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Interlace Polynomials of Cycles with One Additional Chord

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Abstract

In this research, we investigate the interlace polynomial of a certain type of cycle graph with additional edges, called chords. We focus on the graphs resulted by adding one chord to cycle graphs. Consider the cycle C_n with n edges. When adding one chord to it, two sub-cycles were created which share one edge. If the length of one sub-cycle is r ($r \geq 3$), then the other length is $n - r + 2$. All cycles with one chord resulting in a sub-cycle of length r , where $r \leq n - r + 2$, are isomorphic, denoted by $J(n, r)$. When $n \geq 4$ and $r = 3$, we denote $M_n = J(n, 3)$, for convenience. The main results of this thesis include iterative and explicit formulas for the interlace polynomial $q(M_n, x)$ and properties of $q(M_n, x)$ such as its degree, certain coefficients, and special values. An application in linear algebra derived from the adjacency matrix of M_n is explored. The interlace polynomial of $J(n, r)$ is further investigated. Iterative formulas for $q(J(n, r), x)$ are provided in the last chapter of the thesis.

MONTCLAIR STATE UNIVERSITY

Interlace Polynomials of Cycles With One Additional Chord

by

Jhonny Almeida

A Master's Thesis Submitted to the Faculty of

Montclair State University

In Partial Fulfillment of the Requirements

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Department of Mathematical

Sciences

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WITH ONE ADDITIONAL CHORD**

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Chapter 1

Introduction

1.1 History

Interlace polynomials were originally motivated by a problem coming from DNA sequencing by hybridization. The problem requires to count the number of 2-in, 2-out digraphs having a given number of Euler circuits in an Eulerian graph. Solving the problem has led to discoveries of properties of interlace polynomials that can help to count, for any k , the number of k -component circuit partitions. Although it was defined originally for Eulerian graphs, it has been shown that its recursive definition extends to any simple graph. Some popular graphs such as paths, cycles, complete graphs, certain trees, certain bipartite graphs, have been studied. It is known that special values of the interlace polynomial $q(G, x)$ of a graph G can tell structural properties of G . For example, it has an interesting relationship between $q(G, -1)$ and the rank of a matrix derived from the adjacency matrix of G . Certain coefficients of $q(G, x)$ can also provide important information about the graph G . In particular, the coefficient γ of the x -term, called the γ -invariant of G , can help in determining if a vertex of G is a cut-vertex. The evaluation of other values may tell us more distinctive properties of the graph. For example, $q(G, 2)$ tells the number of the induced subgraphs with an odd

number of perfect matchings.

I am interested in working on cycle graphs with one additional chord. The research goals are to develop formulas and to discover properties for the interlace polynomials of such graphs as well as obtaining patterns and applications of special values of the polynomials.

1.2 Definitions

We work with graphs represented by a pair of sets of vertices and edges.

Definition 1. *A graph G is a pair of sets: $G = (V(G), E(G))$ such that:*

1. $V(G)$ is a nonempty set, called the vertex set.
2. $E(G)$ is a set of two-element subsets of $V(G)$, called the edge set, that is, $E(G) \subseteq \{\{u, v\} : u, v \in V(G)\}$.
3. The degree of a vertex v in a graph G , denoted $d(v)$, is the number of edges incident to v . If $d(v) = 1$ then v is called a leaf.
4. A loop is an edge whose endpoints coincide and multiple edges are edges having the same pair of endpoints.
5. $\forall u \in V(G)$, $N(u) = \{v \in V(G) | uv \in E(G)\}$ is called the set of neighbors of u or the neighborhood of u .

Certain well-known graphs appear frequently and their properties are well studied.

Definition 2 (Special Graphs). 1. *A simple graph is a graph with neither loops nor multiple edges.*

2. *A path is a finite or infinite sequence of edges which connects a set of distinct vertices from one another. The path with n edges, denoted by P_n , can be represented as $v_1v_2 \cdots v_{n+1}$, with $V(P_n) = \{v_1, v_2, \dots, v_{n+1}\}$.*

3. A complete graph is a simple graph where every vertex is adjacent to every other vertex. The complete graph with n vertices is denoted K_n .
4. The cycle of length n , denoted C_n , is the resulting graph by adding an edge between the two end vertices of P_n .
5. The complete bipartite graph with two partites of sizes m and n respectively is a graph having two independent sets of vertices of size m and n and there is an edge between every pair of vertices from different partites. It is denoted $K_{m,n}$.
6. A tree is a connected graph without cycle.
7. The star with k edges, denoted S_k , is the tree with one vertex of degree k and other vertices being leaves.

Some well-known properties about these special graphs are given below.

Theorem 1.2.1. *Let G be any graph. Then*

1. $\sum_{v \in V(G)} d(v) = 2|E(G)|$;
2. If G is a tree then $|E(G)| = |V(G)| - 1$;
3. $|E(K_n)| = \frac{n(n-1)}{2}$;
4. $|E(K_{m,n})| = nm$.

1.3 Defining the Interlace Polynomial

The interlace polynomial was defined by Arratia, Bollobás, and Sorkin in [4]. The interlace polynomial of a given simple graph G is recursively defined. We first select one edge from G , say ab , where a and b are two vertices of G . We consider three special neighborhoods related to a and b :

$$V_a = N(a) \setminus (N(b) \cup \{b\}), \quad V_b = N(b) \setminus (N(a) \cup \{a\}), \quad \text{and} \quad V_{a,b} = N(a) \cap N(b).$$

A toggling process is applied to G to create a “pivot” of G based on the neighborhoods for the two end vertices of a fixed edge of G . The toggling process is defined below.

Definition 3 (Toggling Process). *Let G be a graph and ab be an edge of G with a, b being vertices of G . The toggle process of G on ab means to create a new graph G^{ab} such that G and G^{ab} have the same vertex set and for every pair of vertices u, v belonging to different neighborhoods $V_a, V_b, V_{a,b}$ shown above, uv is an edge of G^{ab} if and only if uv is not an edge of G . The resulting graph G^{ab} is called the pivot of G at ab .*

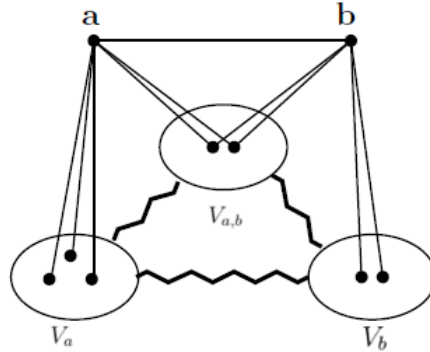


Figure 1.1: Toggling a graph on an edge ab

The interlace polynomial of a graph G is defined recursively by applying the pivot process.

Definition 4 (Interlace Polynomial). [4] *Let G be any undirected graph with n vertices and ab be an edge of G . The interlace polynomial $q(G, x)$ of G is defined by*

$$q(G, x) = \begin{cases} x^n & \text{if } E(G) = \emptyset; \\ q(G - a, x) + q(G^{ab} - b, x) & \text{if } ab \in E(G) \end{cases}.$$

Arratia, Bollobás, and Sorkin in [4] show us that the interlace polynomial is well-defined.

We describe the process by an example. Consider the cycle C_6 with two additional chords. We show the toggling process in figure 1.3 below.

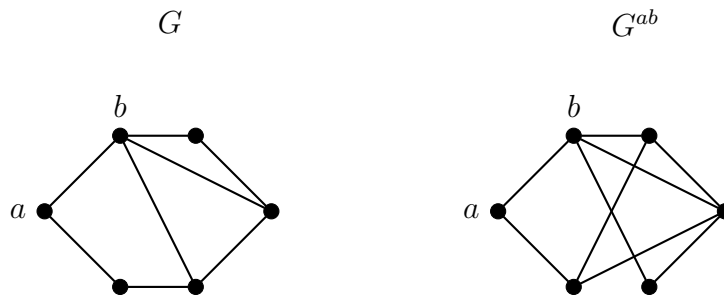


Figure 1.2: Toggling of a chorded cycle on an edge ab

After removing the corresponding vertices, the following graphs are used to give the interlace polynomial $q(G, x)$:

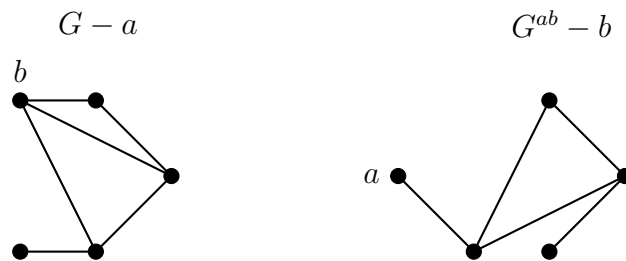


Figure 1.3: The graphs $G - a$ and $G^{ab} - b$

By Definition 4, $q(G, x) = q(G - a, x) + q(G^{ab} - b, x)$.

1.4 Existing Results

The interlace polynomials of several common graphs have been studied by other researchers. The graphs C_n , P_n , K_n , and S_n are common structures found in many other graphs particularly after a graph in consideration is toggled. We list some well-known results below:

Theorem 1.4.1. [4] Consider the graphs P_n , S_n , C_n , K_n , and $K_{m,n}$. Their interlace polynomial are given below.

1. $q(P_0, x) = q(S_0, x) = x$, $q(P_1, x) = q(S_1, x) = 2x$, $q(P_2, x) = x^2 + 2x = q(S_2, x)$;
2. $q(P_3, x) = 3x^2 + 2x$, $q(P_n, x) = q(P_{n-1}, x) + xq(P_{n-2}, x)$ ($n \geq 2$);
3. $q(S_3, x) = x^3 + x^2 + 2x$, $q(S_n, x) = x^n + q(S_{n-1}, x)$ ($n \geq 1$);
4. $q(C_3, x) = 4x$, $q(C_4, x) = 3x^2 + 2x$, $q(C_5, x) = 5x^2 + 6x$;
5. $q(C_n, x) = q(P_{n-2}, x) + xq(P_{n-4}, x) + q(C_{n-2}, x)$ ($n \geq 4$).
6. $q(K_n, x) = 2^{n-1}x$;
7. $q(K_{m,n}, x) = (1 + x + \dots + x^{m-1})(1 + x + \dots + x^{n-1}) + x^m + x^n - 1$ ($n \geq 1$).

The interlace polynomial of a graph G is linked with properties of G . For example, it can tell the number of components and the size of a maximum matching in G . Below are some properties obtained are shown by other researchers.

Theorem 1.4.2. [4, 6] Let G be any graph of order $n > 0$ and $q(G, x)$ be the interlace polynomial of G . Then

1. The power of the lowest-degree term of $q(G, x)$ is precisely equal to $\kappa(G)$, the number of components of G .
2. If G is a forest, then $\deg(q(G, x)) = n - \mu(G)$, where $\mu(G)$ denotes the size of a maximum matching (maximum set of independent edges in G).
3. If G is a connected graph with at least 2 vertices and $v \in V(G)$, then the coefficient of x in $q(G - v, x)$ is 0 if and only if v is a vertex cut in G .

Some special values of $q(G, x)$ may give relative properties of the graph.

Theorem 1.4.3 (Special Values). [5, 6] Let G be a graph with n vertices and $q(G, x)$ be the interlace polynomial of G . Then

1. $q(G, 2) = 2^n$.
2. Let A be the adjacency matrix of G and I be the $n \times n$ identity matrix, then $q(G, -1) = (-1)^n (-2)^{n-r}$, where $r = \text{rank}(A + I)$ over \mathbb{Z}_2 .

1.5 The Graphs of Our Interest

We are interested in graphs produced by adding new edges to the cycle, C_n , where $n \geq 4$.

Definition 5. Consider the cycle C_n of n edges and a graph G containing C_n . An edge $e = uv$ of G is called a chord if $u, v \in V(C_n)$, but $uv \notin E(C_n)$. That is, a chord on C_n is an edge linking two non-adjacent vertices of C_n .

A cycle with chords can not only be distinguished from its number of chords but also by the placement of the chords. If one chord is added to C_n , it results in two sub-cycles within C_n . The sizes of the two sub-cycles are determined by the placement of the chord and the different sizes of the sub-cycles may result in non-isomorphic graphs. Our focus is on the case of one of the sub-cycles being C_3 , the cycle of length 3. We formally define the resulting graph by adding one chord to C_n .

Definition 6. Let $C_n = v_1v_2 \dots v_nv_1$ be the cycle of length n ($n \geq 4$). For $n > 3$, define M_n to be the resulting graph by adding one edge v_1v_3 to C_n (building a C_3 within C_n). More generally, for an integer r with $3 \leq r \leq (n+2)/2$, we define $J(n, r)$ to be the resulting graph by adding an edge v_1v_r to C_n .

Remark 1.5.1. The resulting graph consists of two cycles, $v_1v_2, \dots, v_rv_1 \cong C_r$ and $v_1v_{r+1} \dots v_nv_1 \cong C_{n-r+2}$. We assume $r \leq n-r+2$, that is, $r \leq (n+2)/2$. Also, when $r = 3$, $M_n = J(n, 3)$.

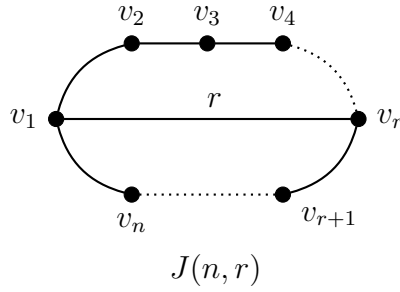


Figure 1.4: The graph of $J(n, r)$

Example 1. By the above definition, $J(n, 3) = M_n$. The graph $M_8 = J(8, 3)$ is given below.

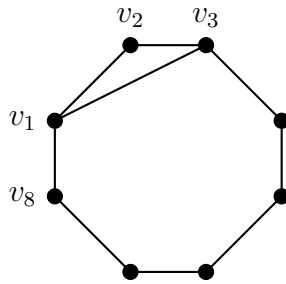


Figure 1.5: $J(8, 3) = M_8$

The focus of this research is on special graphs $J(n, r)$ where $r \geq 3$. The main results are on the simplest case $r = 3$ whose graph is $J(n, 3) = M_n$. For $J(n, 3)$ I will develop explicit formula for the interlace polynomials, study properties of the coefficients and special values of the polynomial, and give an application in linear algebra regarding ranks of some related matrices.

Research Questions. Consider $J(n, r)$, where $n > 3$ and $r \geq 3$.

1. Give iterative or explicit formulas for the polynomial $q(J(n, r), x)$.
2. Identify differences when adding chords in different ways.
3. Find patterns of the coefficients of $q(J(n, r), x)$.

4. Evaluate $q(J(n, r), x)$ at certain values of x .
5. What do the values of $q(J(n, r), x)$ tell about the graph, $J(n, r)$?

Chapter 2

Results on $q(P_n, x)$

We present preliminary results about the interlace polynomial $q(P_n, x)$ of the path P_n such as the coefficients of its terms and some special values. These properties are useful in the study of the interlace polynomials of our graphs $J(n, r)$. We first give a set of properties of $q(P_n, x)$ most of which can be found in the literature.

2.1 General Formulas for $q(P_n, x)$

Theorem 2.1.1. [4] Consider $q(P_n, x)$ where $n \geq 0$. Then

1. $\deg(q(P_n, x)) = \lfloor \frac{n+2}{2} \rfloor$;
2. An explicit formula for $q(P_n, x)$ is given by

$$q(P_n, x) = \sum_{r=0}^{\lfloor n/2 \rfloor} \left[\binom{n-r}{r} + \binom{n-r-1}{r} \right] x^{r+1};$$

3. Let $y = y(x) = \sqrt{1+4x}$. Then

$$q(P_n, x) = \frac{(3+y)(y-1)}{4y} \left(\frac{1+y}{2} \right)^{n+1} + \frac{(3-y)(y+1)}{4y} \left(\frac{1-y}{2} \right)^{n+1}.$$

We further develop some properties for $q(P_n, x)$ which will be used to describe the interlace polynomials $q(M_n, x)$. We give our own proofs for them.

Theorem 2.1.2. *For $n \geq 3$,*

$$\sum_{i=1}^{n-2} q(P_i, x) = \frac{q(P_n, x)}{x} - x - 2.$$

Proof. We use Theorem 1.4.1 and prove it by mathematical induction. For $n = 3$,

$$\sum_{i=1}^1 q(P_i, x) = q(P_1, x) = 2x = \frac{3x^2 + 2x}{x} - x - 2 = \frac{q(P_3, x)}{x} - x - 2,$$

so the formula is true for $n = 3$. Assume $\sum_{i=1}^{n-2} q(P_i, x) = \frac{q(P_n, x)}{x} - x - 2$. Then

$$\begin{aligned} \sum_{i=1}^{n-1} q(P_i, x) &= q(P_{n-1}, x) + \sum_{i=1}^{n-2} q(P_i, x) = \frac{xq(P_{n-1}, x)}{x} + \frac{q(P_n, x)}{x} - x - 2 \\ &= \frac{q(P_{n+1}, x)}{x} - x - 2. \end{aligned}$$

□

For convenience we re-describe $q(P_n, x)$ below.

Definition 7. *Denote the coefficient of the x^i -term of $q(P_n, x)$ as $a_{n,i}$ and $r_n = \lfloor \frac{n}{2} \rfloor + 1$, the degree of $q(P_n, x)$. That is,*

$$q(P_n, x) = a_{n,r_n}x^{r_n} + a_{n,r_n-1}x^{r_n-1} + \cdots + a_{n,2}x^2 + a_{n,1}x = \sum_{i=1}^{r_n} a_{n,i}x^i.$$

2.2 Properties of $q(P_n, x)$

By Theorem 2.1.1(2), $a_{n,i} = \binom{n-i+1}{i-1} + \binom{n-i}{i-1}$. The coefficients $a_{n,1}$, $a_{n,2}$, $a_{n,3}$, and $a_{n,4}$ and other properties of $q(P_n, x)$ are given in this section.

Lemma 2.2.1 (Formulas for Selected Coefficients of $q(P_n, x)$). 1. $a_{n,1} = 2$ for $n \geq 2$,

2. $a_{n,2} = 2n - 3$ for $n \geq 3$,

3. $a_{n,3} = (n - 3)^2$ for $n \geq 4$,

4. $a_{n,4} = \frac{(n-5)(n-4)(2n-9)}{6} = \sum_{i=1}^{n-5} i^2$ for $n \geq 6$.

Proof. By Theorem 2.1.1(2) the corresponding coefficients are:

1. $a_{n,1} = \binom{n}{0} + \binom{n-1}{0} = 2$;

2. $a_{n,2} = \binom{n-1}{1} + \binom{n-2}{1} = 2n - 3$;

3. Similarly, $a_{n,3} = \binom{n-2}{2} + \binom{n-3}{2} = n^2 - 6n + 9 = (n - 3)^2$;

4. The coefficient $a_{n,4}$ is the sum of the consecutive integer squares from 1 to $n - 5$ because

$$\begin{aligned} a_{n,4} &= \binom{n-3}{3} + \binom{n-4}{3} = \frac{(n-3)!}{3!(n-6)!} + \frac{(n-4)!}{3!(n-7)!} \\ &= \frac{1}{6} [(n-3)(n-4)(n-5) + (n-4)(n-5)(n-6)] \\ &= \frac{1}{6} (2n^3 - 27n^2 + 121n - 180) \\ &= \frac{(n-5)(n-4)(2n-9)}{6} = \sum_{i=1}^{n-5} i^2. \end{aligned}$$

□

Next, we describe the leading and the second leading coefficients of $q(P_n, x)$ in detail. By the above notation, these are a_{n, r_n} and a_{n, r_n-1} . Recall that $r_n = \lfloor \frac{n}{2} \rfloor + 1$ and $a_{n, i} = \binom{n-i+1}{i-1} + \binom{n-i}{i-1}$.

Lemma 2.2.2. 1. When n is even, the leading coefficient and the second leading coefficient of $q(P_n, x)$ are

$$a_{n, r_n} = 1 \quad \text{and} \quad a_{n, r_n-1} = \frac{n^2 + 6n}{8}.$$

2. When n is odd, the leading coefficient and the second leading coefficient of $q(P_n, x)$ are given by

$$a_{n, r_n} = \frac{n+3}{2} \quad \text{and} \quad a_{n, r_n-1} = \frac{(n^2-1)(n+9)}{48}.$$

Proof. 1. Assume n is even. By Theorem 2.1.1(3) $r_n = \lfloor \frac{n}{2} \rfloor + 1 = \frac{n}{2} + 1$ which yields the following

$$a_{n, r_n} = \binom{n - \frac{n+2}{2} + 1}{\frac{n}{2}} + \binom{n - \frac{n+2}{2}}{\frac{n}{2}} = \binom{\frac{n}{2}}{\frac{n}{2}} + \binom{\frac{n}{2} - 1}{\frac{n}{2}} = 1 + 0 = 1$$

and

$$\begin{aligned} a_{n, r_n-1} &= \binom{n - \frac{n}{2} + 1}{\frac{n}{2} - 1} + \binom{n - \frac{n}{2}}{\frac{n}{2} - 1} = \binom{\frac{n}{2} + 1}{\frac{n-2}{2}} + \binom{\frac{n}{2}}{\frac{n-2}{2}} \\ &= \frac{(\frac{n+2}{2})!}{(\frac{n-2}{2})!2!} + \frac{(\frac{n}{2})!}{(\frac{n-2}{2})!1!} = \frac{(\frac{n}{2} + 1) (\frac{n}{2})}{2} + \frac{n}{2} = \frac{(n+2)n + 4n}{8} \\ &= \frac{n^2 + 6n}{8}. \end{aligned}$$

2. Similarly, when n is odd, $r_n = \frac{n+1}{2}$ so $r_n - 1 = \frac{n-1}{2}$. This yields

$$\begin{aligned} a_{n, r_n} &= \binom{n - \frac{n+1}{2} + 1}{\frac{n+1}{2} - 1} + \binom{n - \frac{n-1}{2}}{\frac{n-1}{2}} = \binom{\frac{n+1}{2}}{\frac{n+1}{2} - 1} + \binom{\frac{n-1}{2}}{\frac{n-1}{2}} \\ &= \frac{n+1}{2} + 1 = \frac{n+3}{2} \end{aligned}$$

and

$$\begin{aligned}
a_{n,r_{n-1}} &= \binom{n - \frac{n-1}{2} - 1}{\frac{n-1}{2} - 1} + \binom{n - \frac{n-1}{2}}{\frac{n-1}{2}} = \binom{\frac{n+3}{2}}{\frac{n-3}{2}} + \binom{\frac{n+1}{2}}{\frac{n-3}{2}} \\
&= \frac{(\frac{n+3}{2})!}{(\frac{n-3}{2})!3!} + \frac{(\frac{n+1}{2})!}{(\frac{n-3}{2})!2!} = \frac{\frac{n+3}{2} \cdot \frac{n+1}{2} \cdot \frac{n-1}{2}}{3 \cdot 2} + \frac{\frac{n+1}{2} \cdot \frac{n-1}{2}}{2} \\
&= \frac{1}{48} [6(n^2 - 1) + (n^2 - 1)(n + 3)] = \frac{1}{48} [(n^2 - 1)(n + 9)] \\
&= \frac{(n + 1)(n - 1)(n + 9)}{48}.
\end{aligned}$$

□

By the above result we can write,

$$\begin{aligned}
q(P_n, x) &= a_{n,r_n}x^{r_n} + a_{n,r_{n-1}}x^{r_{n-1}} + \dots + \frac{(n-5)(n-4)(2n-9)}{6}x^4 \\
&\quad + (n^2 - 6n + 9)x^3 + (2n-3)x^2 + 2x.
\end{aligned}$$

Note that the second leading coefficient of $q(P_n, x)$ is equal to the sum of the second leading coefficients of $q(P_{n-1}, x)$ and $q(P_{n-2}, x)$.

Corollary 2.2.3. *Applying the above notations,*

$$a_{n,r_{n-1}} = a_{n-1,r_{n-1}-1} + a_{n-2,r_{n-2}-1}.$$

Proof. It is immediately from Lemma 2.2.2. When n is even, $n - 2$ is also even:

$$\begin{aligned}
a_{n-1,r_{n-1}-1} + a_{n-2,r_{n-2}-1} &= \frac{(n-1) + 3}{2} + \frac{(n-2)^2 + 6(n-2)}{8} \\
&= \frac{n^2 - 4n + 4 + 6n - 12 + 4n + 8}{8} \\
&= \frac{n^2 + 6n}{8} = a_{n,r_{n-1}}.
\end{aligned}$$

When n is odd, $n - 1$ is even, and $n - 2$ is odd, hence:

$$\begin{aligned}
a_{n-1, r_{n-1}-1} + a_{n-2, r_{n-2}-1} &= \frac{(n-1)^2 + 6(n-1)}{8} + \frac{[(n-2)^2 - 1](n+7)}{48} \\
&= \frac{6n^2 - 12n + 6 + 36n - 36 + n^3 + 3n^2 - 25n + 21}{48} \\
&= \frac{n^3 + 9n^2 - n - 9}{48} = \frac{(n^2 - 1)(n + 9)}{48} = a_{n, r_n - 1}.
\end{aligned}$$

□

Below we show the values of $q(P_n, x)$ at $x = 1, -1$, and integers in the form of $m(m+1)$, where m is a positive integer. We give our own proofs. The special value $q(P_n, -1)$ is later used in an application in linear algebra involving the adjacency matrix of the graph M_n .

Proposition 2.2.4. [4]

(1) $q(P_n, 1) = F_{n+2}$, the $(n+2)$ th Fibonacci number (with $F_0 = 0$ and $F_1 = 1$),

(2)

$$q(P_n, -1) = \begin{cases} 1, & n \equiv 3, 5 \pmod{6} \\ -2, & n \equiv 1 \pmod{6} \\ -1, & n \equiv 0, 2 \pmod{6} \\ 2, & n \equiv 4 \pmod{6} \end{cases}.$$

(3) If m is a positive integer, then

$$q(P_n, m(m+1)) = \frac{(m+2)m}{2m+1}(1+m)^{n+1} - \frac{m^2-1}{2m+1}(-m)^{n+1}.$$

Proof. (1) is obvious because $q(P_n, 1) = q(P_{n-1}, 1) + q(P_{n-2}, 1)$. For (2),

$$\begin{aligned}
q(P_n, -1) &= q(P_{n-1}, -1) + (-1)q(P_{n-2}, -1) \\
&= q(P_{n-2}, -1) - q(P_{n-3}, -1) - q(P_{n-2}, -1) \\
&= -q(P_{n-3}, -1) = -(q(P_{n-4}, -1) - q(P_{n-5}, -1)) \\
&= -((q(P_{n-5}, -1) - q(P_{n-6}, -1)) - q(P_{n-5}, -1)) \\
&= q(P_{n-6}, -1).
\end{aligned}$$

Thus $q(P_n, -1)$ is periodic with a period of 6. The first 6 values of $q(P_n, -1)$ are: $q(P_0, -1) = q(P_2, -1) = -1$, $q(P_1, -1) = -2$, $q(P_3, -1) = q(P_5, -1) = 1$, and $q(P_4, -1) = 2$. Thus (2) is true.

To obtain (3), we replace x by $m(m+1)$ in Theorem 2.1.1(3). Then $y = 2m - 1$ which results in the desired result. Thus Proposition 2.2.4 is true. \square

Chapter 3

Interlace Polynomials of $q(M_n, x)$

The first graph we study is M_n ($n \geq 3$). It is distinct from the other graphs $J(n, r)$, where $r > 3$. and is the simplest to investigate. The interlace polynomial of M_n reveals P_{n-2} as one of its components. Utilizing the results in chapter 2 we derive M_n 's explicit formula, special values, and coefficients.

3.1 Formulas for $q(M_n, x)$

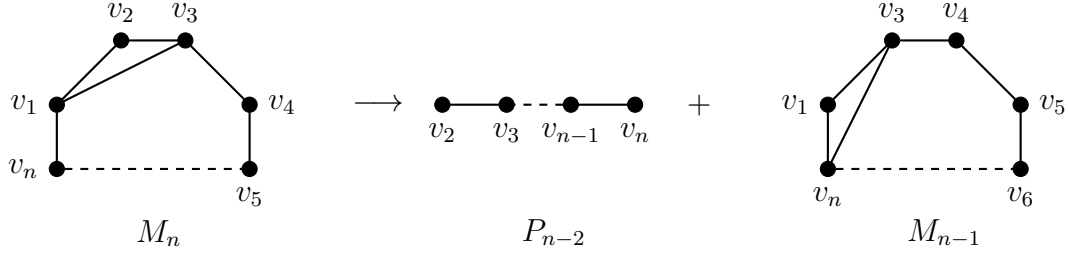
The formulas are derived from the toggle process on the graph M_n , which gives an iterative formula shown below:

Example 2. *The graph M_4 has the vertex set $V(M_4) = \{v_1, v_2, v_3, v_4\}$ and the edge set $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_3\}$. If we toggle M_4 at the edge v_1v_2 , $M_4 - v_1 = C_3$ and $M_4^{v_1v_2} - v_2 = P_1 \cup P_0$. Thus*

$$q(M_4, x) = q(C_3, x) + xq(P_1, x) = 4x + x(2x) = 2x^2 + 4x.$$

Lemma 3.1.1. $q(M_n, x) = q(M_{n-1}, x) + q(P_{n-2}, x)$ for $n \geq 5$.

Proof. The graph M_n is shown below. We perform the toggling process at v_1v_2 of M_n . The decomposition of M_n is as follows:



For $n \geq 5$, the toggling process decomposes M_n into two graphs: M_{n-1} and P_{n-2} . Thus the recursive relation involving P_{n-2} is true. \square

Remark 3.1.2. *The interlace polynomials of M_5, M_6 , and M_7 are given below:*

$$q(M_5, x) = 5x^2 + 6x, \quad q(M_6, x) = x^3 + 10x^2 + 8x \quad \text{and} \quad q(M_7, x) = 5x^3 + 17x^2 + 10x.$$

By the recursive formula in Lemma 3.1.1, we can obtain an explicit formula for $q(M_n, x)$ in terms of M_{n-1} and $q(P_{n-2}, x)$. Later we use the result to obtain useful properties of the polynomial and the ground graph. Repeatedly using Lemma 3.1.1, we obtain

$$\begin{aligned} q(M_n, x) &= q(M_{n-1}, x) + q(P_{n-2}, x) \\ &= q(M_{n-2}, x) + q(P_{n-3}, x) + q(P_{n-2}, x) \\ &\dots \\ &= q(M_4, x) + \sum_{i=3}^{n-2} q(P_i, x). \end{aligned}$$

By Theorem 2.1.2, it gives

$$q(M_n, x) = 2x^2 + 4x + \frac{q(P_n, x)}{x} - x - 2 - (2x + x^2 + 2x) = \frac{q(P_n, x)}{x} + x^2 - x - 2.$$

It results in the following theorem.

Theorem 3.1.3. For $n \geq 4$, $q(M_n, x) = \frac{q(P_n, x)}{x} + x^2 - x - 2$.

Proof. We prove it by mathematical induction. For $n = 4$,

$$\frac{q(P_4, x)}{x} + x^2 - x - 2 = \frac{q(P_3, x) + xq(P_2)}{x} + x^2 - x = 2x^2 + 4x = q(M_4, x).$$

Assume $q(M_n, x) = \frac{q(P_n, x)}{x} + x^2 - x - 2$. Then by Lemma 3.1.1 and the induction hypothesis,

$$\begin{aligned} q(M_{n+1}, x) &= q(M_n, x) + q(P_{n-1}, x) = \frac{q(P_n, x)}{x} + x^2 - x - 2 + q(P_{n-1}, x) \\ &= \frac{q(P_n, x) + xq(P_{n-1}, x)}{x} + x^2 - x - 2 = \frac{q(P_{n+1}, x)}{x} + x^2 - x - 2. \end{aligned}$$

Thus the formula is true for all $n \geq 4$. □

Let $\deg(q(G, x))$ denote the degree of the interlace polynomial of a graph G . We give the degree of $q(M_n, x)$.

Proposition 3.1.4. For $n \geq 4$,

$$\deg(q(M_n, x)) = \left\lfloor \frac{n}{2} \right\rfloor.$$

Proof. From Theorem 2.1.1, $\deg(q(P_n, x)) = \lfloor \frac{n+2}{2} \rfloor$ and since $n \geq 4$ the contributing exponent to the degree of $q(M_n, x)$ is the leading term of $\frac{q(P_n, x)}{x}$. Then $\deg(q(M_n, x)) = \frac{\deg(q(P_n, x))}{x} = \deg(q(P_n, x)) - 1 = \lfloor \frac{n}{2} \rfloor$. □

3.2 Special Values of $q(M_n, x)$

It is known that the values of the polynomial $q(M_n, x)$ at certain integers x can tell about the properties of the graph M_n . In particular, the values at 1, -1 , and 2 play important roles in this regard. In this section we evaluate $q(M_n, x)$ at these values. Additionally, we obtain an explicit formula for $q(M_n, x)$ at a special type of integers.

By Theorem 3.1.3, $q(M_n, 1) = q(P_n, 1) - 2$, $q(M_n, 2) = q(P_n, 2)/2$, and $q(M_n, -1) = -q(P_n, -1)$. Applying special values of $q(P_n, x)$ at $x = 1, 2, -1$ given in Theorems 1.4.3 and 2.2.4, we obtain

Proposition 3.2.1. *For any integer $n \geq 4$,*

1. $q(M_n, 1) = F_{n+2} - 2$, where F_n is the n th Fibonacci number;
2. $q(M_n, 2) = 2^n$;
3. The value $q(M_n, -1)$ is various depending on the remainder of n divided by 6:

$$q(M_n, -1) = \begin{cases} 1, & \text{if } n \equiv 0, 2 \pmod{6} \\ -2, & \text{if } n \equiv 4 \pmod{6} \\ -1, & \text{if } n \equiv 3, 5 \pmod{6} \\ 2, & \text{if } n \equiv 1 \pmod{6} \end{cases}.$$

Remark 3.2.2. *It is known that for any graph G , $q(G, 2) = 2^{|V(G)|}$, where $|V(G)|$ is the number of vertices of G . Since M_n has n vertices, Proposition 3.2.1(2) confirms this result.*

The theorem below gives a formula for the value of $q(M_n, x)$ when x is a positive integer in the form of $m(m+1)$ where m is an integer.

Theorem 3.2.3. *Let $x = m(m + 1)$, where $m \in \mathbb{Z}$, and $m > 0$. Then*

$$q(M_n, m(m + 1)) = \frac{m + 2}{2m + 1}(1 + m)^n + \frac{m - 1}{2m + 1}(-m)^n + (m + 2)(m^3 - 1).$$

Proof. We utilize Theorem 2.1.1 and Proposition 3.1.3 to derive the result. Since $x = m(m + 1)$,

$$\begin{aligned} x^2 - 2x - 2 &= (m^2 + m)^2 - 2(m^2 + m) - 2 \\ &= m^4 + 2m^3 - m^2 - 2m - 2 = (m^3 - 1)(m + 2). \end{aligned}$$

Note that $q(M_n, m(m + 1)) = q(P_n, m(m + 1)) + (m^3 - 1)(m - 2)$. By applying Theorem 2.1.1,

$$\begin{aligned} q(M_n, m(m + 1)) &= \frac{\frac{(m+2)m}{2m+1}(1 + m)^{n+1} - \frac{m^2-1}{2m+1}(-m)^{n+1}}{m^2 + m} + (m^3 - 1)(m + 2) \\ &= \frac{m + 2}{2m + 1}(m + 1)^n + \frac{m - 1}{2m + 1}(-m)^n + (m^3 - 1)(m + 2). \end{aligned}$$

□

Below we show a corollary that confers with the result of $q(M_n, 2) = 2^n$ and a formula for $x = 6$.

Corollary 3.2.4. *For $n \geq 4$, the following is true,*

1. $q(M_n, 2) = 2^n$;
2. $q(M_n, 6) = \frac{1}{5}(4(3^n) + (-2)^n) + 28$.

Proof. Substitute $m = 1$ in the formula in Theorem 3.2.3, $q(M_n, 2) = 2^n$. Substitute $m = 2$ in the formula, we obtain $q(M_n, 6) = (4/5) \cdot 3^n + (-2)^n/5 + 7 \cdot 4$ which gives the result. □

Remark 3.2.5. *In this way we can obtain infinitely many values for $q(M_n, x)$.*

3.3 Coefficients of $q(M_n, x)$

We now investigate useful properties of the polynomial $q(M_n, x)$ where $n \geq 4$. By Proposition 3.1.4, we know that the degree of $q(M_n, x)$ is $\lfloor \frac{n}{2} \rfloor$. For convenience, we re-write the polynomial as follows.

Definition 8. Denote the degree of $q(M_n, x)$ by s_n (so $s_n = \lfloor \frac{n}{2} \rfloor$) and the coefficient of x^i as $b_{n,i}$. We write the polynomial as

$$q(M_n, x) = b_{n,s_n}x^{s_n} + b_{n,s_n-1}x^{s_n-1} + \cdots + b_{n,s_1}x + b_{n,s_0}.$$

This definition is correspondent to the notation $r_n = \deg(q(P_n, x))$ and obviously, $s_n = r_n - 1$.

Theorem 3.3.1. Consider the polynomial $q(M_n, x)$ in the form defined above, where $n \geq 4$. Then an explicit formula for $q(M_n, x)$ is given by

$$q(M_n, x) = \left(\sum_{k=3}^{s_n} \left[\binom{n-k}{k} + \binom{n-k-1}{k} \right] x^k \right) + x^2 - x - 2.$$

Proof. This is an immediate consequence of Lemma 2.1.1 and Proposition 3.1.3.

Note that from the above theorem, the coefficient of x^k is $b_{n,k} = \binom{n-k}{k} + \binom{n-k-1}{k}$ for $2 \leq k \leq s_n$. Furthermore, $b_{n,k} = a_{n,k+1}$.

The next theorem is an application of the coefficients from P_n into M_n 's explicit formula. This yields formulas for each of its coefficients.

Theorem 3.3.2. For $q(M_n, x) = b_{n,s_n}x^{s_n} + b_{n,s_n-1}x^{s_n-1} + \cdots + b_{n,s_1}x + b_{n,s_0}$ where $s_n = \lfloor n/2 \rfloor$. Its coefficients are the following for $n > 6$:

1. $b_{n,0} = 0$;
2. $b_{n,1} = 2n - 4$;

$$3. b_{n,2} = n^2 - 6n + 10;$$

$$4. b_{n,3} = \frac{(n-5)(n-4)(2n-9)}{6}.$$

Proof. The proof is direct from Lemma 2.2.1. For $k = 0$ the summation yields 2 canceling with “ $- 2$ ” hence $b_{n,0} = 0$. For $k = 1$ the summation yields $2n - 3$ but the $-x$ in the expression contributes a -1 making $b_{n,1} = 2n - 4$. Similarly for $k = 2$ the summation yields $n^2 - 6n + 9$ but x^2 contributes a “ $+ 1$ ” to this hence $b_{n,2} = n^2 - 6n + 10$. For $k = 3$ we can directly apply Theorem 3.3.1(3) to obtain $a_{n,4} = b_{n,3} = \frac{(n-5)(n-4)(2n-9)}{6}$. \square

Now we can write $q(M_n, x) = b_{n,s_n}x^{s_n} + b_{n,s_n-1}x^{s_n-1} + \dots + \frac{(n-5)(n-4)(2n-9)}{6}x^3 + (n^2 - 6n + 10)x^2 + (2n - 4)x$.

3.4 Related Matrices

In this section we discuss the ranks of some matrices related to the adjacency matrix of M_n over the field \mathbb{Z}_2 . Let A_n be the $n \times n$ adjacency matrix of M_n and I_n be the $n \times n$ identity matrix. Denote $B_n = A_n + I_n$. We investigate the rank of B_n over \mathbb{Z}_2 . First, we examine cases when n is small.

Example 3. *The matrices A_n and B_n for $n = 4, 5, 6, 7$.*

$$A_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix};$$

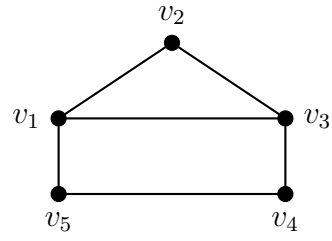


Figure 3.1: M_5

$$A_5 = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_5 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix};$$

$$A_6 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_6 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix};$$

$$A_7 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_7 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Their ranks are:

$$\text{rank}(A_4) = 2, \quad \text{rank}(A_5) = 4, \quad \text{rank}(A_6) = 6, \quad \text{rank}(A_7) = 7,$$

$$\text{rank}(B_4) = 3, \quad \text{rank}(B_5) = 5, \quad \text{rank}(B_6) = 6, \quad \text{rank}(B_7) = 6.$$

In general, the matrices of A_n and B_n have the following forms:

$$A_n = \begin{bmatrix} 0 & 1 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & 1 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B_n = \begin{bmatrix} 1 & 1 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 1 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 1 & 1 \end{bmatrix}.$$

In the matrix B_n there is a tri-diagonal $(n - 2) \times (n - 2)$ submatrix with all 1's along the main diagonal and superdiagonal and subdiagonal positions. We denote such a matrix as Q_{n-2} . It is the resulting matrix after removing the first row, last row, the first column, and the last column from B_n . In general,

Definition 9. For $n \geq 2$, denote

$$Q_n = \begin{bmatrix} 1 & 1 & & & & & \\ 1 & 1 & 1 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & & 1 & 1 & 1 & \\ & & & & 1 & 1 & \end{bmatrix}_{n \times n}.$$

Lemma 3.4.1. Consider Q_n and B_n as above and $n \geq 4$. Then over \mathbb{Z}_2 ,

$$\text{rank}(B_n) = \text{rank}(Q_{n-2}) + 2.$$

Proof. Obviously B_n contains Q_{n-2} as a submatrix. We perform elementary row or column

reductions as follows: first add the second row to the first row so that the new first row is the n -vector $[0, \dots, 0, 1]$. Then add the second column to the first column resulting in the new first column as $[0, \dots, 0, 1]^T$ (n dimensional). This yields the following matrix:

$$B_n \xrightarrow{\text{elementary row/ column reduction}} \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 1 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \vec{0} & 1 \\ \vec{0}^T & Q_{n-2} & \vec{v}^T \\ 1 & \vec{v} & 1 \end{bmatrix}_{n \times n},$$

where $\vec{v} = [0, \dots, 0, 1]_{1 \times (n-2)}$. Therefore the $\text{rank}(B_n) = \text{rank}(Q_{n-2}) + 2$. □

However from the previous chapter we know the value of $q(M_n, -1)$. Since it has a period of 6 and from Theorem 1.4.3 we are able to obtain the ranks of B_n for $n \geq 4$ and the rank of Q_n for all $n \geq 2$.

Theorem 3.4.2. *The ranks of B_n and Q_n over \mathbb{Z}_2 are given below:*

1. For $n \geq 4$,

$$\text{rank}(B_n) = \begin{cases} n, & \text{if } n \equiv 0, 2, 3, 5 \pmod{6} \\ n - 1, & \text{if } n \equiv 1, 4 \pmod{6} \end{cases};$$

2. For $n \geq 2$,

$$\text{rank}(Q_n) = \begin{cases} n, & \text{if } n \equiv 0, 1, 3, 4 \pmod{6} \\ n - 1, & \text{if } n \equiv 2, 5 \pmod{6} \end{cases}.$$

Proof. By Theorem 1.4.3, $(-1)^n(-2)^{n-r} = q(M_n, x)$, where r is the rank of B_n over \mathbb{Z}_2 .

Applying Proposition 3.2.1 we calculate the ranks.

1. If $n \equiv 0$ or $2 \pmod{6}$, then $(-2)^{n-\text{rank}(B_n)} = 1 \implies n - \text{rank}(B_n) = 0$. Thus $\text{rank}(B_n) = n$.

If $n \equiv 3$ or $5 \pmod{6}$, then $-(-2)^{n-\text{rank}(B_n)} = -1 \implies \text{rank}(B_n) = n$ similarly.

When $n \equiv 1 \pmod{6}$, $-(-2)^{n-\text{rank}(B_n)} = 2 \implies n - \text{rank}(B_n) = 1$, again $\text{rank}(B_n) = n - 1$.

If $n \equiv 4 \pmod{6}$ then $(-2)^{n-\text{rank}(B_n)} = -2$ therefore $\text{rank}(B_n) = n - 1$.

2. By lemma 3.4.1 we know that $\text{rank}(B_{n+2}) = \text{rank}(Q_n) + 2$. If $n \equiv 0, 1, 3,$ or $4 \pmod{6}$, $n + 2 \equiv 2, 3, 5,$ or $0 \pmod{6}$ respectively. By (1), $\text{rank}(B_{n+2}) = n + 2$. In these cases $\text{rank}(Q_n) = \text{rank}(B_n) - 2 = n$.

If $n \equiv 2$ or $5 \pmod{6}$, $n + 2 \equiv 4$ or $1 \pmod{6}$. Then $\text{rank}(B_{n+2}) = n + 1$. Therefore, $\text{rank}(F_n) = \text{rank}(B_{n+2}) - 2 = (n + 1) - 2 = n - 1$.

□

Example 4. *It is obvious that $\text{rank}(Q_2) = 1$ and $\text{rank}(Q_3) = 3$. From Example 3, $\text{rank}(B_4) = 3$ and $\text{rank}(B_4) = 5$ which confirms Theorem 3.4.2.*

$$Q_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

In addition, using Matlab we calculated the corresponding ranks for $n = 12, 13$.

$$\text{rank}(B_{12}) = 12, \quad \text{rank}(B_{13}) = 12,$$

$$\text{rank}(Q_{10}) = 10, \quad \text{rank}(Q_{11}) = 10.$$

Note that $12 \equiv 0 \pmod{6}$ and $13 \equiv 1 \pmod{6}$. By Theorem 3.4.2(1), $\text{rank}(B_{12}) = 12$

and $\text{rank}(B_{13}) = 13 - 1 = 12$. In addition, $10 \equiv 4 \pmod{6}$ and $11 \equiv 5 \pmod{6}$. Theorem 3.4.2(2) gives $\text{rank}(Q_{10}) = 10$ and $\text{rank}(Q_{11}) = 11 - 1 = 10$. This example confirms both parts of Theorem 3.4.2.

Chapter 4

Interlace Polynomial of $J(n, r)$

Next we study the graphs $J(n, r)$ where $r > 3$. Within this study we also discuss another related graph we call $B(n, k)$ ($n \geq 3$ and $k \geq 0$). The graph $B(n, k)$ contains two components: the path P_k and the cycle C_n . It is the resulting graph by “gluing” one end vertex of P_k with one vertex of C_n . To derive the formulas for the interlace polynomial of $q(J(n, r), x)$, we must consider those of the graph $B(n, k)$ for some k due to the toggling decomposition of $J(n, r)$. We first define $B(n, k)$ and investigate its interlace polynomial.

4.1 The Graph $B(n, k)$ and its Interlace Polynomial

Definition 10. For $n \geq 4$ and $k \geq 0$, define $B(n, k)$ to be the resulting graph by ‘gluing’ one end vertex of P_k with one vertex of C_n .

Refer to Figure 4.1, where the path $P_k = u_1 \dots u_k u_{k+1}$ is glued to the cycle $C_n = v_1 v_2 \dots v_n v_1$. The vertex u_{k+1} is identified as v_1 .

In particular, the graph $B(8, 4)$ is shown in Figure 4.2.

Remark 4.1.1. $B(n, k)$ is a unicyclic graph with the subgraph, say, $C_n = v_1 v_2 \dots v_n v_1$ and a path P_k is attached to one vertex of C_n . Assume the shared vertex is v_1 . Then $B(n, k)$ has

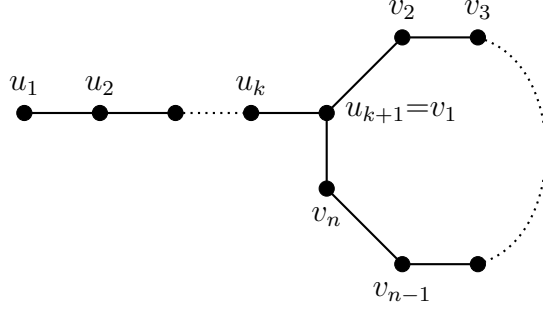


Figure 4.1: $B(n, k)$

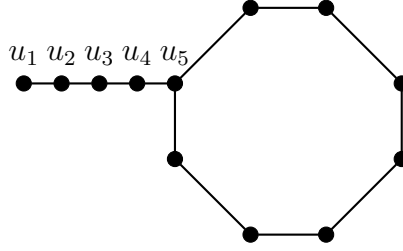


Figure 4.2: $B(8, 4)$

$n + k$ vertices and $n + k$ edges. Also, $B(n, k) - v_1$ is the disjoint union of P_{k-1} and P_{n-2} .

From the definition, $B(n, 0) = C_n$. By simple toggling processes, one can check that

Lemma 4.1.2.

1. $q(B(n, 0), x) = q(C_n, x)$;
2. $q(B(n, 1), x) = q(C_n, x) + xq(P_{n-2}, x)$;
3. $q(B(n, 2), x) = (x + 1)q(C_n, x) + xq(P_{n-2}, x)$;
4. $q(B(n, 3), x) = (2x + 1)q(C_n, x) + x(x + 1)q(P_{n-2}, x)$;
5. $q(B(n, 4), x) = (x^2 + 3x + 1)q(C_n, x) + x(2x + 1)q(P_{n-2}, x)$.

Theorem 4.1.3. 1. For $n \geq 3$ and $k \geq 2$,

$$q(B(n, k), x) = q(B(n, k - 1), x) + xq(B(n, k - 2), x);$$

2. Explicitly for $n \geq 6$ and $k \geq 5$,

$$\begin{aligned} q(B(n, k), x) &= q(C_n, x) [q(P_{k-2}, x) + q(P_{k-4}, x) + 1] \\ &+ xq(P_{n-2}, x) [q(P_{k-3}, x) + q(P_{k-5}, x) + 1]. \end{aligned}$$

Proof. (1) By toggling on the edge u_1u_2 as in Figure 4.1 we are able to show

$$q(B(n, k), x) = q(B(n, k-1), x) + xq(B(n, k-2), x).$$

(2) By mathematical induction on k it is simple to show the recursive relation of $q(B(n, k), x)$ is satisfied. □

Example 5. By Lemma 4.1.2 and Theorem 4.1.3(1), we calculate

$$q(B(8, 5), x) = q(B(8, 4), x) + xq(B(8, 3), x) = x^7 + 18x^6 + 93x^5 + 167x^4 + 133x^3 + 47x^2 + 6x.$$

On the other hand,

$$\begin{aligned} & q(C_8, x) [q(P_3, x) + q(P_1, x) + 1] + xq(P_6, x) [q(P_2, x) + q(P_0, x) + 1] \\ &= (2x^4 + 16x^3 + 21x^2 + 6x)(3x^2 + 4x + 1) + x(x^4 + 9x^3 + 9x^2 + 2x)(x^2 + 3x + 1) \\ &= x^7 + 18x^6 + 93x^5 + 167x^4 + 133x^3 + 47x^2 + 6x = q(B(8, 5), x), \end{aligned}$$

which confirms Theorem 4.1.3(2).

4.2 Interlace Polynomial of $J(n, r)$

Now we perform toggling decomposition on the graph $J(n, r)$ in order to obtain its interlace polynomial ($n \geq 4, r \geq 3$). Recall that $J(n, 3) = M_n$ and $q(B(n, 0), x) = C_n$. Refer to Figure

1.4 for the graph of $J(n, r)$ where $4 \leq r \leq (n + 2)/2$. We perform a toggling decomposition on $J(n, r)$ at the edge v_1v_2 . To obtain an explicit form for $J(n, r)$, we use the results of $B(n, k)$.

Proposition 4.2.1. *Consider $J(n, r)$ where $n \geq 2r - 2$ with $r \geq 3$. Then $q(J(n, 3), x) = q(M_n, x)$,*

$$q(J(n, 4), x) = q(P_{n-2}, x) + q(C_{n-2}, x) + xq(P_{n-4}, x),$$

and for $r > 4$ and $n \geq 2r - 2$, the following recursive formula is true:

$$q(J(n, r), x) = q(J(n - 2, r - 2), x) + q(B(n - r + 2, r - 3), x) + xq(P_{n-4}, x).$$

Proof. Label the cycle C_n as $v_1v_2v_3v_4 \dots v_{r-1}v_r \dots v_{n-1}v_nv_1$. Let the chord be $e = v_1v_r$. By toggling the edge v_1v_2 we obtain

$$J(n, r) - v_2 = B(n - r + 2, r - 3) \quad \text{and} \quad J(n, r)^{v_1v_2} - v_1 = J(n - 2, r - 2) + P_0 \cup P_{n-4}.$$

Figure 4.3 is provided for reference. Thus

$$q(J(n, r), x) = q(J(n - 2, r - 2), x) + q(B(n - r + 2, r - 3), x) + xq(P_{n-4}, x).$$

□

The explicit formulas for $q(J(n, r), x)$, where $r = 5, 6, 7$, require the results on $q(B(n, r), x)$ which we show in the next lemma.

Lemma 4.2.2. *For $J(n, r)$ with $r = 5, 6$, or 7 and $n \geq 2r - 2$,*

1. $q(J(n, 5), x) = q(M_{n-2}, x) + (x + 1)q(C_{n-3}, x) + xq(P_{n-4}, x) + xq(P_{n-5}, x)$
($n \geq 8$);

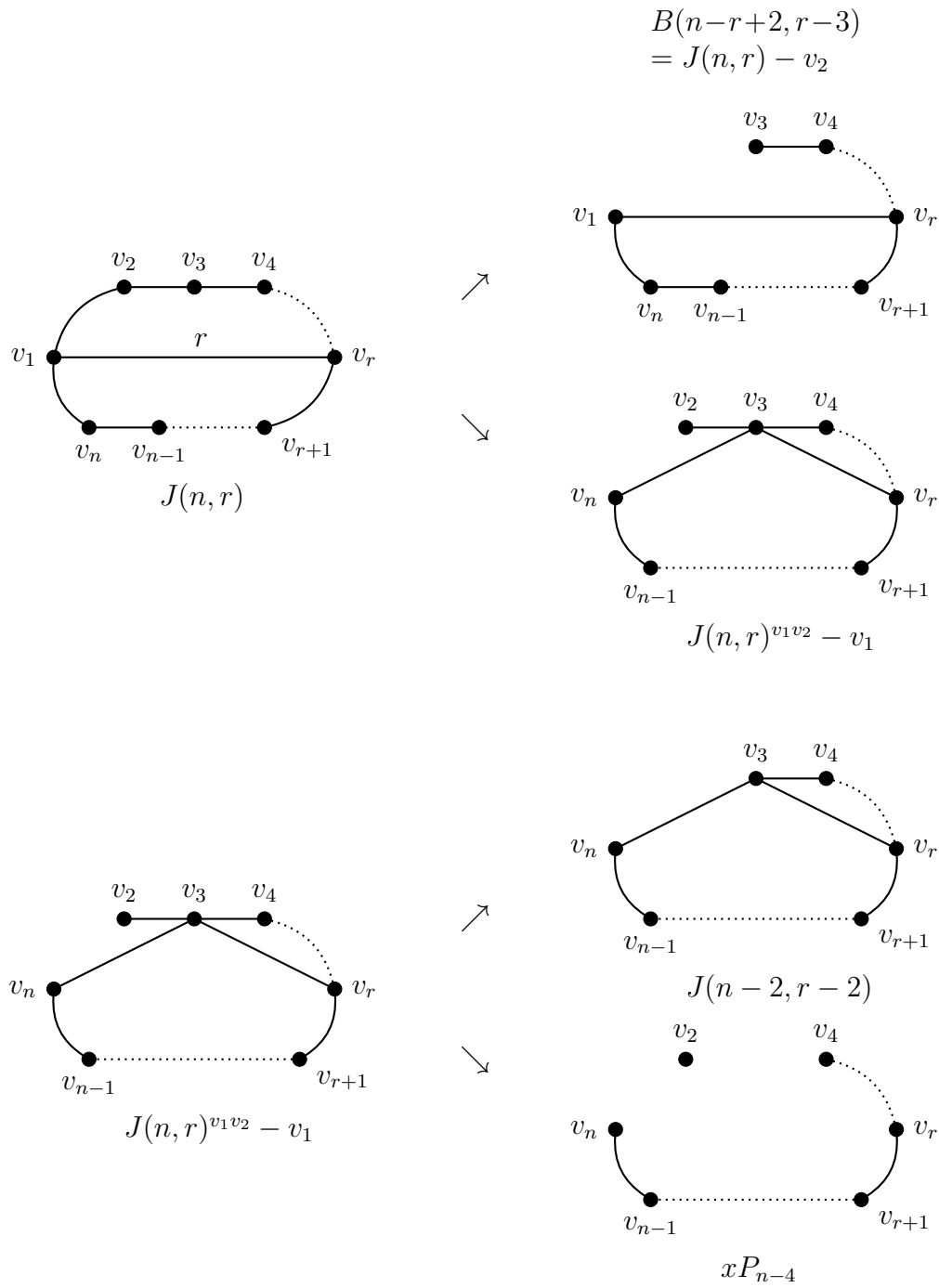


Figure 4.3: Decomposition of $J(n, r)$

$$2. q(J(n, 6), x) = (2x + 2)C_{n-4} + (x + 1)q(P_{n-4}, x) + x(x + 2)q(P_{n-6}, x) \quad (n \geq 10);$$

$$3. J(n, 7) = q(M_{n-4}, x) + (x^2 + 4x + 2)q(C_{n-5}, x) + x(2x + 2)q(P_{n-7}, x) \\ + xq(P_{n-4}, x) + xq(P_{n-6}, x) \quad (n \geq 12).$$

Proof. The proof is an application of Proposition 4.2.1 and Lemma 4.1.2. \square

Now the general recursive formula without $B(n, k)$ is obtained by Proposition 4.2.1(4) and Lemma 4.1.3(2).

Theorem 4.2.3. For $8 \leq r \leq (n + 2)/2$ and $n \geq 14$,

$$q(J(n, r), x) = q(J(n - 2, r - 2) + q(C_{n-r+2}, x)[q(P_{r-5}, x) + q(P_{r-7}, x)) + 1] \\ + xq(P_{n-r}, x)[q(P_{r-6}, x) + q(P_{r-8}, x) + 1] + xq(P_{n-4}, x).$$

Proof. This proof is obtained from Proposition 4.2.1 having Theorem 4.1.3 applied. \square

We end this chapter by examining the interlace polynomials of $J(8, 4)$ and $J(16, 8)$.

Example 6. Refer to the graph of $J(8, 4)$ shown below. By Proposition 4.2.1.

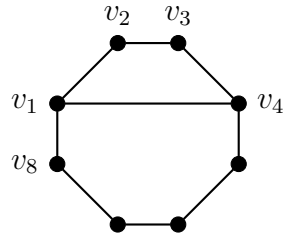


Figure 4.4: $J(8, 4)$

$$q(J(8, 4), x) = q(P_6, x) + q(C_6, x) + xq(P_4, x) = 2x^4 + 16x^3 + 21x^2 + 6x.$$

Example 7. *The interlace polynomial of $J(16, 8)$ by Theorem 4.2.3 and Lemma 4.2.2(2):*

$$\begin{aligned}q(J(16, 8), x) &= q(J(14, 6), x) + q(C_{10}, x) + [q(P_3, x) + q(P_1, x) + 1] \\ &+ xq(P_8, x)[q(P_2, x) + q(P_0, x) + 1] + xq(P_4, x) \\ &= 2x^8 + 52x^7 + 297x^6 + 676x^5 + 797x^4 + 524x^3 + 183x^2 + 26x.\end{aligned}$$

Appendix A

Interlace Polynomials of P_n for

$$0 \leq n \leq 14$$

$$q(P_0, x) = x, \quad q(P_1, x) = 2x$$

$$q(P_2, x) = x^2 + 2x$$

$$q(P_3, x) = 3x^2 + 2x, \quad q(P_4, x) = x^3 + 5x^2 + 2x$$

$$q(P_5, x) = 4x^3 + 7x^2 + 2x, \quad q(P_6, x) = x^4 + 9x^3 + 9x^2 + 2x$$

$$q(P_7, x) = 5x^4 + 16x^3 + 11x^2 + 2x$$

$$q(P_8, x) = x^5 + 14x^4 + 25x^3 + 13x^2 + 2x$$

$$q(P_9, x) = 6x^5 + 30x^4 + 36x^3 + 15x^2 + 2x$$

$$q(P_{10}, x) = x^6 + 20x^5 + 55x^4 + 49x^3 + 17x^2 + 2x$$

$$q(P_{11}, x) = 7x^6 + 50x^5 + 91x^4 + 64x^3 + 19x^2 + 2x$$

$$q(P_{12}, x) = x^7 + 27x^6 + 105x^5 + 140x^4 + 81x^3 + 21x^2 + 2x$$

$$q(P_{13}, x) = 8x^7 + 77x^6 + 196x^5 + 204x^4 + 100x^3 + 23x^2 + 2x$$

$$q(P_{14}, x) = x^8 + 35x^7 + 182x^6 + 336x^5 + 285x^4 + 121x^3 + 25x^2 + 2x.$$

Appendix B

Interlace Polynomials of C_n for

$$3 \leq n \leq 14$$

$$q(C_3, x) = 4x$$

$$q(C_4, x) = 3x^2 + 2x$$

$$q(C_5, x) = 5x^2 + 6x$$

$$q(C_6, x) = 2x^3 + 10x^2 + 4x$$

$$q(C_7, x) = 7x^3 + 14x^2 + 8x$$

$$q(C_8, x) = 2x^4 + 16x^3 + 21x^2 + 6x$$

$$q(C_9, x) = 9x^4 + 30x^3 + 27x^2 + 10x$$

$$q(C_{10}, x) = 2x^5 + 25x^4 + 50x^3 + 36x^2 + 8x$$

$$q(C_{11}, x) = 11x^5 + 55x^4 + 77x^3 + 44x^2 + 12x$$

$$q(C_{12}, x) = 2x^6 + 36x^5 + 105x^4 + 112x^3 + 55x^2 + 10x$$

$$q(C_{13}, x) = 13x^6 + 91x^5 + 182x^4 + 156x^3 + 65x^2 + 14x$$

$$q(C_{14}, x) = 2x^7 + 49x^6 + 196x^5 + 294x^4 + 210x^3 + 78x^2 + 12x.$$

Appendix C

Interlace Polynomials of M_n for

$$4 \leq n \leq 14$$

$$q(M_4, x) = 2x^2 + 4x$$

$$q(M_5, x) = 5x^2 + 6x$$

$$q(M_6, x) = x^3 + 10x^2 + 8x$$

$$q(M_7, x) = 5x^3 + 17x^2 + 10x$$

$$q(M_8, x) = x^4 + 14x^3 + 26x^2 + 12x$$

$$q(M_9, x) = 6x^4 + 30x^3 + 37x^2 + 14x$$

$$q(M_{10}, x) = x^5 + 20x^4 + 55x^3 + 50x^2 + 16x$$

$$q(M_{11}, x) = 7x^5 + 50x^4 + 91x^3 + 65x^2 + 18x$$

$$q(M_{12}, x) = x^6 + 27x^5 + 105x^4 + 140x^3 + 82x^2 + 20x$$

$$q(M_{13}, x) = 8x^6 + 77x^5 + 196x^4 + 204x^3 + 101x^2 + 22x$$

$$q(M_{14}, x) = x^7 + 35x^6 + 182x^5 + 336x^4 + 285x^3 + 122x^2 + 24x.$$

Appendix D

Interlace Polynomials of $B(8, k)$ for

$$0 \leq k \leq 10$$

$$q(B(8, 0), x) = 2x^4 + 16x^3 + 21x^2 + 6x = q(C_8, x)$$

$$q(B(8, 1), x) = x^5 + 11x^4 + 25x^3 + 23x^2 + 6x$$

$$q(B(8, 2), x) = 3x^5 + 27x^4 + 46x^3 + 29x^2 + 6x$$

$$q(B(8, 3), x) = x^6 + 14x^5 + 52x^4 + 69x^3 + 35x^2 + 6x$$

$$q(B(8, 4), x) = 4x^6 + 41x^5 + 98x^4 + 98x^3 + 41x^2 + 6x$$

$$q(B(8, 5), x) = x^7 + 18x^6 + 93x^5 + 167x^4 + 133x^3 + 47x^2 + 6x$$

$$q(B(8, 6), x) = 5x^7 + 59x^6 + 191x^5 + 265x^4 + 174x^3 + 53x^2 + 6x$$

$$q(B(8, 7), x) = x^8 + 23x^7 + 152x^6 + 358x^5 + 398x^4 + 221x^3 + 59x^2 + 6x$$

$$q(B(8, 8), x) = 6x^8 + 82x^7 + 343x^6 + 623x^5 + 572x^4 + 274x^3 + 65x^2 + 6x$$

$$q(B(8, 9), x) = x^9 + 29x^8 + 234x^7 + 701x^6 + 1021x^5 + 793x^4 + 333x^3 + 71x^2 + 6x$$

$$q(B(8, 10), x) = 7x^9 + 111x^8 + 577x^7 + 1324x^6 + 1593x^5 + 1067x^4 + 398x^3 + 77x^2 + 6x$$

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