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Rethinking the Teaching and Learning of Area Measurement

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Abstract: This study focused on exploring an innovative way of teaching and learning measurement, what we refer to as Dynamic Measurement or DYME. Without relying on the common approach of counting square units, our goal was to engage students in contextually rich digital dynamic tasks to visualize area as a continuous quantity and evaluate the area of a rectangular region as a multiplicative relationship between the two lengths of the sides. In this paper, we briefly describe the iterative process of designing, testing and refining the tasks for DYME pointing to the significance of the design for developing students' thinking of area as length times width.

Geometric measurement: What we know and pushing forward

Extensive research on measurement has described the importance of using square units to cover rectangular surfaces and quantify that covering by counting the square units (e.g. Barrett & Clements, 2003; Battista, Clements, Arnoff, Battista & Borrow, 1998; Izsak, 2005; Kamii & Kysh, 2006). For instance, we can use twenty 1 sq. inch tiles to cover a 5 by 4 inches rectangle as in Figure 1a and claim that its area is 20 sq. inches. Similar to these studies, the Common Core State Standards for Mathematics (CCSSO, 2010) in the United States introduce third grade students to area measurement first by counting unit squares in a rectangular surface, thus forming an array (Content standards 3.MD.C.5 and 3.MD.C.6). Next, the standards assume that students will use this tiling experience to "show that the area is the same as would be found by multiplying the side lengths" (3.MD.C.7.A). However, the standards do not provide information on how students will transition from counting individual units to constructing the multiplicative area formula.

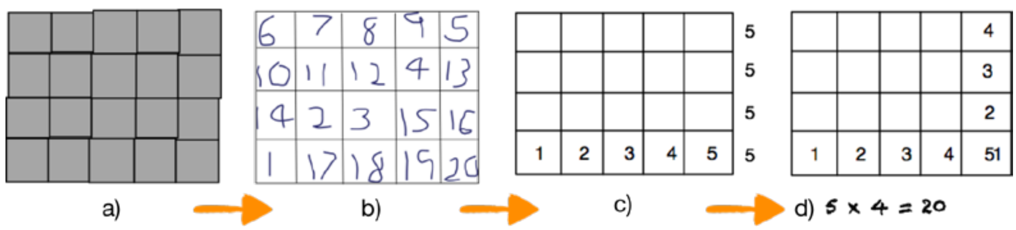


Figure 1. Progression of structuring area based on a synthesis of the measurement literature.

Although the measurement research studies mentioned above suggest a progression of structuring area (Figure 1 a-d), to understand how area is generated by multiplying lengths is a different notion conceptually from the construction of a matrix like shown in Figure 1d. As Piaget et al. (1960) argued, "the difference between the two operational mechanisms is the difference between a matrix which is made up of a limited number of elements and one which is thought of as a continuous structure with an infinite number of elements" (p. 350). Indeed, area, length and width are continuous quantities (Kamii & Kysh, 2006) that are related multiplicatively while covering a surface with discrete unit squares is one-dimensional and additive in nature (e.g. Outhred & Mitchelmore, 2000; Reynolds & Wheatley, 1996).

As a result, this study aimed to go beyond the static perspective of understanding area as the counting of discrete square units and find a more intuitive and accessible approach of illustrating area as a continuous quantity that involves a multiplicative relationship between length and width. To do that, we built on the work of Confrey et al. (2012) and Lehrer, Slovin, Dougherty, & Zbiek (2014) on visualizing area as a 'sweep' of a line segment of length *a* over a distance of *b* to produce a rectangle of area *ab*.

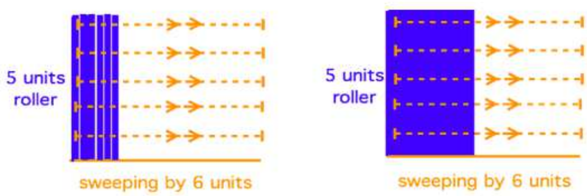


Figure 2. Visualizing area as a continuous structure through 'sweeping.'

For instance, imagine a paint roller with length 5 inches sweeping for a distance of 6 inches and generating a surface of 30 square inches (Figure 2). In this approach, which we refer to as Dynamic Measurement or DYME, area can be visualized as a continuous dynamic quantity which depends on both the length of the roller (length) and the distance of the swipe (width). *DYME* involves engaging students in dynamic experiences of generating 2D surfaces and 3D shapes by iteratively (and multiplicatively) composing lower-dimensional objects (linear measures).

Aims and methods

Our goal was to examine the potential of DYME as an innovative pathway for teaching and learning area measurement. More specifically, we aimed to explore:

a) What type of tasks may be designed for developing students' DYME reasoning?

b) How do these tasks assist students in thinking of area as a continuous quantity?

To provide the experience of visualizing area as a continuous quantity, we used the 'dragging' and 'trace' features of Geometer's Sketchpad (GSP) (Jackiw, 1995) to design a set of tasks. We conducted design experiments (Brown, 1992; Cobb, Confrey, diSessa, Lehrer & Shauble, 2003) with six pairs of third-graders and had 6-10 sessions of 45-90 minutes with each pair of students. The students represented various abilities according to their teacher and all students had some instruction on area as tiling the year before the design experiment. A design experiment starts by formulating some initial conjectures, and these conjectures evolve following an iterative cycle of design, enactment, analysis and redesign:

> On the reflective side, design experiments are conjecture-driven tests, often at several levels of analysis. The initial design is a conjecture about the means of supporting a particular form of learning that is to be tested. During the conduct of the design study, however, more specialized conjectures are typically framed and tested (Cobb et al., 2003, p. 10).

These conjectures evolve throughout the duration of the design study including further iterations that continue in the form of follow-up design experiments. Drawing on the existing literature on area measurement, we gathered measurement constructs identified in previous studies (e.g. indirect/direct measurement, measuring with no gaps or overlaps) and wondered, "How can this construct be interpreted/modified/used in terms of DYME?" We used these wonderings to design some initial tasks and framed humble theories (conjectures) about prospective interactions between our task design and the students' responses. We examined the changes in students' thinking about area when interacting with the DYME tasks, and modified and refined the task design accordingly. During this ongoing analysis (Cobb & Gravemeijer, 2008), our initial conjectures evolved and we modified the tasks in light of iterative examinations of changes in students' thinking about area when interacting with the DYME tasks. The goal of this ongoing analysis (Table 1) was to develop iterative cycles of invention and revision of tasks and humble theories (Cobb et al., 2003). The following section describes in brief this iterative design process aiming to illustrate how our task sequence was constructed.

Table 1: Description of ongoing analysis

Description	Questions to guide the analysis	Evidence	Goal
○ Formulation of research-based initial conjectures that will evolve throughout the duration of the design study (Cobb et al., 2003).	○ How does the task or sequence of tasks engage students in DYME reasoning experiences? ○ What is the nature of students' DYME reasoning in each task or sequence of tasks? ○ How is students' thinking changed, modified and refined in each task or after a sequence of tasks? ○ How does this thinking connect to understanding the area formula?	○ Iteratively examine the changes in students' thinking about measurement when interacting with the DYME tasks and modify the task design accordingly.	○ To produce a series of tasks and a framework for learning DYME resulting from the actual teaching of children that consists of an explanation of students' initial schemes, explanations of changes in those schemes, and analysis of the contribution of the activities involved in those changes (Steffe, 2003).

Designing tasks for DYME

Influenced by the work of Thompson (1993; 1994) on quantitative reasoning, our goal was to design meaningful tasks around a storyline that would illustrate area as an attribute that measures the space covered by a rectangular shape. Aiming to trigger students' interest, we developed an overarching storyline, where students become part of a "Maker Team" that solves a series of DYME challenges embedded in an interactive digital book. To illustrate the continuous nature of area we used the context of painting with paint rollers (Confrey et al., 2012; Lehrer et al. 2014) where we asked students to color surfaces by dragging a roller of a given length over varying distances and also dragging rollers of different lengths over the same distance (Figure 3). The task introduces the quantities of 'length of a paint roller' as the length of a rectangle, 'rolling distance of the paint roller' as the width of a rectangle, and the 'space covered by the paint roller' as the area of the rectangle. Our humble theory was that by providing students with opportunities to create a rectangular surface through dragging and tracing, they would develop an understanding of area as a continuous quantity that depends on two other continuous quantities: the length of roller and the rolling distance.

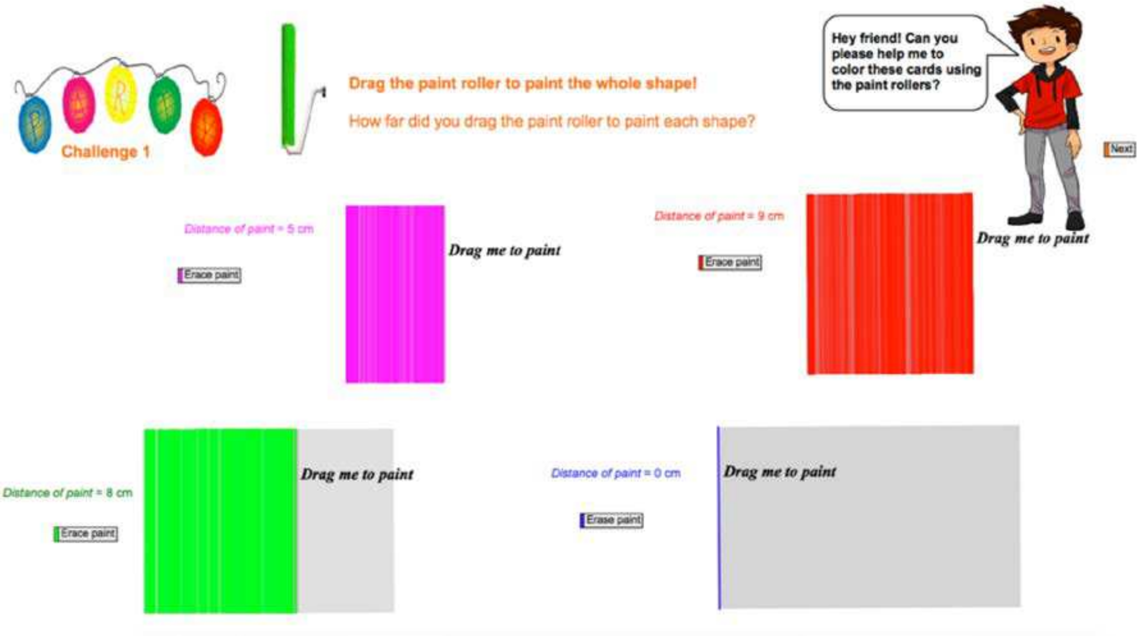
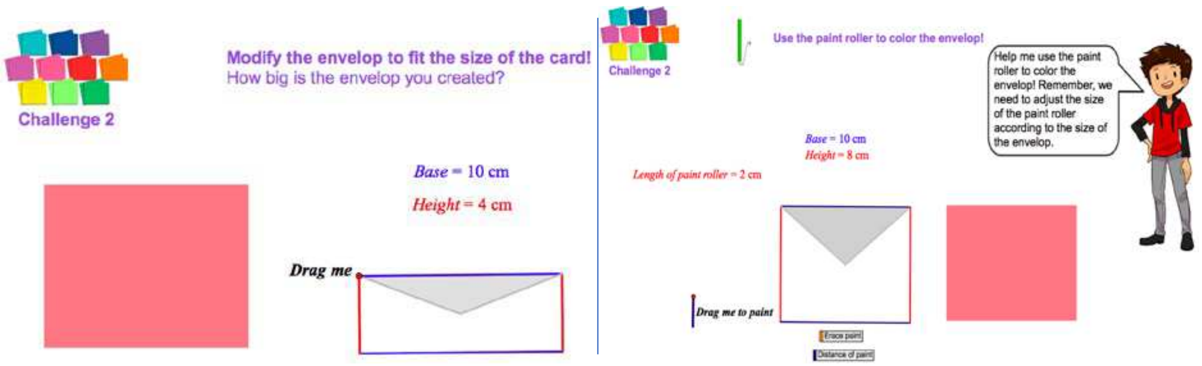


Figure 3. Dragging a roller of a given length over varying distances.

Indeed, the dynamic nature of the task enabled students to visualize area as a continuous structure and helped them recognize that the length of the roller and the rolling distance define the size of a shape. As they could drag the roller as far as they could, students reasoned that, "*the further we drag the roller, the bigger the shape we create.*" Although the task was successful in presenting those quantities as continuous, it did not provide us with evidence that students a) realized that they need to coordinate both quantities (length of roller and rolling distance) in order to make judgments about size and that b) they connected these dynamic experiences of generating area to the more 'static' length and the width measures of a rectangle.

Subsequently, we thought to engage students in experiences where they had to modify a shape to fit another shape, in order to create the need for considering both the base and height when comparing two shapes. (Due to the ambiguity of the word 'length' we used the terms 'base' and 'height' in the beginning, and 'length' and 'width' later.) Thus, in the next set of tasks we asked the students to modify envelops to fit the size of some cards. Students had to change only the base or the height or both on the envelop (Figure 4a). Then students were asked to color each envelop they created by modifying the length of a paint roller and dragging the roller over a distance to color the whole shape (Figure 4b).

Indeed, through task 4a students' articulations showed that they began looking at both the base and the height of the rectangle to compare two shapes. Examples included, "*the height needs to change and the base remains the same, because it's the same* [the base] *as the envelope's.*" Additionally, task 4b assisted students in connecting the dynamic experiences of painting to the static attributes of a rectangle, such as "*If the height is 8, so you are gonna want to make the length of the paint roller 8 cm too so that it can match. If the base is 10 then the distance of paint should be 10 cm.*" Although our design up to this point was successful in assisting students in comparing two shapes by comparing and making inferences about their dimensions, students still could not relate their dimensions multiplicatively.



(a) (b)

Figure 4. (a) Modifying the base and/or height of the rectangle to fit in the card, (b) Coloring the envelop using a paint roller.

This led to a reformulation of our humble theory to include the conjecture that students could reach to the goal of length *times* width if they recognize the proportional relationship between length and area (when one dimension is constant.) Subsequently, we designed tasks that asked students to explore ways to double/half a rectangular parking space designed by GSP (Figure 5). Through this task, students recognized that to double/halve the area they needed to double/halve the length or the width. Although this task helped students to move from non-numeric covariation ("*The bigger the roller the bigger the shape*") to expressing covariation numerically ("*If I double the length, the area is doubled*") providing some evidence of understanding the multiplicative relationship that underlies length and area, still they were not able to identify the multiplicative relationship of the formula.

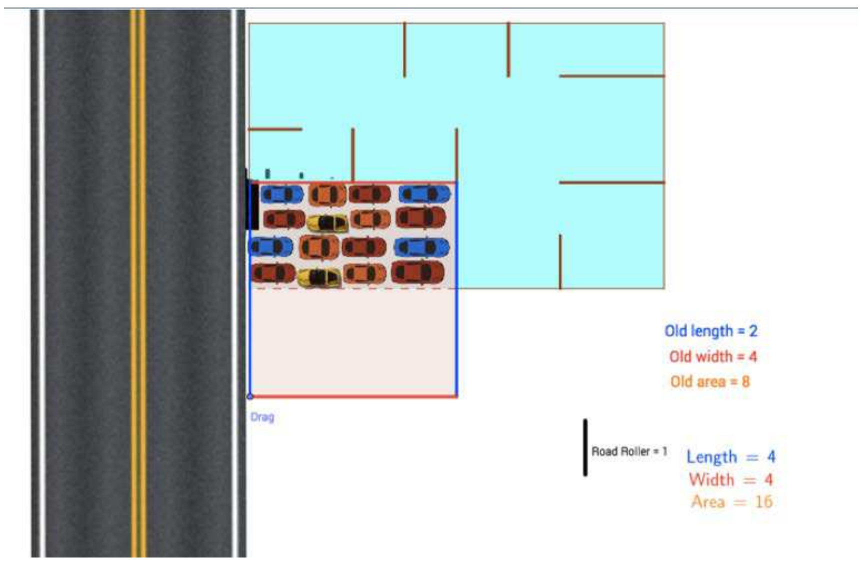


Figure 5. Students double the length of a rectangle to double the area.

Therefore, our focus shifted on research about iterating a fixed unit to find area (Izsak 2005; Lehrer, 2003). We conjectured that if students could iterate a roller of a fixed length (e.g. 1 inch) to cover a surface, they would consider the distance covered in one swipe of the roller with the number of swipes, and construct a repeating pattern for covering the shape (Outhred & Mitchelmore, 2000; Reynolds & Wheatley, 1996). This involved identifying that a length can be partitioned into a number of equal–sized units (Izsak 2005; Lehrer, 2003), which in our case would mean 1-inch rollers. To do that, we designed a task where students had to paint shapes of different lengths and widths using a single 1-inch roller (Figure 6). Students' articulations showed that the task design encouraged students to describe area using the multiplicative 'times' language, such as *"this is 30* [bottom right rectangle in Figure 6] *because the base is 10 and we are going to swipe three times" or* "*we need to do 10 three times."*

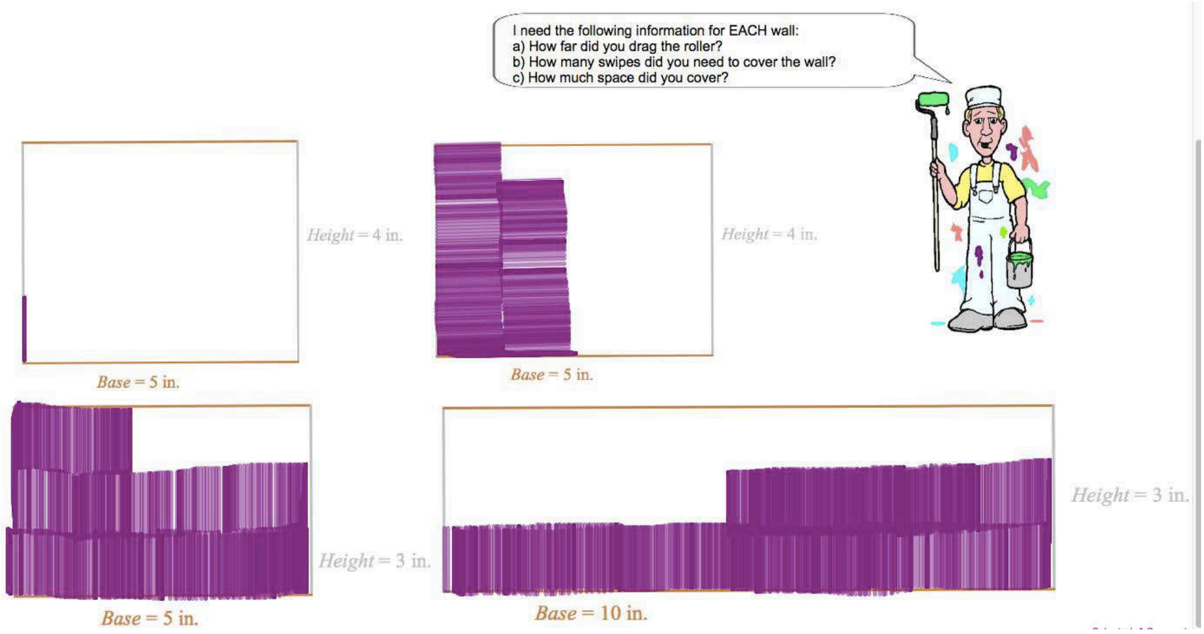


Figure 6. Students use 1-inch rollers to paint walls of different lengths and widths.

Aiming to show that the position of the roller can vary and also illustrate the commutativity of the multiplicative relationship in our design, we constructed tasks where the roller was placed vertically or on the right of the shape (Figure 7). Viewing both the horizontal and the vertical alignment of the roller led to generalizations like "4 swipes of 5 cover the space as 5 swipes of 4."

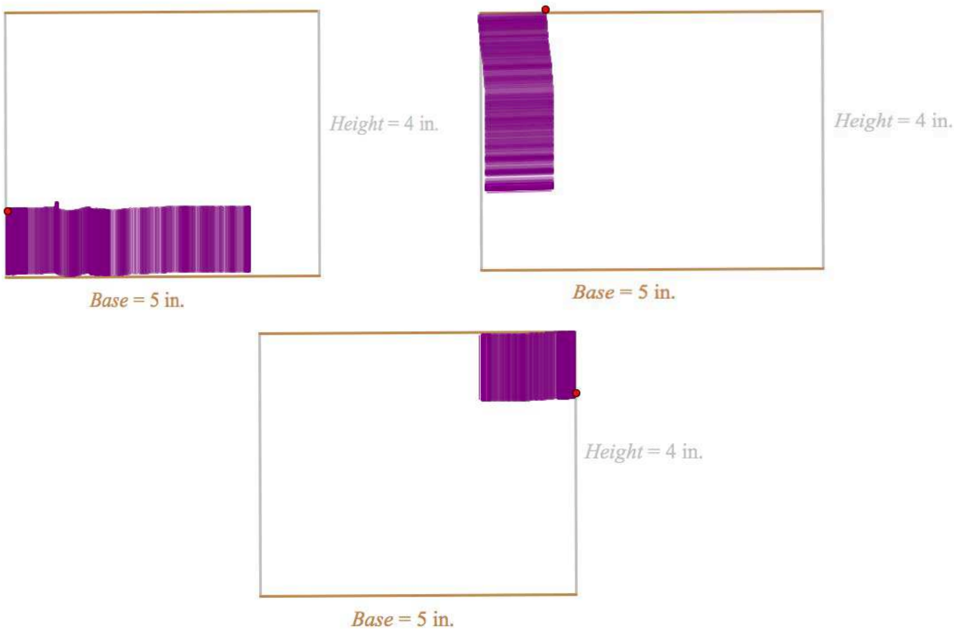


Figure 7. Students use 1-inch rollers in various positions and reason about the space covered.

Although our conjecture of iterating a 1-inch roller along the length of a rectangle was helpful for expressing the space covered multiplicatively, still students were not able to describe area as base *times* height. This led to a reformulation of our humble theory. Our next conjecture was that central to the construction of the area formula would be to give students a roller with the same length as the height of the rectangle and help them identify that a 3-inch roller covers the same space as three 1-inch rollers. For example, students painted a rectangle of base 4 units and height 3 units using a roller of 3 units (Figure 8). We observed that to find the space covered, students began decomposing mentally the large roller to 1-inch rollers. They realized that decomposing the length into unit lengths does not affect the area, for example, *"we could cut a 3-inch roller into 3 parts and go across 3 times for a distance of 4, the shape will cover 12."* Engaging students in these tasks helped them visualize a large swipe as a composite of 1-inch swipes. As we progressed towards the final sessions of the design experiments, students gradually distanced from the terminology of rollers and began using length *times* width intuitively recognizing that the height of a shape shows the number of 1-inch swipes and the base shows the rolling distance.

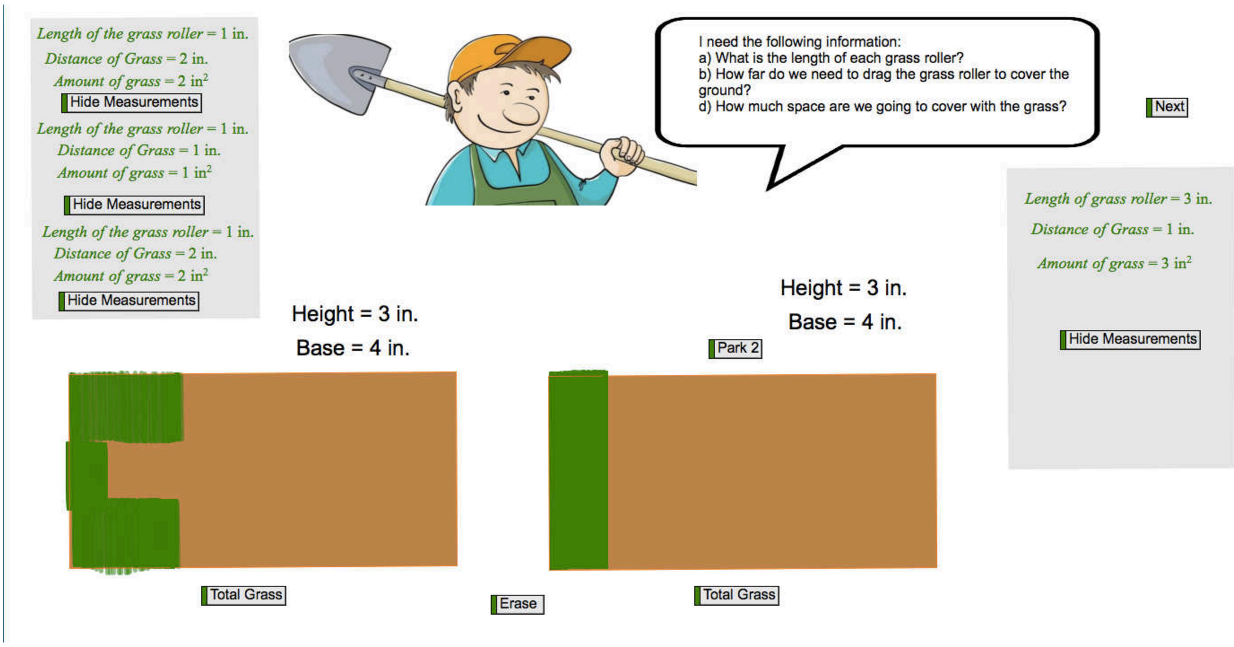


Figure 8. Students cover a rectangular park with grass rollers of different measurements.

Using the 1-inch rollers and iterating was a compromise we had made in our design in order for the students to think multiplicatively about the relationship between length, width and area. As students' multiplicative thinking of area developed further, our next conjecture was that students could use this knowledge to think about the proportional relationship between length and area. We designed tasks, such as the one we had in Figure 6, and also included tasks for students to recognize that in order to split area (fractional thinking), they need to split the length or the width. The tasks engaged students in creating shapes that have a fraction of an area of another shape. For example, students were asked to create a cafeteria which is 1/4 of an 8 by 5 inches garden (Figure 9) and argued that "*If we split this into four parts, then one of the parts will be the cafeteria. It would be 2 inches [the height of the cafeteria] because the if we use only 1-inch roller it would go 8 times across but if you use 2-inch roller then it would go 1,2,3, and that would go 4 parts."*

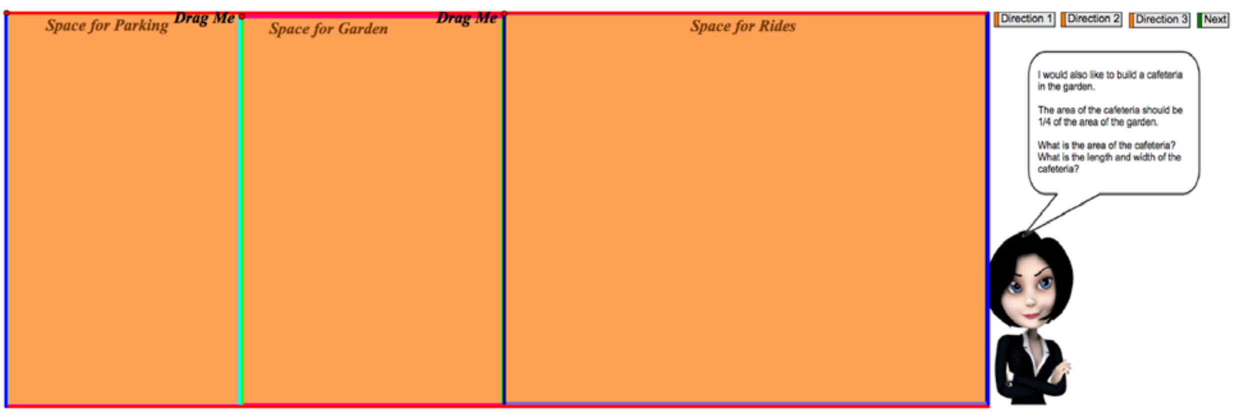


Figure 9. Students create a cafeteria which is ¼ of an 8 by 5 inches garden

Additionally, we wanted to test the conjecture that students could use this dynamic measurement knowledge to recognize area as a multiple of its dimensions and identify factors that give the same area. Our tasks involve asking students to create different rectangles of the same area (e.g. 12 sq. inches) (Figure 10). This connects area measurement to geometry and the concept of congruence by recognizing congruent shapes in different orientations (e.g. 2 x 6 or 6 x 2) and describing congruence by using geometric motions, such as rotation (Huang & Witz, 2011). It also directly relates to the properties of multiplication (e.g. commutative property) as well as factors and multiples, for example during this task students stated, "*length 4 and width 3 is doing 4 swipes of 3. This is same as two swipes of 6, so length 2 and width 6.*" Although students were not asked to paint as they did in previous explorations, we provided paint rollers as a resource. We found that these rollers became a powerful tool to help students visualize the size of covered spaces and for transitioning from sweeping-based reasoning to reasoning about area as length *times* width.

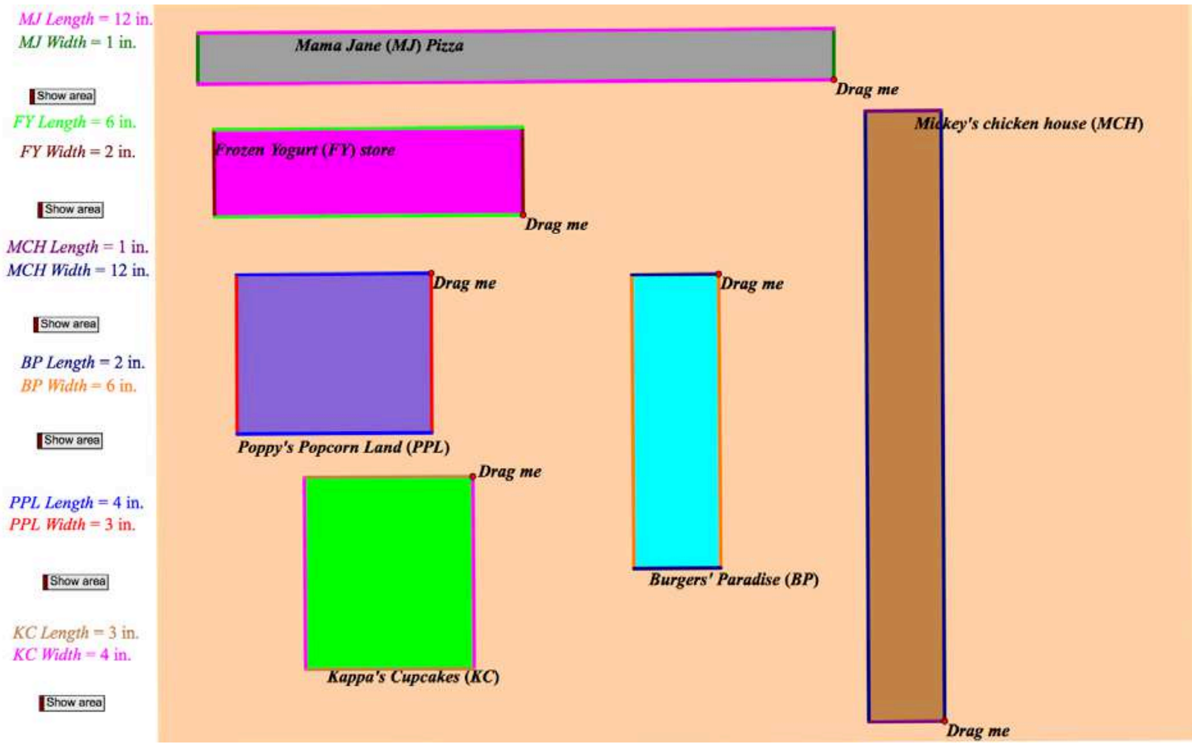


Figure 10. Students create stores that have area of 12 sq. inches and different length and width from other stores.

Concluding remarks

This study examined a dynamic way of learning and teaching measurement. The aim of this paper was to illustrate the iterative process of designing tasks for DYME aiming to show how the task design and the sequence of tasks evolved to assist students in visualizing the multiplicative relationship underlying the area formula. The examples of student behavior presented in this paper illustrate some of the ways that students' thinking of area progressed through the study. Results showed that the paint rollers were a very powerful tool for students in visualizing area as a continuous, dynamic structure defined by two quantities: *length* and *width*. Students also used this knowledge to understand more advanced notions such as scaling, factors and fractions.

This paper presented a snapshot of the first cycle of design experiments. In subsequent cycles, that included whole classroom design experiments, we explored further how students' thinking progresses through the particular tasks and designed a learning trajectory (Simon, 1995), illustrating how students' DYME reasoning may develop over time (Panorkou, 2017). The overall study shows DYME's potential as a route to area measurement that would make the multiplicative formula of area more intuitive and accessible. These findings are also useful for continuing the discussion around the potential of technology to change what is possible to learn. In the future, we plan to examine further how DYME thinking may assist students in making connections between various mathematical ideas, such as multiplication/division, fractions, transformations, and covariation. We are also currently exploring how DYME may be extended to non-rectangular surfaces and volume in the later years of schooling.

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