative aspects of thought associated with quantitative reasoning: constructing measurable magnitudes, partitioning, disembedding, iterating, and forming additive comparisons for the purpose of coordinate amounts of change. Each student exhibited signs of reasoning emergently, but only one student—Annika—did so in a way that dominated figurative aspects of thought without perturbation throughout the entirety of her activity.

Closing their study into students' covariational reasoning, Carlson et al. (2002) explained, "We have provided examples of students who appeared to be able to apply covariational reasoning…in a kinesthetic context but who were unable to use the same reasoning patterns when attempting to construct a graph…for these situations" (p. 376). Compatible with Carlson and colleagues' finding, several of the examples above illustrate students who experienced perturbations constructing a graph to represent relationships they conceived to constitute situations. We extend Carlson et al.'s findings by identifying that our students' perturbations stemmed from their holding meanings for graphing that foregrounded figurative aspects of thought incompatible with meanings rooted in operative thought. This resulted in students ending tasks without reconciling the perturbations they experienced due to perceiving their graphs entailing figurative aspects that caused them to question the efficacy of their actions.

To be clear, we emphasize that we interpret the students' activities to be suggestive of both figurative and operative thought. We point out, however, that a student's engagement in instances of operative and figurative thought over the course of a task does not imply that the student has constructed (meanings or) ways of thinking for graphing that exist in mutual compatibility. Lydia and Patty's actions and experienced perturbations provide novel examples of students holding meanings or ways of thinking that are, in the moment of activity, incompatible due to conflicting elements of operative and figurative thought.

In cases in which participants held incompatible meanings, we can characterize the participants as conceiving or assimilating two different objects (von Glasersfeld, 1995): an object constructed via operative meanings (e.g., an emergent trace of covariation) and an object constructed via figurative meanings (e.g., a set of sensorimotor actions and perceptual results). The extent that our participants experienced perturbations due to incompatibilities between these objects illustrates how an individual's reconciliation of incompatibilities between meanings is not a trivial matter. Importantly, reconciling these incompatibilities requires that they become reflectively aware of their meanings so that they can examine their contents and implications with respect to each other. Such a process is complex, effortful, and of the utmost significance for teachers so that they can identify productive meanings and subsequently support their students in constructing those meanings (Silverman & Thompson, 2008; Simon, 2006; Tallman, 2015; under review; A. G. Thompson & Thompson, 1996). On that note, it is significant that several participants initially constructed what we perceived to be normative graphs through the coordination of quantities' magnitudes, but then questioned the efficacy of their actions due to meanings foregrounding figurative aspects of thought. We can only hypothesize as to our participants' reactions to students who produce such graphs via reasoning emergently, but our results call into question the extent our participants—future teachers—would determine the students' graphs as correct.

We close this section by noting that sensorimotor experience is important to mathematical development, and it forms a foundational basis for children's mathematical experiences (Piaget, 2001; Steffe & Olive, 2010). Furthermore, it is necessarily the case that an individual's meaning for a concept is first constrained to the activity through which it was learned (Cooper, 1991; Simon, Placa, & Avitzur, 2016), leading some researchers to label the initial stage of meaning development as *participatory* (Simon et al., 2016; Tzur & Simon, 2004). At the same time, and as our results illustrate, it is important that individuals develop meanings that become less constrained to those activities first experienced—a stage of learning some researchers have labeled *anticipatory* (Simon et al., 2016; Tzur & Simon, 2004)—especially those activities that are rooted in sensorimotor experience.

Our results corroborate researchers' (Carlson et al., 2002; Moore, Paoletti, & Musgrave, 2013; Moore & Thompson, 2015) claim that operative ways of thinking for graphing, like emergent shape thinking, are productive and generative for accommodating novel quantitative situations and a variety of figurative activity. The mental actions that constitute emergent shape thinking are, at their most fundamental bases, akin regardless of the produced trace and its coordinate system (Moore et al., 2013; Moore & Thompson, 2015); an individual who has constructed graphing meanings consistent with emergent shape thinking understands a drawn graph as the interplay between coordinating quantities and the oriented constraints of a coordinate system. Perceived figurative features of a graph are understood to be a consequence of this interplay, thus enabling an individual to transform their quantitative operations to accommodate novel figurative activity. Such transformations are consistent with an anticipatory stage of learning because the mental actions are not constrained to the specific activity in which they were initially constructed (Tzur & Simon, 2004).

7. Closing remarks

We accomplished three primary goals in this manuscript. First and foremost, we adopted figurative and operative distinctions to characterize important differences in students' graphing meanings, including the extent their meanings foreground sensorimotor actions or covarying quantities. Second, we extended Moore and Thompson's (Moore & Thompson, 2015; Paoletti & Moore, 2017) emergent shape thinking construct in order to underscore the role of reasoning about magnitudes (cf. numbers or values) when graphically representing covarying quantities from some situation. Third, we demonstrated that undergraduate students can experience perturbations that persist and stem from holding incompatible meanings for graphing, essentially resulting in their conceiving a drawn or anticipated graph as two different objects—one that foregrounds figurative aspects and one that foregrounds

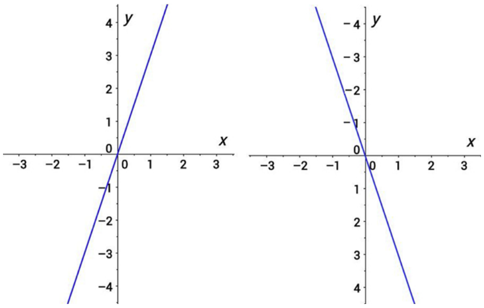

Fig. 13. "Two" graphs of $y = 3x$.

operative aspects. In doing so, we provided evidence that students' meanings for graphing can be problematic if dominated by figurative aspects of thought, resulting in students questioning the efficacy of meanings foregrounding operative thought.

Working with students at the undergraduate level presents both benefits and limitations. Our work with undergraduate students enabled us to gain insights into those meanings they developed during their previous schooling, but we suggest that researchers investigate other populations' responses to tasks designed like the ones in this manuscript. The results of such investigations will provide insights into those populations' meanings for graphing, and they will provide points of comparison that further clarify the findings presented here. Furthermore, we suggest that researchers both within the U.S. and internationally investigate similar populations to the one reported on here. Researchers who extend this work will provide extensions of those meanings reported here and insights into alternative meanings. One interesting point of comparison will involve characterizing differences and similarities among students in various countries, which could provide insights into the different educational experiences students encounter. An important limitation of the present study is that we worked with pre-service teachers, and thus any drawn implication relative to their instruction is at best speculative. Based on that limitation and on those instances in which our pre-service teachers rejected graphs that we perceived as correct, we call for researchers to investigate in-service teachers' actions in similar situations including those instances in which their or hypothetical students produce the solutions (see Moore, Silverman, Paoletti, Liss, & Musgrave, 2018, online).

We close by returning to Piaget's epistemology. Piaget considered mathematical thought to be developmental with roots in figurative activity. He argued that one thing that sets mathematical thought apart from other thought is that it is operational so that it dominates and transforms elements of sensorimotor experience (Piaget, 2001). In this work, we focused on students' in-the-moment meanings. We encourage future researchers to use methodologies, such as design or teaching experiments (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Steffe & Thompson, 2000), that enable researchers to have a systematic way to investigate the *development* of students' meanings. We also encourage researchers to use modifications to the aforementioned tasks to explore how entrenched particular students' meanings are. Using rate of change as an example, researchers might draw on figurative and operative distinctions to characterize shifts in the extent that students construct rate of change as a property of perceptual features (e.g., direction of a line or movement—the graphs in Fig. 13 unquestionably imply a positive and negative rate of change, respectively, because the lines slope upward left-to-right and downward left-to-right, respectively) or as a measure of how quantities change (e.g., the graphs in Fig. 13 unquestionably imply $\Delta y = 3\Delta x$ because x and y are simultaneously increasing such that the change in y is three times as large as the change of x) (see Lobato & Siebert, 2002; Stump, 2001; Zaslavsky, Sela, & Leron, 2002 for productive examples of these distinctions). Researchers undertaking such approaches with other topics will provide insights into students' construction of meanings for not only graphing, but also related topics.

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Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at https://doi.org/10.1016/j.jmathb.2019.01.008.

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Erratum regarding missing Declaration of Competing Interest statements in previously published articles

Declaration of Competing Interest statements were not included in the published version of the following articles that appeared in previous issues of Journal of Mathematical Behavior

The appropriate Declaration/Competing Interest statements, provided by the Authors, are included below.

1 “When itâ€™s on zero, the lines become parallel: Preservice elementary teachersâ€™ diagrammatic encounters with division by zero” [Journal of Mathematical Behavior, 2020; 58C: MATBEH_2018_209] 10.1016/j.jmathb.2020.100760 Declaration of competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.
2 “Spectral analysis of concept maps of high and low gain undergraduate mathematics students” [Journal of Mathematical Behavior, 2019; 55C: MATBEH_2018_102] 10.1016/j.jmathb.2019.01.002 Declaration of competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.
3 “A case for combinatorics: A research commentary” [Journal of Mathematical Behavior, 2020; 59C: MATBEH_2020_1] 10.1016/j.jmathb.2020.100783 Declaration of competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.
4 “Defining and demonstrating an equivalence way of thinking in enumerative combinatorics” [Journal of Mathematical Behavior, 2020; 58C: MATBEH_2019_200] 10.1016/j.jmathb.2020.100780 Declaration of competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.
5 “Computing as a mathematical disciplinary practice” [Journal of Mathematical Behavior, 2019; 54C: MATBEH_2018_28] 10.1016/j.jmathb.2019.01.004 Declaration of competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.
6 “Value-thinking and location-thinking: Two ways students visualize points and think about graphs” [Journal of Mathematical Behavior, 2018; 54C: MATBEH_2018_31] 10.1016/j.jmathb.2018.09.004 Declaration of competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.
7 “Studentsâ€™ reasons for introducing auxiliary lines in proving situations” [Journal of Mathematical Behavior, 2018; 55C: MATBEH_2018_21] 10.1016/j.jmathb.2018.10.004 Declaration of competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.
8 “Gesturing standard deviation: Gestures undergraduate students use in describing their concepts of standard deviation” [Journal of Mathematical Behavior, 2018; 53C: MATBEH_2017_33] 10.1016/j.jmathb.2018.05.003 Declaration of competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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9 “A quadratic growth learning trajectory” [Journal of Mathematical BehaviorJournal of Mathematical BehaviorJournal of Mathematical Behavior, 2020; 59C: MATBEH_2019_188] 10.1016/j.jmathb.2020.100795 Declaration of competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

10 “Facilitating video-based discussions to support prospective teacher noticing” [Journal of Mathematical BehaviorJournal of Mathematical BehaviorJournal of Mathematical Behavior, 2018; 54C: MATBEH_2018_77] 10.1016/j.jmathb.2018.11.002 Declaration of competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

11 “Educative experiences in a games context: Supporting emerging reasoning in elementary school mathematics” [Journal of Mathematical BehaviorJournal of Mathematical Behavior, 2018; 50C: MATBEH_2017_74] 10.1016/j.jmathb.2018.02.003 Declaration of competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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14 “The analysis of the understanding of the three-dimensional (Euclidian) space and the two-variable function concept by university students” [Journal of Mathematical BehaviorJournal of Mathematical Behavior, 2019; 57C: MATBEH_2018_72] 10.1016/j.jmathb.2019.03.004 Declaration of competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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17 “What does â€ÿ is defined as an implicit function of xâ€™ mean?: An application of APOS-ACE” [Journal of Mathematical Behavior, 2019; 56C: MATBEH_2019_89] 10.1016/j.jmathb.2019.100739 Declaration of competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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19 “Preservice teachers learning to teach proof through classroom implementation: Successes and challenges” [Journal of Mathematical Behavior, 2020; 58C: MATBEH_2019_64] 10.1016/j.jmathb.2020.100779 Declaration of competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

20 “Elementary teachersâ€™ beliefs on the role of struggle in the mathematics classroom” [Journal of Mathematical Behavior, 2020; 58C: MATBEH_2018_216] 10.1016/j.jmathb.2020.100774 Declaration of competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

21 “Middle school studentsâ€™ generalizations about properties of geometric transformations in a dynamic geometry environment” [Journal of Mathematical Behavior, 2019; 55C: MATBEH_2018_162] 10.1016/j.jmathb.2019.04.002 Declaration of competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

22 “Disciplinary literacy in mathematics: One mathematicianâ€™s reading practices” [Journal of Mathematical Behavior, 2020; 59C: MATBEH-D-20-00023] 10.1016/j.jmathb.2020.100799 Declaration of competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

23 “On the Construction of Set-based Meanings for the Truth of Mathematical Conditionals” [Journal of Mathematical Behavior, 2018; 50C: MATBEH_2017_49] 10.1016/j.jmathb.2018.02.001 Declaration of competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

24 “Studentsâ€™ understanding of the concepts involved in one-sample hypothesis testing” [Journal of Mathematical Behavior, 2018; 53C: MATBEH_2017_166] 10.1016/j.jmathb.2018.03.011 Declaration of competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

25 “Adolescent perceptions of being known in the mathematics classroom” [Journal of Mathematical Behavior, 2018; 54C: MATBEH_2018_61] 10.1016/j.jmathb.2018.10.002 Declaration of competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

26 “What do teachers need? Math and special education teacher educatorsâ€™ perceptions of essential teacher knowledge and experience” [Journal of Mathematical Behavior, 2020; 59C: MATBEH_2019_121] 10.1016/j.jmathb.2020.100798 Declaration of competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

27 “Collaborative gesture as a case of extended mathematical cognition” [Journal of Mathematical Behavior, 2019; 55C: MATBEH_2018_82] 10.1016/j.jmathb.2018.12.002 Declaration of competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

28 “Pre-service teachersâ€™ figurative and operative graphing actions” [Journal of Mathematical Behavior, 2019; 56C: MATBEH_2018_26] 10.1016/j.jmathb.2019.01.008 Declaration of competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

29 “Conventions, habits, and U.S. teachersâ€™ meanings for graphs” [Journal of Mathematical Behavior, 2018; 53C: MATBEH_2017_205] 10.1016/j.jmathb.2018.08.002 Declaration of competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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31 Strengths and inconsistencies in studentsâ€™ understanding of the roles of examples in proving” [Journal of Mathematical Behavior, 2018; 53C: MATBEH_2018_13] 10.1016/j.jmathb.2018.06.010 Declaration of competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

32 “Reasoning within quantitative frames of reference: The case of Lydia” [Journal of Mathematical Behavior, 2018; 53C] 10.1016/j.jmathb.2018.06.001 Declaration of competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

33 “How elementary and collegiate instructors envision tasks as supportive of mathematical argumentation: A comparison of instructors' task constructions” [Journal of Mathematical Behavior, 2018; 53C: MATBEH_2018_54] 10.1016/j.jmathb.2018.08.004 Declaration of competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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37 “Middle school studentsâ€™ employments of gestures for forming three-dimensional objects using an extrusion or spinning method” [Journal of Mathematical Behavior, 2019; 56C: MATBEH_2019_12] 10.1016/j.jmathb.2019.100737 Declaration of competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

38 “Exploring secondary studentsâ€™ conceptualization of functions in three curriculum contexts” [Journal of Mathematical Behavior, 2019; 56C: MATBEH_2018_131] 10.1016/j.jmathb.2019.100718 Declaration of competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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40 "The role of linguistic features when reading and solving mathematics tasks in different languages" [Journal of Mathematical Behavior, 2018; 51C: MATBEH_2018_62] 10.1016/j.jmathb.2018.06.009 Declaration of competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.