8-2018

Pre-Service Teachers' Engagement in a Discourse about Integers from a Sfardian Perspective

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PRE-SERVICE TEACHERS' ENGAGEMENT IN A DISCOURSE ABOUT INTEGERS FROM A SFARDIAN PERSPECTIVE

A DISSERTATION

Submitted to the Faculty of

Montclair State University in partial fulfillment

of the requirements

for the degree of Doctor of Education

by

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August 2018

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MONTCLAIR STATE UNIVERSITY

THE GRADUATE SCHOOL

DISSERTATION APPROVAL

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FROM A SFARDIAN PERSPECTIVE

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7-23-18

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Abstract

This study describes the way 14 pre-service teachers engaged in discourse about integers. The discourse being examined is framed by Sfard's view of a discourse as being composed of the four elements word use, visual mediators, routines, and narratives (Sfard, 2008). These elements combine to form a means for people to exchange and preserve ideas and coordinate actions within a society. Each of the participants in this study engaged in a seven question, semi-structured interview (Merriam, 2009), which included prompts for the enactment of computations modeled with colored chips and the number line, in addition to written and spoken representations.

Using the constant comparison approach to coding (Corbin & Strauss, 1990) of the transcriptions of their spoken words and use of visual mediators, I identified tendencies in the participants' responses. The first tendency is the procedural manner, unsupported conceptually, with which they approach the expressions presented and questions posed. Secondly, these PSTs tended to base their discourse about integers on narratives which were only partially consistent with common discourse practices. A third tendency is the greater level of success these participants exhibited in the use of the number line as a modeling tool than they had when employing colored chips. This success with the number line was founded on the use of a standard algorithm, which calls for changing the expression into an equivalent form. This change of the expression reverts to the logical structures of whole numbers, rather than the more abstract conceptions required by a discourse about integers as purely mathematical objects.

My findings suggest the need for appreciation of the changes which will be imparted on mathematical discourse when integers are introduced, when course activities with whole numbers, fractions, and decimals are discussed and modeled in mathematics content courses for elementary teachers. It further suggests the need for greater support through provided examples and professional development for the chip and number line models for teachers who are required to teach about integers after participating in an elementary teacher preparation program.

Keywords: Integers, Sfardian view of discourse, chip model, number line, integer expressions, pre-service teacher or preservice (PST), minus sign, integer addition, integer subtraction

Acknowledgement

This study exists because of the lifelong support and encouragement provided by my parents, Douglas A. and Maureen R. Platt and the motivation provided by my brothers James and Thomas. Further, it was significantly aided by the support and indulgence provided by my in-laws in allowing me time during holidays and other occasions to work on this project and other related tasks.

I would like to take the opportunity to acknowledge the support provided, and time and effort spent, by my committee members Dr. Eileen Fernández and Dr. Steven Greenstein. Thank you for pushing me and for providing feedback, which allowed me to improve as a researcher as well as improving this paper. Also thank you for indulging my rather unusual pacing and refocusing of this study between the time of the initial literature review and the accepted proposal.

I would like to thank my second coder for her time and effort, at a point in time when she had several competing obligations.

Special thanks are directed to my dissertation chair Dr. Erin Krupa. I could not have done this without your support, suggestions, guidance, prods, and respect. You allowed this to be my study and me to work at my pace, as erratic as that was, while keeping me from running off the rails irredeemably far.

This study would not have been completed without the strength provided by my Lord, and the encouragement and support, and proofreading, provided by my loving and beloved wife Melissa Platt which was instrumental to my completion of this project, in the long run.

Dedication

I dedicate this study to the future of integer discourse in American schools and the programs that educate teachers to represent that discourse.

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Chapter 1: Introduction

The computations 8 – 5 and 5 – 8 have several things in common. Even so, one may be considered impossible, erroneous, or even absurd, while the other is viewed as being trouble-free, by an individual who is asked to perform these computations (Hefendehl-Hebeker, 1991). The fact that both expressions employ the same set of symbols to indicate subtraction between the numbers eight and five represents the extent of the similarities. The differences between these two expressions are, however, more significant to someone who views only the first expression as solvable. While the first expression can be answered with the whole number 3, the second expression requires the integer -3, a term which may not be in everyone's mathematical vocabulary. Further, the routines utilized to determine the solutions to these expressions contrast, since the second expression contains a subtrahend which is larger than the minuend. The ability to validate whether such a solution is correct, or even permissible, depends on the set of foundational rules about mathematics an individual has available. Thus, an individual's ability to perform these two similar, and rather straightforward, computations is subject to their ability to understand and employ the four elements of mathematical discourse identified by Sfard (2001a). These elements are: word use and meaning, visual mediators, routines, and narratives, tied together by rules that unobtrusively control the nature of the discourse and the nature and behavior of the objects upon which the discourse is directed (Sfard, 2001b). Therefore, an individual who is able to perform the computation on only the first expression can be seen as being a participant in the mathematical discourse about whole numbers, but an outsider to the mathematical discourse about integers (Sfard, 2008). These discourses differ in the core ideas the participants can exchange and those ideas which are understood as being exchanged. Though each individual develops and internalizes their own version of the common discourse, these discourses only exist because they carry ideas whose meaning is defined by the community in ways accepted by the community.

The ideas exchanged in such a discourse can be presented in a written or spoken form, and such an exchange can be among individuals or with oneself. In the latter case, the ideas which an individual can express to others are theorized as taking the same form as they would for a discourse with others. Thus, thinking is considered to be a form of communication, but one which occurs with oneself using the same discourse elements as communicating with others would (Sfard, 2008). A discourse can be viewed as both an activity to be engaged in, through thinking or communicating with others about an object of the discourse, and a corporately developed and sustained entity which provides the content and structure for such communication (Sfard & Kieran, 2001). Because established discourse participants use words and visual mediators in commensurable ways, follow similar routines, and assume a common collection of narratives as validating or refuting their ideas, discourses about a particular topic can only be fully understood by individuals who share a specific understanding of the way the particular key words are used, visual mediators are interpreted, routines are employed, and narratives are accepted. Individuals who do not interpret and understand these elements of a discourse in the way a person who is well versed in the discourse intends can be considered outsiders to the discourse (Sfard, 2001a). New participants enter into an existing discourse, and its community of participants, through a trial-and-error process of engagement in the discourse, with increasing level of proficiency, in the presence a member, or insider, of the discourse community, who will provide examples and feedback until full participant status is reached (Sfard, 2007).

Over the course of a normal day, individuals are routinely shifting between different discourses and participating in several discourse communities (Sfard, 2001a). An individual becomes a participant in some discourses through observation and casual interactions with others, however, more limited-use and specialized discourses require deliberately-created opportunities for engagement. The mathematical discourses about whole numbers and integers represent examples of these extremes. In the case of whole numbers, some components such as the names of the numerals and numbers, the shape and arrangement of the numerals and numbers, routines such as counting, and narratives related to the size and order of numbers are learned through routine interactions (Vygotsky, 1934/1962). Mathematical discourse about integers can be considered a more specialized discourse which is unlikely to be learned through routine interactions with sufficient regularity to allow an outsider to develop their own ability to engage in the discourse through trial-and-error (Sfard, 2000c). Thus, situations need to be created in which the outsider, and developing participant, to a discourse about integers has the opportunity to encounter and employ this discourse while receiving corrective guidance as needed, with such opportunities provided by the teacher in one of many equally effective forms.

As part of this view of discourse development, in the form of internalization by new participants, it is a teacher's obligation to provide their students a valid representation of the discourse employed within the academic discipline which is the topic of instruction (Herbst, 2010; Herbst & Chazan, 2003, 2011, 2012). Teachers assume the responsibility of fulfilling the role as fully-conversant members of a discourse community by accepting the label of "teacher" (Buchmann, 1986). Thus, teachers are responsible for providing a representation of the mathematical discourse about integers that is consistent with the broader mathematical discourse community. By the end of their content coursework in a teacher preparation program, pre-service teachers (PSTs) have attained the capability of providing such a valid representation of the discourse as they will possess upon entry into the profession (Morris & Kiebert, 2017).

A valid representation of mathematical discourse about integers based on the lens offered by Sfard would contain certain components within each of its defining elements (2008). Examples of some of the components which make up word use in such a discourse are a differentiation of the meanings of whole number, integer, positive integer, and negative integer as well as the introduction of the term and concept "opposite of" (Larson, Boswell, Kanold, & Stiff, 2012). While sharing most of its visual mediators with a mathematical discourse about whole numbers, mathematical discourse about integers expands the meaning of the minus sign from simply indicating subtraction to include symbolizing the opposite of a number and the negative nature of a number

(Barber, 1925). The routines for how to read the numbers in an expression, and to perform computations also need to be reconsidered. Finally, the narratives related to the effects of addition and subtraction are significantly modified in a mathematical discourse about integers, when compared with a similar discourse about whole numbers (Bassarear & Moss, 2016). As these points, which will be expanded upon in Chapter 2, suggest, a discourse about integers represents a notably different discourse than one about whole numbers.

This study addresses a collection of PSTs' discourses about integers through examining their responses to interview prompts. This examination consists of descriptions of the components identified within each element which make up these participants' discourses, and their responses to certain critical prompts. This study sought to address the following research question:

> *How can PSTs' mathematical discourse about integers be described based on their word usage, employment of visual mediators, enactment and verbalization of routines, and identification of core narratives of this discourse?*

Chapter 2: Literature Review and Theoretical Framework

In this chapter, I lay the theoretical groundwork for this study by synthesizing existing literature about a nature of mathematical discourse with particular focus on a discourse about integers within that framework. Further, in this chapter I describe what is known about PSTs' ability to engage in such a discourse, as revealed by the existing literature. In the section on the structure of a model of mathematical discourse, I summarize the work of Sfard, as I have found that she presents a clear and compelling operationalization of such a discourse. After having described the structure of this model of discourse, I use this as a tool to unpack the ideas contained in the contrasting discourses about whole numbers and integers. The chapter concludes with an examination of the several discourses a teacher may choose to weave together and draw upon during their act of teaching, and related works about PSTs' ability to engage with integers.

Structure of a Mathematical Discourse

Human community is built and advanced by our ability to communicate and coordinate our efforts (Sfard, 2000b). This communication and coordination can only occur if the individuals involved are able to attend to a common focus and call forth an appropriate response to the object or action which is the focus of that attention. This focus can take many forms, including physical objects which exist outside of the discussion, and discursive objects created through the discourse (Sfard, 2008). The means to accomplish this focusing of attention and negotiation of a set of suitable responses is a key part of the development of a discourse (Sfard, 2008) and forming a discourse community (Sfard, 2007).

Though discourse can be defined in many ways, I choose to follow the framework developed by Sfard based on my perception of its clarity and intuitive structure. In this framework, human thought is considered an intrapersonal discursive process, with properties similar to an interpersonal discussion between pairs of interlocutors (Sfard, 2007). In her definition of discourse, Sfard identified four interacting elements which make a discourse unique, distinguishable from other discourses, and closed to outsiders, namely: its word usage with associated meanings, use of visual mediators, employment of routines, and recognition of defining narratives (2008).

These elements are further drawn together by covert and collectively established rules that govern the structure and application of the discourse. These are referred to as metarules and object-level rules (Sfard, 2008). Metarules serve to guide the participants in a discourse concerning what to accept as an appropriate process or object to include in the discourse. Object-level rules reveal similarities that exist between different objects within a discourse. The objects considered by a discourse may be perceivable, as in the case of natural items, but they need not be, in which case they would be discursively-generated objects. Mathematical objects would tend to fall into this second category of objects, since they, along with their signifiers, tend to have been discussed into existence (Sfard, 2000c). Thus, their existence is the result of recursive and long-term cycles within a discourse and the discourse community, where the result of a process gets treated as a shorthand for the process. This result can then be used as the starting point for additional processes. Over time, individuals and the discourse community tend to employ discursive objects without regard for their origins. An example of this would be the process of counting, which assigns a sequence of words to any collection of objects producing a count of the number of items in the collection. These counts can then be used to compare two collections or to find the total of the combination of the two collections (The National Council of Teachers of Mathematics [NCTM], 1964). Mathematical discourse, which includes mathematical objects and operations, has evolved over time through adjustments to the metarules and object-level rules accepted by practitioners of the discipline. The evolutionary growth pattern experienced within mathematical discourse occurred in such a way that a mathematical discourse about whole numbers and a mathematical discourse about integers, while similar in many ways, contain enough differences in their objects and rules to be distinctive.

The rules of a discourse provide guidance to the participants on how the discourse operates, while its objects serve as the focus of the discourse. The more overt elements of the discourse are found in its words, visual mediators, routines, and narratives. Each of these elements are discussed in turn.

Word usage. Spoken and written words assume a critical role within a discourse, as words make up much of the body of any discussion and, along with visual mediators, allow routines and narratives to be conveyed. In selecting his or her words, a speaker is trying to convey a particular message, with each word or phrase carrying a particular meaning. Thus, a discussant's word-meaning alignment affects "what the user is able to say about (and thus to see in) the world" (Sfard, 2008, p. 133). Often new words are introduced to a language to carry new meanings; in other cases, the meaning of existing words is modified or expanded to embrace the newly assigned meaning. Within mathematical discourse in its broadest sense, the word 'number' experienced evolutionary expansion in historical (Bell, 1945), developmental (Piaget, 1965), and pedagogical terms (National Governors Association Center for Best Practices, 2010). In a mathematical discourse about whole numbers, 'numbers' include 0, 1, 2 …. In contrast, in a mathematical discourse about integers, 'numbers' include … -2, -1, 0, 1, 2 …. In addition to the extension of the term 'number' to include numbers less than zero, the criteria for acceptance of a combination of a numeral and a preceding positive or negative sign as a 'number' signifies the difference between whole numbers and integers. In the latter discourse, 'numbers' are accepted as valid mathematical objects by mathematical criteria alone, while the mathematical discourse about whole numbers accepted its 'numbers' based initially on their ability to describe quantities encountered in the experienced world (Bishop, et al., 2014; Hefendehl-Hebeker, 1991; Sfard, 2007).

Visual mediators. Visual mediators serve to represent the objects which are being discussed in a perceivable way (Sfard, 2008). In the case of mathematical discourse, visual mediators include: numerals and other symbolic represetations, number lines and other graphs, and physically- or digitally-manipulatable objects such as colored chips. As mathematical discourse is a means of communicating ideas, rather than producing or changing physical objects, these discourse-defined visual mediators play an important part in providing durability and focus in the exchange of ideas.

Routines. Regularly occurring sequences of events, or routines, are visible throughout a discourse. Such routines can be identified as functioning at both the object-level and metalevel (Sfard, 2007). An example of an object-level routine involves the process of performing mathematical computations, whereas, metalevel routines are employed in determining which computation routine to perform with the pair of numbers found in an expression (Sfard, 2008). The routines associated with numerical calculations are particularly important within a mathematical discourse, as they serve to determine and verify an equivalent value for the numerical expression involved (Sfard, 2008). For example, the numerical expression eight minus five being equal to three allows 'eight minus five' and 'three' to be used interchangeably.

Discourse participants are frequently shifting between different routines, generally unconsciously. Metalevel structures within a discourse influence the life cycle of each routine, including when to employ a routine, how to execute that routine, and when a routine has been successfully completed (Sfard, 2008). These object-level routines describe the process of evaluating individual expressions, while groups of such object-level routines are consolidated into mid-level routines that link together similar concepts and approaches, which are covered by overarching metalevel routines. Most of the significant changes within a discourse relate to modifications of the accepted metalevel structures of that discourse.

Narratives. Words and visual mediators serve as the means by which focus on a particular object is called forth within a discourse. These objects, which are the subject of the discourse, are knit together into narratives. Such narratives provide the means by which objects can be described, related, or employed in further processes (Sfard, 2008). Within a mathematical discourse, object-level narratives refer to mathematical objects, such as how to identify shapes or the equivalence of a number and the computation that produced that number (Sfard, 2007). Within a discourse, each proposed narrative is held subject to endorsement or rejection in relation to how well it conforms to the rules of the discourse (Sfard, 2007). This process of validation is conducted through reference to metalevel narratives which allow participants to discuss aspects of the discourse itself. Within mathematical discourses, metalevel narratives describe how computations can be modeled or expressed (Sfard, 2008). A collection of mid-level meta-narratives are used to determine which model, and associated routine, would most effectively convey the mathematical object, idea, or computation being considered.

Having identified the key factors that can be used to identify a discourse, it is now time to identify what it means for an individual to learn within a discourse.

Individual Learning within this Discourse Framework

Discourses are accessible to some people but closed to others (Sfard, 2008). This closure is due to the specific way each of the elements is utilized within the discourse community. Thus, within this framework, individual learning is considered to have occurred when an individual gains access to a particular discourse (Sfard, 2000a, 2001a). This is achieved by an individual changing their intrapersonal discourse, or thinking, so that it becomes commensurable with the discourses of established participants in the desired discourse (Sfard, 2007). As an individual develops an increased ability to engage in the new discourse, they become able to evaluate expressions which were previously unsolvable (Sfard, 2001a). However, it is only through engaging in the new discourse that many of the limitations of the existing, more restricted, discourse become apparent, since the existing discourse maintains its utility to the student (Sfard, 2001a) in other situations.

When a teacher introduces an unfamiliar discourse, which differs from that practiced by her students, through spoken or written examples drawn from that discourse, she introduces a commognitive conflict (Sfard, 2007). Sfard defines such a conflict as "the phenomenon that occurs when seemingly conflicting narratives come from different discourses — from discourses that differ in their use of words, in the rules of substantiation, and so on" (Sfard, 2007, p. 577). It is through overcoming this conflict, with the students adjusting their discursive practice to be commensurable with those of the teacher, that students are able to learn a discourse (Sfard, 2007, 2008). The movement of discursive practices within a classroom to mirror those of the teacher is based on the fact that her discursive ways reflect those of the broader community and thus are seen as the "general goal of learning" (Sfard, 2000a, p. 171) for those who wish to enter the teacher's more sophisticated discourse community.

A student's ability to adapt to become a participant in a new discourse develops out of discourses which the student is already capable of engaging in as a full participant (Sfard, 2001a). These existing discourses provide experiences and structures, referred to as templates, which the student employs as they try to communicate with others in the new discourse (Sfard, 2000c). Through a process of socially-mediated trial-and-error, the students choose to retain certain object-template combinations, modify some attempted combinations, while discarding still others, thereby building up their ability to engage in the new discourse (Sfard, 2000c). Ultimately, the goal of such learning in a discourse community is for the student to use the vocabulary and routines of the teacher's discourse as though they are his own (Sfard, 2001a).

Such a transition between pre-existing and developing discourses is present between discourses about whole numbers, and discourses about integers; students are envisioned as entering the latter discourse already fluent in the former. The nature of these two discourses will be discussed in turn.

Discourse About Whole Numbers

Students enter a discourse about integers as established participants in mathematical discourse about whole numbers (Sfard, 2000c, 2001a, 2007, 2008). This discourse provides a set of words, visual mediators, routines, and narratives which need to be retained, modified and expanded upon, or discarded, when the discourse is changed to be one which concerns and employs integers (Sfard, 2000c, 2001a, 2007, 2008). Thus, it seems appropriate to describe some key aspects of this discourse.

Within a discourse about whole numbers, the word 'number', or more specifically 'cardinal number', can be defined as a word or symbol used to represent the cardinality of a set of items greater than or equal to zero (NCTM, 1964). With this meaning assigned to 'number', addition can be defined as "a binary operation on the cardinal numbers associated with two disjoint sets" (NCTM, 1964, p. 57), and "[t]he sum of the cardinal numbers of two disjoint sets is defined to be the cardinal number of the union of the two sets" (NCTM, 1964, p. 57). The cardinal numbers associated with each of the original disjoint sets are called addends. This grounding of the operation of addition validates the use of the terms increasing, joining (Bassarear & Moss, 2016), or combining (Beckmann, 2014) as appropriate terms to operationalize the idea of addition. Subtraction complements addition by being “the operation of finding an unknown addend when the sum and one of the addends are known” (NCTM, 1964, p. 79). In this definition of subtraction, the minuend is the known sum, the subtrahend is the known addend, and the difference represents the unknown addend. Further, this association with the cardinality of physically-existing sets of numbers allows subtraction to be seen as taking away, finding the difference, comparing, and finding the missing addend (Bassarear & Moss, 2016; Beckmann, 2014). Though linked as functionally opposite operations, the narratives for addition and subtraction of whole numbers are governed by contrasting sets of metarules, or properties of the operations, which will be discussed in turn. It should be noted that I have not provided an exhaustive list of possible definitions for addition and subtraction within a discourse about whole numbers.

If two whole numbers, such as 3 and 5 are subjected to the operations of addition and subtraction, several properties of these numbers can be revealed. When three is added to five, the whole number eight is produced; however, when we consider three minus five, a whole number does not result. In fact, for any pair of whole numbers, the addition of those numbers produces another whole number, in accord with the property of closure for the addition of whole numbers. In contrast, the fact that three minus five does not result in a whole number shows that subtraction of whole numbers does not possess the property of closure (NCTM, 1964). Moreover, if the order of the numbers three and five are reversed in the calculations, and three is added to five, the number eight is still produced, while if we subtract three from five, the number two is the result, stemming from the fact that addition conforms to the commutative property, while subtraction does not (NCTM, 1964). As a further point of contrast, addition remains valid under the associative property, but subtraction does not.

When addition and subtraction are performed on whole numbers, different properties need to be employed. However, these operations, though employing different rules, are used on the same collection of mathematical objects. It can also be noted that many of the differences in the structures and properties of addition and subtraction are exemplified by the difference in the wording used to describe the elements in the two operations, namely, an addition expression possesses two addends and a sum, while a subtraction expression, employs the minuend, subtrahend, and difference to indicate the original quantity possessed, the number 'taken away,' and the amount remaining, respectively. This distinction between the minuend and subtrahend when evaluating a subtraction expression is key since subtraction lacks both commutative and associative properties. Further, since the closure property does not hold for subtraction of whole numbers, this operation is only endorsed as being valid when the minuend is greater than or equal to the subtrahend, thus generating a whole number difference (NCTM, 1964).

In the discussion of mathematical discourse about integers which follows, we see that when 'whole number' is replaced by 'integer' subtraction gains the property of closure. Were this to be the only difference between the two discourses, this would be enough to distinguish between mathematical discourse about whole numbers and mathematical discourse about integers. However, as you will see below, the redefining of 'number' to include values less than zero leads to changes throughout the discourse.

Mathematical Discourse About Integers

Mathematical discourse about integers is distinctly different from mathematical discourse about whole numbers. These discourses differ in their word usage, visual mediators, routines, and endorsed narratives, a claim supported by Sfard when she states that "[a]ccording to commognitive analysis, learning about negative numbers involves a transition to a new, incommensurable discourse" (2007, p. 597). However, these changes represent only the visible contrasts between the two discourses, since "[o]ne of the main changes that must happen in this transition is in the metarules of endorsement (and more specifically, of defining)" (2007, p. 597). Yet, such a transition would not be possible without discourse about integers inheriting many of its object- and metalevel attributes from discourses about whole numbers. I will now unpack the key features of a discourse on integers using the commognitive framework of mathematical discourse.

Word use. The words used in a mathematical discourse serve to identify the objects of a discourse as well as the relationship between, and actions on, such objects. The word 'number' can be used to describe either a particular object, such as negative three or positive eight, or an entire class of mathematical objects with shared characteristics, such as negative numbers or positive numbers. The characteristics shared by the group of objects collectively called 'numbers' differ between mathematical discourses about different 'sets of numbers' (Sfard, 2000c, 2008). When applied to discourse about the set of whole numbers, a 'number' refers to the last number named when counting objects, even when the number is not generated by actually performing the act of counting (NCTM, 1964; Sfard, 2007). The mathematical objects that constitute a discourse on integers lack such concrete models (Sfard, 2001a), instead they are based in large part on their ability to serve a mathematically useful role, such as being solutions to quadratic and other higher order equations (Hefendehl-Hebeker, 1991; Whitacre I. , et al., 2011).

Contemporary mathematical discourse about integers affirms 'integers' are defined as any number in the set {…, -3, -2, -1, 0, 1, 2, 3 …} (Carter, et al., 2012). Further, a 'negative integer' is any integer that follows this pattern less than zero, while a 'positive integer' is any integer that follows this pattern greater than zero (Larson, Boswell, Kanold, & Stiff, 2012). When transitioning from a discourse about whole numbers to a discourse about integers, the existing words used to describe 'number' have their meanings adjusted, and a new set of 'number phrases' are introduced. An example of this can be seen as the numbers one, two, and three obtain new identities as 'positive integers', namely positive one, positive two, and positive three as a way to distinguish them from their opposite 'negative integers' which are negative one, negative two, and negative three. These terms enter mathematical discourse with this transition. Along with this change, zero is no longer the smallest number. These are key distinctions between the words used in these two mathematical discourses.

Visual mediators. Within any discourse, the visual mediators provide a durable means of identifying the object of a discussion. Within mathematical discourse about whole numbers, written numerals, a collection of countable objects, and points on a number line serve as visual mediators of numerical values, with other symbols serving to trigger computation routines. Thus, 7 – 4 directs the reader to subtract four from the initial seven. Contemporary discourse about integers borrows, in modified form, the visual mediators employed in discourse about whole numbers. Though integers employ the same system of numeration as that employed to represent whole numbers, discourse on integers utilizes prefixes to indicate positive (+) or negative numbers (-). Thus, the numeral 4, is employed in expressing the whole number four (4), and the integers positive four (+4 or 4) and negative four (-4) (The National Council of Teachers of Mathematics [NCTM], 1969). Therefore, within expressions involving integers, the symbol (+) serves as the visual mediator for both the mathematical operation of addition, and as part of a mathematical object, a positive number. This issue of multiple potential meanings is more prevalent with the symbol (-) which mediates the operation of subtraction, the negative nature of a number, and the operation of identifying the opposite of the number (Barber, 1925, 1926; Gallardo, 2002; Gallardo & Rojano, 1994; Vlassis, 2004, 2008).

Colored chips can be used as a visual mediator representing whole numbers or positive and negative integers; this preserves the cardinal relationship within whole numbers. When different colored chips are used as visual mediators for positive and negative integers, the number of chips of one color represent the magnitude of the positive integer, and the number of chips of another color represents the magnitude of the negative integer, see Figure 1 (VanDeWalle, 2004). This type of visual mediation uses metaphorical relationships between negative values and cardinal values, since negative values do not represent cardinal numbers. This lack of cardinality for the integers represents a complication and limitation of this model (Vig, Murray, & Star, 2014), but one that can be overcome by accepting the notion that each visual mediator requires its own set of routines for modeling a particular expression or narrative.

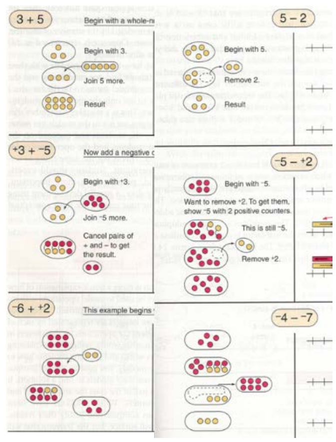


Figure 1 Addition and subtraction of integers with chips *(VanDeWalle, 2004, pp. 460-1)*

The number line serves as a visual mediator in both discourses about whole numbers and integers. In the former case, the number line, marked off and labeled at regular intervals, starts at zero and proceeds to show the sequence of numbers (NCTM, 1964). In contrast, the integer number line uses zero as its midpoint with positive integers marked in an increasing sequence on one side, and negative integers marked in a decreasing sequence on the other side (Carter, et al., 2012). While a dot can serve to indicate the location of a number in its ordinal relation to other numbers on the number line for either discourse, a vector is more informative in representing an integer, the magnitude of the integer is indicated by the number of intervals it extends, while its direction indicates the sign of the number, see Figure 2.

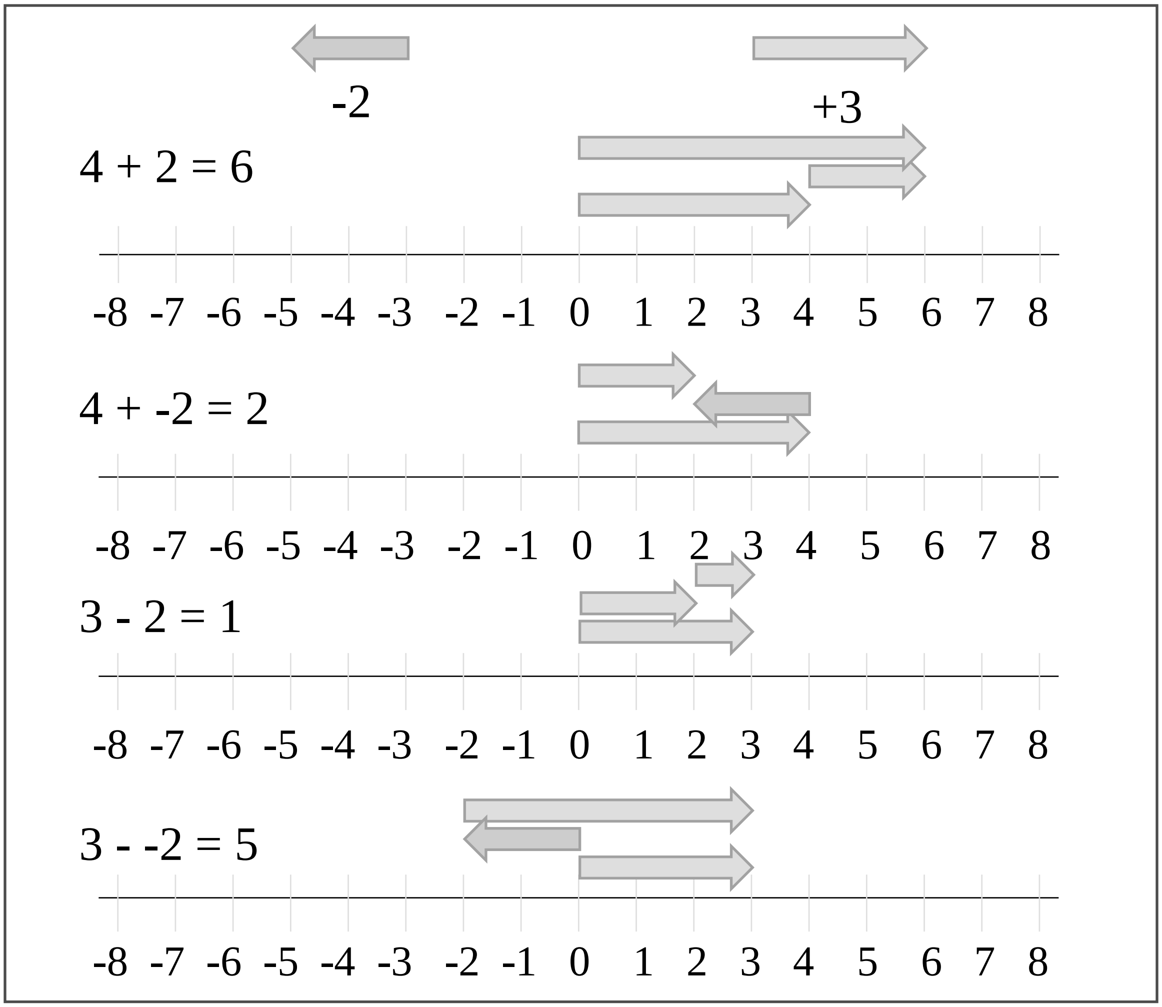


Figure 2 Number line notations for negative and positive integers and addition and subtraction

Routines. Routines are the regularly occurring processes within a discourse. A limited number of the routines found in computation with integers are based firmly in the routines utilized with whole numbers. Many additional routines need to be added to those imported from discourse about whole numbers to allow for computations with integers to be performed. While only certain routines would be applicable for any individual integer addition or subtraction expression, and the selection of such applicable routines is itself a routine, the act of evaluating any expression begins with the identification of the numbers and operation. Once these are determined, the individual's next step, and overall routine are based on which visual mediator he or she will be employing. The routines proposed below represent only a few of many possible routines for each visual mediator presented.

When an individual elects to add or subtract integers using written symbols, or their spoken equivalents, they may follow one of the following routines for these sample cases. When adding two positive or two negative numbers, the magnitudes are added, and the common sign is attached (Carter, et al., 2012). When adding numbers of different signs, the smaller absolute value is subtracted from the larger absolute value, then the sign of the addend with the larger absolute value is attached to the resulting difference, which represents the sum (Bassarear & Moss, 2016). In the case of integer subtraction, when the subtrahend is positive, add the opposite of that value to the minuend (Beckmann, 2014); when the subtrahend is negative, convert the subtraction to addition, and change the subtrahend to its opposite, then simplify as an addition expression (Larson, Boswell, Kanold, & Stiff, 2012). The application of these routines is regulated by the nature of the terms being considered, and the operation being employed. Though the routines described are independent of the use of written numerals and signifiers, the next two routines discussed are made possible by the presence of colored chips or number lines, thus they are discussed here.

Another visual mediator employs colored chips and represents a collection of different routines based on the operation being considered and the relative sizes of the numbers. In this model, chips which are one color, often black, are counted out to match the magnitude of positive numbers, while chips of a second color, often red, are counted out to match the absolute value of the negative numbers (VanDeWalle, 2004). Within this mediator, additive inverses where opposite positive and negative values add to zero, are shown using equal numbers of chips of each color, called zero pairs, see Figure 3. When two numbers are being added, each number is counted out using the appropriate color of chips. Any time one black chip can be paired up with one red chip, they are paired up and thus cancelled out (VanDeWalle, 2004). Once all pairs have been made, the remaining chips will all be one color; these are counted and based on the color identified as positive or negative. If the numbers are to be subtracted, both the minuend and the subtrahend are counted out, then the representation of the minuend is adjusted by adding zero pairs until there are enough chips of the appropriate color to allow for the same color chip to be removed from both the minuend and the subtrahend at the same time; any remaining chips represent the difference between the minuend and subtrahend (VanDeWalle, 2004; Vig, Murray, & Star, 2014).

One of the other key visual mediators employed in mathematical discourse about integers is the number line and vectors or directed arrows. Following a standard representation, positive numbers are placed to the right of zero, and thus the number five would be represented by a vector which is five units long pointed to the right. Similarly, a negative number is symbolized by a left-pointing vector of the appropriate length (Beckmann, 2014). The enactment of a routine using this set of visual mediators requires taking the initial step of identifying the numbers and the operation being computed.

Addition in this model is thought of as joining two sets, or two arrows, with the second arrow starting at the end of the previous one (VanDeWalle, 2004). Thus, the sum of two integers can be determined by looking at the end point of the second arrow. Subtraction of integers can be displayed on a number line using the 'find the missing addend' view of subtraction. In this view, one is asked to determine what number would need to be added to the subtrahend to arrive at the minuend (NCTM, 1969). To show this on a number line, the minuend would need to be represented by a vector of the required length, beginning at zero and projecting left or right, as appropriate, and the opposite of the subtrahend would be represented by a vector that would point in the appropriate direction, beginning at the end of the minuend (VanDeWalle, 2004). An arrow is then used to connect the end of the subtrahend arrow with the end of the minuend arrow. The length and direction of this third arrow indicates the difference, which is the integer that needs to be added to the subtrahend to obtain the minuend (NCTM, 1969). Though other options are available to model addition and subtraction with the number line, this method employs addition and preserves the value of the minuend, but uses the opposite of the subtrahend, see Figure 3. A representation which preserves both the original values and operations is shown in Figure 2 based on Kinach (2002).

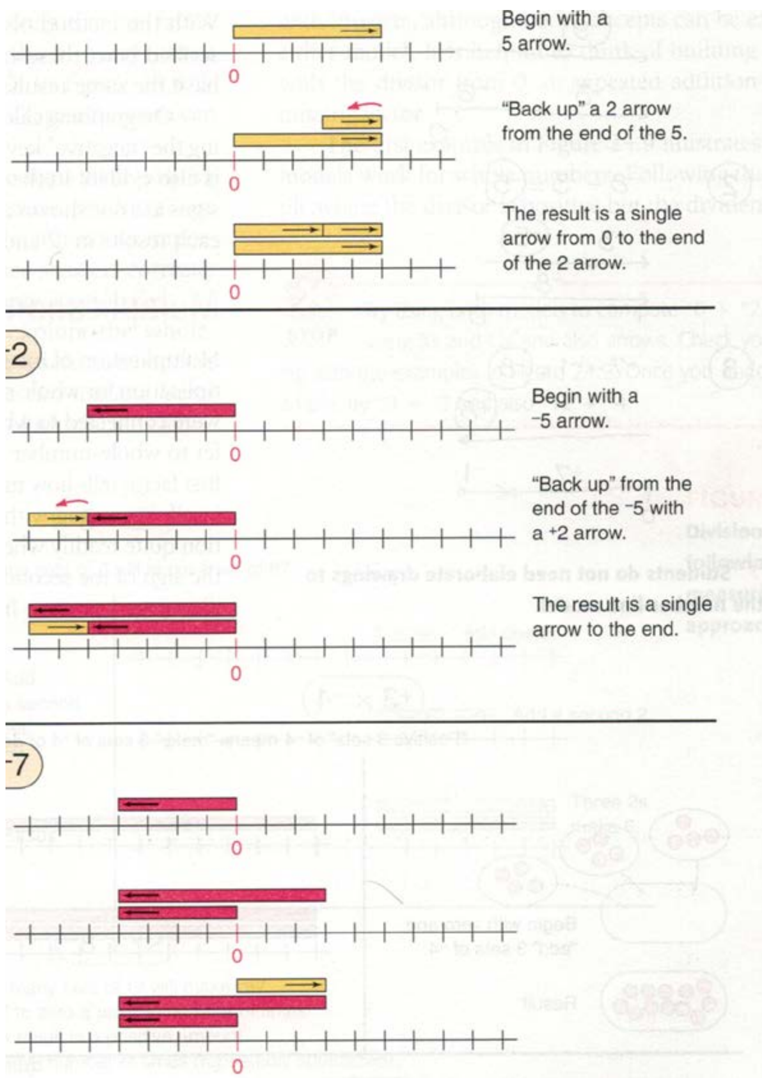


Figure 3 Number line notations and routines employing addition forms *(VanDeWalle, 2004, p. 460)*

As stated above, these routines for solving addition and subtraction expressions involving integers represent only one family of procedural options within each of the visual mediators. They do not represent the only, or even necessarily the best, procedural option of those visual mediators. Since very few of these routines are identical to those one would use with whole numbers, in order to become a full participant a person learning this discourse would need to be exposed to a competent rendition of each of these routines for the two non-symbolic visual mediators.

As the representative of the wider mathematical discourse community, the teacher is obligated to be able to provide her students with a valid and complete rendition of mathematical discourse about integers. Within such a rendition, the teacher, through her own speech or writing, or in her selection of tasks, is called upon to comprehensively reflect the vocabulary and visual mediators that make up the discourse. Additionally, she is called upon to provide, or draw forth from her students, an enacted example of the routines required to complete the desired operations. She must also model those narrative processes, which allow for the acceptance or rejection of the proposed results of following such routines by the students.

Narratives. The changes in the words employed and how they are symbolized serve as the superficial indicators of the changes in the rules of mathematical discourse when it shifts from being about whole numbers to being about integers. The rules which govern acceptable values in mathematical discourse about whole numbers are consistent with the process of determining numbers through the act of counting objects or iterations of a length. Thus, whole numbers remain closely associated with the experienced world (NCTM, 1964). Mathematical discourse about integers required a break from this grounding in the experienced world to attain full acceptance amongst European mathematicians (Bell, 1945; Gallardo, 2002). Integers came to be accepted as numbers because they followed existing mathematical rules when 'numbers' were treated as mathematical objects not subjected to association with the experienced world (Bishop, et al., 2014). In this way, integers can be utilized to describe the number of objects in a collection or the level of a surface with reference to sea level, but they need not be able to do so. Additionally, such relationships as excess and shortage, or above and below ground or sea level, have been, and can be, described quite clearly with the use of descriptive adjectives without needing integers. These aspects of the historically-negotiated basis for integers, form what I believe to be a key narrative about what makes integers and discourse about integers distinct from discourse about whole numbers.

As mathematical objects, positive and negative integers represent both a magnitude and a direction of that value from zero (Bassarear & Moss, 2016). A positive and a negative integer of the same magnitude, are the same distance from zero, but that distance is traveled in opposite directions (Beckmann, 2014). While 'integers' include features of symmetry which are absent in the 'whole numbers,' the binary operations of addition and subtraction retain a consistent meaning across mathematical discourses about whole numbers and integers. However, the opposite nature of positive and negative integers is exploited by new unary operations. Finding the 'opposite of' a number is one such new operation, which identifies the number that has the same absolute value as the given number, but with the opposite relationship to zero as the given number (Beckmann, 2014). Though not based on the existence of opposites, the absolute value operation is also critical to a discourse about integers, since it identifies the magnitude of an integer (Bassarear & Moss, 2016). Pairs of integers with the same magnitude but oriented in opposite directions have a sum of zero, which is called the additive inverse property (Carter, et al., 2012). Thus, I have shown that the nature of integers, and the range of operations with 'integers' are inherently different from those of 'whole numbers' and represent narratives exclusively about integers.

Integers were both born from, and experience their most significant difference from whole numbers in, the realm of computation. Addition within a mathematical discourse about whole numbers validates only two narratives. The first of these involves adding zero and a non-zero value, to obtain or retain that non-zero value, ex. 5 + 0 = 5. The second narrative for addition of whole numbers concerns the addition of two non-zero numbers to produce a third non-zero number that is larger than either of the original numbers, ex. 5 + 2 = 7 (NCTM, 1964). Subtraction with whole numbers results in three narratives. When zero is subtracted from a non-zero number, that same non-zero number results, ex. 5 – 0 = 5. When a number is subtracted from itself, the difference is zero, ex. 5 – 5 = 0. In between these two situations where the minuend is unchanged or entirely removed is the subtraction of a smaller non-zero number from a larger non-zero number to produce a number between the larger number and zero, ex. 5 – 3 = 2 (NCTM, 1964). As narratives provide over-arching structures within the discourse, these relatively few narratives describe an infinite number of different number combinations, as long as each of those combinations fits one of these templates. The addition of negative integers into a mathematical discourse increases the number of narratives required to validate any possible integer addition or subtraction expression.

One such additional consideration that needs to be made before one can select the most appropriate narrative to fit an expression is determining whether the numbers in the expression are whole numbers or integers. Once an expression is identified as only containing whole numbers, or more generally non-negative integers, then only their magnitudes need to be considered. When an expression contains integers, the numbers' magnitudes need to be considered, but in this case that magnitude is a combination of both absolute magnitude and the direction of the number. This additional complexity inherent to integers creates many new narratives to describe addition and subtraction with integers. Those narratives which are added with the shift from whole numbers to integers will be considered in turn.

If one of the addends is positive, the other is negative, and the negative number is less than the additive inverse of the positive addend, then the sum is positive, ex. 4 + -2 = 2. If a positive addend and its opposite are added together, then the sum is zero, ex. 4 + -4 = 0. If a positive number is added to a negative number that is greater than the positive number's additive inverse, then the sum is negative, ex. 4 + -5 = -1 (Larson, Boswell, Kanold, & Stiff, 2012). Though these narratives are written with the positive number first, due to the commutative property, the order of the positive and negative terms does not affect the sum. Further, when both addends are negative, the sum is less than either of the addends, ex. -4 + -3 = -7 (Bassarear & Moss, 2016). Thus, when addition is performed with integers, there are six narratives, including two imported from the discourse about whole numbers, to be considered based on the relative sizes of the absolute magnitude and orientation of each of the addends.

Having considered the narratives associated with addition of integers, it is now time to consider the family of narratives related to subtraction of integers. As will be shown for a given minuend, gradual changes in the value of the subtrahend result in the gradual movement of the difference in the opposite direction. In this case, the subtrahend will be reduced in stages causing an increase in the value of the differences for each successive case. For a given positive minuend, say nine, any subtrahend with a larger value generates a negative difference, ex. 9 – 12 = -3. In those cases, where the minuend and the subtrahend have the same value, the difference is zero, ex. 9 – 9 = 0. When the subtrahend is positive, but smaller than the minuend, the difference is a positive number less than the minuend, ex. 9 – 7 = 2. In the special case where zero is the subtrahend, the difference and the minuend have the same value, ex. 9 – 0 = 9. Further, when the subtrahend is negative the difference is larger than the minuend, ex. 9 – -3 = 12 (VanDeWalle, 2004). A similar pattern develops when the minuend is negative, for example, -5. When the subtrahend is positive, the difference in less than the minuend, ex. -5 – 4 = -9. For the case when the subtrahend is zero, the difference is the same as the minuend, ex. -5 – 0 = -5. When the subtrahend is negative but larger in value than the minuend, the difference is negative but larger than the minuend, -5 – -3 = -2. As with positive numbers, when the minuend and subtrahend are the same number their difference is zero, ex, -5 – -5 = 0. In the case, where the subtrahend is a smaller value than the minuend, the difference is positive, ex. -5 – -9 = 4. A minuend of zero has yet to be considered, but again follows a similar pattern. When a positive number is subtracted from zero, the difference is a negative number. In fact, the difference is the opposite of the subtrahend in this and the final case, ex. 0 – 5 = -5. In the trivial case where both the minuend and the subtrahend are zero, the difference is also zero. For a minuend with a value of zero and a negative subtrahend, the difference is a positive number, ex. 0 – -5 =

5. Thus, we have identified each member of the extensive family of narratives which are held to be valid for addition and subtraction of integers. Each of these members is valid for an infinite number of value combinations, as long as those combinations obey the relative and absolute size relationships indicated for such a narrative. While these narratives can be useful in determining the range of numbers, negative, positive, and zero (NCTM, 1969), which can constitute the equivalent value to an addition or subtraction expression, it is through following a suitable routine that an individual can determine the actual sum or difference of two numbers.

Pre-Service Teachers' Mathematical Discourse about Integers

What has already been learned about pre-service teachers' ability to engage in mathematical discourse about integers? In short, not much, and what we do know does not support any meaningful conclusions relevant to an analysis based on mathematical discourse. In the earliest article considered, Kinach (2002) employed integer addition and subtraction tasks as the content which her pre-service teachers were asked to use in developing their understanding of quality mathematical explanations. During the sequence of lessons described in this study, some but not all of the students progressed from employing tricks which did not distinguish between the subtraction and negative, to being able to both identify and attend to this difference in their explanations in the use of number lines. This focus on tricks to find the solution also appeared to play a significant role in PSTs' thinking in a study by Bofferding and Richardson (2013). Through think-aloud interviews utilizing written expressions, the PSTs' ways of approaching integer addition and subtraction expressions were determined. The PSTs tended to rearrange the terms of these expressions while attempting to arrive at a solution. Each of these articles suggests that the PSTs in these classes failed to exhibit an ability to model competent discourse about integers based in the proposed discourse found above.

This narrow glimpse into PSTs' thinking and ability to engage in discourse about integers suggests that there is room for additional research on this topic. These studies provide limited access to the PSTs' word usage while discussing integers. Further, these studies refer to only one type of visual mediator, either number lines (Kinach, 2002) or written expressions (Bofferding & Richardson, 2013), which is used to represent only a limited range of integer addition and subtraction arrangements.

As is shown in this chapter, Sfard outlines mathematical discourse as representing a clearly defined means of developing and communicating an idea within a community. Such a discourse is distinctive and restricted to only those who are knowledgeable insiders. The differences between a discourse about whole numbers and a discourse about integers have been considered. Thus, I have presented discourse about integers as a focused and identifiable topic, with a fixed set of expectations for participants. The minimal nature of the existing published research on pre-service teachers' knowledge of integers and their ability to engage in a discourse about integers, suggests that a gap exists in the literature. This study seeks to address that gap by drawing out the PSTs' word use, employment of visual mediators, their ability to create and simplify integer addition and subtraction expressions of each of the arrangements presented above, and to justify their routines and define key ideas with their use of narratives.

Chapter 3: Methodology

In an effort to describe how pre-service teachers assume the role of expert participant, or insider, in a mathematical discourse about integers, this study addresses the research question:

> *How can PSTs' mathematical discourse about integers be described based on their word usage, employment of visual mediators, enactment and verbalization of routines, and identification of core narratives of this discourse?*

Sample

Fourteen pre-service teachers were recruited to participate in interviews. These participants were drawn from the population of students who took the first of two mathematical content courses designed for pre-service teachers during the previous spring semester. These PSTs are students at a medium-sized state university in the northeastern region of the United States. The content of this two-semester sequence focuses primarily on the elementary grades, but also serves to prepare those desiring to work at the middle school level.

Potential participants were nominated by the professors who taught these content courses in the spring semester. The professors were asked to provide the names and email addresses of their former students who tended to be engaged and vocal during class, and who represented a variety of levels of mathematical ability, from each of the sections they taught. In providing the nominations, the professors were asked not to provide any indication of the nominated students' perceived level of mathematical ability.

A pool of 65 nominated candidates, including both males and females, was developed in this manner. However, only female PSTs elected to participate.

During the second week in October 2017, during the time in which I received these nominations, I made brief visits to the second content course, in which many of the PSTs were enrolled, to introduce myself and the study. I then contacted each of the nominated students individually via email to inform them that they were nominated to participate, and to provide further information regarding participation in this study. This email included a consent form for the PST to read and sign and asked them to suggest times when they would be available to participate. Thirteen participants accepted the request for their participation after being nominated. Additionally, any student who, though not nominated, desired to volunteer for this study, based on the in-class introduction was provided instructions on how to email me concerning their interest in participating. While the individuals whose interviews were performed first responded to the initial invitation, some of the later participants responded to reminder appeal emails forwarded by their current mathematics professor, or to my reaching out a second time to remind them of their nomination and requesting their participation in the study. One student, Jacqueline, volunteered to participate. In response to her email requesting inclusion, I negotiated a time for the interview to occur. She received the consent form at the time of the interview.

Whether the students responded to the request for participants or volunteered, interview times were determined based on participant availability communicated through an exchange of emails. No potential participant was turned away due to an inability to schedule a suitable time. At the beginning of the interviews, each participant was presented with two copies of the consent form, which they read, discussed, and signed. They kept one copy, while I maintained the other in a secure location. Participants were provided nominal monetary compensation in the form of two $10 gift cards of their choice from a variety of different merchants. The time required to work through the interview protocols varied in length, from 16 minutes up to 43 minutes, with all interview sessions lasting less than an hour inclusive of introductions, conclusions, and brief post-interview instruction, as well as completing the protocol. These interviews occurred between October 30th and November 30th, 2017, in a conference room in one of the academic buildings on campus.

Interview Protocol Structure and Design

The participants participated in semi-structured interviews (Merriam, 2002). The interview consisted of 7 questions, containing between one and 13 separate prompts. Questions 1, 2, 3, and 4 each contained only one prompt, Question 5 had four separate prompts, and Questions 6 and 7 had 13 prompts each. Each prompt sought to explore a different combination of aspects of the participants' discourse about integers based on the exemplar discourse discussed in Chapter 2. The complete interview protocol is found in Appendix A. Each question will be considered in turn.

Question 1. In this item I posed the following prompt to the participant, "If a 7th grade student were to ask you, 'What's the difference between whole numbers and positive integers?' how would you respond?" I sought to expose the participants' way of defining the terms 'positive integer' and 'whole number' as part of the word use element of discourse. Further, it was designed to allow narratives related to what numbers these participants deem acceptable in a discourse about integers.

Questions 2 and 3. In these two questions, each a separate prompt, the participants were first prompted with the statement, "It is a familiar notion that 'addition makes larger'. This is true when non-zero whole numbers are added together. Is this also true when integers are added? Please provide some examples." They were then asked, "What types of numbers can you get when one integer is subtracted from another integer?" Each of these questions sought to explore the participants' narratives about the results of addition and subtraction with integers. They were further meant to allow the participants to provide examples of computations with integers before being presented with a variety of such expressions. However, while seeking examples in either the prompts or in follow up probes, the participants were not asked to provide a comprehensive list of possible expression patterns.

Question 4. In this prompt "What roles can the symbol ' - ' take within a mathematics problem?" I designed a question to explicitly explore the key symbolic visual mediators in a discourse about integers.

Question 5. This question and its follow-up probes are considered to be four separate prompts. In prompt 5 I asked the participants "The numbers 7 and -7 are additive inverses of each other. What does the phrase additive inverse mean?" while asking this question, I presented the numbers negative seven and seven on a slip of paper. Prompt 5A followed with "And why are 7 and -7 additive inverses?", with prompt 5B I explored alternative descriptions by asking "How else would you describe two numbers that are additive inverses?" This question was concluded with prompt 5C, in which I asked, "Can you give me another pair of additive inverses?" The prompts that made up this question were designed to explore the participants' word use and narratives within mathematical discourse about integers.

Questions 6 and 7. This pair of questions each contained four parts and a total of 13 prompts. In Question 6, a quartet of expressions, which all have an equivalent value of eight were presented, namely, '6 – -2', '10 – 2', '-1 + 9', and '6 + 2'. Similarly Question 7 contains a second quartet of expressions with the equivalent value of negative six ('-4 – 2', '2 – 8', '-1 + -5', and '-7 + 1'). These expressions were designed to employ the widest possible arrangement of positive and negative integer addition and subtraction expression types, in a limited number of expressions, and employing numbers with absolute values between one and 10. All participants were presented the expressions typed on individual slips of paper, (the first or second page in Appendix B cut along the black lines); nine participants also had the complete page handed to them. The use of individual slips of paper was done to encourage the participants to work through the expressions in an order that made sense to them. I verbalized the expressions to two participants, before deciding not to read the expressions aloud to the other 12. This was to allow the participants to express the terms and operations indicated in the expression spontaneously, without being influenced by my reading. To preview, neither of the first two participants' ways of stating or approaching these expressions was affected by my verbalization.

These quartets of expressions were employed as part of a series of four prompts. In the first prompt for both Question 6 and 7 they were instructed to "Show and explain how you would evaluate each expression." These prompts were designed to explore the visual mediators they would voluntarily employ, which routines they would follow, and what narratives they would express. They further allowed an unanticipated view into the participants' word use in verbalizing and explaining the process of determining an equivalent value for the expressions, based on the expressions not containing a defining phrase, such as 'positive integer', 'negative integer', or additive inverse. The evaluations of the expressions were followed by the prompt 6B or 7B which asked "In what ways are these expressions similar? In what ways are they different?" These second prompts were designed to draw out the participants' routines and narratives related to these expressions. The remaining prompts C, "Show and explain how you would evaluate each expression using a chip model." and D, "Show and explain how you would evaluate each expression using a number line model." for Questions 6 and 7, directed the participants to execute and describe routines employing the visual mediators of beige and brown chips and integer number lines respectively.

During the interview, the participant and I were seated at the same table facing each other. I placed a stack of plain paper, a white board with an integer number line affixed along with appropriate markers, and a dozen beige and a dozen brown chips on the table. The paper was available throughout the interview, while the premade number line and the chips were only available during items requiring their use. I read aloud each prompt of the interview, except for the ' - ', and expressions which were presented on slips of paper. I made notes related to the participants' use of the chip and number lines models on the pages included in Appendix B.

A video camera mounted on a tripod was placed on the table facing the interview participant, aligned to capture her written work, as well as her actions with the chips and on the number line. A camera connected to my laptop was set up at a different angle as a back-up and to provide a second vantage point of the participant's actions. As an additional precaution, a digital voice recorder was also placed on the table to record dialogue between the participant and myself.

Data Sources

During each interview, I drew images of the participant's set up, action, and final arrangements of the chip model. Similarly, I copied the participant's number line representations onto the appropriate note sheet, shown in Appendix B. I collected any writings or drawings made by the participant during the interview. I collated the participant's written work with my notes and made a digital scan of these collective field notes. At that time, each participant was assigned a pseudonym, based on a list of most popular names from 1995, and all references to the participants made during the creation and analysis of data sources utilized these pseudonyms.

Starting in the last week in November, after having performed 13 of the interviews, I began transcribing the interviews. The transcription process occurred in several rounds. In the first round, one of the video recordings was chosen as primary, the webcam in 13 cases, and the camcorder in the other, and the time on the recording was noted, as was the speaker, and a verbatim transcription of all spoken words was made.

The other recordings were accessed to clarify uncertain statements. Due to the quiet one-on-one setting, the recordings were clear which allowed for unambiguous transcriptions. Verbatim transcription was required to determine the participants' word use, routines, and expression of narratives.

Once all words were transcribed, a second pass was made, in which notes were added describing the actions which occurred in relation to a statement by the participant. On a third pass, portions of the scans of drawings the participants and I made during the interviews were pasted into the transcript. These images, along with the notes on actions, allowed for the analysis of visual mediator use by the participants. During the fourth pass, the prompt number was identified and indicated as part of the transcript, and finally, the transcript was broken into turns in which the participant expressed a complete thought; these replaced the time on recording in sequencing the statements made by the participant.

The transcripts presented in Appendix R include the turn and prompt number as ultimately determined, as well as a verbatim transcription of the participants' speech, actions, and visual mediators. I also made screen captures of the video, to include in the main text, when they better represent the participants' use of visual mediators.

Analysis

After the third pass through the transcription process, when I had the verbatim transcriptions, description of the participants' actions, and images of their visual mediators, I started to develop codes to apply to the resulting transcripts. The process of developing codes used the constant comparison process described by Corbin and Strauss

(1990). Working through the transcripts in alphabetical order by participant pseudonym, I compared the words and short phrases, such as 'whole number', descriptions of actions, and visual mediators, with the discourse about integers discussed in Chapter 2 as the exemplar discourse. This allowed me to produce an initial set of codes for each of the elements of discourse, including things which contrasted with the exemplar discourse presented. The effort to develop a suitable list of codes, caused me to reconsider how I was operationalizing the way I would differentiate the components of the four elements of discourse. This operationalizing and differentiating was a recursive process, along with consolidating related codes and removing codes without clear utility, was one of the distinctions between the rounds of coding. The other distinction related to which element of discourse I was focusing on, especially during the first three rounds. As will be described, the coding scheme I developed was subjected to inter-rater reliability.

First round of coding scheme development. During the first part of this round, I attempted to provide initial guidelines for differentiating between each of the four elements. I also began determining codes within all four elements, based on Amanda's transcript, as hers was first alphabetically. Since the constant comparison process examines both new data in comparison with existing categories and re-examines existing categories considering new data, the starting point has no impact on the final coding scheme. For the word use coding scheme, as it was conceived, I went word by word looking for any word or short phrase capturing a mathematical idea, a name for that idea was written on my list, and an exemplar quote was copied into my coding guide. Statements related to a participant's feeling of uncertainty were included in word use during this round of code development. Visual mediators were identified as being durable representations or initiating recall of the established idea of the number line; hand motions were excluded from the start but pointing to existing expressions was included. This round of code development consisted of examining the word use, description of the participant's actions and scanned field notes. Again, I determined an initial label for this mediator and included it in the appropriate section of my coding guide. Coding for routines during this phase looked for rapid or repeated employment of an approach to determining the equivalent value of an expression. After identifying these routines, I generated a descriptive label and included the label, and an example, in the coding guide. Narratives within a discourse serve to provide the justification of the actions taken or conclusions reached. As such, these somewhat longer phrases were identified as beginning with the word 'because' or a similar structure. Also, I initially included forms of expressions offered and statements about the relationship between 'whole numbers' and 'positive integers' in the narratives category.

Feeling this process was spreading my thoughts too broadly, I decided to focus on word use first, which created the second part of this round. As such, I continued alphabetically through additional transcripts, adding examples to existing codes when a match was found, and adding new codes as required by new word usages. My list of codes for word use appeared to be complete by the time I reached the eleventh transcript. During this part of this round, I formed initial thematic groups of codes. Such groups included, for example, describing a type of number, types of symbols and implied meaning, relationship between numbers, relationship between computation, and associated with the number line.

Second round of coding scheme development. During this round, I focused on expanding the code list for visual mediators and routines, by examining additional transcripts. This resulted in the addition of many new codes to the lists produced at the end of round one, as well as supporting evidence for existing codes. Groups of related codes were also identified within the visual mediators. Within this round, several word use codes were identified as possibly not serving a role within a general discourse. While not analyzing additional data for narratives, I created a theoretical list of possible narratives which could be included in the discourse.

Third round of coding scheme development. The third round of coding involved the development of a suitable operationalization of the narrative element and expanding the list of codes for this element, confined by explicit statements to make it comprehensive. This was followed by my attempt to apply the established codes to all 14 transcripts, which confirmed the comprehensiveness of the codes. I then cleaned up this coding scheme by consolidating terms like fraction and decimal into the single code *other names for sets of numbers*. The expressions codes cross elemental lines, while *participant voiced confusion* was considered to be more of a metacognitive code about the participant's engagement with a component of the discourse. As they lack permanency, I decided to remove the act of pointing from my list of coded visual mediators. This is not meant to minimize the significance of pointing as a tool in synchronous communication, but I had to concede to the occasional lack of clarity as to what was being pointed at, and the transitory nature of pointing within the discourse itself. As I noticed duplicate and overlapping codes being present in several different elements, these were merged into single codes in the element most clearly associated with that component of the discourse. This led to 'spoke of number line' to be coded in the visual mediators, for example. This consolidation left enactment of visual mediators, such as the use of chips of only one color, and routines with chips, such as joining, in different elements of discourse. However, this approach sought to highlight the key features and contributions of the individual elements of mathematical discourse, while recognizing the interaction of those elements within the discourse. This coding scheme was then subjected to an assessment of its clarity and reliability by being employed by a second coder. Details of that process are described below.

Fourth round of coding scheme development. Following the establishment of inter-rater reliability, I reviewed the entirety of the transcripts employing the codes as refined through discussion with the second coder. Several of the codes and transcript formatting included at the end of the third round of coding were subsequently adjusted. However, when this was done efforts were made to maintain the reliability of the coding agreement, including merging codes assigned to any merged turns, Finally, while thematic groups were present in my coding scheme throughout, they were further refined and more fully conceptualized and thus took their final forms. These thematic groups provide a useful, if artificial, level of structure, between the elements of a discourse and the components that make it up. As will be examined more fully in Chapter 4, some codes were later further subdivided into particular uses of the coded word or phrase.

The interview transcripts and notes were broken into turns, a 'turn' is defined as a participant's entire response to an interview prompt. A code was indicated if it applied at least once in the turn. This led, on occasion, to multiple codable utterances being represented by a single code when those utterances occurred in a single turn. As different participants chose to revisit a prompt, or additional probing was applied, the number of turns for each participant varied from 33 up to 40 (see Table 1). The number of turns for each prompt varied from one up to 17, with 14 being most frequent. A brief description of the intention of each of the 17 codes for word use grouped thematically is provided in Table 2, while the same description along with examples drawn from the transcripts is provided in Appendix C. A similar presentation of the 19 visual mediator codes is provided in Table 3 and Appendix D, while the 14 codes in the three thematic groups for routines is found in Table 4 and Appendix E, and narratives' 22 codes in 7 thematic groups are presented in Table 5 and Appendix F. A full table of coding assignment is listed in Appendix S

Table 1 Turns taken during interview

	Amanda	Ashley	Bailey	Brooks	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae	Total
Q1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	15
Q2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	15
Q3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	14
Q4	1	1	1	1	1	1	1	1	1	1	1	1	1	1	14
Q5	1	1	1	1	1	1	1	1	1	1	1	1	1	2	15
Q5A	1	2	1	1	1	1	1	1	1	1	1	0	1	0	13
Q5B	1	1	1	1	1	1	1	1	1	1	1	1	1	1	14
Q5C	1	1	1	1	1	1	1	1	1	1	1	1	1	1	14
Q6A	0	0	0	0	0	0	0	0	1	1	0	1	0	1	4
Q6A '-1 + 9'	1	1	1	1	1	1	1	1	1	1	1	1	1	1	14
Q6A '10 – 2'	1	2	1	1	1	1	2	1	1	1	1	1	1	1	16
Q6A '6 - -2'	1	2	1	1	1	1	2	1	2	1	1	1	1	1	17
Q6A '6 + 2'	1	1	1	1	1	1	2	1	1	1	1	1	1	1	15
Q6B	1	1	1	1	1	1	1	1	1	2	1	1	1	1	15
Q6C	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1
Q6C '-1 + 9'	1	1	1	1	1	1	1	1	1	1	1	1	1	1	14
Q6C '10 – 2'	1	1	1	1	1	1	1	1	1	1	1	1	1	1	14
Q6C '6 - -2'	1	1	1	1	1	1	1	1	1	1	1	1	1	1	14
Q6C '6 + 2'	1	1	1	1	1	1	1	1	1	1	1	1	1	1	14
Q6D '-1 + 9'	1	1	1	1	1	1	1	1	1	1	1	1	1	1	14
Q6D '10 – 2'	1	1	1	1	1	1	1	1	1	1	1	1	1	1	14
Q6D '6 - -2'	1	2	1	1	1	1	1	1	2	1	1	1	1	1	16
Q6D '6 + 2'	1	1	1	1	1	1	1	1	1	1	1	1	1	1	14
Q7A '-1 + -5'	1	1	1	1	1	1	1	1	1	1	1	1	1	1	14
Q7A '2 – 8'	1	1	1	1	1	1	1	1	1	1	1	1	1	1	14
Q7A '-4 – 2'	1	1	2	1	1	1	1	1	1	1	1	1	1	1	15
Q7A '-7 + 1'	1	1	1	1	1	1	1	1	1	1	1	1	1	1	14
Q7B	1	1	1	1	1	1	1	1	1	1	1	1	1	1	14
Q7C	0	1	0	0	0	0	0	1	0	0	0	0	1	0	3
Q7C '-1 + -5'	1	1	1	1	1	1	1	1	1	1	1	1	1	1	14
Q7C '2 – 8'	1	1	1	1	1	1	1	1	1	1	1	1	1	1	14
Q7C '-4 – 2'	1	1	1	1	1	1	1	1	1	1	1	1	1	1	14
Q7C '-7 + 1'	1	1	1	1	1	1	1	1	1	1	1	1	1	1	14
Q7D '-1 + -5'	1	1	1	1	1	1	1	1	1	1	1	1	1	1	14
Q7D '2 – 8'	1	1	1	1	1	1	1	1	1	1	1	1	1	1	14
Q7D '-4 – 2'	1	2	1	1	1	1	1	1	1	1	1	1	1	1	15
Q7D '-7 + 1'	1	1	1	1	1	1	1	1	1	1	1	1	1	1	14
Total	35	40	35	34	35	33	37	36	36	35	34	33	34	36	493

Table 2 Coding scheme for word use

Code	Description
Sets of numbers	Used to identify the type or set of number being named
'Positive integer'	Participant states the phrase 'positive integer'
'Negative integer'	Participant states the phrase 'negative integer'
'Whole number'	Participant states 'whole number'
Integer	Participant states the word 'integer' aside from 'positive integer', 'negative integer'
Other names for sets of numbers	Participant states the name of a set of numbers other than 'whole numbers' and integers
Sign of number read or introduced	Used to identify the use of the exact phrase or any reference to a specific number, i.e. 4, 0, -6, as when offering or reading an expression or its equivalent value. These are not automatically considered as references to positive integers and negative integers, due to participant not explicitly referring to the values as types of integers.
Positive number	Participant states a specific positive number
Zero	Participant state "zero" when representing the solution to an expression or as a separate type of number
Negative number	Participant states a specific negative number
Role of sign verbalized or introduced	Used to denote the operation sign in the context of a spoken expression and the verb forms of the code and thus indicates intended or performed actions
Binary-operational	Used to denote the sign or operation of binary operations.
Addition	Participant states any of the verb forms of add, adding, addition, "do plus"
'Plus' as addition sign	Participant states "plus", addition sign
'Minus' as subtraction sign	Participant states "minus" between two numbers, or "subtraction sign"
Subtraction	Participant states any of the verb forms of subtraction, subtract, subtracting
'Take away' for subtraction	Participant states "take away" while performing subtraction
Signifying or mismatched	Used to denote symbols for the positive and negative nature of an integer or when the wrong word is assigned to a symbol for the role it is playing
Negative sign	Participant state "negative" not associated with a particular number
'Positive sign'	Participant states an expression contains a "positive sign"
Negative for subtraction sign	Participant stated negative prior to a negative number value while reading an expression
'Minus' for negative	Participant states "minus" either follows another operation, or does not follow another number

Some notes on this coding scheme for word use seem appropriate. The codes 'negative number/sign', 'positive number', and 'positive sign explicit' seem to present an inconsistency in this coding scheme. However, though the notation for a negative number requires the explicit use of the negative sign, the notation for a positive number does not require the positive sign, the explicit positive sign was only used for emphasis or in rephrasing an expression. Thus, the structure of this discourse suggests these codes as presented above. Additionally, the terms 'opposite', 'magnitude', and 'direction', which play so important a role in defining the discourse, were not part of the participants' spoken words in the same sense as they are used in the exemplar discourse in Chapter 2. A close read of the transcripts provided in Appendix R will reveal some of these words, but they are used in a different sense or represent artifacts of the interview structure. Any phrases which could be interpreted as being synonymous with these terms will be captured as part of the discussion by theme, in Chapter 5.

Table 3 Coding scheme for visual mediators

Code	Description
Written Expression	Used when the written expression is generated or modified by the participant during the process of solving the expression when first encountering the expression or written on the white board while modeling with the pre-made number line. ▪ *More than one may apply*
Positive number	Participant writes or modified an expression containing any positive number as a term or solution
Zero	Participant writes or modified an expression containing zero as a term or solution
Negative number/sign	Participant writes or modified an expression containing any negative number as a term or solution
Addition	Participant writes or modified an expression so that it contains an addition sign
Subtraction	Participant writes or modified an expression so that it contains a subtraction sign
Positive sign explicitly	Participant writes or modified an expression so that it contains a positive sign
Identification of ‘ - ’	Based on participant response to the Question 4 prompt
Subtraction sign	Participant stated that the “ - ” can be used to indicate subtraction
Negative sign	Participant stated that the “ - ” can be used to indicate a negative number
Chip model notations	Used in association with any time the participant employed drawn or plastic chips
Drawn	The expression is modeled using drawn circles and cross-outs
Beige and brown chips	Used in association the differentiation of the beige and brown plastic chips
One color	Chips with only one color of chips used
Separate colors	Chip model where one of the two colors available was assigned to represent the number of positives while the other color was assigned to represent the number of negatives.
Mixed colors	Chip model where the color of chips is not differentiated by the participant
Number line notations	Used in association with references to number line or symbols drawn along a number line
Participant drawn	Participant draws a number line while solving the expression
Spoke of number line	Participant made verbal reference to a number line, but did not visually employ one
Chips as colored number line	Participant lines up chips and counts along them in a manner that is similar to the use of a number line.
Dot-hops-dot	Participant indicates the movements along the number line with hops which do not clearly indicate the direction of motion after they are drawn
Left or right arrow	Participants drew an arrow with the tip at the left or right side
Numbered steps taken	Participant wrote a sequence of numbers along the number line as she simplified the expression
Dots along number line	Participant indicated steps along number line with dots

Table 4 Coding scheme for routine

Code	Description
Actions with numerals	
Portions of numbers with different signs can cancel to give sum or difference	Participant stated that the used one portion of one term to match with and cancel the other term
Adding to a negative number as subtraction	Participant stated that for an expression which has them adding to a negative number, they subtract the absolute value of the first term from the second term
Subtracting a negative is adding a positive number	Participant states that the subtraction and negative number pair becomes addition with a positive number
Subtraction as adding a negative number	Participated the original subtraction expression in terms of addition and a negative number
Actions with the chip model	
Adding as joining	Participant added chips or circles to their collection or merged two sets to display addition
Opposite colored chips stacked	Participant grouped or stacked light and dark chips to cancel each out or crossing out drawn circles. Requires the joining of sets of chips
Subtraction as taking away	Participant removed chips or circles from their collection to display subtraction
Opposite colors chips swapped	Participant replaces chip of one color with a chip of the other color
Action with number line	
Go up number line	Participant indicated that the equivalent value is up or to the right along the number line from the first value plotted
Go down number line	Participant indicated that the equivalent value is down or to the left along the number line from the first value plotted

Table 5 Coding scheme for narratives

Code	Description
Relationship between positive integers and whole numbers	Used when participant was prompted to compare positive integers and whole numbers
Are synonyms	Participants state that positive integers and whole numbers are the same thing
Mean different things	Participant identifies different types of numbers to be part of the positive integers and the whole numbers.
'Addition makes larger' remains true with integers	Used in association with the participant challenged with the claim addition makes larger
Yes, it remains true	Participant states that addition produces larger numbers
No, not always	Participant states that addition is not always associated with getting a larger number
Types of numbers generated by integer subtraction	Used in association with the participants' responses related to possible differences related to subtraction expressions
Positive number	Participant states that subtraction can result in positive numbers
Negative number	Participant states that subtraction can or cannot result in a negative number
Integer	Participant states that subtraction can result in an integer
Any number	Participant stated that subtraction can result in any type of number
Larger number	Participant states that subtraction can give you a larger number
Smaller number	Participant states that the result of subtraction is smaller than the original value
Properties of additive inverse or the pair '-7 and 7'	Used in association with probes related to additive inverses
Add to zero	Participants state that the sum of -7 and 7 is zero
Are opposites	Participant states that -7 and 7 are opposites or that one is positive while the other is negative
Operation and signifiers both need to be considered	Used in association with participant description of rules for finding equivalent values for expressions presented with addition
Adding a negative number and a positive number can be treated as subtraction	Participant started that rather than adding a negative can be treated as subtracting a positive
Subtracting a negative number can be treated as addition of a positive number	Participants states that when you subtract a negative number you can change the expression to adding the absolute value of the subtrahend
Subtracting a positive number can be treated as adding a negative number	Participants state that whenever you are subtracting a positive you can treat is as adding a negative
When you have a negative number you automatically subtract	Participant stated that the presence of a negative number indicates that subtraction is required
Symbols provide inconsistent meanings	Participant states that operations are dependent on the combination of operational sign and number value
Results of subtraction	Used in association with participant description of rules for finding equivalent values for expressions presented with subtraction
Larger number from a smaller number is not proper	Participant states that subtraction cannot involve subtracting a larger number from a smaller number
Negative number cannot be done	Participant states that subtraction of a negative number is impossible
Similarities of operations on the number line	Used in association with comments relating to the properties of operations performed with a number line
Addition	Participant stated that the number line is used the same way as in other addition expressions
Subtraction	Participant stated that the number line is used the same way as in other subtraction expressions

Reliability and Validity

The reliability of this study is founded on two key principles: thick and rich description of the exemplar discourse, coding scheme, and participants' responses, and the inter-rater reliability of the coding (Merriam, 2009). I have articulated a complete mathematical discourse about integers as framed by Sfard's definition of mathematical discourse, see Chapter 2. Earlier in this chapter I described the construction and rationale of my interview protocol based in that discourse, and my coding scheme developed through the constant comparison method in association with the exemplar discourse presented in Chapter 2. The participants' discourses are described in the next two chapters in a manner designed to provide a rich description of the participants' responses (Rubin & Rubin, 2005), initially by proceeding code-by-code through each element (see Chapter 4), they by examining the same data thematically (see Chapter 5). In each case, the participants' responses are further analyzed through comparison with standard discourse practices as exemplified in Chapter 2. The reader also has access to the code assignment by prompt for each participant in Appendix O and the interview transcript for each participant in Appendix R.

Inter-rater reliability was established by having a second coder read, analyze, and code three of the 14 interview transcripts for 21.4% of the entire interview data, and 88 of the 427 minutes of recordings for 20.6% of the total transcript duration. To ensure responses to each question prompt were given equal consideration the second coder considered complete transcripts. The second coder employed the coding guide produced at the end of the third round of coding scheme development. As noted above, the fourth round of coding involved some consolidation of codes and turns, along with the removal of certain codes.

The second coder, was an individual who is pursuing a Ph.D. in mathematics education at my institution and is an experienced doctoral research fellow. The process of establishing inter-rater reliability occurred in three segments: word use, visual mediators, and routines and narratives. This was done both to distribute the workload more evenly and to attempt to determine the reliability of different aspects of the coding scheme as it was prepared. Before she was introduced to the coding scheme I interviewed her using the full interview protocol, to familiarize her with the questions and give her a perspective on the participants' experience. We then watched the video recording of one of the interviews, read the transcript for that interview and the word use section of the coding guide, and looked at my coding of that transcript for word use. She then read over and coded three transcripts for word use. Then, I provided her with the coding guide for visual mediators, which she reviewed and used to independently code the same three transcripts she had coded for word use, but now coding for visual mediators. The process was repeated a third time for routines and narratives. Meetings between coders were held after she had coded for each element to have a discussion to reconcile the codes for the relevant element. After she had completed coding for all four elements, as well as the three non-elemental codes, she returned her coded documents. I then compared our independent code assignments to these three transcripts. Any turn could have each code unassigned, coded by both coders, coded by me only, or coded by her only. Once this code agreement was determined, we met and discussed anything which was coded by only one coder, and why the mismatch occurred. Codes were either added or removed as we came to a shared understanding as to the nature of the codes. At the end of that meeting our agreements ranged from 95.4% up to 100%. After this agreement was reached, I applied the reconciled coding scheme to the entirety of the data. As presented above, minor changes were made to the coding scheme employed by the second coder. These revisions involved removing some codes, merging other codes, and consolidating turns, in the latter cases the codes applied to the original codes or turns were merged, which in every case involved agreed-upon codes. After adjusting for these changes, inter-rater reliability as shown in Table 6 emerged. Percent agreement was determined by the total number of coded agreements divided by total number of codes assigned for that element.

Table 6 Inter-rater reliability

	Coded by both	Coded by Coder 1	Coded by Coder 2	Percent agreement
Word Use	355	7	2	355/364 = 97.53%
Visual Mediator	145	5	0	145/150 = 96.67%
Routines	72	3	1	72/76 = 94.74%
Narrative	39	0	3	39/42 = 92.86%

Validity in this type of study relates to how authentic or trustworthy the findings appear to be. Internal validity is thus enhanced by certain procedural aspects internal to the creation of the data and findings, and how widely applicable the findings can be. Internal validity was sought through the creation of questions designed in such a way that multiple items examined different aspects of each element of discourse. This alignment between questions and elements of discourse is discussed above. Further, the coding scheme was developed in relation to an externally-defined exemplar discourse while allowing for attention to individual responses. As the data was presented both in relation to each code assigned to it, and based on the prompt that elicited the response, multiple facets of each turn were examined. Thus, the characterization made of the discourse exhibited by these PSTs will be nuanced and aligned with the research framework (Merriam, 2009). External validity which reflects the extent to which this study may be transferable to other settings can best be judged by the reader after considering the similarity and differences between the situation richly described here and any other situations one would seek to relate this study to (Merriam, 2002).

Chapter 4: Results by Element

Discourse is defined in this study by four main elements: word use, visual mediators, routines, and narratives. In this chapter, I present the analyzed data grouped primarily by element of discourse, further partitioned into thematic code groups and the distinct codes used to analyze the interview transcripts for each element. Thus, within each element, the data is broken down by thematic groups with each code then discussed and examples of its use provided.

Word Use

Seventeen distinct codes were used to capture the words used by the participants during their discourses about integers. These codes were applied a total of 1,479 times over the course of 455 turns. A 'turn' is defined as a participant's entire response to an interview prompt. These codes were divided thematically into three groups: *sets of numbers* (five codes assigned 123 total times), *sign of number read or introduced* (three codes assigned 720 total times), and *role of sign verbalized or introduced* (nine codes assigned 633 total times), each of which will be considered below. An overview of the codes for this element is provided by participant (Appendix G) and by question number (Appendix H).

Sets of numbers. The sets of numbers thematic group contains the codes: *Positive integer*, *Negative integer*, *Whole number*, *Integer*, and *Other names for sets of numbers*. The application of each of these codes by participant and by question is shown in Table 7. These codes capture the participants stating the names of these sets of numbers at any point in a turn.

Table 7 Code Use for Sets of Numbers by PST and by Question

Code by PST	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Sets of numbers														
'Positive integer'	1	6	1	1		1	1	4	2	2	1	2	9	2
'Negative integer'	1	7		2			1	2	1	2	2	1	6	5
'Whole number'	1	1	1	3	1	1	5	1	4	1	2	1	3	1
Integer			1	2	1		1		2	2	2		4	1
Other names for sets of numbers		1	2	2		1	1	1	2		2	2	1	3

Code by Question	Q1	Q2	Q3	Q4	Q5	Q6A	Q6B	Q6C	Q6D	Q7A	Q7B	Q7C	Q7D	Total
Sets of numbers														
'Positive integer'	13	2	1		2			3	2	3	2	1	4	33
'Negative integer'	2	5	4	2	2	1	2	1		3	3	3	2	30
'Whole number'	14	4	1	1		2		1		2			1	26
Integer	2	3	4	1	2		1			1	1		1	16
Other names for sets of numbers	10	2	3	1			1						1	18

'Positive integer'. The code '*positive integer'* occurred during 33 turns. Further analysis showed these occurrences could be divided into four types: definition of this type of number (n=13), to describe a general category of numbers (n=10), the cardinality of the number of chips associated with the magnitude of the positive integer (n=4), and the direction and magnitude of a positive integer when modeled on a number line (n=6).

The use of this phrase varied widely between participant, with Rebecca (n=9) and Ashley (n=6) using the phrase most frequently, and Christina (n=0) never using it. The full breakdown of the participants' use of this phrase is shown in Table 7. The most frequent application of the phrase 'positive integer' occurred in response to Question 1 (n=13); this represented the only time the phrase was used for Amanda, Bailey, Brooke, Courtney, Jacqueline, and Monique. As Question 1 directed the participants to compare positive integers and whole numbers, the way each of these participants employ this phrase will be explored in the first section of Chapter 5. In addition to Question 1, Jordan, Kayla, Nicole, and Tae used the phrase in answering one other question. Jordan, Kayla, and Nicole used 'positive integer' to describe a general category of numbers, while Tae used it in association with chip color. This usage suggests that these four PSTs would make some spontaneous use of the phrase during their discourses about integers, which shows a deeper command of the discourse, when compared to those who never spontaneously employed the phrase. Three of the participants, Jamie (n=4), Ashley (n=6), and Rebecca (n=9), used the phrase 'positive integer' freely. These participants show an increasing range and depth of inclusion of this phrase in their discourse about integers. Overall, this limited use of the phrase (P = 33/448 = 7.37% for turns coded for Word Use) suggest that these participants did not make overarching statements about the nature and behavior of positive integers, except where specifically asked, and not always, even when prompted.

The code *positive integer* was used 12 times to name a set of numbers which the participants were asked to define in Question 1. In defining 'positive integer' Amanda, Jacqueline, and Tae each made similar statements that a "positive integer's anything like, on a number line, it's anything after zero" (Amanda), and "a positive integer would be anything that is over, above, zero" (Jacqueline). These examples show that these participants understand positive integers to be numbers greater than, however, these statements do not include the fact that positive integers only exist in contrast to negative integers. In a superficially similar statement by Monique, "positive integers are, numbers that are from zero to a hundred and beyond" we see her incorrectly including zero as one of the positive integers. Bailey and Brooke independently provided this pair of responses: "positive integers are like rational numbers I think, that word, irrational numbers" (Bailey) and "positive integers can have a decimal or a fraction of a whole number" (Brooke). These responses suggest that these PSTs understand positive integers to be the same as rational numbers or decimals, in contrast to whole numbers which are not rational numbers or decimals. While the definitions presented by Bailey and Brooke contain aspects which are accurate for real numbers, they are entirely inaccurate for 'positive integers' since rational numbers are defined in terms of integers. Courtney's statement that with whole numbers you "could still do a fraction, so it would be like twelve out of twelve or like ten out of ten, where positive integers is just like seven" misrepresents both 'whole numbers' and 'positive integer'. Courtney misrepresents whole numbers by indicating that they must be in fractional form, while with positive integers she does not indicate that they exist in association with negative integers. These quotes show the variation in how these participants define 'positive integer', each of which deviates from the way it is defined in the section about Mathematical Discourse About Integers in Chapter 2.

The structure of Questions 2 through 5 explored the PSTs' discourse related to the properties of integers. Within this context, five of the participants (Ashley, Jamie, Jordan, Nicole, and Rebecca) employed the phrase 'positive integer' to describe a general category of numbers, instead of referencing one specific value such as positive seven, being used in an expression. In response to the prompt in Question 2 about how 'addition makes larger' applies to integers, Jordan's statement included:

> when you add one and one together you're making a larger number, but if you add two negatives, you're making a lesser number, so I guess you're not always making a larger number with integers, since integers include positives and negatives.

In this statement, Jordan used the number one as a specific positive integer and concluded with a reference to all integers greater than zero. Nicole uses 'positive integer' similarly when she said "Yes, because no matter if you're adding negative integers or positive integers you're always going to a higher number." In Nicole's response, she indicated that the sum of two negative integers is a larger number when it is in reality a smaller value, though larger absolute value, due to the structure of the integers. Similarly, in her response to Question 3, concerning subtraction with integers, Rebecca used 'positive integer' in this way, "if you're using two positive integers, you can get a positive or a negative number depending on how large the integers are." Jordan's and Rebecca's responses highlight the fact that the type, or magnitude, of the numbers included in an expression affect whether the sum or difference is larger than the original terms in the expression or not, while Nicole's statement indicates that this does not matter. When considered together, these responses suggest that these PSTs are capable of employing 'positive integers' to describe a general category of numbers, while describing properties of numbers or operations while engaged in discourse about integers, without exclusively resorting to specific examples.

A more specific use of the label positive integers, to refer to specific numbers, rather than the general category as provided by Kayla and Rebecca when responding to the question regarding additive inverse using the values -7 and 7. They made the statements: "it means that it's the same integer but one's positive and one's negative" (Kayla, Q5) and "this is a positive integer of seven" (Rebecca, Q5A). These statements by Kayla and Rebecca suggest that numbers of the same magnitude exist both above and below zero. This contrasts with the idea described in the previous paragraph, since in these situations, the participants referenced individual values.

Questions 6 and 7 asked the participants to evaluate and compare two sets of four expressions. While addressing these questions, the participants continued to use 'positive integer' to describe a general category of numbers. While considering her initial approach to '-4 – 2' Ashley stated "four negative integers. One, two, three, four. And I would be subtracting two, and two, yeah, two is a positive integer," identifying the contrasting nature of the two values in the expression. When discussing the expression '-7 + 1', Jamie stated that she noticed "when you add a positive integer and a negative integer it becomes a smaller negative number." Jamie's statement is only true when the negative integer has a larger absolute value than the positive integer, zero and positive integers are possible when the relative absolute values of the positive and negative integers are equal or reversed. Similarly, while considering the expressions '2 – 8' and '-7 + 1', respectively, Rebecca pointed out that, "this ['2 – 8'] only has positive integers involved whereas this ['-7 + 1' has] a negative integer," while neglecting to point out the presence of a positive integer in the second expression. A description of the different ways in which positive and negative integers behave in an expression was provided by Ashley after evaluating the expression '-4 – 2' when she said, "sometimes you would have to do the opposite of what the symbol is asking you to do … depending on if the number you are adding, or subtracting is a negative or a positive integer." In each of these instances, the participant was modeling the process of determining the equivalent value to expressions which were provided. In responding to each prompt, these participants opted to verbally identified the presence of a 'positive integer' among the terms. This verbalization indicates that these PSTs were calling attention to effects that occur any time a positive integer is present in an expression with that operation.

The final two ways that the phrase 'positive integer' was used both involve visual mediators, specifically chips and number lines. In four cases, three different individuals associated a chip color as representing positive integers. While preparing to evaluate '-1 + 9' using the colored chip model, Tae described sorting the brown and beige chips in the following manner, "I am using this to represent negative integers and this to represent positive integers." Similarly, while addressing '6 + 2' with the colored chip model, Ashley described her actions in this way: "I would just add two of the brown chips to get eight brown chips or eight positive integers." These statements suggest that the PSTs used the phrase 'positive integer' in a way that voices their assignment of specific chip colors to the cardinality of either positive or negative integers.

Two participants described movement to the right along the number line a distance by stating the phrase 'positive integer' six times each. This is captured in Rebecca's statement "because it's a plus I'll be moving and a positive integer I'll be moving to the right one" while working to evaluate '-7 + 1' on a number line. Similarly, Jamie approached this same expression by stating, "So we're at negative seven right here and we're adding a positive integer to it so it's going to make the number bigger and closer to zero this one, so we are going to add a positive one which will move it to the right." These responses, when considered along with the frequency with which Jamie and Rebecca used 'positive integer', indicate that their discourse about integers incorporates identification of the presence of 'positive integers' and their associated behavior, such as a shift to the right on the number line, in expressions and models in a way that is consistent with the discourse described in Chapter 2, as well as employing the term more often than their peers.

The number of different ways that 'positive integer' was employed by the participants, combined with the wide range of number of uses of this phrase by participants, from zero to nine such uses, makes it impossible to make a single general statement describing all of the PSTs in this study. However, half of the participants (n=7) only used the expression at most one time, while four others made a second reference to the phrase. This implies that only three of the participants, Ashley, Jamie, and Rebecca, have integrated the phrase 'positive integer' into their discourse about integers in a spontaneous and rich manner. When considering their use of the phrase 'positive integer' throughout the interview, different levels of consistency with established discourse practices were observed. However, these levels of consistency represent observations based on different numbers of occurrences. Ashley, Nicole, Rebecca, and Tae employed 'positive integer' in a way that was consistent with common practice. Most of the uses of 'positive integer' by Jamie, Jordan, and Kayla were determined to be consistent with common practice, after providing definitions which were partially consistent. Amanda and Jacqueline were also partially consistent, while Bailey, Brooke, and Courtney defined 'positive integer' in a manner which is inconsistent with the exemplar discourse described in Chapter 2. Christina did not employ the term and was characterized as not having exhibited this component of a mathematical discourse about integers.

Negative integer. The structure of the interview did not prompt the students to use the phrase 'negative integer,' however 11 of them did, for a total of 30 turns (see Table 7). As with *positive integer*, Ashley (n=7), Rebecca (n=6), and Tae (n=5) used the phrase 'negative integer' most often. Four of the participants, Brooke, Jamie, Kayla, and Monique used the phrase twice, while another four, Amanda, Jacqueline, Jordan, and Nicole only used it once. Bailey, Christina, and Courtney never used the phrase.

The code *negative integer*, when subjected to further analysis, can be broken down into the same uses as were noted for *positive integer*. 'Negative integer' was used as part of a definition of a set of numbers (n=2), to describe a general category of numbers (n=20), the cardinality of the number of chips associated with the magnitude of the negative integer (n=7), and the direction and magnitude of a negative value when modeled on a number line (n=1). These uses of 'negative integer' complement those of 'positive integer' examined above, except in the additional use of negative integer in Question 4. While answering Question 4, about the role ' - ' can take, Amanda and Monique employed 'negative integer' to describe a general category of numbers in the following ways, "It can be in front of a number and it can make the number negative so it can be a negative integer." and "It could just signify a negative integer." These quotes show that these participants grasp the relationship between the symbol and the set of negative integers.

When considering their use of the phrase 'negative integer' throughout the interview, different levels of consistency with established discourse practices were observed. As with 'positive integers', these levels of consistency represent observations based on different numbers of occurrences. Amanda, Ashley, Brooke, Jacqueline, Jamie, Jordan, Kayla, Monique, Nicole, and Tae employed 'negative integer' in a way that was consistent with common practice, any time they used the phrase when only word use was considered. Five of Rebecca's six uses of 'negative integer' were determined to be consistent with common practice, with her one partially consistent use involved her claim that both terms in an expression were 'negative integers' when only one actually was. Bailey, Christina, and Courtney did not employ the term, and will be characterized as not having exhibited this component of a mathematical discourse about integers.

Whole numbers. The phrase '*whole number*' was used during 26 turns in four different ways. The most frequent way was in defining the term 'whole number', which was done by all 14 participants. It was also employed to describe a general category of numbers (n=10), the cardinality of the number of chips associated with the magnitude of a whole number (n=1), and the direction and magnitude of a positive integer when modeled on a number line (n=1). With five participants (Brooke, Jacqueline, Jordan, Monique, and Rebecca) using the phrase in one or more of the other ways. This includes Jordan's use of it with the chip model, and Jacqueline's use of it while modeling with a number line (see Table 7).

The use of the phrase '*whole number*' as part of a definition can be seen in Tae's statement, "I would say that whole numbers include all the numbers on the number line, but positive integers are only to the right of zero on the number line," in which she is comparing the properties of whole numbers and positive integers. This statement can be misleading based on what is meant by the two uses of number line, since, for example, if they both refer to an integer number line, her definition of 'whole number' is incorrect. The other participants used the phrase in a widely varying manner as part of their definitions of this set of numbers, which I have described in more detail in Chapter 5.

Another use of this phrase is to identify the presence of a general category of the cardinal numbers greater than or equal to zero in a computation. Rebecca did so in the first part of her response to the prompt regarding addition making larger, which is shown in this passage:

> addition makes larger is misleading when we're working with integers. For instance, if we are working with simply whole numbers, whole numbers need to be positive so if we're working with four plus three we see that they're both positive numbers and they're going to equal seven.

This passage includes '*whole numbers*', to describe a general category of numbers greater than or equal to zero, which are then translated into numbers within an example expression. This use of the phrase was distributed over responses to seven different prompts. The use of the code 'whole number' in this manner occurred 10 times during the 14 interviews. This shows that these participants did not employ a discourse which frequently includes a statement about 'whole numbers' being included in the expression, which would be considered as consistent with a discourse about integers, if the phrase 'positive integer' was used more frequently for this purpose.

The other two uses of 'whole number' were seen while participants employed visual mediators. As Jordan was modeling '-1 + 9' using colored chips, she stated "I add this whole number from the nine, well, one, it takes place of the negative, and then I only have, since I used that one, I have eight." Jordan represented the negative one with a beige chip, while representing the whole numbers with brown chips. While making this statement, she swapped a brown chip for the beige chip, which canceled them out, leaving her with eight brown, 'whole number' chips. In addition to being hard to follow, this description inappropriately mixes negative integers with whole numbers, since negative integers properly pair with positive integers in a discourse about integers. Similarly, while addressing '-7 + 1' using the number line Jacqueline stated, "Negative seven plus one … you start at negative seven and you add one whole number. So that would be negative six." While being easier to follow, Jacqueline's description, again, inappropriately mixes negative integers with whole numbers, since the names for these sets of numbers are drawn from different discourses. This misuse, if more frequent, could lead to these PSTs engaging in discourse that is misleading about both the nature of integers and 'whole numbers'.

These 26 identified uses of '*whole number*' include using the phrase as part of a statement where a definition of the phrase was provided, to describe a general category of numbers, in the identification of the meaning of the brown chips, or movement a certain distance to the right on the number line. While an assessment of the definitions provided by these PSTs will be offered in Chapter 5, we can see that the 'whole numbers' were identified as terms in expressions in far fewer than the already limited number of opportunities presented. In addition to any expressions the PSTs chose to introduce, only '6 + 2' and '10 – 2' could be considered as expressions within a discourse about whole numbers as well as a discourse about integers. Further, the uses of the phrase while describing the workings of the visual mediators were shown to be misaligned with the integer values, and cardinal or ordinal representations of those values. Thus, when considering their use of the phrase 'whole numbers' throughout the interview, different levels of consistency with established discourse practices were observed. These levels of consistency represent observations based on different numbers of occurrences. Amanda and Christina employed 'whole number' in a way that was consistent with common practice, any time they used the phrase. Both Brooke and Jordan were noted as using the phrase in ways that were consistent, partially consistent, and inconsistent at different points in their interviews. Jacqueline defined 'whole number' in a consistent manner, then referred to numbers in expressions in an inconsistent manner, while Monique and Rebecca reversed this order. The seven remaining PSTs used the phrase in a manner that was inconsistent with the discourses defined in Chapter 2.

Integers. The participants use 'integer' without reference to positive or negative in 16 turns. This includes its use in defining the term (n=2), as a general category of numbers (n=13), and the direction and magnitude of an integer when modeled on a number line (n=1). The term was used most frequently by Rebecca (n=4), with Brooke, Jordan, Kayla, and Monique each using the term twice. Bailey, Christina, Jacqueline, and

Tae each used the expression once. Amanda, Ashley, Courtney, Jamie, and Nicole never used the term. As can be seen in Table 7, this term is used most frequently in Questions 2 (n=3) and 3 (n=4) where the participants were asked to describe the results of adding or subtracting integers. One such use of 'integer' to describe a general category of numbers was by Kayla as part of this exchange where we are discussing the types of numbers that can result when integers are subtracted in Question 3:

> Kayla: "What types of numbers do you want? Other integers, like smaller integers?"
>
> Doug: "Ok, so when integers are subtracted you said smaller, does that mean other smaller positive integers?"
>
> Kayla: "Well it could be either, because if you subtract a larger integer from, like, if you subtracted six from two it would be negative four, so it would be negative, a negative integer.
>
> Doug: "Ok, can you get a positive integer when you subtract integers?"
>
> Kayla: "Yeah, like if you subtract a smaller number from a bigger number. It is still positive."

In this passage, Kayla provided accurate examples of expressions and arrangements of relative absolute values which produce positive or negative integers, however, she does not provide an expression with a difference of zero, or a comprehensive list of integer subtraction expressions which result in either positive or negative integers.

The use of this term while responding to other questions was primarily to represent any integer value and the participants employed the term mainly to describe a general category of numbers. One of the times Rebecca used 'integer' in this way was when she stated that "just because the integers are positive, that does not necessarily mean you will be getting a positive result" while discussing the expression '2 – 8'. When considering the PSTs' use of the term 'integer' throughout the interview, different levels of consistency with established discourse practices were observed. As noted Amanda, Ashley, Courtney, Jamie, and Nicole never used the term and are considered to have not exhibited the term integer. Jacqueline and Tae used the term in a manner which was consistent with discourse practices. Monique and Nicole used the term in a manner consistent with the discourse community in one response, while being only partially consistent with such practices in another. Jordan and Kayla used the term in a consistent manner in one response then in an inconsistent manner in another. Bailey, Brooke, and Christina used the term only in a manner that was inconsistent with the wider community. These levels of consistency represent observations based on different numbers of occurrences.

Other names for sets of numbers. This code (n=19) captures the wide range of number types named by the participants. These include references to fractions, decimals, rational numbers, irrational numbers, and real numbers, as well as created types of numbers such as "rational whole number" by Nicole, "whole positive numbers" by Jordan, and "whole integers" by Bailey. Ten of the uses of 'other names of sets of numbers' were part of the participants' responses to Question 1 and will be discussed in Chapter 5.

In response to Question 2, PSTs had a range of responses that referred to other sets of numbers. Nicole described addition with positive integers as moving the sum towards "positive infinity," an appropriate type of number in a discourse about integers. Tae, however, offered fractions in her expressions "adding one seventh and three sevenths you get four sevenths which is more than both" and "add three sevenths plus negative one seventh so this would actually be two sevenths." Both of Tae's expressions contained numbers which would be appropriate in a mathematical discourse about rational or real numbers, but do not represent integers. When asked about the results of subtracting integers, Tae responded that "any number" was possible, Brooke stated, "you can also have integers meaning decimals or fractions of a whole number," and Monique independently answered, "you can get fractions" and "you can get a decimal." Each of these responses include numbers which are not included in a discourse about integers and would be most appropriate in a discourse about rational and irrational numbers or real numbers. While a complete characterization of these PSTs' use of these terms would be inappropriate at this point, I will note that many of their uses of these terms are consistent with a discourse about real numbers. When consideration is limited to a discourse about integers Nicole, had a pair of partially consistent uses in her responses, while Tae had an inconsistent as well as a consistent response. Three participants Amanda, Christina, and Kayla did not exhibit a term included within this code. The other nine participants made statements which contained types of numbers which are not typically considered with discourse about integers. In this situation, the non-use of a term indicates that the participants were not inconsistent, for this component, with a discourse about integers.

Summary. In this section I have discussed the ways the phrases ‘positive integer,’ ‘negative integer,’ and ‘whole number,’ as well as phrases coded as *integer*, and *other names for sets of numbers* are used in similar and notably contrasting ways. The presence of each of these sets of numbers in the responses to Question 1, links them to their use in defining ‘positive integer’ and ‘whole number’ which is further explored in Chapter 5.

While it would be compelling to make strong claims about the participants’ use of the phrases in this thematic group, that is not possible since only the use of each term or phrase was considered within each turn, meaning the turns were not compared holistically, an analysis which is performed in Chapter 5. Additionally, while these terms and phrases range from being consistent to inconsistent in relation to word use, they may form part of routines or narratives which display their own degree of consistency. However, it can be noted that no PST was uniformly consistent, and Bailey and Courtney were never consistent, nor even partially consistent, in their use of these phrases and terms. All the other PSTs employed at least one term or phrase in a manner that was consistent with common practice, as I describe it in Chapter 2.

Sign of number read or introduced. This code group contains the codes *positive number* (n=371), *negative number* (n=321), and *zero* (n=28). The codes *positive number* and *negative number* are the most frequently identified word use codes, representing 25.14% (P=371/1476) and 21.75% (P=321/1476) of the word use codes assigned respectively, with zero contributing 1.90% (P=28/1476). These codes represent the participants’ statement of any number while responding to a prompt, which can take the form of them suggesting an expression or value, verbalizing the expression presented, providing a response to that expression, or rephrasing the expression presented in the prompt. Table 8 shows the distribution of these three codes by participant and by question.

Table 8 Code Use for Sign of Number Read or Introduced by PST and by Question

Code by PST	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Sign of number read or introduced														
Positive number	27	26	25	26	28	21	27	30	31	27	24	28	28	23
Zero	3	3	2	1		1	3	2	2	2	2	5	2	
Negative number	25	23	20	25	22	17	24	24	28	26	20	26	24	17

Code by Question	Q1	Q2	Q3	Q4	Q5	Q6A	Q6B	Q6C	Q6D	Q7A	Q7B	Q7C	Q7D	Total
Sign of number read or introduced														
Positive number	6	10	13	4	34	55	14	49	55	44	8	39	40	371
Zero	1	3	2	0	16	1	1	1	0	1	2	0	0	28
Negative number	5	15	13	3	32	23	11	22	31	52	12	50	52	321

Positive number and/or negative number. These codes will be considered together, as they are closely related and intertwined. One or both of these codes appears in 420 of 448 turns coded for word use, (P=93.75%). They appear together in 270 turns, with *positive number* occurring alone in 101 turns and *negative number* occurring alone in 51 turns. These phrases occurred in response to every question and were stated by all 14 participants. However, the participants’ use of the phrase ‘positive number’, or their reading or stating a positive number, ranged from Courtney’s 21 times up to Jordan’s 31 times. Similarly, the use of ‘negative number’ ranged from a minimum of 17 uses by Courtney and Tae, to a maximum of 28 by Jordan. These ranges suggest that while each participant used the terms effectively to convey the composition of the expressions and communicate their thinking about those expression, some read the expression presented, offered expressions containing particular values, and spoke their responses much more freely than others.

Further analysis suggests three ways in which the participants employed *positive numbers* or *negative numbers*. One was as a phrase representing a particular number or all numbers greater than zero (n=25) or less than zero (n=49). A second use was in verbalizing a particular positive number (n=301) or negative number (n=217) offered as part of the question task; the latter two counts include 175 times when both types of numbers were stated in the same turn. The participants also introduced positive numbers (n=132), negative numbers (n=132), or both (n=65) in a single turn to provide expressions which satisfied certain conditions (Questions 2 and 3), to introduce particular values (Question 5), to modify the expression provided as part of the solution process (Questions 6 and 7), or to provide the equivalent value to an expression (Questions 2, 3, 6, and 7).

The participants employed the phrase 'positive number' or 'negative number' to represent numbers above and below zero, in a way that roughly parallels the way 'positive integer' and 'negative integer' were used. Jamie's response to Question 3, about the results of subtraction, provides an example which shows the contrast between the use of the phrase to describe a general category of numbers, with the provision of specific values above or below zero. Jamie said, "If you were doing two positive numbers, so you're doing like ten minus three you'd get seven and that's a positive number," and continued with "three minus seven and you would get a negative number. You would get a negative four."

In Questions 5, 5A, and 5B, the participants were presented with the numbers seven and negative seven, and asked questions about the properties of additive inverses. In response to this question, Brooke explained that, "the opposite of the negative seven would be a positive seven," clearly stating both numbers. While responding to these items, the participants either mentioned both numbers (n=19), or neither number (n=23). This use of both numbers is consistent with established discourse practices, in relation to the term additive inverse within a discourse about integers.

The participants chose to verbalize the numbers in the expressions for Questions 6 and 7 at almost every opportunity (n=311 of 336 opportunities, p=92.56%). Note, in three of the cases the expression provided contained only positive numbers, ('6 + 2', '10 – 2', and '2 – 8'), four of the expressions contained both positive and negative numbers ('-1 + 9', '6 – -2', '-7 + 1', and '-4 – 2'), and the final expression contained only negative numbers, ('-1 + -5'). Each of these cases will be considered in turn.

Typical exemplars of verbalizing an expression that only contained positive numbers as terms, "ten minus two is already a positive eight" by Courtney, and "if you add six and two that would be eight again" (Kayla). Courtney is quoted implicitly indicating two positive numbers as her terms, then she indicated the sum explicitly as being positive. Kayla implicitly identified positive numbers for both the terms and the sum. Since it is a conventional practice in discourse about integers to assume a number is positive unless negative is stated with the number, this implicit indication of the positive nature of the numbers is appropriate. These correct exemplars contrast with the occasional misstatement like this one by Courtney, which she quickly rectified, "You would do two and negative eight, two minus eight would be negative six." Courtney's slip was atypical, and the participants were able to clearly articulate the numbers presented in these examples.

The participants were less consistent when verbalizing expressions containing both a positive number and a negative number as they were written, since they often converted the expressions into a different form. This inconsistency may be derived in part from the fact that while they were directed to model the expressions as they would to a class of seventh grade students, they were not directly asked to read each expression aloud. An example of the rephrasing of an expression without verbalizing the written form is provided by Jacqueline when she declared, "when we have subtraction and a negative together they combine to make positive, so six plus two," when considering the expression '6 – -2'. While the expression was on the table between Jacqueline and myself, her lack of verbalization of the initial expression would provide no indication of context for the subtraction and negative she is referring to. Further, her statement is incorrect when she refers to the subtraction and negative making a positive, as she left off the operation now being considered, an omission partially remedied in her final clause. More commonly, the participants verbalized the expression as written, as when Amanda stated, "Negative seven plus one, really it is just seven minus one." In this example, and almost all the turns where the participant verbalized the written expression, they stated the expression properly, then as Amanda's example highlights, they frequently rephrased the expression into a form which was easier for them to handle. More discussion on their approaches to determining the equivalent values, including modifying the proffered expression, will be provided in Chapter 5.

Only with the expression '-1 + -5' were both terms negative numbers. Similar to other participants, Bailey read the expression correctly on her first encounter, "you have negative one, you add negative five and then you have negative six." However, later she misstates the expression in this way, "Negative one, plus five, start here and go up negative five numbers." Though there were a few cases where the participant misidentified the numbers in the expression, the participants were remarkably consistent in reading and verbalizing the expressions properly. This lays a solid foundation for their future students' discourse about integers to be built.

The final situation in which *positive numbers* and *negative numbers* were found was when the participant introduced them. This was done in rephrasing and identifying an equivalent value to an existing expression, to suggest expressions or individual values in response to Questions 1, 2, 3, and 4, or stating a pair of numbers with the same properties and relationship as negative seven and seven in Question 5C. Exemplars and discussion of each of these situations are provided in turn here, while a fuller investigation of the expression suggested and the nature of the rephrasing of existing expressions is found in Chapter 5.

Recall, the participants were asked to show how they would model the expressions presented in Questions 6 and 7, and they often rephrased the expression provided. This can be seen in the quotes from Amanda and Jacqueline provided above.

Additionally, the participants verbalized an equivalent value for each of the expressions they interacted with, except for a few times where they were unable to proceed with the model. Some examples of this can be found in the quote from Courtney, seen above, as well as the value eight in this quote by Bailey, "For negative one plus nine, you start at negative one, and you go up nine. And that is eight."

When asked to provide examples of addition expressions with integers, seven of the participants included both positive and negative numbers in the expressions they introduced. One example of how this was done is provided in Brooke's response, "negative six plus four, … which would give me negative two." In contrast with the presence of positive and negative numbers in the expressions offered by these seven participants (Amanda, Brooke, Jamie, Monique, Nicole, Rebecca, and Tae), Jordan and Christina provided examples which used only positive numbers. Christina's statement, "two plus two equals four," is an example of the use of only positive numbers. As exemplified by Jacqueline's offering, "negative five plus negative three you would get negative eight", Ashley, Courtney, Jacqueline, and Kayla, included only negative numbers in their expressions. Additionally, Bailey chose to write the expression "-1 + 1 = 0." Each of the participants who employed only 'positive numbers' or 'negative numbers' effectively communicated their thinking within the range of values they utilized. In response to being asked for examples to support their claims about what types of numbers can result from subtraction, nine of the participants (Amanda, Bailey, Christina, Courtney, Jamie, Kayla, Nicole, Rebecca, and Tae) provided expressions which contained both a positive number and a negative number, the other five (Ashley, Brooke, Jacqueline, Jordan, and Monique), did not articulate a response containing numbers.

While most of the time each participant employed 'positive number' and 'negative number' in a manner consistent with standard discourse practices, there were times when participants made mistakes, some of which are noted above. The participants' readiness to include both 'positive number' and 'negative number' in expressions they introduced as examples, which were only partially consistent with full discourse about integers or were considered not to have been exhibited. Those who only included both positive and negative numbers in response to one prompt regarding the results of adding or subtracting integers were considered as partially consistent, those who chose to provide examples to neither were considered not to have exhibited this aspect of the discourse. This later aspect is critical, as selection of examples is a significant part of introducing a mathematical discourse, and a key part of PSTs' discourse about integers. Due to the small number of mistakes in identifying the sign of the number which they read or introduced in relation to the number of times these PSTs' read the numbers properly, I decided not to identify any participant as displaying a discourse which is inconsistent with common practice.

Zero. In contrast to *positive numbers* and *negative numbers*, *zero* entered this conversation only when the participant introduced it. All participants except Christina and Tae introduced zero at some point in the interview (see Table 8). Sixteen of these occurrences were in response to Question 5, eight of which came when the participants associated the sum of the additive inverses seven and negative seven and zero. For example, “if you add the positive to the negative you get the zero” (Nicole). In addition, Amanda noted that seven and negative seven are, “both seven away, they are both the same distance away from zero … on the number line,” which is an idea also held by Bailey and Rebecca. In general, the few uses of zero indicate that the participants were able to see it as the sum of additive inverses in the context of Question 5, and as a location on the number line that separates the negative numbers from the positive numbers. Thus, the PSTs were rated as either introducing zero in a manner that was consistent with discourse practices, or they did not exhibit zero in their discourse.

Summary. In this analysis I have shown that while *positive numbers, negative numbers,* and *zero* were used throughout the interviews, their distribution is not uniform over either the participants or questions. More importantly, utterances coded in this way take several different forms: as phrases, as introduced values, and in verbalizing what was offered to the participant. Within each of these uses, the participants were seen to employ *positive numbers* and *negative numbers* in a manner consistent with community practices while engaging in discourse about integers. Some of the PSTs were rated as partially consistent with standard discourse practices, because of their use of only ‘positive number’ or ‘negative number’ in the expressions they chose to offer, as they were able to read the expressions presented properly, except in a very limited number of instances. These instances are further examined in Chapter 5 and can be found in the transcripts in Appendix R.

Role of sign read or introduced. This code group contains the codes *addition*, *‘plus’ as addition sign*, *‘minus’ as subtraction sign*, *subtraction*, and ‘*take away’ for* *subtraction*, '*negative sign*', *'positive sign', negative for subtraction sign, and 'minus' for negative*. The first five of these codes can be considered as binary-operational codes, while the last four, '*negative sign'*, *'positive sign', negative for subtraction sign, and 'minus' for negative*, serve as signifying or mismatched codes. The binary-operational codes are used to identify cases where an operation is to be performed using the numbers before and after the signifier. The signifying or mismatched codes represent the sign attached to a numeral to indicate a positive integer or negative integer, or the naming of a sign as indicating a role which is different from convention, or unambiguous identification. The binary-operational codes were used in 374 turns, signifying or mismatched codes in only 20 turns, see Table 9. Each of these subgroups will be considered in turn: the binary-operational codes mostly in a quantitative manner, while the signifying or mismatched codes will be expounded upon more richly.

Binary-operational codes for role of sign read or introduced. The addition of a pair of numbers was clearly indicated in Questions 2 and 4, and in association with the four expressions '-1 + 9', '6 + 2', '-1 + -5', and '-7 + 1', which provided at least 18 opportunities for each participant to use '*plus*' or *addition* for a total of at least 252 turns. Similarly, subtraction is implied or requested in Question 3, and in association with the expressions '6 - -2', '10 – 2', '-4 – 2', and '2 – 8', which provided at least 15 opportunities for each participant to use '*minus*', *subtraction*, or '*take away*' for a total of at least 210 turns. As seen in Table 9,*'plus' as addition sign* (n=150), *'minus' as subtraction sign* (n=132), and *subtraction* (n=99) were used by every participant to varying degrees. The more numerous *addition* (n=167) was used by all participants except Courtney, while '*take away*' (n=59) is used by 11 participants at least once.

Table 9 Code use for Role of Sign Read or Introduced by PST and by Question

Code by PST	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Role of sign read or introduced														
Binary-operational														
Addition	13	16	9	9	9		11	20	16	15	10	17	11	11
'Plus' as addition sign	11	12	9	10	13	7	16	14	7	13	9	12	10	9
'Minus' as subtraction sign	11	10	9	8	8	4	13	15	4	9	4	11	14	10
Subtraction	10	12	5	8	7	1	3	4	9	13	1	11	9	6
'Take away' for subtraction	2	6	4	1	6		7	9	11	2	10		1	
Signifying or mismatched														
Negative sign	1	2		1	1		3				2	2	5	
'Positive sign'											1	2	1	
Negative for subtraction sign								1				2	1	
'Minus' for negative									1					

Code by Question	Q1	Q2	Q3	Q4	Q5	Q6A	Q6B	Q6C	Q6D	Q7A	Q7B	Q7C	Q7D	Total
Role of sign read or introduced														
Binary-operational														
Addition		10	1		16	15	9	24	21	12	11	25	23	167
'Plus' as addition sign		11	1		2	29	5	12	23	27	5	20	17	152
'Minus' as subtraction sign			7			25	5	14	16	26	4	17	16	130
Subtraction		1	8		4	13	10	13	9	13	5	9	14	99
'Take away' for subtraction				3	1	11	1	14	7	6	3	6	7	59
Signifying or mismatched														
Negative sign				1	3	6	2	1		1	2	1		17
'Positive sign'					2	1					1			4
Negative for subtraction sign					1	1	1						1	4
'Minus' for negative					1									1

Though these terms were used by almost every participant at one point or another, their use was not uniform. The use of the word *addition* to signify an operation ranged from zero utterances for Courtney to 20 by Jamie. The word '*plus*' was used as an addition sign, with a range of seven times each by Courtney and Jordan, up to 16 times by Jacqueline, while the word '*minus*' was used as a subtraction sign, least often by Courtney, Jordan and Monique (n=4), while Jamie used it most often (n=15). The operational signifier *subtraction* was used once each by Courtney and Monique while Kayla (n=13) used it most often; the phrase '*take away*' was never used by three participants, while being used 11 times by Jordan, who represented the maximum. These numbers indicate that the participants regularly verbalized the expression they were considering. While trying to not over-state the point, those PSTs who more frequently verbalized the operation being considered, and thereby stated the expression in full, tended to engage in discourse which was easier to follow with regards to which expression was being considered, though not necessarily how they were approaching the expression.

The codes in the *role of sign read or introduced* thematic group were used by the participants to either verbalize the nature or role of the symbols they were seeing in a pre-existing expression, or the operation they were introducing; this distinction is shown in Table 10. In Questions 2, 3, and 5 only the latter case was present (n=21), as no expression was provided. Jamie's response to Question 2 can serve as an example of how '*plus*' was used to verbalize an addition sign: "let's say you were adding negative two plus negative two, you get negative four." In an analogous way, Tae's response to Question 3 is typical, "you can do seven minus nine, would give you a negative integer" and "eleven minus nine and you get two." These show that the participants were able to introduce an expression including an operation.

Table 10 Primary Codes for Role of Sign for Questions 2, 3, and 5.

	Role Introduced in New Expression			Introduced operation		
	'Plus', and 'Minus'	'Plus', only	'Minus', only	Addition and Subtraction	Addition	Subtraction
Question 2		11			9	1
Question 3		1	7	1	1	5
Question 5		1		2	5	
Question 5A				1	5	
Question 5B				1	1	
Question 5C		1				

The participants introduced '*plus*' or '*minus*' while addressing a different expression than was originally presented in responding to Questions 6 and 7. The terms or phrases indicated by these five codes were most often used in Questions 6 and 7 while reading the expression I presented (n=351). However, the participants used *addition, 'plus,' 'minus,' subtraction,* and/or *'take away*', while introducing a different but equivalent expression within their responses to Questions 6 and 7 on occasion (n=59). In the following passage where Ashley is comparing and contrasting '-1 + 9,' '10 – 2,' '6 + 2,' and '6 - -2' she represented '10 – 2' and '6 + 2' in their original form, while verbalizing '-1 + 9' and '6 – -2' as different but equivalent expressions.

> I mean essentially with the with this one ['-1 + 9'] … you are taking away one ones from nine ones so this is essentially subtraction because you are taking away one from nine this ['10 – 2'] is subtraction, you are taking away two ones from ten ones and this ['6- -2'] is addition so you'd add two ones to six ones and this ['6 + 2'] is also addition so you add two ones to the six ones again. So, they are similar, these two ['6 - -2' and '6 + 2']

> because you're adding two to six ones and these two ['10 – 2' and '-1 + 9'] are similar because they are both subtraction. I mean, you're adding negative one ['-1 + 9']. It would just be different phrasing.

In this quote we see her flexibly working with some expressions she is comfortable with, while recasting others. For each expression, she clearly identifies the operation being performed. This is typical of the manner in which the participants verbalized alternative forms of the expressions they were being asked to consider. A fuller discussion about how the participants went about modifying the expressions to produce these alternate forms and employ different operators is found in the analysis of these expressions in Chapter 5.

In addition to the separate roles of signs being used to verbalize the expressions as presented and introducing new versions of the expressions, at times, the different expressions such as '*plus*' and *addition* were used together while describing the nature of the expression and how to simplify the expression. Examples of the use of *subtraction*, '*minus*' and '*take away*' can be seen in Ashley's initial effort to "explain ten minus two without using just mental math, "if I were to have ten ones and then two ones you'd be subtracting them taking, so you'd take away two of these ten ones to get eight ones." We see the use of '*plus*' and *addition* in the same passage in this statement by Jamie while modeling '-1 + 9' on the number line: "negative one plus nine. We would start at negative one and we're adding so we go to the right, so we will count up nine from negative one, … then we'll get eight for our answer." Thus, we see participants displaying versatile behavior in naming the role of the addition sign and the subtraction sign.

Signifying or mismatched. The codes *negative sign*, *'positive sign', negative for subtraction sign, and 'minus' for negative*, are terms or phrases used for signifying the positive or negative nature of an integer, or which represent a mismatch between the spoken word and the desired function. The codes *negative sign* (n=17) and *positive sign* (n=4) appear in statements where the participant stated the phrase. The codes *negative for subtraction sign* (n=4) and *'minus' for negative* (n=1), both represent cases where the code was used in a manner that does not align with rigorous communal discourse standards. As these terms either represent proper employment of a key concept, or what I consider a breach of discourse norms, I will be spending a bit of time looking at the turns coded in this way.

For each of the four times where '*positive sign*' was stated, the phrase '*negative sign*' was also stated. While considering the meaning of the phrase additive inverse,' Rebecca noted that negative seven and seven are "the same value or the same quantity but this sign, so the positive sign indicates that it's seven wholes to the right of zero and then this sign indicates left, so it's seven wholes to the left of zero for negative seven." In neither this nor similar situation within the data, were the participants fully explicit with regards to verbalizing the nature of either the negative sign or positive sign, or both, but in each statement, they identified the presence of each sign. When working with the expression '6 – -2' for the first time, Nicole expressed the idea that when confronted with "a subtraction sign and a negative sign you make it a positive sign and then you add them together." In this quote Nicole rephrased the expression during the process of which, she included both the positive and negative sign. While in these responses Nicole and

Rebecca used the 'positive sign' and 'negative sign' in a manner which is consistent with common practice, Monique did not. When Monique was considering the similarities and differences between '-1 + -5', '-4 – 2', '-7 + 1', and '2 – 8' she stated:

> These two ['-4 – 2' and '-1 + -5'] are similar because they have negative signs as well as … They have negative signs, this one ['-7 + 1'] is similar too, because it has a negative sign in it. They are different though because this ['-4 – 2'] here is not necessarily a negative sign but a subtraction sign and this ['2 – 8'] one is different from the rest because there's no negative sign at all there, just the subtraction sign. These two ['-7 + 1' and '-1 + -5'] are similar because they have the positive sign as well as the negative sign, negative integers.

In this quote we see Monique making repeated appropriate references to the negative sign. However, when she referred to the positive sign, she was pointing to the addition sign in '-7 + 1' and '-1 + -5'. The four turns in which 'positive sign' was employed were notably fewer than the 17 turns in which 'negative sign' occurred to signify the sign of a negative integer, a discrepancy which can be explained by the fact that positive signs are assumed, while negative signs are more novel and explicit. However, it does allow for each misuse of the phrase to cause a higher percentage of inconsistent use then is the case with 'negative sign', while providing fewer opportunities for explicit consistent employment.

In a manner intertwined with rephrasing expressions, introduced in the discussion of primary codes for roles of signs, the sign of the number was often changed. This occurred three times with the expression '-1 + 9', four times with '6 - -2', and twice with '2 – 8'. In each of the cases where the participant addressed the expression '-1 + 9' they changed the role of the negative sign into that of a subtraction sign, as stated by Amanda, "since it has a negative in it you have to subtract, so it would actually be nine minus one equals eight". In three of the four cases where the participant addressed '6 – -2' they stated 'negative sign' while combining a subtraction sign and a negative sign so it became addition with a positive. For example, Jacqueline did this, as shown in this quote, "when we have subtraction and a negative together they combine to make positive, so six plus two." A few minutes later in her interview, Jacqueline again used the term '*negative number*', this time to make sense of her answer for '2 – 8', she stated, "Two is a whole number and we take away eight from it, so it would be in the negatives, so it would technically be eight minus two which would be a negative number with a negative in front." In each of these cases the phrase 'negative sign' seemed to occur spontaneously. However, the roles of the minus sign and negative sign can get obscured in challenging ways. This is evident in Brooke's comment while working on '2 – 8' with colored chips:

> I am going to represent my positive two, with two light brown chips and I am going to represent my eight with the dark brown chips. So, the dark brown chips are going to represent negative eight. I am not sure exactly how I would explain why that is, when at first, when I was solving before, I just described the subtraction sign as movement on the number line whereas now I am using it as a negative sign, but that's what makes sense to me right now, so like I did before, my positive two and negative two

> would cancel each other out, so now since now I know my darker chips represent negative, this would be negative six.

In this passage we see Brooke referring to how she treated '2 – 8' as subtraction on a number line, but now wants to treat it as '2 + -8' while using chips. This displays her flexibility with these mathematical terms, where several expressions containing forms of the same value can be written so they all have the same equivalent value. While this can be seen as a virtue, such shifts in representation with the brown and beige chips, without an explicit change in written notation, remove the clear distinction in the roles of the minus sign and the negative sign, which is important to establish early in the process of learning or developing a discourse about integers.

The lack of precision with which Jamie, Jordan, Nicole, and Rebecca, in particular, held the distinction between the roles of the minus sign, is seen when they used negative to indicate subtraction, or the case where 'minus' was used to indicate negation. An example of the former case is explicitly shown in this quote from Nicole, when she was working on '-1 + -5' using the number line: "you start here when you have a plus and a negative together, you cancel out the positive and you just use the negative or the subtraction sign, or the negative sign as your subtraction sign." Jordan provided the sole example of '*minus' for negative*, "One is basically plus seven while one is minus seven" which also shows how plus is sometimes used as a synonym for the positive sign, and in each case demonstrates these PSTs lack precision when distinguishing between binary operators and positive/negative symbols.

Summary. In order to determine the level of consistency with established discourse practices exhibited by these participants, a scan across all nine codes in the thematic group, *role of sign read or introduced*, was completed. This was necessary, because if I had labeled only the uses of the binary-operational codes which were consistent with common discourse practices, a false impression of consistency would result. The same can be said about 'negative sign' and 'positive sign'. It is the participants' use of the mismatched codes, *negative for subtraction sign* and *'minus' for negative* as well as the use of the plus sign to denote positive, where inconsistencies are identified. Using these considerations, the use of the codes from this group within the discourses of Amanda, Ashley, Brooke, Christina, and Jacqueline, were seen to be consistent with the discourse described in Chapter 2. Bailey, Courtney, Kayla, and Tae displayed partially-consistent discourses, due to lack of inclusion of the 'negative sign', while making consistent, though sometimes limited, use of the phrases in this group. In contrast, Jamie, Jordan, Monique, Nicole, and Rebecca are considered to have displayed both consistent and inconsistent discourse patterns, since they explicitly used mismatched phrases within discourses that were otherwise consistent.

Summary of word use. Careful attention to the words the participants used revealed two patterns which become clear by the presence of certain words, and another by the absence of certain words. One pattern revealed by the words used was that these PSTs' discourse was heavily biased towards identifying individual numbers and operations within expressions. This can be seen in the high number of code assignments in the middle of Table 11 (and Appendix G) and in the right side of Table 12 (and

Appendix H). As was indicated above, most of the codes which were assigned were related to the participant reading aloud the expressions which were provided as part of the interview protocol. They did this, almost universally, in a manner that is consistent with common practice. Additionally, but in a more inconsistent, or non-comprehensive, manner was their introduction of equivalent values or expressions. The second pattern revealed that these PSTs made few attempts to make general statements about the nature of the terms making up an expression before, or in association with, evaluating the expression. This is revealed by the concentration of values for the codes in the 'sets of numbers' thematic group occurring early in the interview. These patterns may have resulted, in part, from the structure of the mathematical discipline and the interview protocol, which both provide more exposure to computation than to conceptual items.

Table 11: Code Use for Word Use by PST

	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae	Total
Sets of numbers															
'Positive integer'	1	6	1	1		1	1	4	2	2	1	2	9	2	33
'Negative integer'	1	7		2			1	2	1	2	2	1	6	5	30
'Whole number'	1	1	1	3	1	1	5	1	4	1	2	1	3	1	26
Integer			1	2	1		1		2	2	2		4	1	16
Other names for sets of numbers		1	2	2		1	1	1	2		2	2	1	3	18
Sign of number read or introduced															
Positive number	27	26	25	26	28	21	27	30	31	27	24	28	28	23	371
Zero	3	3	2	1		1	3	2	2	2	2	5	2		28
Negative number	25	23	20	25	22	17	24	24	28	26	20	26	24	17	321
Role of sign read or introduced															
Binary-operational															
Addition	13	16	9	9	9		11	20	16	15	10	17	11	11	167
'Plus' as addition sign	11	12	9	10	13	7	16	14	7	13	9	12	10	9	152
'Minus' as subtraction sign	11	10	9	8	8	4	13	15	4	9	4	11	14	10	130
Subtraction	10	12	5	8	7	1	3	4	9	13	1	11	9	6	99
'Take away' for subtraction	2	6	4	1	6		7	9	11	2	10		1		59
Signifying or mismatched															
Negative sign	1	2		1	1		3				2	2	5		17
'Positive sign'											1	2	1		4
Negative for subtraction sign								1				2	1		4
'Minus' for negative									1						2
Total	106	125	88	99	96	54	116	127	120	114	92	122	129	88	1476

Table 12: Code Use for Word Use by Question

	Q1	Q2	Q3	Q4	Q5	Q6A	Q6B	Q6C	Q6D	Q7A	Q7B	Q7C	Q7D	Total
Sets of numbers														
'Positive integer'	13	2	1		2			3	2	3	2	1	4	33
'Negative integer'	2	5	4	2	2	1	2	1		3	3	3	2	30
'Whole number'	14	4	1	1		2		1		2			1	26
Integer	2	3	4	1	2		1			1	1		1	16
Other names for sets of numbers	10	2	3	1			1						1	18
Sign of number read or introduced														
Positive number	6	10	13	4	34	55	14	49	55	44	8	39	40	371
Zero	1	3	2		16	1	1	1		1	2			28
Negative number	5	15	13	3	32	23	11	22	31	52	12	50	52	321
Role of sign read or introduced														
Binary-operational														
Addition		10	1		16	15	9	24	21	12	11	25	23	167
'Plus' as addition sign		11	1		2	29	5	12	23	27	5	20	17	152
'Minus' as subtraction sign			7			25	5	14	16	26	4	17	16	130
Subtraction		1	8		4	13	10	13	9	13	5	9	14	99
'Take away' for subtraction				3	1	11	1	14	7	6	3	6	7	59
Signifying or mismatched														
Negative sign				1	3	6	2	1		1	2	1		17
'Positive sign'					2	1					1			4
Negative for subtraction sign					1	1	1						1	4
'Minus' for negative					1									1
Total	53	66	58	16	118	183	63	155	164	191	59	171	179	1476

The pattern in participant word use which was revealed through the absence of codes were references to 'opposite', 'magnitude', and 'direction'. These terms were identified in Chapter 2 as being associated with key ideas within a mathematical discourse about integers, especially when that discourse is compared with a mathematical discourse about whole numbers. As noted in Chapter 3, the participants did not employ any of these terms in a way that clearly matches with the use of those terms in the discourse described in Chapter 2. I interpret this absence as exposing a significant inconsistency between the discourse about integers expressed by these PSTs and common and comprehensive discourse practices. The consequence of such an omission from the discourse provided by these PSTs would be that it would lead to an incomplete representation of this discourse, with the missing components being those that serve to establish integers as a separate discursive object than the whole numbers. However, some of these ideas, if not this vocabulary, are captured within idea driven and less vocabulary-dependent narratives which are discussed later in this chapter, and in the more holistic analysis of the participants' responses, provided in Chapter 5.

Visual Mediator

While word use provides a critical means of conveying ideas, this is often supported by, or replaced with, visual mediators. These visual mediators can take on many forms, but I will be focusing on how the participants employed writing, identified previously written forms, and modeled expressions using colored chips and number lines. These four types of enactment capture the 19 codes, assigned 673 times, to 367 turns, identified within this element of discourse. An overview of these codes can be obtained by referring to the tables in Appendix I which shows codes by participant and in Appendix J for codes by question. These codes were formed into four thematic groups, *written expressions* (six codes assigned 317 total times), *identification of* ' - ' (two codes assigned 26 total times), *chip model notations* (four codes assigned 142 total times), and *number line notations* (seven codes assigned 188 total times). Each type of employment of a visual mediator will be described below by thematic group.

Written Expression. This group of codes includes: p*ositive number*, *zero*, *negative number/sign*, *addition*, *subtraction*, and *positive sign explicitly* (see Table 13). These codes form the representations of numerals, signifiers, and operators, which together form mathematical statements and expressions. Due to the structure of the interview protocol, these written notations rarely occurred alone, thus the number of turns in which any of these were utilized is more informative than the number of times the individual codes were applied. As seen in Table 13, the number of turns in which the participants chose to make a written notation varied widely from Nicole (n=2) to Rebecca (n=15).

Table 13 Code use for Written Expressions by PST and by Question

Code by PST	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Written Expression														
Positive number	9	8	11	5	10	7	4	11	8	1	7	1	11	4
Zero	1		1							1				
Negative number/sign	9	8	13	4	7	4	4	10	8	4	6	2	11	7
Addition	9	6	7	4	6	5	3	9	4		4	2	4	1
Subtraction	5	6	6	4	5	4	1	6	4	1	4	2	4	1
Positive sign explicitly					1		1						1	
Number of turns	11	10	14	5	11	8	8	13	10	4	8	2	15	8

Code by Question	Q1	Q2	Q3	Q4	Q5	Q6A	Q6B	Q6C	Q6D	Q7A	Q7B	Q7C	Q7D	Total
Written Expression														
Positive number		7	5		3	34			15	25			8	97
Zero		2								1				3
Negative number/sign		7	5		3	15			8	46			13	97
Addition		8	3			18			8	20			7	64
Subtraction			4			16			6	19			8	53
Positive sign explicitly			1			2								3

The structure of the interview also led to these written forms only occurring in response to certain questions. The participants were prompted to suggest numbers or expressions to support their claims in Questions 2, 3, and 5, and to model how they would approach a set of expressions in Questions 6 and 7. In responding to these questions, the participants wrote numbers, signifiers, and operators while offering new expressions (n=14), equivalent (n=92) or suitable values (n=2), and while rewriting (n=83) or rephrasing an expression (n=22). These totals included times when multiple uses of such written notations were made during the same turn (n=81) for a total of 127 turns. The codes for *positive number* and *negative number/sign* were used, either together or separately, primarily in rewriting the expression as originally presented onto a different surface, paper or white board, or writing rephrased expressions into an equivalent form. Similarly, written expressions containing *addition* and/or *subtraction* were used in a like manner.

Within this element of discourse, the written signifier-numeral combination conveys visually the quantities being considered, while the operation sign indicates the operation to be performed. As mentioned above, one use of written positive and negative numbers is to suggest an equation that supports the claims made while addressing Questions 2 and 3. An example of this is seen in Figure 4, which contains written expressions including *positive numbers* ('2' and '3'), *negative number/signs* ('-10', '-7', and '-2'), *zero* ('0'), and *addition.* While responding to Question 3, Rebecca wrote down several expressions including the two shown in Figure 5, the second of which was originally '-4 – (-1)' but which she changed to include a *positive sign explicitly*. Ashley provided a nice example of both rewriting the original expression and rephrasing that expression in Figure 6. In these three examples we see how the participants wrote the expressions and equivalent values they were referencing.

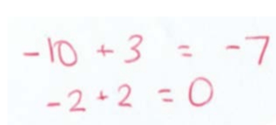


Figure 4 The expressions Amanda offered in response to Question 2

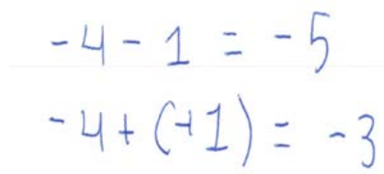


Figure 5 Two of Rebecca's expressions in response to Question 3

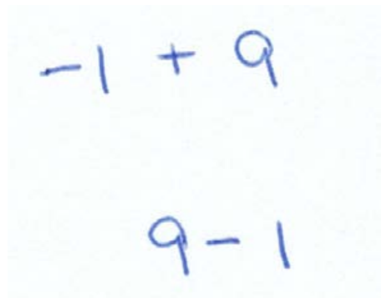


Figure 6 Ashley's written expressions for '-1 + 9'

An examination of the expressions the participants provided in response to Questions 2, 3, and 5, and how they approached Questions 6 and 7, will be provided in Chapter 5 because a more holistic examination of the participants' responses is required. The participants' written expressions for these purposes were done in a manner which is entirely consistent with discourse practices, as verified by comparison with the printed expressions I provided, or the participants' verbalized expressions.

Identification of ' – '. In the participants' responses to the prompt: "What roles can the symbol ' – ' take within a mathematics problem?", all 14 participants identified it as taking on the role of 'subtraction sign' and 12 stated that it took on the role of negating the number that follows it. The following example of one of these responses was provided by Tae, "That could indicate that you're subtracting. It could indicate that a number is negative." This response includes both of the roles that were coded for roles the ' - ' symbol can take. On the other hand, Jacqueline identified subtraction as its only role in her brief response, "What roles? Subtraction." In a manner similar to Jacqueline, Ashley only described the subtraction role for the symbol ' - '. It should be noted that no participants made reference to the third role the ' - ' symbol can take, that of the unary operation 'find the opposite of'. As a result, 12 PSTs revealed aspects in their discourse about integers which is partially consistent with comprehensive common practices, with Ashley and Jacqueline being inconsistent. These levels of consistency are based on the PSTs' identification of only two, or as few as one, of the three roles, unary operator for opposite of, binary operator for subtraction, and negation signifier for ' - '.

Chip model notations. When the participants were presented with the prompts which made up Questions 6A and 7A, they were given the liberty to choose how they wanted to model the expressions. As seen in Table 14, *drawn* chip models were introduced by five of the participants as they were responding to the initial prompts in Questions 6 and 7. Later, when prompted to use the chip model, the participants were presented with a dozen beige- and a dozen brown-colored chips, and all the participants attempted to employ the chip model for those prompts, part C of Questions 6 and 7.

Table 14 Code Use for Chip Model Notations by PST and by Question

Code by PST	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Chip model notations														
Drawn		8	4						1				3	3
Beige and brown chips														
One color	6	9	4	5	1	2	2	2	3	4	3	3	9	7
Separate colors		5	3	3	5	2		6	4	3	4	4	1	3
Mixed colors	2		5		2	4	6		1	1	1	1	1	1

Code by Question	Q1	Q2	Q3	Q4	Q5	Q6A	Q6B	Q6C	Q6D	Q7A	Q7B	Q7C	Q7D	Total
Chip model notations														
Drawn						9				10				19
Beige and brown chips														
One color						7		25		4		23		59
Separate colors								16		3		24		43
Mixed colors								15				10		25

Drawn. The 19 cases where a chip model was *drawn* were unevenly distributed over five participants and took two different forms. In one case, Tae drew a strip of base ten blocks then removed two while modeling '10 – 2', see Figure 7. The more common drawn representation is that of circles, which may be drawn as two groups of addends followed by a drawn sum, or circles from the minuend being crossed out to model subtraction. Each of these cases can be seen in Figure 8, where Ashley models subtraction in the top two expressions, '10 – 2' and '-1 + 9', and the bottom left expression '-1 + -5', and models addition in the middle two, '6 - -2' which became '6 + 2' on the left and '6 + 2' unlabeled on the right, with an incomplete attempt for '-4 – 2' at the bottom right corner. In Ashley's drawing in Figure 8, we see her using the same shape, open circles for both positive and negative values. Ashley's use of this notation was inconsistent for '-4 – 2', '6 – -2', '-1 + -5', and '2 – 8' since she did not complete the process by modeling a single set of circles to represent the equivalent value. This contrasts to her use of drawn chips in '10 – 2', '6 + 2', and '-7 + 1' which are independently consistent with common practice, by providing a clear notation of the equivalent value. However, Ashley's approach to '-1 + 9' is a bit more complicated to analyze, since her drawn chip model reflects the rephrased expression '9 – 1' in a manner that is consistent with common practice. Further, her use of open circles to represent positive integers in some expressions and negative integers in others makes her overall approach inconsistent to rigorous common practice. Jordan's use of single colored dots presents the same lack of distinction, but she does clearly identify the equivalent value, resulting in her notation being deemed partially consistent. Bailey used two colors, blue for negative and red for positive, which allowed for differentiation between the sign of the number modeled by each chip; additionally, she clearly identified the resulting equivalent value, making her use of this notation entirely consistent with the exemplar discourse modeled in Chapter 2, see Figure 9. Tae used circles (negatives) and squares (positive), allowing for differentiation, but did not provide distinct equivalent values, and is thus partially consistent with common discourse practices. Rebecca added further variety by only referring to the magnitude of the integer which she did by using open circles, while employing the negative sign to identify negative integers, see Figure 10. This, and the fact that Rebecca only clearly noted with circles one equivalent value while creating three drawings, makes her usage inconsistent with common discourse practices.

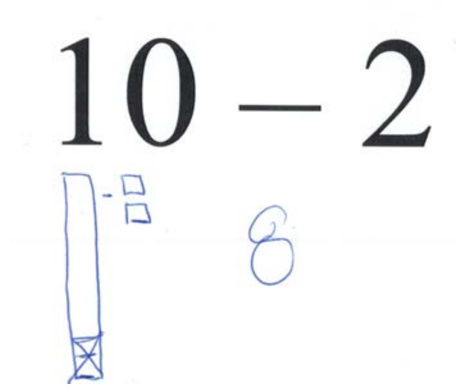


Figure 7 Tae's model for '10 - 2'

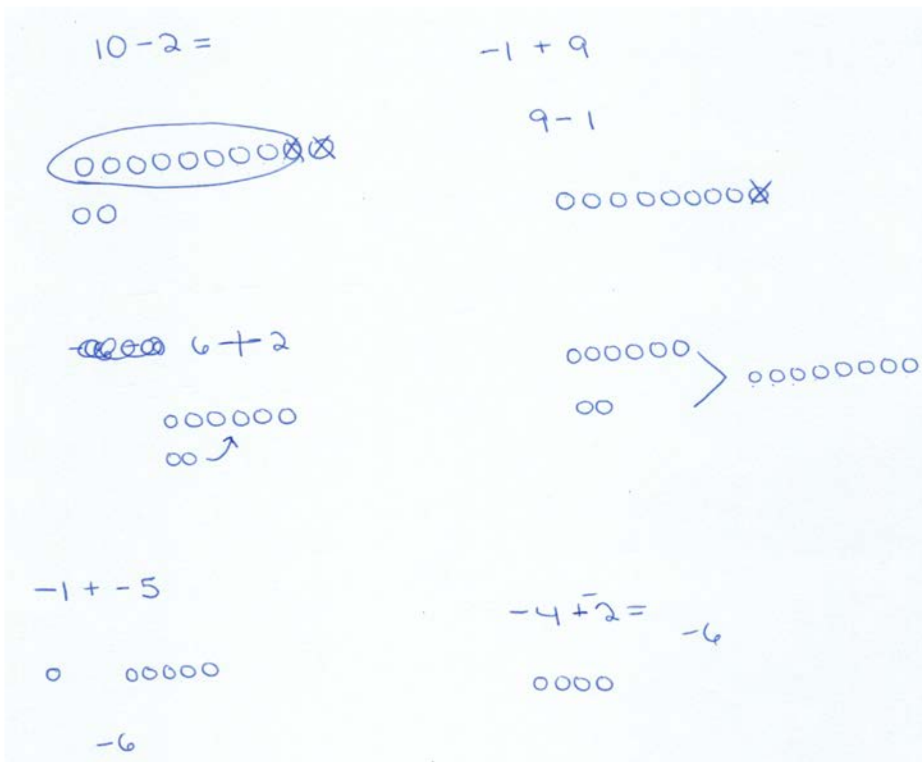


Figure 8 Ashley's drawn chip models from several expressions

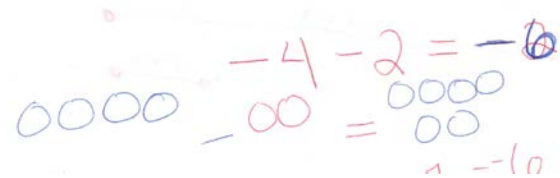


Figure 9 Bailey's drawn chip model for '-4 - 2'

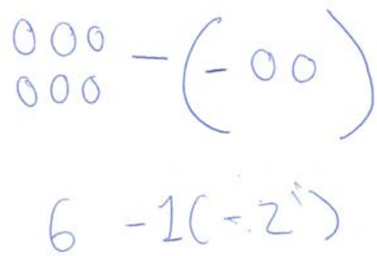


Figure 10 Rebecca's drawn chip model for '6 - -2'

Beige and brown chips. While employing the chip model to represent the expressions discussed in Questions 6 and 7, the participants used chips of *one color*, *separate colors*, and *mixed colors*. When using *one color,* the participant employed chips of only one color. In contrast, when they used *separate colors* or *mixed colors,* both colors were used, but only in the case of *separate colors* did the participant assign the positive integers to either the beige or brown chips, with the other color representing the negative integers. With chips of *mixed colors*, beige and brown chips were used without an association being made between each color and the positive or negative integers. Examples of each of these will be provided in turn. The mixing of colors is never consistent with communally-developed chip models for integers, as it fails to represent the positive or negative aspect of an integer, leaving only cardinal magnitude. The applicability of these color assignments depended in large part on the positive or negative nature of the numbers in the expression, and the operational approach taken in modeling it.

One color. The employment of *one color* of chips required the participant to identify only beige or brown chips and use them throughout their modeling of the expression. An example of this is seen in Figure 11, where both numbers were positive integers. The inappropriate use of *one color* is shown in Figure 12 where the original expression '-4 – 2' contains both positive and negative integers. The use of a single color for this expression requires the rephrasing of the expression to '-4 + -2', which changes the role of the minus sign to that of a negative sign and requires a change in operation to addition. Each of the participants modeled at least one expression using one color. Thirteen of the participants used a one-color model successfully, whereas Jacqueline used it unsuccessfully as she was unable to proceed after setting out one value for '2 – 8'.

Figure 11 Tae modeling '10 - 2'

Figure 12 Jordan modeling '-4 - 2'

Separate colors. Through a combination of spoken phrasing and visual patterns, 12 of the participants were able to indicate that they were representing positive and negative values with chips of *separate colors*. Christina provided a good example of assigning different colors to represent positive and negative values in her statement accompanying Figure 13 in which she indicated that the seven beige chips represented negative seven and the one brown chip represented the positive one being added, and that a beige and brown pair are, "going to cancel, these two out right here. And you have negative six as well." In a similar manner, Kayla stated "I am going to use these to represent negative numbers, the lighter chips, and then dark ones are going to be positive" while modeling '2 – 8' along with what is shown in Figure 14. However, in this example, Kayla is modeling '-8 + 2', when she was prompted to model '2 – 8', which is a different, but equivalent, expression. Kayla's rephrasing notwithstanding, the use of chips with separate colors in modeling expressions with integers can provide a clear distinction between the positive and negative terms contained in an expression.

Figure 13 Christina modeling '-7 + 1'

Figure 14 Kayla modeling '2 - 8'

Mixed colors. The use of *mixed colors* shows a lack of association between the presence of two colors and the presence of positive and negative integers. One such example is seen in Jacqueline's modeling of '6 + 2' shown in Figure 15, where she represented six with three brown and three beige chips, and the two with a pair of beige chips. In contrast, Bailey used beige for six and brown for two in her equivalent values after modeling of '6 + 2' and '6 - -2' (see Figure 16). This makes it appear that either no difference in number type is being indicated by the color of the chips or that pairs of numbers should be canceling out since chips of different colors are present. Eleven of the participants employed mixed colors of chips as part of their initial or final representation when they modeled with chips.

Figure 15 Jacqueline modeling '6 + 2'

Figure 16 Bailey's chip models for '6 + 2' (bottom) and '6 - -2' (top)

One virtue of the chip model is that it offers the participants in a discussion the opportunity to visually differentiate positive and negative numbers, by assigning a color to positive numbers and a separate color to negative numbers. This differentiation by color was not universally noted in the approach taken by these PSTs. In fact, the use of mixed colors by 11 of the participants makes their use of colored chips inconsistent with common discourse practices about integers as defined in Chapter 2. Ashley and Brooke made use of the chip colors in a manner that is consistent with common practices. As will be explained shortly, Jamie's use of the colored chips was both unorthodox and inconsistent with the practice of others.

Number line notations. During most of the interviews, the participants were free to approach and model the expressions which they introduced, or which I presented, in any manner they chose. However, the participants were asked specifically to model eight expressions using the number line (see Appendix A). In order to complete this task, a number line was provided in the form of a number line taped onto a white board. The seven participants (Amanda, Ashley, Bailey, Brooke, Jamie, Rebecca, and Tea) voluntarily introduced the number line while responding to other prompts. These took the form of *participant drawn* number lines, *number line spoken of* as a mental image, and *colored chips as number line* (see Table 15). The *participant drawn* number line occurred as the code suggests, whenever the participant chose to draw and label a number line, see Figure 17 for an example. The code *number line spoken of*, is a special case, since no number line is visible, instead it relies on an interpretation of the participant's intent for the listener to visualize a number line. Ashley's response: "if you were to look at it on a number line, if you found where negative seven was on a number line and you added seven to it, you'd get zero," while providing additional explanation about why negative seven and seven are additive inverses, is one such situation. The use of *colored chips as number line* is seen in Figure 18, where Jamie made a long line of chips, positioning one chip to serve as zero, and counted along that line to determine her equivalent value.

Table 15 Code Use for Number Line Notations by PST and by Question

Code by PST	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Number line notations														
Introduced														
Participant drawn	1	1	3	4				5					6	6
Spoke of	2	5	1	5	1		1	4				1	2	2
Colored chips as number line								5					2	
Requested														
Dot-hops-dot	8	9		10	7	8	8	11	1	8	7		14	11
Left or Right arrow			10					3	1					
Numbered steps taken												8		
Dots along									7					

Code by Question	Q1	Q2	Q3	Q4	Q5	Q6A	Q6B	Q6C	Q6D	Q7A	Q7B	Q7C	Q7D	Total
Number line notations														
Introduced														
Participant drawn	1	1			3	12				9				26
Spoke of	2				9	2	1	2		4	2	2		24
Colored chips as number line								2				5		7
Requested														
Dot-hops-dot		1			1	7	1		42	7			43	102
Left or Right arrow						3			7				4	14
Numbered steps taken									4				4	8
Dots along									4				3	7

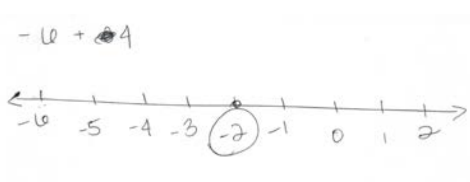


Figure 17 Brooke's response to Question 2

Figure 18 Jamie pointing to her chips as number line model for '-1 + 9'

While using either the premade or drawn number line, the participants employed the following representations to model the solution: *dot-hops-dot*, *left or right arrow*, *numbered steps taken*, and *dots along* (see Table 15). Most of the participants consistently used one representation while modeling using the number line. However, Jamie used the arrow to indicate the direction the hops advance (see Figure 19), while Jordan, shifted to the *dot-hop-dot* and arrow approaches from her more common *dots along number line* for reasons that were not elicited during the interview. Examples of each of these representations will be considered below.

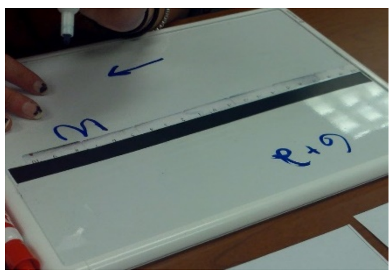


Figure 19 Jamie's solution to '6 + 2'

Dot-hops-dot. By far, the most commonly used representation was *dot-hops-dot* (n=103), which always began at the numerical value for the first term in the expression, and ended at the equivalent value, as shown in Figure 20. This representation effectively communicated moving right for addition and left for subtraction while being drawn and accompanied by the spoken telling of the expression. However, as a static representation, it is ambiguous, as far as which of a set of expressions was modeled. For example, Tae's model seen in Figure 20 succeeds in modeling '2 – 8' but could also show '-6 + 8', '-6 - -8', or '2 + -8' properly. This flexibility can be seen as a strength; however, it does not provide distinctive representations to different expressions and as such tends to weaken the discourse about integers exhibited by the 11 PSTs who relied on this method.

Figure 20 Tae's number line model for '2 - 8'

Left or right arrows. A visual mediator which allows for less ambiguity in the expression being represented would be the use of left or right arrows. Bailey was the most frequent user of this approach, see Figure 21, which she employed 10 times. Thus, we see her modeling '-1 + -5' in a manner that could also be used to model '-1 – 5', as is suggested by her arrows to the left for '2 – 8', '-4 – 2', and '-1 + -5'. However, she started her arrows at the first term and ended them at the equivalent value without regard for the nature of the operation and integers involved in the expression.

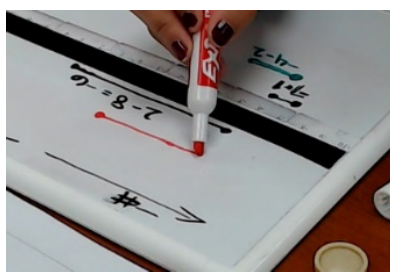


Figure 21 Bailey modeling '-1 + -5'

Numbered steps taken. The approach *numbered steps taken* a number line, which only Nicole used, displays the same issue of only showing the first term and equivalent value while marking the steps taken to travel the distance. An example of Nicole's work while doing this is shown in Figure 22.

Figure 22 Nicole modeling '2 - 8'

Dots along number line. The same lack of distinctiveness in modeling expressions with different combinations of positive and negative integers being added or subtracted was present in the use of *dots along number line,* exhibited by Jordan. Though clear about the first term and equivalent value associated with an expression (for example '-1 + 9' shown in Figure 23), the operation and second term are still left undefined.

Figure 23 Jordan modeling '-1 + 9'

Summary. The use of the number line offers a means to visually convey the properties of integers, and to model addition and subtraction with integers, in a way that shows their distinctiveness from whole numbers. These PSTs did not embrace this opportunity to display the addition and subtraction of integers in ways that preserved the operation and the integers involved. This was most evident for '6 - -2', but was true for all the expressions except '10 – 2', '6 + 2', and '-1 + -5'. This led me to consider the models exhibited by all participants except Bailey and Nicole to be inconsistent with the discourse modeled in Chapter 2. Bailey's use of arrows or directed lines, more clearly showed the direction of movement from the first term presented to the equivalent value, thus she is considered to display a notation which is consistent with expert practices. Nicole's use of numbers to mark her steps is consistent with the intention to represent the expressions in a such a way that the original expression can be determined by considering the number line model. However, they are considered only partially consistent, as she, along with all of the other participants did not maintain the distinctiveness of positive and negative integers as well as addition and subtraction on the number line.

Summary of Visual Mediators. Visual mediators are employed as part of a discourse about integers as a means of referring to an object, in this situation, an expression containing positive or negative integers, or both. As described above, the participants were able to generate and copy expressions in a manner which is consistent with common practices. However, since full representation of these expressions link spoken words with visual mediators and motivation for introducing the expression, a detailed examination of the alignment between these means of communication and the routines followed to find the equivalent value will be discussed in Chapter 5. The participants' responses to my prompt about the roles ' - ' can assume were considered to be either partially or fully inconsistent with comprehensive discourse practices. As previously stated two of the participants identified subtraction as the only role ' - ' can assume, while the remaining 12 only expanded this list of roles to include signifying a negative integer. None of the participants were considered as displaying consistent discourse about integers, because they all omitted the third role, that of 'opposite of'. Similarly, though nine of the participants did not exhibit a drawn chip model. Of the five that did, only Bailey was considered to exhibit models which are consistent, Jordan's and Tae's models were partially consistent, and Ashley's and Rebecca's were inconsistent with common representations. For the prompted modeling with the beige and brown chip model, 12 participants exhibited inconsistent, and two exhibited partially-consistent representations, based heavily on the use of mixed colors while employing this model. The suggested common practice described in Chapter 2 calls for the use of directed arrows to show the nature of each term and distinctively identify the operation within the expression, when modeling with the number line, and none of the participants exhibited such a model, they were all considered as displaying either a partially or completely inconsistent model on the number line.

The uses of the visual mediators for integers which these participants struggled with are specialized representations, closely aligned and developed for a discourse about integers. This means that those approaches that these PSTs used, which represented their attempts to extend models they learned for whole numbers, ended up being incompatible, and inconsistent in this new discourse, no matter how well aligned they would be in a mathematical discourse about whole numbers.

Routines

Within a discourse, a routine represents a way of approaching an expression to find its equivalent value. The 10 codes identified in describing the participants' methodologies for approaching expressions with integers were thematically grouped based on the nature of the mediator being employed. These codes stated within their thematic groups are shown in Appendix K as they are assigned by participant, and in Appendix L as they are assigned by question. These mediators involve spoken or written numerals and expressions, thematically grouped as a*ctions with numerals* (four codes assigned 82 total times), actions with the colored chip visual mediator, thematically grouped as *actions with the chip model* (four codes assigned 150 total times), and use of the number line, thematically grouped as *actions with number line* (two codes assigned 149 total times). These codes were identified during 288 turns and were coded a total of 382 times. A single routine code was applied on 216 occasions; with multiple codes being applied during 72 turns. Frequently the participants provided responses, such as Christina's "six plus two equals eight," which are not considered to have exhibited any routine, and thus was not assigned a routine code. However, the use of visual mediators allows the actions that make up the routines to be displayed, a possibility which is absent with expressions that only take on a spoken form, unless the participant chose to elaborate on her evaluation process.

Actions with Numerals. This group of codes relates to the spoken and written words a participant employed while modeling her approach to finding the equivalent value to an expression. These codes represent only those techniques which were used more than eight times, and which are uniquely or especially relevant to addition or subtraction with integers. Codes of this group, p*ortions of numbers with different signs can cancel to give sum or difference*, *adding to a negative as subtraction*, *subtracting a negative number is adding a positive number*, and *subtraction as adding a negative* *number*, were applied on 82 occasions, to responses provided by each of the 14 participants. The frequency with which these participants made statements to which these codes are applicable varied from a single statement by Courtney, up to 12 by Amanda, see Table 16 for the distribution over the participants. These codes were most frequently applied to responses related to Questions 6 and 7, as can also be seen in Table 16. An example, and some discussion will be provided for each of these codes.

Table 16 Code Use for Routines Involving Actions with numerals by PST and by Question

Code by PST	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Actions with numerals														
Portions of numbers with different signs can cancel to give sum or difference		1		3	2				2	3	3	4		1
Adding to a negative as subtraction	5	2			1		2		3	1		1	1	
Subtracting a negative number is adding a positive number	4	5	2		1	1	3	1	1	2		1	4	1
Subtraction as adding a negative number	3			2	1		3	1	1	2		5	1	2

Code by Question	Q1	Q2	Q3	Q4	Q5	Q6A	Q6B	Q6C	Q6D	Q7A	Q7B	Q7C	Q7D	Total
Actions with numerals														
Portions of numbers with different signs can cancel to give sum or difference						2		6		4		7		19
Adding to a negative as subtraction		1				5	1	3		2		4		16
Subtracting a negative number is adding a positive number			2			13	2	5	4					26
Subtraction as adding a negative number			1					1	1	5	2	11		21

Portions of numbers with different signs can cancel to find sum or difference. The code *portions of numbers with different signs can cancel to find sum or difference* describes the participants' having worked from the larger magnitude to zero, then having proceeded the rest of the way to the equivalent value. These 19 responses occurred in response to prompts included in Questions 6 and 7. Of the eight participants whose work was coded in this way, the frequency with which it was applied varied from one response identified for Ashley and Tae, up to four responses by Nicole. One such response was provided by Brooke who, while modeling '-7 + 1' with chips described her actions as "I know one negative one and a positive one cancels each other out, which leaves me with negative six," see Figure *24*. Thus, we see Brooke employing the idea of canceling portions of each number, which she narrates while pairing chips, representing a positive and a negative number to form a zero pair, while approaching this expression. Brooke used this approach with the addition of integers, whereas Kayla used it while engaged in subtraction. When solving '2 – 8' Kayla said, "two of the eight that you're subtracting would cancel out to zero. So, then you would have six more to subtract, so it would be zero minus six and that would be negative six" (see Figure 25). When used with addition, this routine makes use of the additive inverse property, which is a key aspect of addition with integers. Further, it embraces the signs of the numbers as presented in these expressions. In describing the actions associated with this code, the eight PSTs whose responses exhibited this action were applying a routine which is consistent common practice, while the other six did not exhibit such a routine.

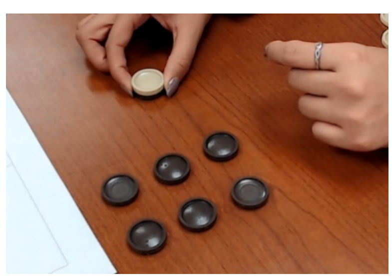


Figure 24 Brooke modeling '-7 + 1'

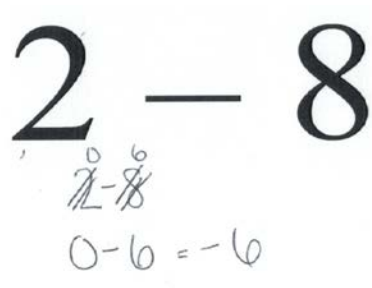


Figure 25 Kayla's written notes while modeling '2 – 8'

Adding to a negative as subtraction. Amanda's approach to '-1 + 9' provides a good example for the method coded as *adding to a negative as subtraction*. Amanda describes her thinking as "Since it has a negative in it, you have to subtract, so it would actually be nine minus one equals eight." She recorded this as seen in Figure 26, where after recording the expression offered, she rewrote it vertically as '9 – 1' prior to determining the equivalent value. This approach involves a shift in the roles indicated by the minus sign, from that of negation to that of subtraction, in the process of rephrasing the expression. In the 16 uses, involving eight of the participants, this shift, while representing common practice, does not preserve the original distinctions between the several roles of the minus sign or respect the nature of the integers in the expression. Thus Amanda, Ashley, Christina, Jacqueline, Jordan, Kayla, Nicole, and Rebecca, who each utilized this approach at least once, exhibited an activity that was consistent with common discourse practices which themselves involve rephrasing expressions such as '-1 + 9' as '9 – 1', while Bailey, Brooke, Courtney, Jamie, Monique, and Tae, did not exhibit this activity.

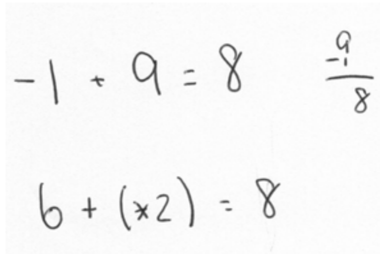


Figure 26 Amanda's written work for '-1 + 9' and '6 - -2'

Subtracting a negative number is adding a positive number. The lower portion of Amanda's written work, shown in Figure 26, shows her using the most frequently employed routine where *subtracting a negative number is adding a positive number* (n=26). This was used by 12 of the participants, all except Brooke and Monique, during their responses to five different prompts, many of them in Question 6, where it was exclusively used with '6 - -2'. In discussing her actions, Amanda said, "if you had six minus negative two these cancel, this becomes a plus and this cancels, so this also becomes a positive eight" while she drew vertical slashes through the minus sign to make it a plus, and the negative sign to make it a positive sign. This approach, while effective and often appropriate, does not preserve the initial structure of the expression as it was presented. Though uncommon as a learning goal with symbolic notations, this routine is especially at odds with comprehensive and rigorous common practice with the number line and chip models. This variation is notable, as it fails to accentuate the defining features of integers and how integers behave under different models of subtraction or the role of the additive inverse in the chip model. However, the 10 PSTs who exhibited this routine did so in a manner which is consistent with common discourse practices, the other four did not exhibit this approach.

Subtraction as adding a negative number. In the 21 cases where a positive number is subtracted, for example '2 – 8' and '-4 – 2', 10 of the participants, all except Ashley, Bailey, Courtney, and Monique, treated these as *subtraction as adding a negative number*, primarily with prompts within Question 7. Jacqueline's response to '-4 – 2' provides a suitable example, "Negative four minus two you would do … negative four plus negative two is negative six." This is shown in Figure 27, where we see her inserting a plus sign between the negative four and the minus sign rephrasing the expression as '-4 + -2'. In this approach, these PSTs shifted the roles of the minus sign, without comment, and not attending to the ordinal nature of the integers. This procedure is recognized as acceptible, by the sources listed in Chapter 2, and thus represents common practice in general terms. However, its suitability for the modeling of individual expressions with particular visual mediators will be considered in Chapter 5.

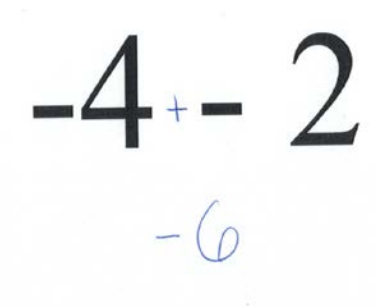


Figure 27 Jacqueline's written notation for '-4 - 2'

Summary. The four codes discussed in this section capture only those clearly-employed and narrated routines which these PSTs utilized at least nine times, while considering expressions independent of or accompanying the number line and chip models. These independent implementations involved only spoken or written notations. The supporting use of these routines allowed the PST to think through the expression while using the chip or number line models. While the examples described above could be considered as being consistent with common discourse practices, the utilization of these four codes need to be taken into consideration as a whole, since they represent four of the more common routines for evaluating expressions involving one or more negative values. Since Christina, Jordan, Kayla, and Nicole were coded as exhibiting each of these four routines, they are deemed to be fully consistent with common discourse practices. As Amanda, Ashley, Brooke, Jacqueline, Jamie, Rebecca, and Tae were coded to have exhibited two or three of these routines, they are considered to be partially consistent, while Bailey, Courtney, and Monique only used one of these routines explicitly, they are considered to have exhibited inconsistent discourse practices when compared with a comprehensive and exemplar mathematical discourse about integers.

Action with the chip model. The four codes used to describe the participants' actions with the chip model (*adding as joining*, *opposite colors stacked*, *subtraction as taking away*, and *opposite colored chips swapped*) were used primarily, but not exclusively, with the beige and brown chips provided, see Table 17, another use was for drawn chips. Such concentrated usage can be expected since this visual mediator was specifically requested in Questions 6C and 7C. All 14 participants used an approach I coded as *adding as joining* at least once, and up to eight times, including drawn notations. Along with the *adding as joining* approach, 10 of the participants, all except Amanda, Jacqueline, Jamie, and Rebecca, paired opposite colored chips in a way which led me to coded them as *opposite colored chips stacked*. *Subtraction as taking away* was used by all 14 participants between one and four times. While six participants, Ashley, Bailey, Christina, Courtney, Jordan, and Nicole, used a technique I coded *opposite colored chips swapped* up to three times, as a chip-based equivalent to rephrasing the expressions.

Table 17 Code Use for Actions with the chip Model by PST and by Question

Code by PST	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Actions with the chip model														
Adding as joining	5	7	8	6	7	5	4	1	7	7	6	7	7	8
Opposite colored chips stacked		3	5	3	3	2			3	3	4	3		3
Subtraction as taking away	4	3	1	1	1	2	2	1	2	1	1	1	3	2
Opposite colored chips swapped		1	1		1	3			1			2		

Code by Question	Q1	Q2	Q3	Q4	Q5	Q6A	Q6B	Q6C	Q6D	Q7A	Q7B	Q7C	Q7D	Total
Actions with the chip model														
Adding as joining						5		33		5		42		85
Opposite colored chips stacked								9		4		19		32
Subtraction as taking away						4		17				4		25
Opposite colored chips swapped								4		1		4		9

Adding as joining. When performing the operation of addition with the chip model participants either formed an arrangement of chips to represent each of the numbers in the expression and then joined the two collections or formed a single arrangement which they supplemented from the unused chips. This action was the most frequently used of any code in this thematic group, however it was used in a few distinct ways. One way was the joining of two values which contained all positives, as in ‘6 + 2’, or all negatives, as in ‘-1 + -5’; the latter case is shown in Figure 28. It was also used in expressions rephrased to include only positives, as in ‘6 - -2’ becoming ‘6 + 2’, or all negatives, as in ‘-4 – 2’ becoming ‘-4 + -2’. While this rephrasing conceals features of the original expression, it allows this routine to be used appropriately. Since these uses can be interpreted as exhibiting versions of standard practices, each participant is deemed to use this approach in a manner which is consistent with common practice.

Figure 28 Jordan sliding in the last of five chips while modeling '-1 + -5'

Opposite colored chips stacked. The other common use of *adding as joining* is linked to the pairing of chips so they cancel out, coded as *opposite colored chips stacked*. Such a stacking of beige and brown chips to cancel both of them out was assumed to utilize the additive inverse property, where the addition of opposite values equals zero, see Figure 29. Though is it unclear that this was the goal of these PSTs, it is the true that the zero-pairs idea was only clearly used in this manner. It was used to good effect in modeling '-1 + 9' (n=8), '2 – 8' (n=8), and '-7 + 1' (n=10). However, Tae rephrased the expression '2 – 8' by stating, "It's easier for me to almost think of minus eight as plus negative eight. So these would represent … negative eight and this is the two and you add two positives to the negatives you're left with negative six." While this rephrasing yields the proper equivalent value, it fails to reflect an appropriate method to depict subtraction of this type with this particular model. Ten PSTs exhibited this approach, over the total of 32 turns, in a manner consistent with common discourse practices. However, I caution the reader that this does not account for the expression-based analysis of the use of this approach which will follow in Chapter 5.

Figure 29 Christina pairing a negative and a positive chip while modeling '-7 + 1'

Subtraction as taking away. Modeling subtraction with the chip model was done in a manner coded as *subtraction as taking away*. In this method, one or several of the chips, as indicated by the subtrahend, in an existing arrangement were removed. This approach was taken by all 14 participants on up to four occassions, for a total of 25 enactments. Of the expressions provided, only '10 – 2' was regularly approached in this manner (n=14). This expression was also the only expression including subtraction that exclusively involved positive numbers, and no examples were provided that contained subtraction with both numbers being negative. This method was, however, applied to '-7 + 1' and '-1 + 9', as seen in Figure 30, which required the expressions to be rephrased to include subtraction in order for this method to be employed. Those who were able to model '6 - -2' rephrased it as '6 + 2' and thus did not show the subtraction of a negative from a positive. As these are standard ways of approaching each of these expressions, all of the participants were considered to have exhibitied this routine in a manner which is consistent with common practice when analyzed in this way.

Figure 30 Jacqueline removed one chip from nine while modeling '-1 + 9'

Opposite colored chips swapped. While modeling certain expressions, six of the participants, Ashley, Bailey, Christina, Courtney, Jordan, and Nicole, deliberately started the expression with one color of chips representing either a positive or negative number, then changed them out on a one-to-one basis with chips of the opposite color to represent the opposite of the value in the original expression. Nicole described how this worked in this passage, which is accompanied by Figure 31,

> negative four minus two, again it's like a negative, and technically it's a positive, but if you want to ignore the negative, [treat] the subtraction sign as a negative sign, it would be a negative four and then two, but because you're subtracting, you flip them. And then you have negative six in total because you're subtracting two positives, which technically make them negative.

This approach allowed the participant to proceed with the expression in a manner she was comfortable modeling. It further serves as a means of representing the rephrasing of an expression using chips in a clear way. The use of these methods is consistent with common practices where appropriate. Also, eight participants did not use this approach at any time as a way to represent the rephrasing of particular expressions as mentioned above.

Figure 31 Nicole modeling '-4 - 2'

Summary. The participants made use of one or several of these approaches on almost any occasion where a chip model was drawn or required. Overall, Ashley, Bailey, Christina, Courtney, Jordan, and Nicole used all four of these approaches at some point in the interview, and thus are considered to have exhibited a consistent model when compared with common practices. Amanda, Brooke, Jacqueline, Jamie, Kayla, Monique, Rebecca and Tae each used two or three of these approaches and are considered as displaying partially-consistent modeling.

Action with the number line. The two codes *go up number line* and *go down number line* are partially influenced by the structure of a number line. Another influence is the range of options the discussant possessed as to how to model her thinking on a number line, which I described in the results about visual mediators as merely ways to show the initial terms and the final value. Thus, the participants' use of *dot-hops-dot*, *left or right arrows*, *numbered steps*, or *dots*, showed the path of the movement from the initial value to the equivalent value, as either going up or down the number line. The total number of times the participants' responses were identified as *go up number line* was 75, while that of *go down number line* was 74, the frequency with which these codes were used is captured in Table 18. All 14 participants used these notations, with Kayla and Monique using these actions least (n=7) and Rebecca using them most (n=19). The participants with more than eight instances of these codes used number line representations in response to prompts other than those in Questions 6D and 7D, as can be seen in the lower half of Table 18.

Table 18 Code Use for Action with Number Line by PST and by Question

Code by PST	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Action with number line														
Go up number line	4	5	5	10	4	4	4	9	4	3	3	4	9	7
Go down number line	4	5	5	7	4	4	4	8	5	4	4	4	10	6

Code by Question	Q1	Q2	Q3	Q4	Q5	Q6A	Q6B	Q6C	Q6D	Q7A	Q7B	Q7C	Q7D	Total
Action with number line														
Go up number line		1			3	8	2	1	40	4	1	1	14	75
Go down number line					2	3	1		17	7	1	3	40	74

Go up number line. When a participant enacted a routine coded as *go up number line*, she would mark the starting position, based on the form of the expression, either the arrangement which was originally presented, or in a rephrased form. The participants would then employ addition of a positive value as presented in such expressions as '6 + 2' (n=14), '-1 + 9' (n=13), and '-7 + 1' (n=13) or as adjusted into the rephrased form, such as in '6 – -2' (n=13). A result of one such enactment is shown in Figure 32, where Tae is seen making the final dot after making two hops to the right while modeling '6 + 2'. This approach is consistent with standard computational algorithms aligned with written, spoken, or thought of numbers. Further, as it employs the addition and positive number represented in the original form, it is the appropriate second step in modeling '6 + 2', '-1 + 9', and '-7 + 1' with the number line, while serving to identify the point on the number line which represented the equivalent value. However, when employed in modeling '6 – -2' it requires the modification of the original subtraction and negation within the expression and requires rephrasing the expression into addition of positive numbers. Though this is a standard algorithm, it does not make use of the number line's ability to reflect positive and negative numbers, as well as both addition and subtraction. Thus, without having considered, at this time, the accuracy of the participants' enactment of this approach, all 14 of these PSTs' use of this routine as being consistent with standard practices.

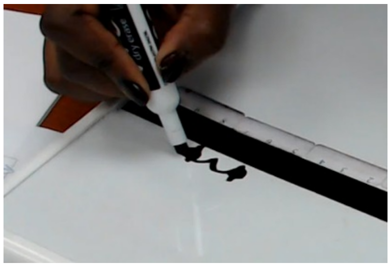


Figure 32 Tae modeling '6 + 2'

Go down number line. These 14 participants all enacted routines coded as *go down number line*, in a way exactly analogous with their approaches coded as *go up number line*. This approach is consistent with both common practice and the original presentation of the expression for '10 – 2' (n=14), '2 – 8' (n=13), and '-4 – 2' (n=13). The use of this approach for '-1 + -5' (n=14) required rephrasing the expression to include subtraction of a positive number. One such example is shown in Figure 33. While this is a standard algorithm, it did not employ the number line's ability to show positive and negative numbers, as well as addition and subtraction. Thus, each of these 14 PSTs consistently employed a valid technique in a communally-sanctioned way, however, the accuracy of their execution of this approach will be presented in Chapter 5.

Figure 33 Amanda modeling '2 - 8'

Summary. These 14 PSTs each enacted a routine involving going up or down the number line in a way which is consistent with common algorithmic expectations. As noted, this maintains the original expression's positive or negative numbers and addition or subtraction for six of the expressions. Taking these approaches required a rephrasing of the expresssions in '6 – -2' and '-1 + -5' in ways that remove the original operation and the sign of the second term, a process which is unnecessary with the number line model.

Summary of routines. The PSTs' use of routines in modeling operations with integers was consistent with routines they might employ when dealing with whole numbers or numeral-based algorithms. As noted above, these participants expressed, in verbal or written form, routines which involve actions with numerals in a manner that was consistent (n=4), partially-consistent (n=7), or inconsistent (n=3) with common discourse practices for these means of representation. Their enactment of routines involving actions with the chip model, was consistent for six PSTs and partially-consistent for the other eight, without taking account of accuracy of the equivalent value or clarity in the use of beige and brown chips. Each of the 14 PSTs' actions with the number line are consistent with common algorithms and practices, if as noted, not always making full use of the number line as a visual mediator for modeling operations with integers. More critical analysis of the application and execution of these routines will be described in Chapter 5.

Narratives

A narrative is a general statement related to the structure of the discourse in which one is engaged. The 21 codes related to the narrative element of discourse were applied to participants' statements 169 times, and have, after further analysis, been formed into seven thematic groups. A table showing these code assignments in included as Appendix M with codes listed by participant, and Appendix N with codes listed by question. The code groups are: *relationship between positive integers and whole numbers* (two codes assigned 14 total times), *'addition make larger' remains true with integers* (two codes assigned 14 total times), *types of numbers generated by integer subtraction* (six codes assigned 31 total times), *properties of additive inverse or the pair '-7 and 7'* (two codes assigned 35 total times), *operations and signifiers both need to be considered* (five codes assigned 32 total times), *results of subtraction* (two codes assigned 7 total times), and *similarity of operations on the number line* (two codes assigned 36 total times). Many of these thematic groups are associated with a specific part of the interview (see Appendix A). Though some of these codes were used fewer than the nine times specified for the previous two elements (visual mediators and routines), the importance of the statements captured by these coded narratives justified having included them. The distribution of the codes by participant within each thematic group is shown in Table 19 and Table 20, and examples of each coded narrative are discussed below by thematic group.

Table 19 Code use for Narrative by PST with Thematic Groups

Codes	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae	Total
Relationship between positive integers and whole numbers															
Are synonyms	1				1				1						3
Mean different things		1	1	1		1	1	1		1	1	1	1	1	11
'Addition makes larger' remains true with integers															
Yes, it remains true				1	1							1			3
No, not always	1	1	1			1	1	1	1	1	1		1	1	11
Types of numbers generated by integer subtraction															
Positive number	1		1	1		1	1	1	1	1	1	1	1		11
Negative number	1		1	1	1	1	1	1	1	1	1	1	1		12
Integer					1					1					2
Any number											1			1	2
Larger number		1													1
Smaller number		1								1	1				3
Properties of additive inverse or the pair '-7 and 7'															
Are opposites	1		1	2	2	1	2	3		3	1	1	2	1	20
Add to zero	2	3	2				2		1		2	1	2		15
Operation and signifiers both need to be considered															
Adding a negative number and a positive number can be treated as subtraction	3				1		1			1			2		8
Subtracting a negative number can be treated as addition of a positive number		1		1			2	2	1	1		3	3	1	15
Subtracting a positive number can be treated as adding a negative number	1						1	1	1	1			1	1	7
When you have a negative number you automatically subtract							1								1
Symbols provide inconsistent meanings		1													1
Results of subtraction															
Larger number from a smaller number is not proper	2	1					1								4
Negative number cannot be done	1							1	1						3
Similarity of operations on the number line															
Addition		2		2	1			4	1	2		1	2	1	16
Subtraction	1	4		3	1			4	1		1	1	3	1	20
Total	15	16	7	12	9	5	14	19	10	14	10	11	19	8	169

Table 20 Code use for Narrative by Question with Thematic Groups

Codes	Q1	Q2	Q3	Q4	Q5	Q6A	Q6B	Q6C	Q6D	Q7A	Q7B	Q7C	Q7D	Total
Relationship between positive integers and whole numbers														
Are synonyms	3													3
Mean different things	11													11
'Addition makes larger' remains true with integers														
Yes, it remains true		3												3
No, not always		11												11
Types of numbers generated by integer subtraction														
Positive number			11											11
Negative number			12											12
Integer			2											2
Any number			2											2
Larger number			1											1
Smaller number			2						1					3
Properties of additive inverse or the pair '-7 and 7'														
Are opposites					20									20
Add to zero					15									15
Operation and signifiers both need to be considered														
Adding a negative number and a positive number can be treated as subtraction		1			1	2	2					1	1	8
Subtracting a negative number can be treated as addition of a positive number			1			6	3	2	3					15
Subtracting a positive number can be treated as adding a negative number										2		5		7
When you have a negative number you automatically subtract							1							1
Symbols provide inconsistent meanings													1	1
Results of subtraction														
Larger number from a smaller number is not proper										1	1	2		4
Negative number cannot be done						1		1	1					3
Similarity of operations on the number line														
Addition						2			6				8	16
Subtraction						2			6			1	11	20
Total	14	15	31	0	36	13	6	3	17	3	1	9	21	169

Relationship between positive integers and whole numbers. The participants made statements coded as *are synonyms* and *mean different things* in response to Question 1 which asked the participants to respond to the prompt 'What's the difference between whole numbers and positive integers?' Three of the participants, Amanda, Christina, and Jordan, provided responses which suggested that positive integers and whole numbers *are synonyms*. Christina's answer "I thought they were the same thing. Integers are whole numbers" can serve to represent all three participants' responses. The other 11 participants provided answers which were coded as *mean different things*. Many of these responses reveal that the participant did not articulate, and therefore would not be considered to maintain, definitions for these phrases which are consistent with exemplary discourse practices, as specified in Chapter 2. Jamie's response; "If you were talking about a positive integer specifically, you are talking about a positive number, whereas a whole number could be positive or negative," displays this inconsistency well. Only those three PSTs (Amanda, Christina, and Jordan) whose responses were coded *are synonyms* are considered to have expressed a narrative consistent with common practice; the other 11 are considered to have expressed an inconsistent narrative.

'Addition makes larger' remains true with integers. In response to Question 2, which prompted the participants to consider whether the notion 'addition makes larger' is applicable to integers, Brooke, Christina, and Nicole made, and supported with examples, claims that it does remain true, while the other 11 participants justified their claim that it is not valid when integers are added. Question 2 was the only point during these interviews in which a claim related to this narrative was made by any participant. Nicole's response is a noteworthy example of believing that 'addition makes larger' remained true "no matter if you're adding negative integers or positive integers, you're always going to a higher number," while this is correct when referring to absolute values or magnitudes of the sums, she is incorrect regarding actual values. This is in contrast to Kayla's response "No, [because] you can add negative numbers and then if you add two negative numbers it would become less, like [a] higher negative number which would … is further below zero" which she followed up by providing the example "negative two plus negative six your answer's negative eight." The responses that 'addition makes larger' remains true, which were provided by Brooke, Christina, and Nicole, are inconsistent with the narratives of discourse about integers, while those provided by the other 11 PSTs that it does not, is, in principle, consistent with common practice.

Types of numbers generated by integer subtraction. Each of the participants suggested some range of numbers which subtraction with integers can generate. These responses can be seen in the list of codes grouped as *types of numbers generated by integer subtraction: positive number, negative number, integer, any number, larger number, and smaller number* (see Table 21). The number of different types of numbers coded in the participants' responses range from Tae (n=1) up to a maximum of four (Kayla and Monique). With one exception, Monique's response to Question 6D, these codes occur in association with Question 3.

Table 21 Code Use for Types of Numbers Generated by Integer Subtraction by PST and by Question

Code by PST	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Types of numbers generated by integer subtraction														
Positive number	1		1	1		1	1	1	1	1	1	1	1	
Negative number	1		1	1	1	1	1	1	1	1	1	1	1	
Integer					1					1				
Any number											1			1
Larger number		1												
Smaller number		1								1	1			

Code by Question	Q1	Q2	Q3	Q4	Q5	Q6A	Q6B	Q6C	Q6D	Q7A	Q7B	Q7C	Q7D	Total
Types of numbers generated by integer subtraction														
Positive number			11											11
Negative number			12											12
Integer			2											2
Any number			2											2
Larger number			1											1
Smaller number			2						1					3

A quote showing an example of how each of these codes was applied will be provided here. In her response to Question 3, which asked “What types of numbers can you get when one integer is subtracted from another integer?”, Courtney provided an example of both *positive number* and *negative number*, as seen in this quote, “you can get positive or negative, depending on the question given.” Jacqueline’s response to this same item was, “You would only get positive because … you can’t have a negative integer” was coded as *negative number*, yet it is unique among the statements coded as *negative number* as, in it, Jacqueline claimed that a negative number cannot result from subtraction involving integers. This is inconsistent with the core narratives of a discourse about integers. The code *integer* was applied to “You’ll still get an integer” which was the first part of Christina’s response, supplemented by “if you did, four minus six, you get negative two," which was coded as *negative number*. Monique's response "you can get positive numbers, you can get negative numbers, you can get fractions… you can get a decimal" was coded as *any number,* due to its extensive list of possibilities. In a final example of how these codes are used, Ashley stated "When one integer is subtracted from another integer you can get a number that is bigger than what you started with, or a number that is smaller than you started with" which was coded as both *larger number* and *smaller number*. Thus, we see that, in most cases, the participants provided answers which included several distinct types of numbers being generated when integers are subtracted. Only Jacqueline, Monique, and Tae, who indicated that you can get *any number* or that you cannot get a *negative number,* would be considered as providing a response which is inconsistent with common practice. The responses provided by the other 11 would be deemed as consistent with common practice.

Properties of additive Inverse or the Pair '-7 and 7'. The two codes *are opposite* (n=20) and *add to zero* (n=15) occur only in the participants' responses to the first three prompts in Question 5, which collectively sought out the participants' discourse about the properties of the phrase additive inverse, along with their responses to the question "Why are 7 and -7 additive inverses?" In their responses to this series of prompts all participants except Ashley and Jordan made a statement which was coded as *are opposites*. One example of a statement which was coded as *are opposites* was when Jacqueline said, "They are opposite, negative and positive, so opposite integers." Further, this quote includes both uses of the code *are opposite*: 'opposite' and the paired phrases 'positive integer' and 'negative integer'. Amanda, Ashley, Bailey, Jacqueline, Jordan, Monique, Nicole, and Rebecca all made at least one statement which indicated that additive inverses or the pair of numbers '-7 and 7' *add to zero*. An example of this narrative was provided by Bailey when she stated, "they're additive inverses of each other, that means that if you add these numbers together you'd get zero."

The idea that additive inverses, or the pair of numbers '-7 and 7', *are opposites* and *add to zero* represent a key structural and operational component of integers. While all of the participants used at least one of these two ideas, only six of them used both ideas. This leads me to conclude only Amanda, Bailey, Jacqueline, Monique, Nicole, and Rebecca displayed a discourse about integers which is consistent with common practice; the other eight PSTs omitted one of these key ideas and therefore presented answers which are deemed only partially-consistent with such a discourse.

Operation and signifiers both need to be considered. In each of these five codes, *adding a negative number and a positive number can be treated as subtraction* (n=8), *subtracting a negative number can be treated as addition of a positive number* (n=15), *subtracting a positive number can be treated as adding a negative number* (n=7), *when you have a negative number you automatically subtract* (n=1), and *symbols provide inconsistent meanings* (n=1), the participant suggests that, as a fundamental rule of adding or subtracting with integers, the size of the numbers and the operation need to be considered jointly while determining ways to arrive at the equivalent value. Such statements were distributed among Questions 2 (n=1), 3 (n=1), 5 (n=1), 6A (n=8), 6B (n=6), 6C (n=2), 6D (n=3), 7A (n=2), 7C (n=6) and 7D (n=2). All participants except Bailey, Courtney, and Monique provided a statement which was identified by one of the

32 uses of the codes within this thematic group. The distribution of how frequently each participant used these narratives is shown in Table 22. An example of the statements made by the participants associated with each of these five codes is presented here, however, a fuller appreciation of the impact these narratives had within the contexts of these PSTs' responses to these prompts is in Chapter 5.

Table 22 Code Use for Operation and Signifiers Both Need to be Considered by PST and by Question

Code by PST	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Operation and signifiers both need to be considered														
Adding a negative number and a positive number can be treated as subtraction	3				1		1			1			2	
Subtracting a negative number can be treated as addition of a positive number		1		1			2	2	1	1		3	3	1
Subtracting a positive number can be treated as adding a negative number	1						1	1	1	1			1	1
When you have a negative number you automatically subtract							1							
Symbols provide inconsistent meanings		1												

Code by Question	Q1	Q2	Q3	Q4	Q5	Q6A	Q6B	Q6C	Q6D	Q7A	Q7B	Q7C	Q7D	Total
Operation and signifiers both need to be considered														
Adding a negative number and a positive number can be treated as subtraction		1			1	2	2					1	1	8
Subtracting a negative number can be treated as addition of a positive number			1			6	3	2	3					15
Subtracting a positive number can be treated as adding a negative number										2		5		7
When you have a negative number you automatically subtract							1							1
Symbols provide inconsistent meanings													1	1

Adding a negative number and a positive number can be treated as subtraction. When providing or evaluating an addition expression containing a negative addend, five PSTs, over eight turns, indicated that subtraction can be performed. Christina, Jacqueline, and Kayla expressed this idea once each, Rebecca indicated it twice, while

Amanda noted it most frequently (n=3). For example, Christina stated, "then negative one plus nine equals eight because that's negative, if you add a negative number to a positive number you're subtracting." This narrative was stated by at least one participant when they introduced an expression (Questions 2 and 5), when they first encountered an expression (Question 6A), when recapping and comparing expressions (Question 6B), with the chip model (Question 7C), and with the number line (Question 7D). This shows that collectively, these five participants who mentioned this narrative find the use of this narrative appropriate over a wide variety of mathematical situations. Thus, Amanda, Christina, Jacqueline, Kayla, and Rebecca exhibited in these an idea which is common to discourse about integers, while the other nine participants did not exhibit such a narrative.

Subtracting a negative number can be treated as addition of a positive number. The code, *subtracting a negative can be treated as addition of a positive* (n=15) was used most frequently, and by more participants (n=9), than any of the others in this thematic group. An example of the idea represented by this code is found in Rebecca's statement while responding to Question 3, where she offered several expressions as examples of the different types of numbers that can result from subtracting integers which included, "if you would subtract a negative from a negative, so negative four minus negative one, you will get a negative number also, but the signs will change and the result will be negative three." This was the only example where this code was applied that did not relate to the expression '6 - -2'. Five participants (Ashley, Jacqueline, Jamie, Jordan, and Nicole) relied on this narrative to attempt this expression on each of their first encounters with it, while Nicole continued to employ this narrative with the colored chip and number line models as well. In an example from Nicole's first encounter she stated, "when you have two negatives next to each other, or a subtraction sign and a negative sign you make it a positive sign and then you add them together". This exemplifies her approach to all three representational formats. In making statements of this nature, Ashley, Brooke, Jacqueline, Jamie, Jordan, Kayla, Nicole, Rebecca, and Tae were exhibiting a component of a discourse about integers which was consistent with common practice, while Amanda, Bailey, Christina, Courtney, and Monique made no statements which exhibited such an idea.

Subtracting a positive number can be treated as adding a negative number. The narrative *subtracting a positive can be treated as adding a negative* was included in statements made by half of the participants (Amanda, Jacqueline, Jamie, Jordan, Kayla, Rebecca, and Tae). The first six participants listed used it with the expression '-4 – 2' and one, Tae, used it with '2 – 8'. Each participant only used it once; perhaps more telling is that five of the times the narrative was employed was with the chip model. In each of these instances, the participant was asked to take away either positive integers from negative integers or a larger positive integer than is present in the minuend. This use corresponds to rephrasing the expression to allow for a commonly employed approach, and thus provides a justification for the related routine. Therefore, while perhaps not the most sophisticated approach possible, each of these seven PSTs did exhibit a narrative which is consistent with common practice, while the remaining seven (Ashley, Bailey, Brooke, Christina, Courtney, Monique and Nicole) did not exhibit such a narrative.

When you have a negative number you automatically subtract. The narrative that *when you have a negative number you automatically subtract*, was only verbalized by one participant. This occurred as part of Jacqueline's response to Question 6B, where she stated:

> both of these ['6 - -2' and '–1 + 9'] are different than the two above ['10 – 2' and '6 + 2'] because they include negative numbers, but they're also the same because you once you have a negative number you automatically subtract them, so that ['-1 + 9'] would be the same as ten minus two,

when considering the four expressions in relation to each other. However, the frequency with which the participants rephrased addition with a negative number as subtraction, suggests that this narrative may be more common than is suggested by its single recorded use. Regardless of how widely held this narrative actually is, this statement exhibits a narrative which is inconsistent with the communally-held idea that different signs signify different objects or operations within an expression.

Symbols provide inconsistent meanings. Though the code *symbols provide inconsistent meanings* was used only once, it seems to capture the frustration which Ashley, at least, felt in trying to make sense of addition and subtraction with integers. Ashley made this statement which seems to have summarized her thinking at the end of her interview, "sometimes you would have to do the opposite of what the symbol is asking you to do … depending on if the number you are adding, or subtracting is a negative or a positive integer." While potentially unsophisticated, her statement conveys the increased complexity and interdependence found in operations with integers, especially when compared to the same operations with whole numbers.

Summary. Eleven participants used these five narratives 32 times to justify their routines in evaluating the three expressions where both positive and negative integers were provided, as well as some additional expressions suggested by the participants. Given the number of times these expressions were considered, these values appear small, however it is important to remember that only in those cases where these narratives were clearly stated, are they coded as such; additional probing regarding the participants' employed narratives which were not clearly spoken may have uncovered many more examples. Be that as it may, the presence of these narratives, especially when applied to the colored chip and number line models, suggests that the discourse about integers engaged in by these PSTs lacks a firm foundation built upon narratives. However, the narratives employed, while few in number, were consistent with comprehensive and rigorous discourse practice, with only Bailey, Courtney, and Monique being considered to have not exhibited any of these narratives, and thus not revealing anything about the narratives they may be capable of expounding upon. As with other codes in this chapter, the analysis of these narrative statements as part of their larger context will be provided in Chapter 5.

Results of Subtraction. The two codes which form this group are important, not for their number of occurrences, which is quite low, but because of the claims being made. The codes *subtracting a larger from a smaller is not proper* (n=4) and *subtracting a negative number cannot be done* (n=3) were collectively used by five of the participants

(Amanda, Ashley, Jacqueline, Jamie, and Jordan), with only one, Amanda, using any of these narratives more than once.

Subtracting a larger from a smaller is not proper. The statement associated with the code *subtracting a larger from a smaller is not proper* was used on four occasions where the participant (Amanda twice, Ashley, and Jacqueline once each) was considering the expression '2 – 8'. When approaching this expression, Ashley first attempted to show it using a drawn chip model. While attempting to model this expression with chips she stated, "two minus eight … I don't have enough to subtract two minus eight." Ashley then shifted to a number line and was able to model the expression (see Figure 34). Amanda and Jacqueline ran into similar expressions while trying to model '2 – 8' with chips, which they revealed in their narrations to their actions. Amanda said, "if you have two, you can't subtract eight from two" while Jacqueline stated, "two minus eight … if we start with two and know that we can't take away eight, so if we have two, we need six more, that would be negative six that we don't have." Additionally, while considering '2 – 8', Amanda stated "you can't subtract eight from [two] so you get negative six because it not a proper subtraction." In each of these cases, the statements would hold true if whole numbers were being considered, however, as part of a discourse about integers these claims are inconsistent with common practice and incorrect.

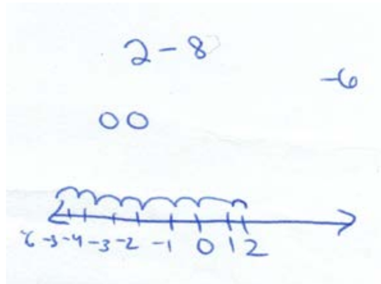


Figure 34 Ashley explanation of '2 - 8'

Subtract a negative number cannot be done. While considering the expression '6 - -2', three participants (Amanda, Jamie, and Jordan) stated that *subtract a negative number cannot be done*. Amanda seemed pretty confident in her statement, "six minus negative two, but you really … you can't take away negative two from here, so you end up adding two, so you have eight." In contrast, Jordan stated, "six minus negative two would also be eight. Negative eight … This is tricky, because you can't take away a negative there, so it is like saying plus, six plus a negative" and Jamie when she stated, "So here we are again, six minus negative two … It's the same thing as adding a positive. Because you can't take away a negative number" In both of these responses the participants were both more thoughtful and conflicted. This is true when positive and negative numbers are treated only in terms of their magnitudes in a way paralleling whole numbers, but this is not true when both their magnitude and direction as integers are considered. While each of these participants were able rephrase '6 – -2' as '6 + 2' and proceeded to arrive at the equivalent value of eight, the existence of this narrative in their discourse about integers, combined with their relative uncertainty of how to approach such an expression, indicates that these participants' discourse about integers involving subtraction of a negative is underdeveloped and, at most, partially-consistent with discourse practices' suggested use with the chip model.

Summary. While there is a lack of direct common usage of the ideas expressed in these codes (subtracting a *larger number from a smaller one is improper,* and subtraction of a *negative number cannot be done),* they contrast with all of the fundamental narratives within a mathematical discourse about integers as outlined in Chapter 2. As such their use by Amanda, Ashley, Jacqueline, Jamie, and Jordan is inconsistent with common discourse practices. The other PSTs never expressed these ideas.

Similarities of operations on the number line. While creating drawn number lines, or using those provided, eleven of the participants, all except Bailey, Courtney, and Jacqueline, made statements to the effect that addition and subtraction expressions on the number line are done the same way. These are coded as *addition* and *subtraction* to denote the operation being considered within the thematic group *similarities of operations on the number line.* These 16 statements for *addition* and 20 for *subtraction* were distributed over three addition prompts within Questions 6A, 6D, and 7D and four subtraction prompts within Questions 6A, 6D, 7C, and 7D. Each of these codes will be considered separately.

Addition. Nine of the participants expressed statements which were coded as narratives describing the *similarity of addition on the number line*. Jamie (n=4) was the most frequent user of this narrative, followed by Ashley, Brooke, Kayla, and Rebecca who each used it twice, and Christina, Jordan, Nicole, and Tae, who each used it once. Kayla's statement while modeling the expression '-7 + 1': "I would start at negative seven and since we are moving, this way is positive, and this way is negative, to the right is positive, I move up one spot because you're adding one, and you're at negative six," serves as an example of when this narrative might be true. However, her description of how to model '-1 + -5' on the number line, "I would start at negative one and then since you're adding a negative number, …. because it's negative so I'd move to the left five spots." suggests that this might not always be so.

Subtraction. Eight participants expressed the idea that there is a similarity in the way to approach subtraction expressions on the number line 20 times. Ashley and Jamie (n=4) had the most turns coded, with Brooke and Rebecca stating this idea on three occasions, and Amanda, Christina, Jordan, Monique, Nicole and Tae each using it once. Ashley expressed this idea when she was considering the expressions '10 – 2', '2 – 8', and '-4 – 2', in stating "the subtraction symbol would tell me to go to the left, and it is a positive number", which is a valid rule for these expressions. However, she struggled to adjust this rule to '6 – -2', as can be seen in this quote, "the subtraction symbol is telling me to go left but since I'm subtracting a negative integer, it cancels it out, so I am essentially adding two to six. …I mean you can't just turn it into a positive integer."

Summary. While the variety of narratives to support the limited number of options for routines to model the evaluation of expressions containing integers is small, it is broader and more nuanced than what was captured by these two codes representing the existence or absence of similarity in modeling all expressions containing addition or subtraction of integers. Thus, the 11 participants who used one or both of these narratives are only partially-consistent with common discourse practices. Bailey, Courtney, and

Jacqueline never expressed such a narrative, which prevents us from identifying either the presence or absence of these narratives in their discourse about integers.

Summary of Narratives. When viewed in their entirety, the narratives provided by these PSTs indicate that their discourse about the narratives and the rules that support integers is, at best, either inconsistent or partially-consistent, as revealed. This is mostly influenced by their reserved use of narratives, collectively, and very few uses of narratives by Bailey (n=7) and Courtney (n=5), who expressed no ideas coded as narratives aside from those required to address Questions 1, 2, 3, and 5. None of the other participants was deemed to have a discourse which was consistent with common discourse practice across all seven of these thematic groups for integers. Further, being more prolific in providing narratives did not align with expressing narratives which are consistent with common discourse practices. Amanda (n=15 codes) was considered as expressing narratives which are consistent with common practice in five thematic groups, which represents the maximum. Jordan (n=10) had four, Ashley (n=16), Bailey (n=7), Christina (n=9), Jacqueline (n=14), Jamie (n=19), Kayla (n=14),Nicole (n=11), and Rebecca (n=19) each were credited with consistent usage in three thematic groups, while Brooke (n=12), Courtney (n=5), Monique (n=10) and Tae (n=8) had the least thematic groups considered as consistent, with two each. Each of the other coded turns was either considered as inconsistent or partially-consistent with common discourse practices. While numbers do not say everything, especially in a qualitative study, these values support my observation that these PSTs enacted discourses which tended to lack narratives consistent with those of the exemplar discourse described in Chapter 2.

Summary of Results by Element

In this chapter, I showed the composition of responses provided by the 14 participants in my interviews, as considered in terms of the four elements which make up discourse. The full text of these interviews is provided in Appendix R and the full table of coding assignments is listed in Appendix S. The elements of word use, and visual mediators provide the basis for conveying computational routines and structural narratives. The participants were able to use the words which make up the discourse to effectively read and rephrase expressions and offer equivalent values. While these PSTs did not venture far from what was written to provide a more general view of the expressions, they also tended to not make incorrect statements or use words improperly. Their use of visual mediators included writing numbers, signifiers, operators, expressions, drawn chips and number lines, as well as making use of colored chips and a premade number line. These PSTs tended to copy or write their expressions clearly, however, the presence of pre-written expressions for all of the items they were asked to consider makes it difficult to assess how much they would have written under other circumstances. The use of colored chips and number lines appeared to be grounded in discourse about whole numbers. With this background for the use of these visual mediators, the participants were not able to adjust to representations which preserve the values of the integers and the operations being considered. Even more so than with visual mediators, and directly intertwined with them, was their struggle to come up with routines which maintained the original values and operations included in an expression to model the finding of an equivalent value to an expression. However, many of their approaches are consistent with commonly-employed algorithmic approaches. These issues both converge at, and emerge from, the absence of a suitable narrative structure about integers as the foundation of their discourse about integers. However, these conclusions are based on only one way of looking at the data, that of assessing individual word usages, the use of certain drawn forms and sorting decisions with visual mediators, assessment of the types of routines described and the surface appearance of those narratives expressed, which tended to fragment each response, but allowed for an examination across all codes. In the next chapter I examine the same data thematically, while looking at the participants' use of words, visual mediators, routines and narratives in a holistic manner.

Chapter 5: Thematic Results Across Elements

In Chapter 4, I examined the participants' discourse by looking at the components of their discourse by codes and thematic groups within elements. In this chapter I examine their discourse as it unfolded within the entirety of their responses across all four elements, based on themes within the 32 interview prompts. The first several of these themes are the participants' descriptions of narratives, including 'the difference between whole numbers and positive integers', the nature of equivalent values while performing operations with integers, and properties of additive inverses or the pair '-7 and 7'. These are followed by a discussion of the participants' engagement with expressions, including their approaches to modeling the determination of equivalent values for expressions and similarities between expressions. As each of these themes are described, the prompts which elicited each theme are stated. This chapter concludes with a description of the participants' self-expressed uncertainly related to the ideas examined during the interviews. This supports the approach taken in Chapter 4 which examined the codes used across questions, by knitting together the ideas across elements. These two ways of looking at the data allow for a more complete and richer way to characterize these PSTs as participants of a mathematical discourse about integers, than either would allow in isolation.

The Difference Between Whole Numbers and Positive Integers

In responding to the prompt "If a seventh-grade student were to ask you, 'What's the difference between whole numbers and positive integers?' how would you respond?" Each participant responded to this question in what I consider to be one turn, or a continuous dialogue in response to this prompt, except Tae, who chose to elaborate her initial response after Question 3, in a second turn during which she did not duplicate any of the ideas coded in her initial response. These 15 turns were coded 81 times using 16 distinct codes. The participants can be divided into two groups based on the narrative they provided: those who suggested that whole numbers and positive integers are synonyms, and those who did not.

Three of the participants provided narratives indicating that positive integers and whole numbers are synonyms. The most straightforward comment was made by Christina, when she stated that "integers are whole numbers." Amanda went a bit further in her statement, "a positive integer's anything like, on a number line, it's anything after zero and a whole number is the same thing." Jordan provided a clearer and more specific answer, when she said that, "I understand that integers mean all positive and negative numbers also zero, but no fractions, [no] decimals, but I also see a whole number as being the same thing." Thus, we see that each of these participants considered positive integers and whole numbers to be the same thing, while providing different descriptions of what they believed that shared meaning to be. While this idea is held by many, a careful comparison of the two discourses, as they are described in Chapter 2, will show that they differ in several fundamental ways, including the existence of a negative integer which corresponds to each positive integer, a relationship which is not present within the whole numbers. Thus, Amanda and Christina are only partially consistent with common and rigorous discourse practices, while Jordan's definition of an integer is consistent, her definition of whole numbers is inconsistent with such discourse practices.

The other 11 participants' narratives provided descriptions of whole numbers and positive integers as different. Within this group of participants, seven described whole numbers as being both positive and negative, while positive integers are only greater than zero. The idea that these two sets of numbers do not include fractions or decimals was included in the responses provided by four of these seven, namely Ashley, Kayla, Nicole, and Rebecca. We see this in Ashley's statement, which she made while moving her hands along the table over a visualized number line, "numbers on the right side of the zero are positive integers and then a number on the left side of the zero would be negative integers and then all together, they're all whole numbers" to which she added that "there are numbers in between whole numbers that aren't whole numbers." The other three of these seven, Jamie, Monique and Tae, would allow for the presence of fractions or decimals in their definitions. This is shown in Tae's statement which she made while pointing to her number line shown in Figure 35, "So I would show them that these are all whole numbers but only this group are the positive integers and these ones are the negative integers." The arrow in Tae's drawing shows the positive integers, while the entire line reflects her definition of whole numbers. Monique went beyond allowing for decimals or fractions, and specifically included them, as seen in this quote, "positive integers are numbers that are from zero to a hundred and beyond, and that whole numbers can include negative numbers, fractions, decimals." These seven participants all define the positive integers as a subset of the whole numbers, while differing on the inclusion of the decimals and fractions in the whole numbers. This break with standard discourse practices seems consistent with Monique's observation that "I might have confused the whole numbers with the set of real numbers." The discourse provided by all seven of these PSTs would be inconsistent with customary practices in mathematical discourse about integers. Further, it should be noted that these PSTs provided a definition of whole numbers which is entirely inconsistent with common practices in mathematical discourse about whole numbers.

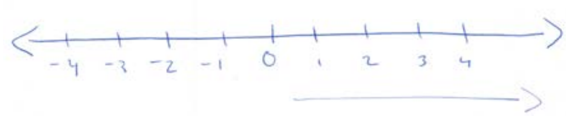


Figure 35 Tae's number line for Question 1

The other four participants, who provided descriptions of whole numbers and positive integers as different, did not mention negative numbers in their responses. Instead they used the inclusion of fractions or decimals as their criteria. Brooke and Jacqueline provided very similar definitions, when Brooke stated, "positive integers can have a decimal or a fraction of a whole number," and Jacqueline responded with, "a whole number is non-decimals and a positive integer would be … a positive integer would be anything that is over, above zero." These definitions contrast with Courtney's statement that with whole numbers "you could still do a fraction, so it would be like twelve out of twelve or like ten out of ten, where positive integers is [sic] just like seven." Bailey's statement "Well, I kind of remember whole numbers as being different because positive numbers are, positive integers are like rational numbers I think, that word, irrational numbers I guess you could say," while clearly showing that she thinks they are different, is in other ways harder to reconcile, because she uses two contrasting sets of numbers, the rational and irrational, in her definition. This would suggest that during her response to this question, Bailey was searching for the words to use to describe her thinking. These four PSTs provided definitions of positive integers and whole numbers which refer to the same region of the number line, specifically that greater than zero. However, their integration of fractions and decimals makes their discourse about integers inconsistent with common practice.

Overall, the responses provided by these 14 PSTs can be characterized as revealing a discourse founded on nonstandard, or more simply incorrect, meanings for the terms 'whole numbers' and 'positive integers'. Seven participants claimed that 'whole numbers' stretch from negative infinity to positive infinity, while positive integers are only that portion of 'whole numbers' above zero. These participants have the terms 'whole number' and integers reversed, which is an issue that might be easily reconciled with a bit of review or reference to the textbook. Additionally, four PSTs linked 'whole numbers' or 'positive integers' with fractions or decimals. Two of the three who described 'positive integers' and 'whole numbers' as being the same things, or synonyms, failed to reference the other half of the integers, whose existence necessitates the idea of 'positive integers' as a non-duplicate term for 'whole numbers'. Thus, 11 of the PSTs displayed discourse which is inconsistent with common discourse practices. Two of the PSTs, Amanda and Christina, displayed partially-consistent discourse practices, while only Jordan provided a definition which is consistent with a rigorous interpretation of communal mathematical discourse practices about integers.

The Nature of Equivalent Values while Performing Operations with Integers

As part of the interviews, the participants were prompted to describe the nature of the equivalent values produced when integers are added or subtracted. These prompts were delivered as Questions 2 and 3, which were stated as "It is a familiar notion that 'addition makes larger'. This is true when non-zero whole numbers are added together. Is this also true when integers are added? Please provide some examples." and "What types of numbers can you get when one integer is subtracted from another integer?" The responses to Question 2 occurred during 14 turns and were coded 112 times using 21 distinct codes. The responses to Question 3 occurred during 14 turns and were also coded 112 times using 25 distinct codes.

For the prompt 'addition makes larger', three of the participants felt that this remains true when integers are being added, while the other 11 felt that is not true. Similarly, for the prompt regarding the types of numbers one can get when integers are subtracted, seven distinct narratives were identified, with all participants except Tae providing two or more of these narratives. The most numerous example of the use of multiple narratives included the indication that both 'positive number' and 'negative number' are generated when integers are subtracted, which was provided by 10 of the participants. In addition, 'negative number' resulting from subtraction was provided by Christina, after indicating that an integer would be produced. Jacqueline accepted positive numbers as a difference for an integer subtraction expression, while she rejected the possibility of negative numbers when she stated, "You would only get positive because negative, you can't have a negative integer." While these 12 PSTs indicated that 'positive numbers', 'negative numbers' or both can result from subtraction, two PST indicated different ranges of numbers. Ashley claimed that both larger and smaller numbers can result but did not mention the positive or negative nature of the numbers, while Tae suggested any number as a possible result. These narrative claims were supported by sample expressions by all the participants. These took the form of complete expressions being verbalized by all participants except Bailey and Ashley. Bailey wrote expressions which she did not verbalize in the case of addition, and Ashley did not present an expression for subtraction. Five participants put their subtraction expressions in writing while responding verbally to this prompt. The participants' claims about the types of equivalent values which can be produced by addition and subtraction will be further discussed presently.

Eight of the participants indicated that 'addition makes larger' is not true for integers, and that both positive and negative numbers can be generated when integers are used in subtraction. This portion of Jamie's interview provides an example for these eight, in response to the prompt regarding 'addition makes larger', she stated, "It is not true because integers can be negative so if you are adding negative numbers together you are not going to get a larger number" which she followed with the examples "negative two plus negative two you get negative four" and "adding two and two." In response to the prompt about the types of numbers subtraction can generate, she stated, "positive or negative numbers" which she was able to support with the examples "ten minus three you'd get seven and that's a positive number" and "three minus seven and you would get … a negative four." Jamie, and the other seven PSTs who rejected the notion 'addition makes larger' for integers and who identified both positive and negative numbers as outputs for integer subtraction, displayed a mathematical discourse about integers which is consistent with commonly accepted norms.

The other three participants who rejected the claim 'addition makes larger' for integers each provided a different description of the type of numbers as resulting from subtraction. Ashley indicated that subtraction could generate both larger and smaller numbers without specifying the type of number when she said, "when one integer is subtracted from another integer … you can get a number that is bigger than what you started with, or a number that is smaller than you started with." Tae's statement "You can get any number I think" was supplemented with the following expressions in response to follow-up probes "seven minus nine would give you a negative integer" and "eleven minus nine and you get two," and claims no restriction on the types of numbers that could be obtained; but she only included integers in her examples. In contrast, Jacqueline only allowed for positive numbers for subtraction, as described above. While Ashley and Tae appear to have engaged in a discourse about integers which aligns with common practices, Jacqueline's discourse must be identified as having been inconsistent with common discourse. In addition to her false claim about integer subtraction, Jacqueline only provided a single example of integer addition in which both numbers were negative, which suggests that integer addition always results in a sum which is smaller than either addend.

Brooke, Christina, and Nicole each suggested that addition continues to result in larger numbers, while providing various types of numbers as possible outputs for integer subtraction. Christina claimed that subtraction can yield an integer, which was supplemented in response to a probe with the expression "four minus six, you get negative two." In contrast, Brooke and Nicole both claimed that subtraction can result in either positive or negative numbers. In response to the 'addition makes larger' prompt, Brooke initially offered the expression, "negative six plus negative four" but followed this with the correction "that wouldn't be plus negative. That would just be plus four … which would give me negative two which is bigger than where I started at negative six, and it is closer to a whole number as well." Brooke's response to the prompt "What types of numbers can you get when one integer is subtracted from another integer?" was "you can have whole numbers whether it be positive or negative, and then you can also have integers meaning decimals or fractions of a whole number." This violates discourse norms in terms of the definition of whole numbers and integers, as well as suggesting that answers outside of the integers are possible. While Christina and Nicole displayed discourses about integer subtraction which appeared to be partially-consistent with common discourse practices, their discourses about integer addition were inconsistent with such discourse practices. Brooke's discourses about integer addition and integer subtraction are inconsistent with the practices of the discourse community, and involved some significant errors, such as allowing for rational number differences.

It is impossible to use a single level of consistency with communal discourse practices for the discourse about addition and subtraction with integers for these 14 PSTs. In total, it is possible to consider the responses of 10 of the PSTs as being consistent with common discourse practices. This includes the eight PSTs who rejected the claim that 'addition makes larger' and who suggested that subtraction with integers can yield both positive and negative numbers, and Ashley and Tae, who provided responses describing different collections of numbers then these eight. Brooke's and Jacqueline's discourses must be identified as inconsistent with that common discourse. As noted, Christina and Nicole provided responses which were either only partially-consistent or fully inconsistent with communally-employed discourse practices about subtraction or addition with integers.

Properties of Additive Inverses or the Pair '-7 and 7'

As discussed in Chapter 2, the idea of additive inverse enters mathematics as part of the mathematical discourse about integers. It describes the fact that a number and its opposite have a sum of zero. Though the concept of additive inverse may not be used explicitly very often, it provides the metarule narrative supporting many routines for computing with numbers of different signs. The first three prompts in Question 5 related to the meaning of the term additive inverse or the properties of the pair of integers '-7 and 7'. In the 14 turns taken in response to the initial prompt, codes were assigned 80 times using 22 distinct codes. While the most numerous were '*positive number*' (n=9) and '*negative number', zero, and addition* (n=8), the more important codes were the narratives *add to zero* (n=7) and *are opposites* (n=7). The second prompt, 'why are 7 and -7 additive inverses?', was responded to during 12 turns, with codes assigned 68 times, using 15 distinct codes. As with the initial prompt from this question '*positive number*' (n=10), '*negative number'* (n=9), *zero* (n=6), and *addition* (n=6) continued to occur most frequently, while '*opposites*' (n=6) and *spoke of number line* (n=6) gained in relative abundance. The narrative codes *add to zero* (n=7) and *are opposites* (n=6) continued to most clearly convey the main idea under consideration. For the third prompt, 'how else would you describe two numbers that are additive inverses?' the participants responded during 12 turns to which codes were assigned 34 times using 17 distinct codes. The narratives *add to zero* (n=1) and *are opposites* (n=7) continued to be identified, and to serve to convey the key idea of this phase of the discourse. The word use codes '*opposites*' (n=4) and '*same number*' (n=4) also contributed to their responses to this prompt.

In responding to these three prompts, eight of the participants initially explicitly stated that they were unfamiliar with the phrase, three provided definitions which focused on one aspect of the phrase without stating that the term was unfamiliar, two focused on the pair of numbers provided while they avoided the phrase additive inverse, and one, Rebecca, provided a response aligned with common discourse practices. Brooke's response to the first probe provided an example of the type of response indicating unfamiliarity with the phrase when she stated, "When I think about it, I am not familiar with the entire term, but just using context clues, I think inverse means opposite, so here you have a negative seven so the opposite of that would be positive seven." In developing her response, Brooke was able to use the integers provided in the prompt to form a response which aligned with the *'opposite'* aspect of additive inverses. In an analogous manner Bailey, Jordan, Kayla, and Nicole used the values provided to form responses that attended to either the *'opposite'* or *add to zero* aspect of additive inverses. In response to the additional probes offered in Questions 5A and 5B, two of these participants expanded on her initial response just described. Bailey, who initially described that negative seven and seven *add to zero* without using the phrase additive inverse was able to add that this pair of numbers are opposites. Conversely, Jordan added *add to zero* to her initial response of negative seven and seven being opposites. Kayla and Nicole did not expand on their initial ideas in any of their later responses. Over the course of their responses to the three prompts being considered, these five PSTs exhibited partially-consistent discourse about this aspect of integers.

The other three participants who expressed uncertainty or who were unsure of how to address the topic did not make use of both of the numbers provided. Tae's attempt, in which she started to answer, but lost confidence and stopped after stating, "if I were to add seven, that's not true. I know" revealed an initially high level of confidence that disappeared as she responded. This initial confidence was not as evident in this response provided by Monique:

> I believe it means that if you can, inverse. If you add, no, not zero. I do know they're opposite each other and their additives possibly if you add a negative seven here negative seven to seven that would give you zero. I'm not sure, can we skip this question?

In this response, Monique was able to describe the fact that the two numbers provided add to zero, but did not connect that idea with the phrase, and questions her own quite well-stated response. Christina's progression through the prompts had her providing an initial response of "no idea," after which she was able to describe this pair of numbers as being the same distance from zero, but did not label them as opposites. Tae, who was initially stumped, was later able to provide the description that this pair of numbers was '*opposites*'. In her responses to the later prompts, Monique did not model a discourse that expanded on those in her initial prompt. These three, though expressing a lack of awareness on how to engage in suitable communal discourse practices about this aspect of mathematical discourse about integers, provided responses which are partially-consistent with those practices.

In contrast to Christina, Monique, and Tae, who, though they doubted their discourse were able to provide suitable responses, Ashley, Courtney, and Jamie proceeded as though they could define the topic, while providing responses that were only partially-consistent with common discourse practices. Ashley responded with, "Added inverse? (Interviewer: Additive inverse) Additive inverse, so if you were to add these two together, you would get the number zero". She did not initially recognize the phrase, but was able to state that the sum of negative seven and seven is zero. Courtney's response of "you know, when I was taught about inverses, it is like the flipped version of seven or negative seven. Negative seven can turn into a positive seven. Adverse [sic] I don't remember going over ever" again expressed an unawareness of the phrase but indicated that the numbers are *opposites*. However, Courtney's use of "flipped" for what I am assuming to indicate *opposite*, employs a term more often used to describe reciprocals of fractions. In contrast to these succinct responses, we have this passage by Jamie:

> Ok, so additive inverse, I think it means…. So, let's say you're starting at… First thing I thought of, if you're starting at zero and you're adding seven you're going to get positive seven and then if you're starting at zero and you're taking away seven you get negative seven, so I'm seeing them sort of as they're the same number technically, but they're inverses because one is positive and one is negative, I guess, I guess additive

> because, yeah, I guess I am just going to go with you can add seven and then you can also add I guess, like, I don't want to say add negative seven to zero but they are sort of just like on opposite sides of the zero and they're the same distance from zero. So that is what I was thinking.

In this response, Jamie talked her way into putting the ideas together, which she finally achieved by stating that they are opposites. Of these three, only Courtney's responses to the later prompts, where she added that most numbers have the property of having an opposite, increased the scope of the content they introduced in responding to the initial prompt.

Two participants, Amanda and Jacqueline, avoided the phrase additive inverse entirely and focused on the pair of numbers provided. In doing so Jacqueline, identified both relationships in this pair of numbers in her statement, "The opposite of each other, so you would have negative seven and positive seven, so they would equal, negative seven plus seven is zero." The same can be said of Amanda who identified the fact that negative seven and seven are opposites and that they have a sum of zero as well. The fact that they were able to identify the relationships, but not the phrase, makes their discourse about integers partially-consistent with common discourse practices.

The only participant to provide a clear definition which is consistent with common discourse practices, as described in Chapter 2 was Rebecca, who started off by saying, "So the phrase additive inverse means that if you add these two values together the result would be zero." She goes on to describe their opposite relationship as viewed on a number line in this way, "seven and negative seven coming into zero is an equivalent value. Seven bumps this way … and then seven bumps this way. So, these values are able to meet in the middle at zero." That by itself was a well-constructed representation of this idea, which was enhanced when she continued to describe how when "you add, for instance, the negative seven to the seven you're really subtracting the value of seven from seven, so you're getting to zero" and "if you add the value seven to negative seven, you're adding seven, seven towards zero, to the right, so that gets you to zero when you add seven to negative seven." While providing a complex description, she makes explicit the fact that as examples of a pair of additive inverses, the numbers negative seven and seven add to zero, she also described them as being opposites without using this phrase.

The last prompt of Question 5 asked the participants "Can you give me another pair of additive inverses?" Each of the 14 participants contributed to the total of 16 spoken and two written pairs of opposites provided. This occurred over 14 turns and included the assignment of codes 30 times requiring six distinct codes. The 18 pairs provided included 'negative two and positive two' (n=3), 'negative three and positive three' (n=1), 'negative four and positive four' (n=3), 'negative five and positive five' (n=3), 'negative six and positive six' (n=3), 'negative nine and positive nine' (n=1), '-10 and 10' (n=2), '-11 and 11' (n=1), and '-20 and 20' (n=1). While most of the pairs were presented a form exemplified by Bailey's statement "Negative nine and nine", Jacqueline stated her response in the form of "negative two plus two, you get zero." These responses show that these participants were able to exhibit pairs of opposite values that mirror '-7 and 7' in a manner that is consistent with common discourse practices.

When the responses to the four prompts which comprise Question 5 are analyzed together, Rebecca remains the only participant who exhibited mathematical discourse about integers which is consistent with common discourse practices. The other 13 PSTs, while deemed partially-consistent with discourse practices when including '-7 and 7', would be considered as presenting inconsistent discourse practices if I limit my focus to the term additive inverse as representing the sum of a number and its opposite are zero, see Chapter 2.

Up to this point in this chapter, the alignment of the participants' narratives with commonly accepted discourse practices has been the primary focus. When using narratives related to the differences in the structure of 'whole numbers' and 'integers', the results of integer addition and subtraction, and the nature of the phrase additive inverse and numbers which are additive inverses, I found the participants displayed a range of different levels of consistency with common discourse practices. Jordan was the closest to consistent, having been deemed consistent for two themes, and partially consistent for one theme discussed so far in this chapter. Rebecca is the only other PST deemed consistent for two themes, but whose use was inconsistent with common discourse practices for the other theme. Amanda displayed discourse which is consistent with common practice for one theme, and partially-consistent for two themes. Ashley, Bailey, Courtney, Jamie, Kayla, Monique, and Tae were considered to be consistent for one of the themes discussed in this chapter, but were deemed partially-consistent for one theme and inconsistent for the other. Christina exhibited a discourse which was partially-consistent for each of the three themes discussed. Nicole, with two partially-consistent and one inconsistent, and Brooke and Jacqueline with one partially-consistent and two inconsistent represent the participants who were least consistent with common discourse practices among these PSTs. Thus, all PSTs except Brooke, Jacqueline, and Nicole, who I consider to have displayed discourse which is inconsistent with common practice, would be considered to have displayed partially-consistent discourse about these narratives about integers. When considering these levels of consistency with the general discourse about integers with reference to the larger community, it is fair to recognize that integers are not part of the curriculum for the mathematical content course they had taken at that time, nor is it part of the second course in that sequence.

In the next section of this chapter, my focus is on the participants' ability to determine equivalent values and model routines within a discourse about integers. This takes the form of describing the expressions they provided to support their claims related to possible results of addition or subtraction of integers. Further, I will analyze their use of visual mediators and routines to model the expressions I provided, including the alignment of any verbalized or visually-presented equivalent value to common discourse practices. This is followed by an examination of the similarities and differences the participants attended to when considering quartets of questions which all had the same equivalent value. It will close with a look at the participants' self-identified confusion with aspects of this discourse.

Providing or Modeling Expressions

As discussed previously, the participants introduced expressions in response to certain prompts (Questions 2, 3, and 5), or were asked to model expressions as part of other prompts (Questions 6 and 7), see Appendix A.

Expressions introduced to explain situations. The prompts for Questions 2, 3, and 5, led the participants to suggest expressions to explain and justify their claims. Each of the expressions presented contained integers ranging between negative ten and positive ten. Eleven participants contributed 22 examples of types of expressions, of which 19 included numbers in response to the prompt whether 'addition makes larger' with integers. For Question 3, which asked about the types of numbers resulting from integer subtraction, 10 participants offered 23 examples, including four which were originally phrased as addition expressions and two that become addition expressions upon rephrasing. Rephrasing occurs when the participant introduces an expression which differs in operation and values than that which was initially presented. Whenever an expression is properly rephrased it will have the same equivalent value as the initial expression, which is part of common discourse practices. In contrast, an improper rephrasing results in a new expression, with a different equivalent value than the original. Most of the rephrasing produced by the participants in my study provided properly rephrased expressions. An example of one such proper rephrasing was captured in Amanda's response to this question when she initially provided the expression "negative ten minus three", which to simplify she rephrased as negative ten plus negative three for an equivalent value of negative thirteen. In responding to the four prompts within Question 5, six participants offered 11 expressions, all of which contained numbers.

In their responses to Question 2, 11 of the participants were able to provide a complete example of an integer expression with defined terms and equivalent values. When they did so, they were able to provide the equivalent value in 16 of the 19 attempts. Brooke presented "negative six plus negative four … I'm sorry that wouldn't be plus negative. That would just be plus four … negative six plus four, one, two, three, four, which would give me negative two" where she started with adding a pair of negative integers together, then stopped before providing an equivalent value, then changed to adding a negative integer and a positive integer, which has a negative integer as the equivalent value. In Brooke's example, we see both an unsuccessful and a successful modeling of an expression including integers. Amanda provided two accurately solved expressions, one with a negative plus a positive equaling zero and the other with a negative plus a positive equaling a negative; however, for the later expression, she verbalized solving it as a positive minus a larger positive equals a negative integer. Ashley (n+n=N, for the two examples "-7 + -1 = -8" and "-6 + -6 = -12"), Bailey (n+p=0, "-1 + 1 = 0"), Christina (p+p=P, "2 + 2 = 4"), Courtney (n+n=N, "-7 + -6 = -13"), Jacqueline (n+n=N, "-5 + -3 = -8"), Rebecca (p+p=P, N+p=n, and n+n=N for the expressions "4 + 3 = 7," "-5 + 3 = -2," and "-5 + -3 = -8"), and Tae (p+p=P and P+n=p, "1/7 + 3/7 = 4/7" and "3/7 + -1/7 = 2/7") each provided complete number-based expressions along with the equivalent value. However, for none of these 12 expressions was the evaluation process verbalized or modeled.

Jamie presented an expression and its equivalent value for the sum of two negative numbers, but only presented the expression without the equivalent value for a pair of positive numbers. Similarly, Jordan provided a pair of numbers added together, but she did not provide the equivalent value. She then followed that with an expression based on types on numbers, in this statement, "add one and one together you're making a larger number, but if you add two negatives, you're making a lesser number." Nicole only presented examples of the latter type with two positives or two negatives being added together. Monique never attempted to construct such an example, either with numbers or with number types.

Ten participants offered sample expressions in support of their responses to Question 3 discussed above. In doing so, Christina (P – p = p and p – P = n, for the expressions "4 – 2 = 2" and "4 – 6 = -2"), Jamie (P – p = p and p – P = n, "10 – 3 = 7" and "3 – 7 = 4"), and Tae (p – P = n and P – p = p, "7 – 9 = -2" and "11 – 9 = -2") were able to provide complete examples. Jordan (P – p = p and N – n = n, "3 – 2 = 1" and "-5 – -3 = n") and Kayla (p – P = n and big – small = p, "2 – 6 = -4" and "if you subtract a smaller number from a bigger number, it is still positive") each provided two expressions one as a complete example, with numbers, and the other as a description of a rule, without numbers. Amanda and Rebecca presented three and four separate expressions, respectively, including three which they rephrased while presenting. Amanda determined the equivalent value in the form she presented the expression for "10 – 3 = 7" as did Rebecca for "3 – 2 = 1," "3 – 4 = -1," and "-4 – 1 = -5". After Amanda presented "-10 – 3" she rephrased it as "-10 + -3" which gave a sum of "-13", similarly Amanda's "10 – -

3" became "10 + 3" and produced a sum of "13," and Rebecca's "-4 – -1" became "-4 + $^{+}$1" which she assigned an equivalent value of "-3." Each of these rephrasings are consistent with standard discourse practices related to subtraction of integers, and are indicated as such in Chapter 2. The four other addition expressions noted in this table were provided by Bailey (n + n = N and n + p = p, "-1 + -1 = -2" and "-2 + 1 = 1") and Courtney (0 + n = n and p + p = P, "0 + -7 = -7" and "7 + 7 = 14") of which three of the four are valid expression/equivalent value pairs, but do not address the question of what type of numbers can result from integer subtraction. Ashley, Brooke, Jacqueline, and Monique each never offered an expression in support of their claim in addressing Question 3.

Question 5 asked the participants to describe the meaning of additive inverse and offered the pair of integers '-7 and 7' as additive inverses, this was followed by associated probes, see Appendix A. In total, this question with it four prompts, resulted in 11 expressions being suggested. Five of these expressions, provided by Ashley, Jacqueline, Monique, Nicole, and Rebecca, were presented as a negative integer plus a positive integer equals zero; three other expressions were offered in the form of a positive integer plus a negative integer equals zero. In using the latter arrangement of negative seven and seven, Monique and Jordan were both able to associate the value zero with 'seven plus negative seven' but Rebecca who started with this arrangement needed to rephrase this as 'seven minus seven'. Each of these approaches are consistent with common discourse practices, as is Jacqueline's expression of "negative two plus two, you get zero" when asked for another pair of additive inverses. Nicole's statement that "if you subtracted the negative from the positive you get zero" however is inconsistent with common practices of mathematical discourse about integers.

Provided expressions. During the course of the final two questions, Questions 6 and 7, the participants were asked to consider eight questions four times. These expressions were grouped in Question 6 as '10 + 2', '6 + 2', '-1 + 9', and '6 – -2' which each has an equivalent value of eight, while in Question 7 the equivalent value of negative six was associated with the expressions '-4 – 2', -1 + -5', '-7 + 1', and '2 – 8'. When the participants initially saw the expressions, they were asked to model them, in whatever way they desired. Secondly, they were asked to describe similarities and differences in the expressions just considered. During their third and fourth time considering the expressions they were prompted to model the determination of the equivalent value by using beige and brown chips and with a number line. While there are ways to model the expressions as presented in each of these cases, see Chapter 2, the standard approach for many of them is to rephrase them into a more limited number of option.

Four forms of representation were used while working through these four prompts. The most frequently enacted was spoken or verbalized words used to state the expression and the equivalent value. Of greater durability were the visual mediators of written expressions, beige and brown chip models, and the number line. The use of these last two were directly prompted for, while the use of the first was suggested but not required. When the participants performance is measure within each prompt in terms of any representation of the correct equivalent value, only 28 opportunities to indicated the equivalent value to the expressions are seen to be incorrectly answered (n=14) or not answered at all (n=14). However, as an example a closer look at the responses reveals that while Tae may have verbalized and wrote the equivalent value of eight for '10 – 2' she drew a representation similar to a chip model based on a ten-rod and unit squares but did not use that visual mediator to clearly model the equivalent value. Since the approaches taken by the participants has been explored in Chapter 4 when I discusses visual mediators and routines, I have focused on those responses which resulted in erroneous responses for either one or all the representations the participant opted to employ. The full code list for these prompts is found in Appendix Q. When working through any of the prompts in these two questions the participants had the expressions they were considering on the table in front of them. Thus, they were not required to repeated state the expression while working through it or write it down to have access to it while addressing it.

Initial exposure to the expressions. During their initial engagement with these expressions, each attempted model for '2 – 8' and '-1 + -5' were completed successfully. The expressions '10 – 2', '-1 + 9', and '-7 + 1' each had one representation marked as incorrectly addressed, while '-4 – 2' had two participants model it completely incorrect, '6 + 2' had three incorrect responses, while three participants provided incorrect responses and another three did not respond to '6 – -2' during this attempt. '6 – -2' is the only expression for which participants felt unwilling or unable to respond for the initial prompt in Questions 6 and 7. In Table 23 the asterisks (*) represent when the expression was considered to have correctly modeled the expression as presented, or one of its equivalent forms, for the expressions in response to the initial prompt in the respective question.

As mentioned Tae's attempted to use the chip model was noted as the only representation of the equivalent value which was not considered correct for '10 – 2'. Tae also had trouble modeling '-1 + 9', though for this expression she utilized the number line. While she was able to express verbally, and in the form of a written numeral, the equivalent value of eight, her number line contained ten hops from negative one to nine, in what I interpret to be an unintended modeling of '9 – -1', with this interpretation based on her approach to '6 – -2' for both her first attempt when she draws a number line, and when prompted to use the number line. Monique's responses to '-7 + 1' were both six which she intended to represent the amount of money still owed when she started seven dollars in debt and paid off one dollar.

Table 23 Correctly provided responses during the first exposure the expressions.

Initial expression prompt	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
'10 – 2'														
Verbalized		*	*	*	*	*		*	*	*	*	*	*	*
Written	*			*	*		*		*		*		*	*
Chips		*											*	X
Number line			*					*						
'6 + 2'														
Verbalized	*	*		*	*	*	*	*	*	*	*	*	*	*
Written	*						*		*		*		*	*
Chips		*											X	X
Number line			*					X						
'6 – -2'														
Verbalized	*	*	*		X	*			X	*	X		*	*
Written	*						*		X		X		*	*
Chips		*												
Number line														*
'-1 + 9'														
Verbalized	*	*	*	*	*	*	*	*	*	*	*	*	*	*
Written	*				*		*		*		*		*	*
Chips		*												
Number line			*	*				*					*	X
'-1 + -5'														
Verbalized	*	*	*	*	*	*	*	*	*	*	*	*	*	*
Written	*	*			*		*	*		*	*		*	*
Chips			*											
Number line													*	
'-4 – 2'														
Verbalized	*		X	*	*	*	*	*	*	*	X	*	*	
Written	*	*	X		*		*	*	*	*	X		*	*
Chips														
Number line													*	*
'-7 + 1'														
Verbalized		*	*	*	*	*	*	*	*	*	X	*	*	
Written	*						*	*	*	*	X		*	*
Chips		*												
Number line													*	*
'2 - 8'														
Verbalized	*	*	*	*	*	*	*	*	*	*	*	*	*	*
Written	*		*		*		*	*	*	*	*		*	*
Chips			*						*					
Number line		*		*									*	*

The * represents correctly expressions for the model, while X represents an incorrectly modeled expression

For the expression '-4 – 2' both Bailey and Monique arrived at the same value of '-2' for their first attempts at this expression. In both cases this was arrived at by subtracting two from the four. Monique kept negative two are her remaining debt in the calculation shown in Figure 36, which she wrote while staying "if I have … negative four dollars and I take away two of that, so maybe that would be like somebody taking that debt away from me so negative four take away negative two would be negative two." Bailey returned to this expression and by employing color coded drawn circles was able to determine that correct equivalent value should be negative six.

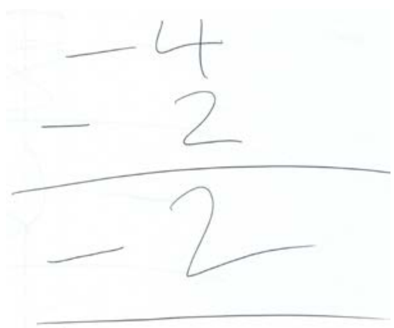


Figure 36 Monique's initial approach to modeling '-4 - 2'

The expression '6 + 2' elicited three inconsistent responses, where the verbalized and written equivalent values were the correct eight, the participants struggled with the drawn number line and chip models. Jamie in drawing her number line skipped the number eight and thus landed on nine, which she noticed as being incorrect and fix by inserting the missing number on the line and correcting her drawn representation. Rebecca and Tae each experienced trouble with their drawn chip models, by drawing the collections representing the addends but not representing the sum with drawn chips but use the numeral eight to indicate the equivalent value. While these represent minor mistakes in displaying this discourse, they appear more significant because the participant opted to enact these approaches.

The expression '6 – -2' was only correctly modeled every time by eight of the participants. Three of the other participants did not provide an equivalent value after identifying the procedure of changing subtraction of a negative as addition of a positive. Since they did not understand how the procedure works they did not chose to follow this procedure to generate a value, as is shown in Jamie's statement when addressing this expression, "I learned like that keep change, change procedure, but that really doesn't show any understanding of what you're actually doing, because if I did that it would be what six plus two because it would be keep change, change." Christina and Monique, each provided an incorrect equivalent value of four, while Jordan initially stated that negative eight was the equivalent value for '6 – -2'. Christina, after initially indicating the value of four second guesses herself, thinks more about it and concludes that the equivalent value should be eight. Jordan comes to a similar conclusion after expressing initial doubts and recognizing the common equivalent value of negative six for the expressions in Question 7. Monique arrived at four after converting the expression to "six take away two" while applying a debt model on items.

Over the course of these eight expressions the participants attempted 197 distinct representations employing either verbalizations, written values, drawn chips, or drawn number lines. They ended up not providing equivalent values on three occasions and providing and incorrectly modeled equivalent value for 16 representations or 11 occasions. These incorrect of models were made by for one expression by Bailey, Christina, Jamie, and Jordan, while Monique and Tae each provided incorrectly stated or modeled equivalent values for three expressions. As noted Tae's mismodeled equivalent values contrasted to her correctly stated equivalent values, while Monique's mistakes were complete for the three expressions she incorrectly answered. For each of the other expressions where an equivalent value was provided the PSTs provided the correct equivalent value and represented it in a manner which is consistent with common discourse practices. Thus Amanda, Ashley, Courtney, Jacqueline, and Kayla exhibited discourse which was consistent with common discourse practices. Bailey, Brooke, Christine, Jamie, Jordan, Nicole, and Rebecca, who each either provided one improper equivalent value, were unwilling to provide an equivalent value, or both exhibited partially-consistent discourse. I consider that Monique and Tae each exhibited mathematical discourse which is inconsistent with common discourse practices due to their three incorrect responses.

Identification of similarities and differences within the quartets of expressions. During the interviews, as part of Questions 6 and 7, the participants were initially presented with a quartet of expressions which they were asked to model as if they were teaching the topic to seventh graders. This was followed by them being prompted to identify similarities and differences in the expressions they had just considered. This was followed by them being asked to use the chip and number line models explicitly. As presented above, not all the participants determined the correct equivalent values at this point in the interview. The participants' discussion of each quartet of expressions will be considered in turn.

For the first quartet of expressions ('10 – 2', '6 + 2', '6 – -2', and '-1 + 9'), the participants tended to look at the type of integers that made up the terms, and the nature of the operation indicated or performed, without making general statements about the structure of computations with integers. This is shown in the high counts for the categories of word use, and the limited number of other codes shown in Table 24. However, this does not tell the whole story, though it does cover most of it. Further analysis of the responses shows that nine of the participants mentioned that all four expressions have eight as the equivalent value, with the other five not indicating this fact. Ashley, Courtney, and Jamie indicated that similar routines are suggested by different phrasing, while Amanda suggested that even when an operation is indicated, it is not always performed the same way. This can be seen in Amanda's statement:

> They are similar because they all have, they all come up with the same answer so they all equal positive eight. But they're different, well these two are both addition, but it's different because this one ['6 + 2'] is just a regular addition it is just basic, but this ['-1 + 9'] is actually, you have to do subtraction to find the answer. And the same thing with this, these are both subtraction but this ['10 - 2'] is just basic subtraction, but this one ['6 – -2'] you have to actually do addition in order to find the answer because of the negative number.

In this response, we see Amanda representing typical behavior in indicating that the equivalent values are each eight and recapitulating the operations which she had just performed on these expressions. Christina, Jordan, Rebecca, and Tae each commented on the level of ease or difficulty they felt a student would experience when encountering these expressions.

Table 24 Codes assigned to the PSTs' determination of similarities and difference in '10 – 2', '6 + 2', '6 – -2', and '-1 + 9'

Code	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae	Total
Word Use															
Set of numbers															
'Negative integer'				*									*		2
Integer				*											1
Other names for sets of numbers									*						1
Sign of number read or introduced															
Positive number	*	*	*	*	*	*	*	*	*	*	*	*	*	*	14
Negative number	*			*	*	*	*	*	*	*		*	*	*	11
Zero												*			1
Role of sign read or introduced															
Addition	*	*	*				*	*	*	*		*		*	9
"Plus" as addition sign				*	*			*			*			*	5
Subtraction	*	*	*				*	*	*	**		*		*	10
"Minus" as subtraction sign				*	*		*						*	*	5
"Take away" for subtraction		*													1
Negative sign											*		*		2
Visual Mediator															
Number line notations															
Number line dot hops dot								*							1
Drawn number line								*							1
Spoke of number line													*		1
Routine															
Actions with numerals															
Adding to a negative as subtraction		*													1
Subtracting a negative number is adding a positive number		*												*	2
Actions with number line															
Go up number line								*					*		2
Go down number line								*							1
Narrative															
Operation and signifiers both need to be considered															
Subtracting a negative number can be treated as addition of a positive number							*			*				*	3
Adding a negative number and a positive number can be treated as subtraction							*			*					2
When you have a negative number you automatically subtract							*								1
Participant voiced confusion								*							1
Total	4	6	3	6	4	2	8	10	5	8	3	5	7	8	80

A similar examination of the second quartet, which contains the expressions '-1 + -5', '2 – 8', '-4 – 2', and '-7 + 1', shows that the participants tended to focus on the components of the expressions, and similarities in the operations performed within the expressions, see Table 25. However, Jamie and Nicole provided more general trends in the operations by referring to the type of numbers employed rather than the specific numbers. This can be seen in Jamie's response:

> Things I notice: I got negative six for all my answers. I noticed that when you add two positive integers, I mean when you add two negative ones together you get a negative answer because you're just increasing, you're making a number smaller therefore making it a bigger negative number. I notice when you add a positive integer and a negative integer it becomes a smaller negative number because it's getting bigger and the smaller the negative number the bigger it is and also with two minus eight… No, well, when you take away a bigger positive integer from a smaller positive integer you are going to get a negative number because it's bigger than the small one so you're going to go past zero. To the negative zone.

This passage included an indicator that there is a common equivalent value for this quartet, namely negative six, a fact which was identified by nine participants. These nine participants include three of the participants who did not indicate that fact for the first quartet of expressions. Only Ashley and Jacqueline failed to observe the presence of a common equivalent value for at least one of the two quartets of expressions, which suggests a focus on the expressions and the computational approaches required to the exclusion of a broader examination across the expressions and their resulting equivalent values. Further, as is seen in this quote of Jacqueline's response:

> I would say the negative four plus negative two is the same, is very similar to negative one plus negative five the just, basically this is like you saying negative one minus five you would get the same response from that. And then negative seven plus one carries the negative and two minus eight this is the same thing as saying negative eight plus two. So, it would kind of be in the same category as that

Jacqueline rephrased these expressions, in an almost random way, with little regard for uniformity across routines or efforts to establish consistent symbol meanings.

In considering this latter quartet of expressions, more participants critiqued the expressions. These critiques took a number of forms, including Amanda's statements that "this one ['-1+ -5'] is just normal addition because they're both, they have the same symbol" and "for this one ['2 – 8'] you can't subtract eight from two, so you get negative six because it's not a proper subtraction." Brooke commented about the movement in different directions on the number line, which is the visual mediator Tae thought would be the best way to represent these expressions. Christina provided an extensive list of which aspects in these expressions made things complicated, while Bailey's response started with the rephrasing, "So, in negative seven plus one you are actually subtracting" and continued with, "This one, ['-1 + -5'] you're adding. This one ['-4 – 2'] I don't know what you're doing, it just happened, and then this one ['2 – 8'] you're also kind of adding on to the first number." Bailey's comment followed her stating that '-4 – 2' equals negative two initially, which she corrected to show an equivalent value of negative six.

Table 25 Codes assigned to the PSTs' determination of similarities and difference in '-1 + -5', '2 – 8', '-4 – 2', and '-7 + 1'

Code	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae	Total
Word Use															
Set of numbers															
'Positive integer'								*					*		2
'Negative integer'								*			*		*		3
Integer											*				1
Other names for sets of numbers									*						1
Sign of number read or introduced															
Positive number	*	*	*	*	*	*	*	*	*	*	*	*	*	*	14
Negative number	*			*	*	*	*		*	*		*	*	*	10
Zero												*			1
Role of sign read or introduced															
Addition	*	*	*				*	*	*	*		*		*	9
"Plus" as addition sign				*	*			*			*			*	5
Subtraction	*	*	*				*	*	*	*		*		*	9
"Minus" as subtraction sign				*	*		*						*	*	5
"Take away" for subtraction			*												1
Negative sign											*		*		2
Visual Mediator															
Number line notations				*										*	2
Spoke of number line															
Routines															
Actions with numerals															
Subtraction as adding a negative number							*					*			2
Actions with number line															
Go up number line				*											1
Go down number line				*											1
Narrative															
Results of subtraction															
Larger number from a smaller number is not proper	*														1
Participant voiced confusion			*					*							2
Total	5	3	5	8	5	2	6	7	5	4	5	7	6	7	75

Thus, when presented with an opportunity to describe these quartets of expressions, the participants tended towards listing the components of the expressions and how they would go about modeling them or determining their equivalent values. These responses may have been suggested by the nature of the prompts, see Appendix A, which might have been viewed by the participants as narrowing their freedom to choose what they wanted to comment on. From these responses, the discourse about integers exhibited by these PSTs can be characterized as highly focused on the structure of individual expressions and the routines which the PSTs would employ to model or to determine the equivalent value while they were not being directed to provide narratives which arch over the several examples. These PSTs provided a summarizing discourse about each quartet of expressions which was inconsistent with those aspects of a discourse about integers focused on narratives. They did exhibit a discourse which is consistent with the word use, and partially-consistent with the routine aspects, of a mathematical discourse about integers. They collectively did not exhibit, as part of their responses to this pair of prompts, the visual mediator aspect of a mathematical discourse about integers.

When prompted to use the chip model. The chip model was the first of the two visual mediators I prompted the participants to use. As is shown in Table 26, none of the participants was able to model every expression properly using the chip model. This was true even though the participant was able to verbalize the appropriate equivalent value on all but three occasions when the incorrect equivalent value was modeled. In responding to the prompts to employ the chip model, the differentiation of chip color and the routines employed represented the most significant variations in approaches between participants over these four expressions.

Evidence of the challenge presented to the participants by this means of modeling computation with integers can be seen by the fact that they were not universally able to employ this model. Those who were unable to proceed to the point of showing an equivalent value stated their inability to proceed. This is captured when Jamie said, "Negative four minus two, one two three four, so again, I am not really sure how to show that one, so I am just going to skip it." The most frequent participants to not enact the chip model were Ashley and Bailey, who each did not provide equivalent values to three expressions, while Jamie did not provide equivalent values for two expressions, and Brooke, Jacqueline, Jordan, Monique, and Tae each opted to not model one expression.

Another aspect of the challenge was the number of incorrect responses when the participants did proceed. Jacqueline modeled the most incorrect responses with the chip model, with five expressions modeled incorrectly, Courtney (n=4), Bailey (n=2), Christina (n=2), Monique (n=2), Amanda (n=1), Jordan (n=1), Kayla (n=1), Nicole (n=1), and Rebecca (n=1) each had diminishing numbers of incorrectly modeled expressions. While not modeling every expression, Brooke, Jamie, and Tae correctly represented each expression for which they provided an equivalent value.

While not having been modeled by every participant, the expression '-7 + 1'was the only one of these eight expressions which was not modeled incorrectly, however, it was not shown with the chip model by Jacqueline, see Table 26. The expression '2 – 8' was left unmodeled twice and only incorrectly modeled once. The expressions '-1 + -5' and '-1 + 9' were each improperly modeled twice; '-1 + -5' was attempted by all participants, while '-1 + 9' was not. Though '10 – 2' and '-4 – 2', were attempted by 11 and all 14 participants, respectively, each was modeled incorrectly by three of them. '6 + 2' was likewise attempted by every participant, but incorrectly represented by seven of them. For the expression '6 – -2', six participants exhibited proper models, five did not exhibit a model, and the other three provided an incorrect model.

Special comment should be made here about Jamie's approach to '-1 + 9', '-1 + -5', '-7 + 1', and '2 – 8'. As mentioned in Chapter 4, rather than treating the chips as counters for positive and negative integers, she lined up the chips using brown chips for the negative integer locations, and beige chips for the positive integer locations. She identified zero as the location where the brown and beige chips met. She then counted along the line of chips from the first term's location in the direction indicated by the operation, and the distance indicated by the second term's magnitude. This might appear to be contradicted for '-1 + -5', but she identified that both terms were negative and thus the sum would be farther from zero on the negative side. While this allowed her to be successful for these expressions, she had previously employed chips as individual counters which could be joined or taken away for '6 + 2' and '10 – 2'. However, neither method enabled her to comfortably and confidently proceed with the expressions '6 – -2' and '-4 – 2' which she admitted to not being able to model.

Table 26 Correctly provided responses when prompted to use the beige and brown chips

Chip model prompted	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
'10 – 2'														
Verbalized	*	*		*	*		*	*	*	*	*		*	*
Chips	*	*	*	*	X	*	X	*	*	*	*	X	*	*
Number line														
'6 + 2'														
Verbalized	*	*	*	*	*		*	*	*	*	*	*	*	*
Chips	X	*	X	*	X	X	X	*	X	*	X	*	*	*
Number line														
'6 – -2'														
Verbalized	*		*		*		*			*	X	*		
Chips	*					*	X			X	X	*	*	
Number line														
'-1 + 9'														
Verbalized	*	*		*	*	*	*	*	*	*	*	*	*	*
Chips	*	*		*	*	*	X		*	*	*	*	X	*
Number line								*						
'-1 + -5'														
Verbalized	*	*		*	*	X	*	*	*	*	*	*	*	*
Chips	*	*	*	*	*	X	X		*	*	*	*	*	*
Number line								*						
'-4 – 2'														
Verbalized	*		*	*	*		*		*	*		*	*	
Chips	*		X	*	*	X	X		*	*		*	*	*
Number line														
'-7 + 1'														
Verbalized	*	*	*	*	*	X	*	*	*	*	*	*	*	
Chips	*	*	*	*	*	*			*	*	*	*	*	*
Number line								*						
'2 – 8'														
Verbalized	*		X	*	*	*	*	*	*	*	*	*	*	*
Chips	*		X	*	*	*			*	*	*	*		*
Number line								*					*	

The * represents correctly expressions for the model, while X represents an incorrectly modeled expression

These participants had the most success of any of the eight expressions presented to them with '-7 + 1', as 13 of them were able to successfully model this expression with chips. This includes Courtney, who ended her verbalized statement with "six" and after several swappings of chip colors and the forming and removing of a pair was left with six beige chips, which were the same color she used to represent the negative seven in the expression. Jacqueline was unable to proceed any further then setting out the single positive chip and identified the need for seven positive chips to take away which she did not have. Each of the other PSTs who modeled this expression successfully either simultaneously used the two routines coded as *adding as joining* and *opposite colored chips stacked* (n=8) forming zero pairs, *subtraction as taking away* (n=3), or chips as a number line (n=1), see Appendix Q.

The expression '2 – 8' was attempted by 12 of the 14 participants. Of these 12, the most common approach was taken by eight participants who rephrased the expression as '2 + -8' and used *adding as joining* and *opposite colored chips stacked* to successfully model this expression. In her enactment, which was typical of these PSTs, Brooke used different colored chips to represent the rephrased expression '2 + -8' which after she removed two zero pairs, left her with six brown, or negative, chips representing '-6'. The only incorrect modeling was provided by Bailey when she set out two beige chips which she placed two brown chips on top of, forming inadvertent zero pairs, she then added four more brown chips, showing an equivalent value of four, which accompanied the verbalized "four" at the end of her counting sequence.

Because every participant tried to model this expression in her own way, '-1 + -5' appears to have been one of the more comfortable expressions for participants to approach. In doing so, 11 of the PSTs took the most frequently used approach of joining one and five chips of a single color to show six chips of the same color. The two PSTs who incorrectly modeled '-1 + -5' each selected one color to be the one and the other color to be the five, which when joined, common discourse practices would identify zero pairs having been formed, leaving four unpaired chips for an equivalent value of negative four, which is in contrast to the participants' suggested values of six (Courtney) and negative six (Jacqueline).

This level of comfort cannot be said to have been present in the expression '-1 + 9', which was only left unattempted by Bailey. Ashley was seen to have verbalized the equivalent value as eight and properly paired up chips to cancel out a brown with a beige to leave eight brown, positive, chips, even as she expressed uncertainty about how to model the expression throughout her turn. Rebecca mentioned that she was visualizing her line of chips to be like a number line, and while she spoke of getting to eight, she had ten chips in the line. While the presence of a zero pair would allow her representation to be considered as showing eight unpaired chips, the fact that she spoke of the number line while not specifically pointing to a chip in the line, makes it indicate an inappropriate equivalent value.

The complexities of modeling integers with chips went unresolved by Christina and Jacqueline while attempting to model '10 – 2', which was modeled incorrectly by three PSTs. Incorrect modeling occurred for two of these three PSTs because, while they removed two chips from a collection of 10, the original and final groupings consisted of both beige and brown chips. This resulted in the eight chips including pairs which allowed for five unpaired chips in Christina's model and two unpaired chips in Jacqueline's model. Additionally, Nicole removed a pair of chips from her original collection, however she had counted out 11 chips, and by removing two she was left with nine chips. She tried a second modeling for this expression, setting out 11 beige chips, and in attempting to rephrase the expression as '10 + -2' swapped out two beige chips and added two brown chips which she then removed leaving nine chips again.

The expression '-4 – 2', while being modeled correctly by eight participants and left unmodeled by three participants, was modeled incorrectly by Bailey, Courtney, and Jacqueline. Each of these incorrect models displayed six chips of different colors, which the participants treated as showing six. Bailey's actions in addressing this expression began with four brown chips to which she added two beige chips, at which point she claimed to have displayed negative six, but she then continued by placing a brown chip on each of the beige chips, showing a result of two beige-brown zero pairs and four brown chips, or four. Courtney joined four beige and two brown chips to create a mixed group of six chips which, by common discourse with chip interpretations, showed negative two. Similarly, Jacqueline enacted this approach, using five beige chips and one brown chip to model an equivalent value of four rather than negative six. Monique's money model did not allow her to represent negative four dollars in modeling this expression.

In using the chip model for the expression '6 + 2', Ashley, Brooke, Jamie, Kayla, Nicole, Rebecca, and Tae each joined chips of a single color, either all beige or all brown, creating a group of eight chips, a proper model for this expression. In contrast six participants (Amanda, Bailey, Christina, Jacqueline, Jordan, and Monique) each used one color for the 'six', and a second color for the 'two' which when joined was claimed to be eight, however, this would represent four, as the beige-brown pairs formed would cancel out as zero pairs in common discourse practices. Courtney employed only brown chips as the two positive values, but never joined the two terms to form the sum. Thus, what may have been a familiar expression when considered in a discourse about whole numbers, becomes much more complicated in a discourse about integers.

While '6 + 2' was modeled incorrectly most frequently (n=6) out of these eight expressions, the expression '6 – -2' had the fewest correct modelings (n=4). This apparent paradox results from only four participants (Amanda, Courtney, Nicole, and Rebecca) correctly modeling '6 – -2', while three (Jacqueline, Kayla, and Monique) modeled it incorrectly, and seven (Ashley, Bailey, Brooke, Christina, Jamie, Jordan, and Tae) having claimed to have not completed a model for this expression by employing the chips. However, when considering their movement with the chips, Ashley got to the point of joining two sets of brown chips, which would create a single group of eight, thereby successfully modeling this expression. Likewise, when looking at the way Bailey and Jordan ended up arranging the chips, we have six browns and two beiges, which with zero pairs, leaves four unpaired chips rather than eight.

When considered in its entirety over all eight expressions, the PSTs exhibited equivalent values being correct for 165 of the 190 answered suggested, along with 10 turns not being taken. Since none of the participants correctly modeled all eight expressions using chips, none of these PSTs exhibited fully-consistent discourse practices in utilizing this model. Amanda, Brooke, Kayla, Nicole, and Rebecca were each successful with seven expressions and are thus considered to be partially-consistent with common discourse practices, while Ashley, Bailey, Christina, Courtney, Jacqueline, Jamie, Jordan, Monique and Tae would each have displayed a discourse which is inconsistent with these discourse practices, primarily through the inability to accept the claim of having modeled the expressions or to proceed far enough for the results to be recognized as representing a model, or through the frequency of errors, see Table 26.

When prompted to use the number line. The participants' use of the number lines resulted in a much greater degree of success then found in either of the models considered. This may be due to it being the fourth time the participant had considered the expression, but it is perhaps more likely that it relates to the use of only two routines, that of going up or going down the number line the magnitude of the second term from the first term. While this contrasts with the approaches described in Chapter 2, it was used effectively to model seven of the eight expressions without error. This includes modeling by all 14 participants for six of the seven expressions exhibited without error, while one PST proved unprepared to use the number line to model '-4 – 2'. All seven of the mistakes noted for these prompts were associated with '6 – -2', which include four verbalized errors and three drawn errors. Table 27 shows that in addition to using the number line, most often with dot-hops-dot, see Chapter 4 and Appendix Q, the participants also verbalized, or noted in written form, the expression and the associated equivalent value.

Table 27 Correctly provided responses when prompted to use the number line

Number line prompt	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
'10 – 2'														
Verbalized	*	*	*	*	*		*	*		*	*	*	*	*
Written														
Number line	*	*	*	*	*	*	*	*	*	*	*	*	*	*
'6 + 2'														
Verbalized	*	*		*	*		*	*	*	*	*	*	*	*
Written		*												
Number line	*	*	*	*	*	*	*	*	*	*	*	*	*	*
'6 – -2'														
Verbalized	*	*	*	*	*	X	X	*	X	*	X	*	*	*
Written			*											
Number line	*	*	*	*	*	*	X	*	X	*	X	*	*	*
'-1 + 9'														
Verbalized	*	*	*	*	*	*	*		*	*	*	*		
Written			*					*						
Number line	*	*	*	*	*	*	*	*	*	*	*	*	*	*
'-1 + -5'														
Verbalized	*	*		*	*		*	*	*	*	*		*	
Written			*											
Number line	*	*	*	*	*	*	*	*	*	*	*	*	*	*
'-4 – 2'														
Verbalized	*			*	*		*		*	*		*	*	
Written														
Number line	*	*	*	*	*	*	*	*	*	*		*	*	*
'-7 + 1'														
Verbalized	*	*		*	*		*	*	*	*	*	*	*	
Written								*						
Number line	*	*	*	*	*	*	*	*	*	*	*	*	*	*
'2 -8'														
Verbalized	*	*	*	*	*	*			*	*	*	*	*	
Written														
Number line	*	*	*	*	*	*	*	*	*	*	*	*	*	*

The * represents correctly expressions for the model, while X represents an incorrectly modeled expression

Monique was unable to proceed with '-4 – 2' beyond placing a dot at negative four, which accompanied her statement: "We're at negative four here take away two. It is a possibility that you are taking away negative four and negative. Yeah, this one is hard to do, to model." This suggests that her discourse about integers does not embrace the abstract and ordinal nature of the integers, and the fact that integers are not founded on physical objects.

Several causes were seen as to why incorrect equivalent values were verbalized or modeled for '6 – -2'. Courtney's statement, "Ok, you have negative one, six then you have negative two" was marked as conveying an incorrect equivalent value, since it ends with negative two. However, it could be considered that she was only verbalizing her steps and did not verbalize an equivalent value. Courtney was able to place a dot on her number line at eight by starting at six and adding two. Jacqueline and Monique each initially identified four as the answer, but took different paths to arrive at this value. Jacqueline started at negative two and made six hops to the right, while Monique started at six and made two hops to the left. As such Jacqueline appears to have rephrased the expression as '-2 + 6', while Monique rephrased it as '6 – 2'. Jordan also initially indicated an incorrect equivalent value, by starting at negative six and moving two places to the left to show a value of negative eight. Thus, Jordan modeled '-6 – 2'; in such a situation she would seem to have assigned the negative to the wrong term. Later in the turn in which she indicated the value of four, Jacqueline also tried going up two from six, to place a mark at eight. Similarly, though in a later turn, Jordan identified her mistake and properly modeled '6 – -2' as being equivalent to eight.

When considered in its entirety over all eight expressions, the PSTs modeling with the number line displayed a high degree of accuracy, with 200 of 207 exhibited equivalent values being correct, and only one turn not being taken. Additionally, the mistakes identified seem to be revealing more in relation to the participants' momentary inattention to detail rather than to foundational problems in their discourse about integers. Thus, for this approach to modeling, Amanda, Ashley, Bailey, Brooke, Christina, Jamie, Kayla, Nicole, Rebecca, and Tae each exhibited discourse that was consistent with common discourse practices, while Jacqueline and Jordan both appear to be partially-consistent, due to their initially incorrect responses. Courtney and Monique each exhibited discourse that best aligns with a rating of inconsistent with common discourse practices, due to the nature of the errors and omissions.

Expressions of Uncertainty

Throughout the course of the interviews, some of the participants' responses included statements which expressed a lack of ability, or the presence of uncertainty about how to properly engage in an aspect of this discourse. Such statements are not part of any of the elements of discourse considered previously, but are suggestive of some aspects of the participants' self-characterization of their discourse about integers. These statements are also suggestive of aspects of the discourse about integers that the participants perceived as more challenging, or further removed from their personal discourse about integers, see Table 28. To be considered as expressing uncertainty, the participant did not need to explicitly state 'I am uncertain', but this sentiment was suggested by "This is a tough one," stated by Jordan while considering Question 1, or Christina's "I think it's true. I think, I don't know, I'm trying to, like, visualize it and I can't think of a way to explain it so, so I guess, yeah," while responding to Question 2 which contained the phrases "I guess," "I think," and "I don't know."

Table 28 Expressions of uncertainty by student for a given question

	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae	Total
Q1									*						1
Q2			*		*				*						3
Q5		*	*	*	*	*			*	*	*	*		*	10
Q5A						*					*				2
Q6A '-1 + 9'								*							1
Q6C '-1 + 9'		*	*								*				3
Q6A '6 – -2'			*	*	*			*	*						5
Q6C '6 – -2'		*		*	*		*	*	*	*	*			*	9
Q6D '6 – -2'							*	*	*	*					4
Q6B								*							1
Q6C								*							1
Q7A '2 – 8'									*						1
Q7C '2 – 8'		*		*											2
Q7A '-4 – 2'		*						*							2
Q7C '-4 – 2'		*					*	*			*				4
Q7D '-4 – 2'			*					*			*				3
Q7B			*												1
Q7C								*							1
Total	0	6	6	4	4	2	3	10	7	3	6	1	0	2	54

The extent to which the participants expressed uncertainty varied markedly across different prompts. This sentiment was most frequently applied in reaction to the prompts regarding the meaning of additive inverse (n=10) and modeling '6 – -2' using chips (n=9). In fact, the expression '6 – -2' elicited 18 of the expressions of uncertainty, or a third of the total. Some of the uncertainty the participants felt with additive inverse was presented above. However, it is clearly seen in Bailey's lament, "Sorry, I haven't done negative numbers in such a long time" to which she added "Ok, so if you … I'm going to take a guess, so they're additive inverses of each other, that means that if you add these numbers together you'd get zero" after I reassured her that "I just want your natural reaction if you were being asked to teach it today." Jordan's statement shows less despair but does indicate the presence of uncertainty: "One is basically plus seven while one is minus seven. They're inverses because … I don't know how to describe it," while responding to this same prompt.

The expression '6 – -2' elicited the most comments about the participants' uncertainty of any prompt over the course of its three different representations. While not all participants made such a comment, six of the 10 who did, expressed it on multiple occasions. Brooke was one of these six, as she expressed uncertainty in her responses to both the initial prompt and when asked to model with chips. This is seen in her comments in response to the initial prompt:

> I guess, maybe, if I were to draw a number line. So, if we start at six, and we are going to subtract negative two, Yeah, I am not quite sure how I would explain to a student how, even though its subtracting a negative you are still going to the right of the number line because when I see subtraction I always think you have to go to the left of the number line.

While making this statement she drew the upper, incomplete number line in Figure 37 which contrasts to her complete number line modeling '-1 + 9' also shown as the lower of the two number lines in Figure 37.

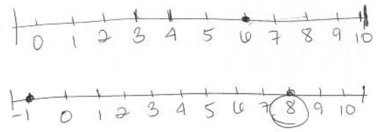


Figure 37 Brooke's drawing while considering '6 - -2' and '-1 + 9'

Brooke's uncertainty remained as she attempted to use chips while modeling this expression, which was indicated in her statement, "Six minus negative two, like I said before when I was solving it, I am not sure how I would show a student how to do that … do the problem using chips as well." While making this statement, she set out six beige chips, after which she was unable to proceed. However, she was able to approach '6 – -2' using the number line as seen in her statement:

> I am thinking about it, thinking about our conversations … I can also explain this as an inverse of the equation, so instead of six minus negative two, I would take the inverse of negative two which would be a positive two which would also mean I would take the inverse of subtraction, which would mean addition. Perhaps, that is how I could explain it, and that way at least I know the answer to this would be positive eight, so if you move over two spots, it gets you positive eight.

Though she used the term "perhaps" in this latter quote, there is a much more confident feel in this statement than was seen in her previous two quotes. This suggests that some combination of repeated exposure to this expression, the nature of the models, and progression of the discourse allowed Brooke to gain, perhaps only slightly, in her confidence in modeling this expression. As can be seen in Table 28, Kayla, who did not express uncertainty initially, did in the case of the chip and number line models, while Jamie and Jordan expressed uncertainty upon every contact with this expression.

The number of occasions on which participants felt uncertain ranges from zero for Amanda and Rebecca, up to 10 for Jamie. In addition to Jamie's difficulty with the expression '6 – -2', she also struggled with the other expression involving subtraction and containing both a positive and a negative number, '-4 – 2'. Since examples of a participant expressing confusion have been provided for the former expression, I will be focusing on Jamie's expression of confusion with the latter. On her initial exposure to '-4 – 2' she stated, "So you were taking away a positive number from a negative number … I really, my gut reaction is to just do this the procedural method of keep change, change. I just don't know why I'm doing it," which represents uncertainty about how to represent this expression in a manner she deemed to be consistent with expectations, though not with a valid representation or the equivalent value. When considering modeling this expression with the chip model she stated "Negative four minus two, … so again, I am not really sure how to show that one, so I am just going to skip it because we're not going to get anything, and it is going to leave me confused" as she counted out four chips then stopped. Her more numerous expressions of uncertainty should be ascribed to her thoughtfulness in approaching each prompt, as can be seen in this passage where Jamie's initial uncertainty is replaced with solid comprehension:

> Umm, so the next one … oh yeah, this one again. I am really just not sure how to show it. I don't know how to show it that I would be counting up like this, well, I wouldn't be doing that … oh my gosh, never mind, I'm not. Oh, wait, I think I actually just had an epiphany. Oh, I actually understand this one now, ok, wow, all I had to see was it here on the number line so it works the same way that this works, so if you have negative four and you want to take a value of two away from it, it's making the number smaller the same way that if it was, oh, the same way that if it was negative four plus two it would make it ah, umm, oh, now I

> am just going to confuse myself, but negative four minus two would give us a smaller value because we're taking a value of two numbers away from it, so negative four taking two numbers away from it would be one, two, because I am making it a smaller number. Yeah, yeah, if I was adding two it would go this way because I'm making the number bigger. Ok. I completely understand that now, wow, that was an epiphany.

Thus, we see that while some of the expressions of uncertainty exposed weaknesses in the participants' discourse about integers which would limit these PSTs' ability to engage in this discourse, in at least Jamie's case, such uncertainty is based on a thoughtful desire to engage in a discourse about integers founded on appropriate narratives and which employs suitable routines. It should be noted that the participants' uncertainty was only noted when they verbalized the sentiment. This may suppress an unknown degree of uncertainty in these participants. Further, as may be noted in the discussion of the participants' success with modeling expression, just because the individual expresses confidence, that does not automatically translate to a quality representation of the discourse they are engaged in.

Conclusion

In this chapter I described the way the participants' discourses unfolded through the combination of the distinctly coded components within the elements discussed in Chapter 4. I found that only one of these PSTs was able to describe the relationship between whole numbers and positive integers in a way that is consistent with common discourse practices. In describing the possible results of addition and subtraction of integers, several different combinations of numbers were offered, including several which are consistent with common mathematical discourse about integers. It was only based on the presence of the values '-7 and 7' that all but a few of the PSTs were able to make sense of the relationship between additive inverses. Since that pair of numbers was present, they were able to indicate, with some regularity, the fact that they are opposites and add to zero, but this was not universal. Several PSTs remained unsure of what the term meant and the relationship which exists between negative seven and positive seven, even if they were able to provide another pair of values possessing a similar relationship.

The correctness of the PSTs' responses to the eight expressions was then described and was shown as being very procedural in nature, and dependent on the form of representation being considered. Thus, while Monique provided correct answers on only 17 of the 24 opportunities, each of the other PSTs did so at least 21 times over the three modeling prompts in Questions 6 and 7 (initial, chip, and number line). These turns with correct responses obscure both the turns which contained no modeled equivalent value and those that contained the correct value represented in one or two ways but being incorrectly shown in another. The presence of both unmodeled expressions and statements of uncertainly indicate the degree of thoughtfulness with which these PSTs participated in the interview.

Chapter 6: Discussion

In this chapter, I will provide a brief summary of the preceding chapters, followed by a set of conclusions I have drawn from my findings and the relationship of these findings to existing research. Finally, I will present some implications derived from this study, limitations, and suggestions for future research, as well as some final remarks.

Overall Summary

In this study, I described and applied a definition of discourse based on the work of Sfard, and examined the components that make up mathematical discourse about whole numbers and mathematical discourse about integers in terms of Sfard's work. I then performed semi-structured interviews with 14 PSTs and coded their responses using constant comparison methods. The results were then presented based on codes identified within the elements of this discourse, namely word use, visual mediators, routines, and narratives and based on themes among the responses. This was done to address the question:

> *How can PSTs' mathematical discourse about integers be described based on their word usage, employment of visual mediators, enactment and verbalization of routines, and identification of core narratives of this discourse?*

When considering the results presented in Chapter 4, I observed that these PSTs freely expressed the numbers and operations within computations, but made far less use of names for core ideas within the discourse. They willingly engaged, to varying degrees, in writing expressions and numbers, drew and/or utilized both the chip model and number lines. Though these uses of visual mediators presented a clear image of the PSTs' ideas, the ideas expressed were not always well-aligned with common discourse practices. Further, while these PSTs tended to be able to execute routines and reference narratives in addressing prompts and modeling the process of determining the equivalent value for expressions, they did not always verbalize the routines and narratives executed and referenced while engaged with the discourse. Those routines which were revealed were highly procedural in nature, and while conforming to standard practice, often resulted in rephrasing the expression. This reprhasing of expressions tended to remove distinguishing features of the original expression, while resulting in the PSTs being highly successful at determining the correct equivalent value. In stark contrast, the narratives stated, and thus endorsed, by these PSTs tended to lack alignment with common and rigorous discourse practices for a discourse about integers. For example, when asked to distinguish positive integers from whole numbers only three of the 14 PSTs provided a description which was consistent with the discourse identified in Chapter 2. Further, none of the PSTs identified all three meanings of the minus sign when prompted. The relationship between the terms and results of the operations of addition and subtraction with integers described by these PSTs ranged from those who accepted the continued applicability of 'addition makes larger' and 'subtraction makes smaller' to those who accepted any integer as the result of either operation.

While I did not determine the participants' level of experience with colored chip models, each of them expressed a willingness to try to model integer expressions with this visual mediator. In doing so, these PSTs employed single, separated, and mixed colors in ways which did not always ensure a modeling that was consistent with expected practices. Within this range of chip color uses, some PSTs modeled the expressions in a manner which had a high rate of consistency with common practices, while others displayed a very low rate, and one PST employed a nearly unique approach to using chips to model these expressions.

The approaches taken by these PSTs to model the expressions on the number line were most frequently to make a dot, draw some hops while moving to the right or left as indicated by the expression being modeled, then make a second dot at the equivalent value. In making the second dot at the equivalent value for the expression, this modeling, though conforming with common practices, is not as representative of the nature of the integers and operations in the original expressions as the use of arrows or counted steps along the number line tended to be.

Conclusions

The discourses provided by these PSTs, when viewed holistically, indicated wide variations, with some PSTs exhibiting discourses which were nearly consistent with common discourse practices, while others were almost entirely inconsistent with those same practices. Within this range, I would describe the discourse about integers presented by those PSTs who fall in the middle of this range as only partially-consistent with common discourse practices. Examples of how the discourse exhibited by these PSTs relates to observations made by other researchers is described below.

I was unable to uncover any research studies in which the participants were asked to consider the relationship between whole numbers and positive integers, the types of numbers which can result from computation with integers, the meanings of the minus sign, or to describe the nature of additive inverses. I found that the participants in my study exhibited discourses which described these terms in a way that was most often partially-consistent, while a few can only be described as inconsistent with common discourse practices.

The descriptions of the discourses exhibited by these PSTs are accentuated by the nature of their responses to the prompt about the meanings of the minus sign. Within a discourse about integers, there are a trio of meanings a minus sign can have when appearing in a mathematical expression: binary operator (subtraction), unary operator (opposite of), and predicative signifier of value (negative). The PSTs in this study only explicitly identified the roles of subtraction (all 14), and negation (12) when asked to indicate the roles ‘ - ’ can take. In fact, they referred to the binary nature of subtraction during 168 of the 294 interactions with expressions, and identified working with negative numbers, the predicative signifier, during 228 of the 294 interactions with the expressions; while never indicating the unary operation ‘the opposite of’. It should be remembered that the expression ‘6 + 2’ was the only one considered which did not contain either a negative or subtraction, but it accounted for 42 interactions by the participants, which are not included in the totals above.

An existing research study by Bishop and her colleagues (Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2014), described the way a second-grade student, who they referred to as Violet, considered expressions involving the following: subtraction which yielded negative numbers, addition not making larger, and subtraction yielding a larger number. During Violet's first interview she rejected these ideas, however, after thinking about the expressions over a six-week period, she accepted the notion of negative equivalent values for subtraction, subtraction yielding a number which is larger than the minuend, and that addition can result in a smaller sum than either of the addends. In considering the results of computations with integers within my study, these PSTs tended to fall between Violet's initial and follow-up interviews, inclusive, as she gradually accepted the need to reconsider the notions that 'addition makes larger' and 'subtraction makes smaller'.

While the participants in my study tended to be able to compute equivalent values accurately, they did so in a manner which did not uncover the key conceptual ideas inherent in the expressions. This approach can be compared with one of the participants in a study conducted by Charalambos Y. Charalambous (2010), in which teachers with various scores on the Mathematical Knowledge for Teaching tests, and who were using different versions of a textbook, were described. One of the individuals, referred to as Karen, is described as having 37 years of experience in teaching and a high assessment score; she was able to display the connections and justification for the procedures needed to evaluate '+5 – -4'. The procedural approach modeled by the PSTs in my study do not show such connections; however, due to Karen's much longer period of interaction with this content, this is perhaps not a fair comparison.

In his 1985 article, Murray presented success rates for computation exercises for 9th grade students who had received instruction on computation with integers. The participants in Murray's study differed from the participants in this study primarily in age, with my participants being of college age rather than high school. The participants in Murray's study responded to a series of expressions which align closely with those presented in my study in terms of structure and level of difficulty, with the expressions and percentage correct shown in Table 29.

Table 29 Comparison of percentage of correct responses between the participants in Murray (1985) and this study

Expression in Murray (1985)	Percentage of students providing a correct answer	Expression in this study	Percentage of PSTs providing a correct answer
'-8 + 3'	78%	'-7 + 1'	100%
'-4 + 7'	77%	'-1 + 9'	100%
'-7 + -5'	74%	'-1 + -5'	100%
'3 – 8'	69%	'2 – 8'	100%
'-7 – 4'	50%	'-4 – 2'	86%
'8 – -3'	46%	'6 – -2'	64%

In each case, my participants were able to display a higher success rate than participants in Murray's study.

The PSTs in my study were generally able to model '10 – 2', '6 + 2', '-1 + '5', '-1 + 9', and '-7 + 1', with beige and brown chips, while not always expressly verbalizing and justifying the cancellation occurring with expressions having negative values or subtraction. My participants did not spontaneously add the necessary zero pairs and therefore were not able to smoothly model '2- 8', '-4 – 2', and '6 – -2'. The complication with this aspect of the chip model has been argued in Vig, Murray, and Star (2014).

Peled, Mukhopadhyay, and Resnick (1989) described the nature of the integer number line, as an extension of the 'whole number' number line held by grade school age children. They describe two models, the "continuous number line" and the "divided number line" (p. 108). With the continuous number line, a student is able to move smoothly from the positive to the negative numbers while considering a subtraction expression such as '2 – 8', while with a divided number line model, the individual would go to zero, then determine how much of the subtrahend remains in the negative region. When considering the PSTs' number line modeling of the integer expressions in my study, most moved smoothly along the number line and the transition from positive to negative numbers in a way which suggests they think of integers in a manner similar to those who employed the "continuous number line". However, among the participants in my study Jamie and Kayla, in particular, displayed actions suggestive of possessing or seeing merit in the "divided number line model" (Peled, Mukhopadhyay, & Resnick, 1989, p. 108).

Implications

This study has implications for several different groups, including individuals teaching and developing curriculum for mathematics content courses for teachers, the authors of textbook support materials, and professional developers and department chairs. The implications for each of these groups will be discussed in turn.

Based in part on my personal history, the primary implications I have observed are directed to the instructors and curriculum developers for mathematics content courses for teachers. The PSTs' responses to Question 1, in which they were asked to compare 'whole numbers' and 'positive integers', seemed to indicate that the mathematics content courses they had experienced did not lead them to develop a clear definition of whole numbers, decimals, and fractions. Thus, a closer focus on the nature of these sets of numbers and the relationship between them should be built into courses designed for the teaching of mathematics content than currently exists.

Monique’s employment of the debt model was unique among these PSTs in attempting to regularly associate the expressions presented with a practical example. This contrasts with the frequent use of practical situations found in practitioner and research journals. Examples include, using money (Gregg & Gregg, 2007; Stephan & Akyuz, 2012; Whitacre I. , et al., 2015; Whitacre, Bishop, Philipp, Lamb, & Schappelle, 2014), temperature (Wessman-Enzinger & Tobias, 2015), emotions (Petrella, 2001; Whitacre I. , et al., 2012), balloons and weights (Reeves & Webb, 2004), scores/forfeits and numbers entering/leaving (Liebeck, 1990; Linchevski & Williams, 1999), and references to water levels (Kent, 2000; Tillema, 2012). This suggests that these PSTs were not exposed to practical examples of mathematics in their coursework, nor do they explore current or recent journals related to the subjects they will be teaching. Therefore, if we want PSTs to embrace teaching models based on practical experiences, publications and practical examples need to be more fully integrated into their coursework and the general culture of PSTs.

Further, responses from these PSTs indicate that they had not internalized the multiple meanings of the minus sign, as well as the range of numbers to which the notions ‘addition makes larger’ and ‘subtraction makes smaller’ are accurately applicable. While these ideas hold true for arithmetic with numbers greater than zero (the numerical realm these PSTs are being prepared to teach (K – 5)), they lose their applicability with the introduction of integers which comes shortly after (Grades 6 and 7) (National

Governors Association Center for Best Practices, 2010). Thus, there appears to be a need to anticipate the modification that these established narratives will experience among elementary level teachers, even without the expectation that they will be teaching about integers. This claim receives additional support from my study when one considers these PSTs' approaches to modeling with chips and the number line, which was performed in such a way as to frequently disregard the color of the chips, with color potentially serving to aid in distinguishing positive and negative numbers, or ways to identify addition and subtraction in a manner which can be easily extended to support modeling integers.

The implications for textbook authors are related to supplemental resources which support the teaching of the content in the textbook. These resources need to place a particular focus on showing complete and comprehensive examples of how to employ chip and number line models with integers in a way which preserves the form of the expressions. While these forms are introduced in Chapter 2, the need for them and support for the claim that many of these models are not intuitive, even for PSTs who are familiar with their use with whole numbers, is provided in Chapter 4 and Chapter 5. Additionally, it seems appropriate that the models initially presented during the early elementary grades for whole number addition and subtraction should evolve by the upper elementary grades to be better suited for a smooth transition to their use with integers.

Professional development providers and mathematics department chairs need to provide in-service teachers an opportunity to learn to think beyond 'keep-change-change' when dealing with subtraction of integers, though this thinking may align with the standard algorithm. While doing so, the teachers should also be encouraged to explore the use of visual mediators and their associated routines while modeling operations with integers, thereby allowing the teacher to develop their discourse in employing these models to include clear and powerful representations, as well as discretion on when to enact different approaches.

Limitations

Several factors serve as limitations to this study. Amongst them is the fact that while an interview provides multiple opportunities for an individual to enact a discourse, it is impossible to gain a comprehensive view of a person's intrapersonal and interpersonal discourses (Sfard, 2008). The size of the sample included in this study acts as a limitation to generalizability, as does the fact that all participants are drawn from the same university. Further, since 13 of the 14 participants accepted their professor's nomination, they tended to be the more verbal, though not necessarily the best performing students, in their respective classes. Thus, the average PST at this institution may have engaged in such a discourse in a different manner.

Future research

This study sought to describe, in terms of levels of consistency, the discourses about integers provided by PSTs, with a focus on the content embedded in such a discourse. These PSTs had completed mathematics for teaching coursework designed for elementary teachers, which did not focus on integers. As such, future studies employing this approach, and referencing the common discourse about integers established here, which explore the exhibited and enacted discourses of PSTs preparing to teach grades six through nine, or in-service teachers in that grade range would be a good addition to the existing literature. Additionally, this study did not attend to how the PSTs would engage in the pedagogical aspects of teaching this discourse. Such an approach to this discourse, especially with active teachers or PSTs who have been exposed to a course sequence which includes integers as a content for teaching topic, appears to be a significant extension of the ideas and methods introduced in this study. Studies of the nature just indicated would ideally be conducted in a more authentic setting, including direct observation of teaching actual or sample lessons.

Final remarks

In completing this study, I intended to determine the extent to which pre-service teachers were able to engage in mathematical discourse about integers. They were provided the opportunity to do so through one-on-one semi-structured interviews, attending to key narratives, identification and representation of visual mediators, and enactment of routines while engaging verbally in a discussion. This interview served to uncover those components of this discourse which were most immediately available. I hypothesize that with additional time to review and reflect, these PSTs would be able to present a discourse about integers more nearly consistent with common practices than was the case in this study. This is suggested by their use of a variety of ways to enact the chip model using zero pairs and use of the number line model, employing directed arrows with the second addend's arrow emerging from the end of first addend's arrow in modeling addition, and both arrows originating from zero to model subtraction. The participants' ability to execute these approaches was not assessed, though they made statements which suggested they were grasping the ideas being presented.

While the PSTs in my study performed as well or better than the high performers described in Peled, Mukhopadhyay, and Resnick (1989), and Murray (1985), they did not compare as well to Karen (2010) or even to Violet (Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2014). This suggests that if integers are considered to be a significant topic, which has been suggested as an important building block to the learning of algebraic thinking (Gallardo, 2002), additional, and focused, attention needs to be given to integers in teacher preparation mathematics classes.

References

Barber, H. C. (1925). Real improvement in algebra teaching. *The Mathematics Teacher, 18*(6), 364-374.

Bassarear, T., & Moss, M. (2016). *Mathematics for elementary school teachers*. Boston: Cengage Learning.

Beckmann, S. (2014). *Mathematics for elementary school teachers* (4th ed.). Pearson Education, Inc.

Bell, E. T. (1945). *The Development of Mathematics.* New York: McGraw-Hill Book Company.

Bishop, J. P., Lamb, L. L., Philipp, R. A., Whitacre, I., & Schappelle, B. P. (2014). Using order to reason about negative numbers: the case of Violet. *Educational Studies in Mathematics, 86*(1), 39-59.

Bishop, J. P., Lamb, L. L., Philipp, R. A., Whitacre, I., Schappelle, B. P., & Lewis, M. L. (2014). Obstacles and affordances for integer reasoning: An analysis of children's thinking and the history of mathematics. *Journal for Research in Mathematics Education, 45*(1), 19-61.

Bofferding, L., & Richardson, S. E. (2013). Investigating integer addition and subtraction: A task analysis. *Proceedings of the 35th Annual Meeting of the North American Chapter of the International Group of the Psychology of Mathematics Education*, (pp. 14-17). Chicago.

Buchmann, M. (1986). Role over person: Morality and authenticity in teaching. *Teachers College Record, 87*(4), 529-543.

Carter, J. A., Cuevas, G. J., Day, R., Malloy, C., Molix-Bailey, R. J., Price, J., & Willard, T. (2012). *Glencoe Pre-Algebra.* New York: McGraw-Hill Education.

Charalambous, C. Y. (2010). Mathematical knowledge for teaching and task unfolding: An exploratory study. *The Elementary School Journal, 110*(3), 247-278.

Corbin, J., & Strauss, A. (1990). Ground theory research: Procedures, canons and evaluative criteria. *Qualitative Sociology, 13*(1), 3-21.

Gallardo, A. (2002). The extension of the natural-number domain to the integers in the transition from arithmetic to algebra. *Educational Studies in Mathematics, 49,* 171-192.

Gregg, J., & Gregg, D. U. (2007, August). A context for integer computation. *Mathematics Teaching in the Middle School, 13*(1), 46-50.

Hefendehl-Hebeker, L. (1991). Negative numbers: Obstacles in their evolution from intuitive to intellectual constructs. *For the Learning of Mathematics: An International Journal of Mathematics Education, 11*(1), 26-32.

Herbst, P. (2010). Practical rationality and the justification for actions in mathematics teaching. *Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* (pp. 46 - 54). Columbus, OH: The Ohio State University.

Herbst, P., & Chazan, D. (2003). Exploring the practical rationality of mathematics teaching through conversations about videotaped episodes: the case of engaging students in proving. *For the Learning of Mathematics, 23*(1), 2-14.

Herbst, P., & Chazan, D. (2011). Research on practical rationality: Studying the justificaiton of actions in mathematics teaching. *The Mathematics Enthusiast, 8*(3), 405-462.

Herbst, P., & Chazan, D. (2012). On the instructional triangle and sources of justification for actions in mathematics teaching. *ZDM Mathematics Education, 44*, 601-612.

Kent, L. B. (2000). Connecting integers to meaningful contexts. *Mathematics Teaching in the Middle School, 6*(1), 62-66.

Kinach, B. M. (2002). Understanding and learning-to-explain by representing mathematics: Epistemological dilemmas facing teacher educators in the secondary mathematics "methods" course. *Journal of Mathematics Teacher Education, 5*(2), 153-186.

Larson, R., Boswell, L., Kanold, T. D., & Stiff, L. (2012). *Larson Pre-Algebra.* Orlando, FL: Holt McDougal.

Liebeck, P. (1990, June). Scores and forfeits: An intuitive model for integer arithmetic. *Educational Studies in Mathematics, 21*(3), 221-239.

Linchevski, L., & Williams, J. (1999). Using intuition from everyday life in 'filling' the gaps in children's extension of the number concept to include negative numbers. *Educational Studies in Mathematics, 39*(1-3), 131-147.

Merriam, S. B. (2002). Assessing and evaluating qualitative research. In S. B. Merriam (Ed.), *Qualitiative Research in Practice Examples for Discussion and Analysis* (pp. 18 - 33). San Fransisco, CA: Jossey-Bass.

Merriam, S. B. (2009). *Qualitative research: A guide to design and implementation.* San Fransisco, CA: Jossey-Bass.

Morris, A. K., & Kiebert, J. (2017). Effects of teacher preparation courses: Do graduates use what they learned to plan mathematics lessons? *American Educational Research Journal*, *54*(3), 524-567.

National Governors Association Center for Best Practices, C. o. (2010). *Common Core State Standards (Mathematics).* Washington DC: National Governors Association Center for Best Practices, Council of Chief State School Officers.

Peled, I., Mukhopadhyay, S., & Resnick, L. B. (1989). Formal and informal sources of mental models for negative numbers. *Proceedings of the 13th international conference for the Psychology of Mathematics Education*, *3*, pp. 106-110.

Petrella, G. (2001, November). Subtracting integers: An affective lesson. *Mathematics Teaching in the Middle School*, *7*(3), 150-151.

Piaget, J. (1965). *The cild's conception of number.* New York: W.W. Norton.

Reeves, C. A., & Webb, D. (2004, May). Balloons on the rise: A problem-solving introduction to integers. *Mathematics Teaching in the Middle School*, *9*(9), 476-482.

Rubin, H. J., & Rubin, I. S. (2005). *Qualitative Interviewing* (2nd ed.). Thousand Oaks, CA: Sage Publications.

Sfard, A. (2000a). On reform movement and the limits of mathematical discourse. *Mathematical Thinking and Learning*, *2*(3), 157-189.

Sfard, A. (2000b). Steering (dis)course between metaphors and rigor: Using focal analysis to investigate an emergence of mathematical objects. *Journal for Research in Mathematics Education, 31*(3), 296-327.

Sfard, A. (2000c). Symbolizing mathematical reality into being - or How mathematical discourse and mathematical objects create each other. In *Symbolizing and communicating in mathematics classrooms: prespectives on discourse, tools, and instructional design.* (pp. 37-98). Hoboken, NJ: Taylor and Francis.

Sfard, A. (2001a). Learning mathematics as developing a discourse. In R. Speiser, C. Maher, & C. Walter (Ed.), *Proceedings of the 21st conference of PME-NA* (pp. 23-44). Columbus, Ohio: Clearing House for science, mathematics, and environmental education.

Sfard, A. (2001b). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics, 46*, 13-57.

Sfard, A. (2007). When the rules of discourse change, and nobody tells you: Making sense of mathematics learning from a commognitive standpoint. *The Journal of the Learning Sciences, 16*(4), 567-615.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourse, and mathematizing.* New York: Cambridge University Press.

Sfard, A., & Kieran, C. (2001). Cognition as communication: Rethinking learning-by-talking through multi-faceted analysis of students' mathematical interactions. *Mind, Culture, and Activity, 8*(1), 42-76.

Stephan, M., & Akyuz, D. (2012). A proposed instructional theory for integer addition and subtraction. *Journal for Research in Mathematics Education, 43*(4), 428-464.

The National Council of Teachers of Mathematics [NCTM]. (1964). *Topics in mathematics for elementary school teachers.* Washington, D.C.: The National Council of Teachers of Mathematics.

The National Council of Teachers of Mathematics [NCTM]. (1969). *More topics in mathematics for elementary school teachers.* (NCTM, Ed.) Washington, DC: The Council.

Tillema, E. S. (2012, April). What is the difference? Using contextualized problems. *Mathematics Teaching in the Middle School, 17*(8), 472-478.

VanDeWalle, J. A. (2004). *Elementary and Middle School Mathematics Teaching Developmentally* (5th ed.). New York: Pearson Education, Inc.

Vig, R., Murray, E., & Star, J. R. (2014). Model breaking points conceptualized. *Educational Psychology Review, 26*(1), 73-90.

Vlassis, J. (2004). Making sense of the minus sign or becoming flexible in 'negativity'. *Learning and Instruction, 14*, 469-484.

Vlassis, J. (2008). The role of mathematical symbols in the development of number conceptualization: The case of the minus sign. *Philosophical Psychology, 21*(4), 555-570.

Vygotsky, L. (1934/1962). *Thought and language.* (E. Hanfmann, & G. Vakar, Trans.) Cambridge, MA: MIT Press.

Wessman-Enzinger, N., & Tobias, J. M. (2015). Preservice teachers' temperature stories for integer adition and subtraction. *Proceedings of 39th Psychology of Mathematics Education Conference. 4,* pp. 289-296. Hobart, Australia: PME.

Whitacre, I., Bishop, J. P., Lamb, L. L., Philipp, R. A., Bagley, S., & Schappelle, B. P. (2015). 'Negative of my money, positive of her money': secondary students' ways of relating equations to a debt context. *International Journal of Mathematical Education in Science and Technology, 46*(2), 234-249.

Whitacre, I., Bishop, J. P., Lamb, L. L., Philipp, R. A., Schappelle, B. P., & Lewis, M. (2011). Integers: History, textbook approaches, and children's productive mathematical intuitions. In L. R. Weist, & T. D. Lamberg (Ed.), *Proceedings of the 33rd Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 913-920). Reno, NV: University of Nevada.

Whitacre, I., Bishop, J. P., Lamb, L. L., Philipp, R. A., Schappelle, B. P., & Lewis, M. L. (2012). Happy and sad thoughts: An exploration of children's integer reasoning. *The Journal of Mathematical Behavior, 31*, 356-365.

Whitacre, I., Bishop, J. P., Philipp, R. A., Lamb, L. L., & Schappelle, B. P. (2014, September). Dollars & sense: Students' integer perspectives. *Mathematics Teaching in the Middle School, 20*(2), 84-89.

Appendix A: Interview Task Protocol

Hello, my name is Douglas Platt, and I want to start by thanking you for meeting with me today.

You are participating in an interview to collect data for my study entitled "Pre-Service teachers' discourse on integers: transforming discourse on whole numbers to discourse about integers." Such a discourse relates to what words are used and how they are used, how things represented in a visual or related form, what routines are followed, and how statements are confirmed or rejected. Such a discourse occurs in a specific location, but tie into a large tradition in how ideas are displayed and conveyed.

That being said, try to relax and answer each of the following questions as fully as you can. If you do not feel comfortable answering a question, we can skip that item. Though your responses will be evaluated along with those of the other participants, you are not being graded. If time permits as the end of the interview, we can discuss some of these ideas more fully, but in an effort to capture your line of thinking, I will not be providing guidance or instruction about this topic during the interview. However, as some points I may make examples related to certain ideas available to you to look over.

The recordings and transcriptions of this interview will be stored in a password protected laptop. Your name will be replaced with that of a pseudonym or alias. You may suggest such a pseudonym if you would like.

You are free to ask for a break or request that the interview be ended at any point.

Are you ready to begin?

1. If a 7th grade student were to ask you, "What's the difference between whole numbers and positive integers?" how would you respond?

2. It is a familiar notion that "addition makes larger". This is true when non-zero whole number are added together. Is this also true when integers are added? Please provide some examples.

3. What types of numbers can you get when one integer is subtracted from another integer?

4. What roles can the symbol " - " take within a mathematics problem? (This symbol will be expressed on a card.)

5. The numbers 7 and -7 are additive inverses of each other. What does the phrase additive inverse mean? (The numbers will be expressed on a card.)

   a. And why are 7 and -7 additive inverses?

   b. How else would you describe two numbers that are additive inverses?

   c. Can you give me another pair of additive inverses?

6. Please consider the following expressions: "6 – -2", "10 – 2", "-1 + 9", and "6 + 2". (Each expression would be written on a separate card.)

   a. Show and explain how you would evaluate each expression.

   b. In what ways are these expressions similar? In what ways are they different?

   c. Show and explain how you would evaluate each expression using a chip model. (If the participant is unsure how to proceed offer to move on to the next item.)

d. Show and explain how you would evaluate each expression using a number line model.

(If the participant is unsure how to proceed offer to move on to the next item.)

7. Please consider the following expressions: "-4 – 2", "2 – 8", "-1 + -5", and "-7 + 1". (Each expression would be written on a separate card.)

a. Show and explain how you would evaluate each expression.

b. In what ways are these expressions similar? In what ways are they different?

c. Show and explain how you would evaluate each expression using a chip model.

(If the participant is unsure how to proceed offer to move on to the next item.)

d. Show and explain how you would evaluate each expression using a number line model.

(If the participant is unsure how to proceed offer to move on to the next item.)

Do you have any questions you would like me to answer about what we discussed today?

Conclusion: Thank you again for participating in this interview. I hope that you found this time informative and enjoyable, at least to a limited extent. To preserve the spontaneous nature of other participants' responses, I ask you to please refrain from discussing any of the questions posed during this interview, for the next few weeks.

You have provided me valuable information.

Appendix B: Interview Accessories

$10 - 2$

$6 + 2$

$6 - {-2}$

$-1 + 9$

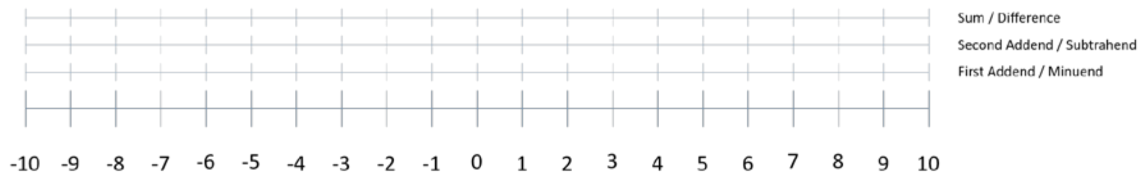
$2 - 8$ $-7 + 1$

$-4 - 2$ $-1 + -5$

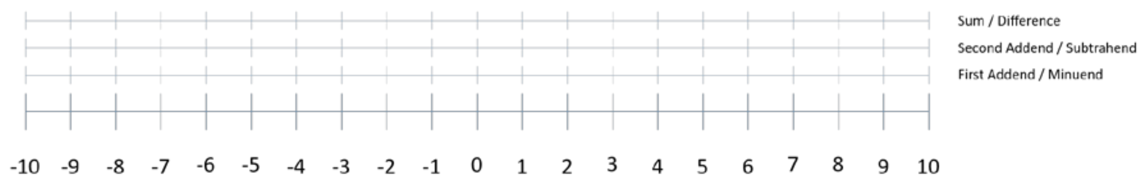
-

-7, 7

Expression to be computed: Sum / Difference:



Expression to be computed: Sum / Difference:



Expression to be computed: Sum / Difference:

Initial Expression	Equivalent form arranged for computational actions	Final Arrangement after computational actions

Expression to be computed: Sum / Difference:

Initial Expression	Equivalent form arranged for computational actions	Final Arrangement after computational actions

Appendix C: Word Use Coding Scheme

Things are coded as word use based on spoken words. Any code in "quotes" is used to identify when the participant <u>used the phrase explicitly</u>.

- Set of numbers
    - This use is not assumed by reading a positive or negative number.
    - 'Positive integer'
        - Participant states the phrase 'positive integer'
            - Quote: "a **positive integer's** anything like, on a number line, it's anything after zero" (Amanda, Q1)
            - Quote: "**positive integer** would be anything that is over, above zero." (Jacqueline, Q1)
    - 'Negative integer'
        - Participant states the phrase 'negative integer'
            - Quote: "a number on the left side of the zero would be **negative integers**" (Ashley, Q1)
    - 'Whole number'
        - Participant states 'whole number'
            - Quote: "a **whole number** is the same thing. (Amanda, Q1)
    - Integer
        - Participant states the word integer aside from 'positive integer', 'negative integer'
            - Quote: "However, when working with **integers**, **integers** can be positive or negative so if you're adding for instance negative five plus three the result will be negative two." (Rebecca, Q2)
            - Quote: "So they're all **integers**." (Brooke, Q6B)
    - Other names for sets of numbers
        - Participant states the name of a set of numbers other than 'whole numbers' and integers
            - Quote: "they're not a **fraction** or a **decimal**" (Ashley, Q1)
            - Quote: "integers are like **rational numbers** I think, that word, **irrational numbers** I guess you could say." (Bailey, Q1)
            - Quote: "the whole numbers with the set of **real numbers**. (Monique, Q1)
            - Quote: "there are numbers in between whole numbers that **aren't whole numbers**" (Ashley, Q1)
            - Quote: "So you start at negative seven, **whole integers** would just over one" (Bailey, Q7D '-7 + 1')
- Sign of number read or introduced
    - *Used to identify* the use of the exact phrase or *any reference to a specific number, i.e. 4, 0, -6, as when offering or reading an expression or its equivalent value.*

        - *These are not automatically considered as references to positive integers and negative integers, due to participant not explicitly referring to the values as types of integers.*
    - Positive number
        - Participant states a specific positive number
            - Quote: "negative ten minus **three** you have to change that to addition, and you get negative thirteen." (Amanda, Q3)
    - Zero
        - Participant state 'zero' when representing the solution to an expression or as a separate type of number
            - Quote: "If you get two minus two you'd get zero." (Nicole, Q3)
            - Quote: "If you do this you get **zero**." (Amanda, Q2)
    - Negative number
        - Participant states a specific negative number
            - Quote: "**negative ten** minus three you have to change that to addition, and you get negative thirteen" (Amanda, Q3)
- Role of sign read or introduced
        - *Used to denote the operation sign in the context of a written expression, but not the action of the operation*
    - Binary-operational Thematic sub-group
        - *Used to denote the sign or operation of binary operations.*
    - Addition
        - Participant states any of the verb forms of add, adding, addition, "do plus"
            - Quote: "Negative four plus, oh wait, so if it is minus two then you are really going to **do plus** a negative two so that's a negative six." (Amanda, Q7A '-4 – 2')
            - Quote: "negative ten minus three you have to change that to **addition**, and you get negative thirteen." (Amanda, Q3)
    - 'Plus' as addition sign
        - Participant states "plus", addition sign
            - Quote: "negative six **plus** negative four" (Brooke, Q2)
            - Quote: "for this one {'-1+9'} it kind of makes more sense for me cause when **I see addition** I always think you move to the right of the number line." (Brooke, Q6A '-1 + 9')
    - 'Minus' as subtraction sign
        - Participant states *"minus" between two numbers, or "subtraction sign"*
            - Quote: "negative ten **minus** three" (Amanda, Q3)
            - Quote: "this is already a negative and that the **subtraction sign** means to take away." (Christina, Q7B)
    - Subtraction
        - Participant states any of the verb forms of subtraction, subtract, subtracting
            - Quote: "then negative four minus two you would just umm, I don't know, to me I think of it as like just four plus two and then it's negative cause like

if you **subtract** a negative number from a negative number it's kind of like adding two positive numbers so I would just kind of count back like in the negatives, so it would be negative six" (Kayla, Q7A '-4 – 2')

- Quote: "and others {'6 + 2' and '10 – 2'} are basic adding and subtracting." (Bailey, Q6B)

- 'Take away' for subtraction
  - Participant states 'take away' while performing subtraction
    - Quote: "you have negative one plus nine, so you really have eight, because if you that's like saying nine **take away** one." (Jordan, Q6A '-1 + 9')
- Signifying or mismatched
  - *Used to denote symbols for the positive and negative nature of an integer or when the wrong word is assigned to a symbol for the role it is playing*
- 'Negative sign'
  - Participant state "negative" not associated with a particular number
    - Quote: "this one {'-1+ -5'} is just normal addition because they're both, they have the **same symbol.** So, they're both **negative**." (Amanda, Q7B)
    - Quote: "It could tell you to take away a certain number or it can tell you that a **number is negative**" (Bailey, Q4)
- 'Positive sign'
  - Participant states an expression contains a 'positive sign'
    - Quote: "negative one minus a positive five, but you don't necessarily need the **positive sign** there it is just assumed" (Brooke, Q7A '-1 + -5')
- Negative for subtraction sign
  - Participant stated negative prior to a negative number value while reading an expression
    - Quote: "That's **negative** and negative, basically just makes this a plus." (Jordan, Q6C '6 - -2')
- 'Minus' for negative
  - Participant states "minus" either follows another operation, or does not follow another number
    - Quote: "you had **minus seven**, ok so if you add seven to one you get zero and if you subtract seven from one you also get zero." (Jordan, Q5A)
    - Quote: "[Negative one] plus **minus** five is like minus five so negative four minus two" (Rebecca, Q7C '-4 – 2')

Appendix D: Visual Mediators Coding Scheme

When one considers a mathematical discourse, visual mediators take the form of written expressions, the chip model, and number lines. The use of visual mediators allow for permanence for the item being discussed. However, this permanence can be enhanced or suppressed by in the way the visual mediator is employed.

- Written expression
    - Used when the written expression is generated or modified by the participant during the process of solving the expression when first encountering the expression or written on the white board while modeling with the pre-made number line.
    - *More than one may apply*
    - Positive number
        - Participant writes or modified an equation containing any positive number as a term or solution
            - Action: wrote -5 + 3 then -2 then crossed out -5 + 3 = -2 (Rebecca, Q2)
    - Zero
        - Participant writes or modified an equation containing zero as a term or solution
            - Action: Writing the expression and solution -2 + 2 = **0** (Amanda, Q2)
    - Negative number/sign
        - Participant writes or modified an equation containing any negative number as a term or solution
            - Action: wrote -5 + 3 then -2 then crossed out -5 + 3 = -2 (Rebecca, Q2)
    - Addition
        - Participant writes or modified an equation so that it contains an addition sign
            - Action: Wrote the expression "-1 + 9" then 8 (Jordan, Q6A '-1 + 9')
    - Subtraction
        - Participant writes or modified an equation so that it contains a subtraction sign
            - Action: Wrote "4 – 6 = -2" (Christina, Q3)
    - Positive sign explicitly
        - Participant writes or modified an equation so that it contains a positive sign
            - Action: wrote the expression -4 – (-1) then changes the subtraction and negative sign to addition and positive signs before writing -3 (Rebecca, Q3)
- Identification of ' - '
    - Based on participant response to question with ' - ' not associated with number or expression
    - Subtraction sign
        - Participant stated that the ' - ' can be used to indicate subtraction
            - Quote: "it could indicate either negative or subtraction." (Brooke, Q4)
    - Negative sign
        - Participant stated that the ' - ' can be used to indicate a negative number
            - Quote: "it could indicate either negative or subtraction." (Brooke, Q4)

- Chip model notations
    - Used in association with any time the participant employed drawn or plastic chips
  - Drawn chip model
    - The expression is modeled using drawn circles and cross-outs
      - Action: Wrote 2 filled in dots and 6 open dots having crossed off the first 2 dots. The 6 open dots give -6 (Jordan, '2 – 8')
  - Beige and brown chips
    - Used in association the differentiation of the beige and brown plastic chips
  - Chip model with one color
    - Chips with only one color of chips used
      - Action: Counted out 10 light chips by pairs, lined up in two columns. Slid 2 light chips away from the others and counted up to 8 on the remaining chips (Brooke, Q6C '10 – 2')
  - Chip model with separate colors
    - Chip model where one of the two colors available was assigned to represent the number of positives while the other color was assigned to represent the number of negatives.
      - Quote: "So I would say the dark color would be negative, um, then you would start adding positives so one two three four five six seven eight nine and because you have one versus the negative it would cancel each other out and then you would have eight positives left." (Nicole, Q6C '-1 + 9')
      - Action: Slid 1 dark chip over and counted out 9 light chips which she made it to a square pattern paired up 1 light and 1 dark chip and removed them leaving 8 light chips. (Nicole, Q6C '-1 + 9')
  - Chip model with mixed colors
    - Chip model where the color of chips is not differentiated by the participant
      - Quote: "So you'd have six, one, two, three, four, five, six. And then you are going to add two, I'm going to represent these {2} as the darker color. So, six plus two which is eight." (Bailey, Q6C '6 + 2')
      - Action: Counting out 6 light colored chips and 2 darker colored chips creating a set of 8 chips with two different colors. (Bailey, Q6C '6 + 2')
- Number line notations
    - Used in association with references to number line or symbols drawn along a number line
  - Drawn number line
    - Participant draws a number line while solving the expression
      - Action: Wrote -6 + -4 then drew and labeled a number line (Brooke, Q2)
  - Spoke of number line
    - Participant made verbal reference to a number line, but did not visually employ one
      - Quote: "I'd show them a number line" (Ashley, Q1)

        - Action: Pointing to the left and right of a common center point as if on a number line (Ashley, Q1)
- Colored chips as number line
    - Participant lines up chips and counts along them in a manner that is similar to the use of a number line.
        - Quote: "So I am actually gonna show it kind of like a line … I am just going to line them up as if this was a number line" (Jamie, Q6C '-1 + 9')
        - Action: Placed a dark chip out, then left a gap held by her finger and moved in a light chip closed the gap then opened it again and made a long line of light chips. Pointed to one of the light chips to represent zero and added a pair of dark chips slide the zero light chip out of line pointing to -1 dark chip counts 9 chips right and counts the number of light chips to the right of zero (Jamie, Q6C '-1 + 9')
- Number line dot hops dot
    - Participant indicates the movements along the number line with hops which do not clearly indicate the direction of motion after they are drawn
        - Action: Drew a dot at -1, while counting made 9 hops to the right from -1 to 8. Makes a dot at 8. (Brooke, Q6D '-1 + 9')
- Left or Right arrow
    - Participants drew an arrow with the tip at the left or right side
        - Action: Drew a black dot at 2 then **drew an arrow to the left** 8 units. Placed a dot at -6 (Bailey, Q7D '2 – 8')
        - Action: Made a mark at 6 and a mark at -2 made 8 hops to the right to get from -2 to 6 and **drew an arrow to the right** pointing at 6 (Tae, Q6D '6 - -2')
- Numbered step taken up or down number line
    - Participant wrote a sequence of numbers along the number line as she simplified the expression
        - Action: Made a dot at -1 and **numbered her way up** 9 spaces [1 2 3 4 5 6 7 8 9] then drew and arrow pointing to 8 (Nicole, Q6D '-1 + 9')
- Number line dots along number line with markers at beginning and end
    - Participant indicated steps along number line with dots
        - Action: Makes a mark at -1 then makes an arrow with her finger pointing to the left counts out and makes 5 marks to the left and an x at -6 (Jordan, Q7D '-1 + -5')

Appendix E: Routines Coding Scheme

Routines are the patterns of actions an individual used or described verbally to simplify recurring or highly similar expressions. If an individual makes an immediate or spontaneous action on first encountering an expression it is considered as a routine.

Evidence for the assignment of a routine includes spoken words, written notations, movements with chips, and symbols used on the number line.

When a participant expresses an answer to the written expression, or an expression they present, quickly without explanation or they labor over how to approach a expression no routine is evident.

- Actions with numerals
  - Portions of numbers with different signs can cancel to give sum or difference
    - Participant stated that the used one portion of one term to match with and cancel the other term
      - Quote: "You take one away from the nine which is <u>a positive one and you add it to the negative one to create zero</u> and you have eight left." (Nicole, Q6A '-1 + 9')
      - Quote: "Ok, so if you subtract eight from two you'd go down, if you start at two you would so <u>two of the eight that you're subtracting would cancel out to zero</u>." (Kayla, Q7A '2 – 8')
  - Adding to a negative as subtraction
    - Participant stated that for a expression which has them adding to a negative number, they subtract the absolute value of the first term from the second term
      - Quote: "then negative one plus nine equals eight because that's negative, if you <u>add a negative number to a positive number you're subtracting</u>." (Christina, Q6A '-1 + 9')
  - Subtracting a negative number is adding a positive number
    - Participant states that the subtraction and negative number pair becomes addition with a positive number
      - Quote: "ten minus negative three then that cancel and then that becomes positive thirteen" (Amanda, Q3)
      - Action: 10 - -3 = 13 changed to 10 + 3 = 13 (Amanda, Q3)
  - Subtraction as adding a negative number
    - Participated the original subtraction expression in terms of addition and a negative number
      - Quote: "Two minus eight equals, so to me <u>it is like you are doing plus a negative</u>, so it becomes negative six." (Amanda, Q7A '2 – 8')
      - Action: Wrote 2 – 8, then converted it to 2 + -8 and found -6. (Amanda, Q7A '2 – 8')
- Actions with chip model
  - Adding as joining
    - Participant added chips or circles to their collection or merged two sets to display addition

- Quote: "So now we have six plus two. So, we start with six chips, and then we're adding two more, so now we have eight chips in total." (Brooke, Q6C '6 + 2')
- Action: Starting with 6 light chips in two columns of three then slides in a pair of light chips to make two columns of four, shows the 8 chips (Brooke, Q6C '6 + 2')
- Quote: "I have negative four, and negative two … they're the same color chips, essentially, I add them together. And I get negative six." (Brooke, Q7C '-4 – 2')
- Action: Made a group of 4 dark chips and a pair of dark chips, slid the pair to join the 4 dark chips Forming two rows of three dark chips of 6 dark chips (Brooke, Q7C '-4 – 2')
- Quote: "Negative one plus negative five, just going to join the negative one with the negative five and have a negative six." (Christina, Q7C '-1 + -5')
- Action: Set out 1 light and counted out 5 light chips slid the 1 to join the 5 to make 6 light chips (Christina, Q7C '-1 + -5')
- Quote: "they're both technically, positive values so two eight, these cancel when you subtract the two, so you're going to remove the two from the eight, like this and you're still gonna have negative six." (Christina, Q7C '2– 8')
- Action: Set out 2 dark chips and counted out 8 light chips point to the top pair of light chips and both dark chips then removes the 2 dark and 2 light chips (Christina, Q7C '2 – 8')

- Opposite colored chips stacked
  - Participant grouped or stacked light and dark chips to cancel each out or crossing out drawn circles. Requires the joining of sets of chips.
    - Quote: "they're both technically, positive values so two eight, these cancel when you subtract the two, so you're going to remove the two from the eight, like this and you're still gonna have negative six." (Christina, Q7C '2– 8')
    - Action: Set out 2 dark chips and counted out 8 light chips point to the top pair of light chips and both dark chips then removes the 2 dark and 2 light chips (Christina, Q7C '2 – 8')
    - Quote: "I have negative one and I have a positive one they would cancel out each other since negative one plus positive one would give me zero" (Brooke, Q6C '-1 + 9')
    - Action: counted out 9 light chips in pairs and formed an angled pair of rows. Then reangles the light chips. And took out a single dark chip. Pointed to the light chips while referring to them. Picked up one of each color chips and placed them together. Set the mixed color pair aside, and rearranged the light chips to show 8 (Brooke, Q6C '-1 + 9')

        - Quote: "So this represents negative seven adding one which is positive. So, I would, hum. So, I would cancel these out. To get negative six." (Ashley, Q7C '-7 + 1')
        - Action: Set out 7 light colored chips and pulled out 1 dark chip, paired a light and dark chip and moved them away (Ashley, Q7C '-7 + 1')
  - Subtraction as taking away
    - Participant removed chips or circles from their collection to display subtraction
      - Quote: "that would be ten take away two and you have eight." (Amanda, Q6C '10 – 2')
      - Action: Sorted out ten dark chips and two light chips. Then as a correction go rid of the two light colored chips and removed two dark chips to show the subtraction. (Amanda, Q6C '10 – 2')
      - Quote: "Ok, so for ten minus two you would start with ten chips. Two, four, six, eight, ten. And then since it is subtraction it means you take away, so you take away two and then you're left with two, four, six, eight chips. (Brooke, Q6C '10 – 2')
      - Action: Counted out 10 light chips by pairs, lined up in two columns. Slid 2 light chips away from the others and counted up to 8 on the remaining chips (Brooke, Q6C '10 – 2')
  - Opposite colored chips swapped
    - Participant replaces chip of one color with a chip of the other color
      - Quote: "Negative four and you're gonna subtract negative two, no, you're just going to subtract regular two, but since you don't have a positive value to remove the two from, you're really just going to end up adding two more to the negative value" (Christina, Q7C '-4 - 2')
      - Action: Set out 4 light and 2 dark in separate piles slid in 2 more light chips and got rid of the 2 dark chips leaving 6 light chips (Christina, Q7C '-4 -2')
      - Action: Made an array of 11 light chips then slid away 2 light chips and slid in 2 dark chips then slid away the 2 dark chips leaving 9 light chips (Nicole, Q6C '10 – 2')
- Action with number line
  - Go up number line
    - Participant indicated that the equivalent value is up or to the right along the number line from the first value plotted
      - Quote: "If you have six, and then you go one, two, then you get eight." (Amanda, Q6D '6 + 2')
      - Action: Pointing at 6 then placing a dot, then doing two hops to the right and placing a dot at 8 this was in blue (Amanda, Q6D '6 + 2')
      - Quote: "I see addition I always think you move to the right of the number line. (Brooke, Q6A '-1 + 9')
      - Action: Placing a dot on -1 counting to the right, then putting a dot at and circling 8 (Brooke, Q6A '-1 + 9')

- Quote: "Negative seven plus one start on negative seven right here and you're adding positive one, so you're going in the positive direction right one to negative six." (Christina, Q7D '-7 + 1')
- Quote: "So for negative one plus nine, you start at negative one and you add nine, so you go one, two, three, four, five, six, seven, eight, nine. So, for that one, you would also get eight." (Bailey, Q6A '-1 + 9')
- Action: Put a dot at the -1 on the number line drew an arrow to the right nine units long. (Bailey, Q6A '-1 + 9')

- Go down number line
  - Participant indicated that the equivalent value is down or to the left along the number line from the first value plotted
    - Quote: "If you have, if you're at ten, and you have to go back two, you go one, two and you're at eight." (Amanda, Q6D '10 – 2')
    - Action: Placed a red dot at ten, making two hops to the left and placing a dot at 8 (Amanda, Q6D '10 – 2')
    - Quote: "Negative four, here, subtracting positive two, so you're still going in the negative direction, since it's two you do one two. Negative six." (Christina, Q7D '-4 – 2')
    - Action: Drew a blue dot at -4 then made two hops to the left and put a dot at -6 (Christina, Q7D '-4 – 2')
    - Quote: "Negative one, plus five, start here and go up negative five numbers." (Bailey, Q7D '-1 + -5')
    - Action: Drew a red dot at -1 and a line 5-units to the left and drew a dot at -6 wrote -1 + -5 = -6 (Bailey, Q7D '-1 + -5')

Appendix F: Narratives Coding Scheme

Narratives in a discourse represent those ideas which are communicated and can be affirmed or rejected. Within a discourse about integers, I am defining such narratives as including the relationship between number sets, and the nature of mathematical computations.

- Relationship between positive integers and whole numbers
    - ▪ Used when participant was prompted to compare positive integers and whole numbers
  - o Are synonyms
    - ▪ Participants state that positive integers and whole numbers are the same thing
      - Quote: "a positive integer's anything like, on a number line, it's anything after zero and a whole number is the same thing, it's anything greater than zero. I think, so I think **they're the same thing**." (Amanda, Q1)
  - o Mean different things
    - ▪ Participant identifies different types of numbers to be part of the positive integers and the whole numbers.
      - Quote: "I would tell them that **positive integers can have a decimal or a fraction of a whole number**" (Brooke, Q1)
- 'Addition makes larger' remains true with integers
    - ▪ Used in association with the participant challenged with the claim addition makes larger
  - o Yes, it remains true
    - ▪ Participant states that addition produces larger numbers
      - Quote: "I would say yes, because even if you add together negative integers, umm, and they'd still be a negative number, but it can be larger or closer to zero closer to a whole number then it would be before" (Brooke, Q2)
  - o No, not always
    - ▪ Participant states that addition is not always associated with getting a larger number
      - Quote: "Not all the time because when you add two negative integers you get a number that's smaller than what, then both of them, those numbers, so like if you were to do negative seven plus negative one, that would be negative eight which is something which is smaller than what you started with negative seven, so no addition is not always associated with getting a larger number" (Ashley, Q2)
- Types of numbers generated by integer subtraction
    - ▪ Used in association with the participants' responses related to possible differences related to subtraction expressions
  - o Positive number
    - ▪ Participant states that subtraction can result in positive numbers

            - Quote: "You can have whole numbers whether it be **positive** or negative, and then you can also have integers meaning decimals or fractions of a whole number." (Brooke, Q3)
    - Negative number
        - Participant states that subtraction can or cannot result in a negative number
            - Quote: "You can have whole numbers whether it be positive or **negative**, and then you can also have integers meaning decimals or fractions of a whole number." (Brooke, Q3)
            - Quote: "When an integer, you can get. You would only get positive because negative, **you can't have a negative integer**." (Jacqueline, Q3)
    - Integer
        - Participant states that subtraction can result in an integer
            - Quote: "Well you'll still get You'll still get **an integer**." (Christina, Q3)
    - Any number
        - Participant stated that subtraction can result in any type of number
            - Quote: "You can get **any number** I think." (Tae, Q3)
    - Larger number
        - Participant states that subtraction can give you a larger number
            - Quote: "**you can get a number that is bigger than what you started** with" (Ashley, Q3)
    - Smaller number
        - Participant states that the result of subtraction is smaller than the original value
            - Quote: "a **number that is smaller than you started with**" (Ashley, Q3)
            - Quote: "Other integers, like **smaller integers**" (Kayla, Q3)
- Properties of additive inverse or the pair '-7 and 7'
        - Used in association with probes related to additive inverses
    - Are opposites
        - Participant states that -7 and 7 are opposites or that one is positive while the other is negative
            - Quote: "**they're also opposite from each other** on the number line so from zero to seven and from zero to negative seven," (Bailey, Q5A)
            - Quote: "seven or negative seven. Negative seven can turn into a positive seven." (Courtney, Q5)
            - Quote: "The same distance away from zero just on different sides, like **one is positive and then one can be negative**. So, if you want, I guess different from what I said I would say the **same distance from zero, but one is on the negative side one is one the positive side**." (Jamie, Q5B)
            - Quote: "And this was zero they are **both seven away, but one's negative and one's positive. So, they are on different sides of a number line**." (Amanda, Q5)
    - Add to zero
        - Participants state that the sum of -7 and 7 is zero

        - Quote: "**when you add seven to negative seven you would get zero** and when you subtract seven from seven you would get zero." (Jordan, Q5A)
        - Quote: "If you **add them together you get a number, you get zero**" (Bailey, Q5A)
- Operation and signifiers both need to be considered
    - Adding a negative number and a positive number can be treated as subtraction
        - Participant started that rather than adding a negative can be treated as subtracting a positive
            - Quote: "then **negative one plus nine equals eight because that's negative, if you add a negative number to a positive number you're subtracting.**" (Christina, Q6A '-1 + 9')
    - Subtracting a negative number can be treated as addition of a positive number
        - Participants states that when you subtract a negative number you can change the expression to adding the absolute value of the subtrahend
            - Quote: "**when you have two negatives next to each other, or a subtraction sign and a negative sign you make it a positive sign and then you add them together**." (Nicole, Q6A '6 - -2')
    - Subtracting a positive number can be treated as adding a negative number
        - Participants state that whenever you are subtracting a positive you can treat is as adding a negative
            - Quote: "**so negative four minus two, I can think of that as adding two negative values** so then that would be negative five and then negative six after that" (Rebecca, Q7C '-4 – 2')
    - When you have a negative number you automatically subtract them
        - Participant stated that the presence of a negative number indicates that subtraction is required
            - Quote: "**once you have a negative number you automatically subtract them**, so that '-1 + 9' would be the same as ten minus two, and then when you're adding them, when you're subtracting, and you have a negative number it would be the same." (Jacqueline, Q6B)
    - Symbols provide inconsistent meanings
        - Participant states that operations are dependent on the combination of operational sign and number value
            - Quote: "I am trying to say that sometimes you would have to do the opposite of what the symbol is asking you to do depending on if, depending on if the number you are adding, or subtracting is a negative or a positive integer" (Ashley, Q7D '-4 – 2')
- Results of Subtraction
    - Larger number from a smaller number is not proper
        - Participant states that subtraction cannot involve subtracting a larger number from a smaller number
            - Quote: "Umm, so two minus eight. Umm if we start with two, if we start with two and know that we can't take away eight so if we have two we

need six more that would be negative six that we don't have." (Jacqueline, Q7C '2 – 8')

            - Quote: "And then two minus eight I don't have two, I don't have enough to subtract two minus eight so then I'll." (Ashley, Q7A '2 – 8')
    - Negative number cannot be done
        - Participant states that subtraction of a negative number is impossible
            - Quote: "Alright and if you did, six, so if they, ah, so if it is six minus negative two, but you really, you can't take away negative two from here, so you end up adding two, so you have eight." (Amanda, Q6C '6 - -2')
            - Quote: "When you're taking away a negative they sort of cancel each other out, I just can't. … It's the same thing as adding a positive. Cause you can't take away a negative number." (Jamie, Q6D '6 - -2')
- Similarity of operations on the number line
    - Addition
        - Participant stated that the number line is used the same way as in other addition expressions
            - Quote: "So this one, so you're still this is an addition problem, so you can just use the number line the same way you would in the other problems." (Jamie, Q6A '-1 + 9')
    - Subtraction
        - Participant stated that the number line is used the same way as in other subtraction expressions
            - Quote: "Even though its subtracting a negative you are still going to the right of the number line cause when I see subtraction I always think you have to go to the left of the number line." (Brooke, Q6A '6 - - 2')

Appendix G: Code Assignments for Word Use by Participant

Codes recheck number by number.	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae	Total
Sets of numbers															
'Positive integer'	1	6	1	1		1	1	4	2	2	1	2	9	2	33
'Negative integer'	1	7		2			1	2	1	2	2	1	6	5	30
'Whole number'	1	1	1	3	1	1	5	1	4	1	2	1	3	1	26
Integer			1	2	1		1		2	2	2		4	1	16
Other names for sets of numbers		1	2	2		1	1	1	2		2	2	1	3	18
Sign of number read or introduced															
Positive number	27	26	25	26	28	21	27	30	31	27	24	28	28	23	371
Zero	3	3	2	1		1	3	2	2	2	2	5	2		28
Negative number	25	23	20	25	22	17	24	24	28	26	20	26	24	17	321
Role of sign read or introduced															
Binary-operational															
Addition	13	16	9	9	9		11	20	16	15	10	17	11	11	167
'Plus' as addition sign	11	12	9	10	13	7	16	14	7	13	9	12	10	9	152
'Minus' as subtraction sign	11	10	9	8	8	4	13	15	4	9	4	11	14	10	130
Subtraction	10	12	5	8	7	1	3	4	9	13	1	11	9	6	99
'Take away' for subtraction	2	6	4	1	6		7	9	11	2	10		1		59
Signifying or mismatched															
Negative sign	1	2		1	1		3				2	2	5		17
'Positive sign'											1	2	1		4
Negative for subtraction sign								1				2	1		4
'Minus' for negative									1						1
Total	106	125	88	99	96	54	116	127	120	114	92	122	129	88	1476

Appendix H: Code Assignments for Word Use by Question

Codes recheck number by number.	Q1	Q2	Q3	Q4	Q5	Q6A	Q6B	Q6C	Q6D	Q7A	Q7B	Q7C	Q7D	Total
Sets of numbers														
'Positive integer'	13	2	1		2			3	2	3	2	1	4	33
'Negative integer'	2	5	4	2	2	1	2	1		3	3	3	2	30
'Whole number'	14	4	1	1		2		1		2			1	26
Integer	2	3	4	1	2		1			1	1		1	16
Other names for sets of numbers	10	2	3	1			1						1	18
Sign of number read or introduced														
Positive number	6	10	13	4	34	55	14	49	55	44	8	39	40	371
Zero	1	3	2		16	1	1	1		1	2			28
Negative number	5	15	13	3	32	23	11	22	31	52	12	50	52	321
Role of sign read or introduced														
Binary-operational														
Addition		10	1		16	15	9	24	21	12	11	25	23	167
'Plus' as addition sign		11	1		2	29	5	12	23	27	5	20	17	152
'Minus' as subtraction sign			7			25	5	14	16	26	4	17	16	130
Subtraction		1	8		4	13	10	13	9	13	5	9	14	99
'Take away' for subtraction				3	1	11	1	14	7	6	3	6	7	59
Signifying or mismatched														
Negative sign				1	3	6	2	1		1	2	1		17
'Positive sign'					2	1					1			4
Negative for subtraction sign					1	1	1						1	4
'Minus' for negative					1									1
Total	53	66	58	16	118	183	63	155	164	191	59	171	179	1476

Appendix I: Code Assignment for Visual Mediators by Participant

Codes recheck number by number.	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae	Total
Written Expression															
Positive number	9	8	14	5	10	7	4	11	8	1	7	1	11	5	101
Zero	1		1							1					3
Negative number/sign	9	9	14	4	7	4	4	10	8	4	6	2	11	7	99
Addition	10	6	9	4	6	5	3	9	4		4	2	4	1	67
Subtraction	6	6	7	4	5	4	1	6	4	1	4	2	4	1	55
Positive sign explicitly					1		1						1		3
Identification of ' - '															
Subtraction sign	1	1	1	1	1	1	1	1	1	1	1	1	1	1	14
Negative sign	1		1	1	1	1		1	1	1	1	1	1	1	12
Chip model notations															
Drawn		5	4						1				3	2	15
One color	6	9	4	5	1	2	2	2	3	4	3	3	8	7	59
Separate colors		5	3	3	5	2		6	4	3	4	4	1	3	43
Mixed colors	2		5		2	4	6		1	1	1	1	1	1	25
Number line notations															
Participant drawn	1	1	3	4				5					6	6	26
Spoken of	2	5	1	5	1		1	4				1	2	2	24
Colored chips as number line								5					2		7
Dot-hops-dot	8	9		11	8	8	8	11	1	8	7		14	12	105
Left or right arrow			10					3	1						14
Numbered steps taken												8			8
Dots along number line									7						7
Total	56	64	77	47	48	38	31	74	44	25	38	26	70	49	687

Appendix J: Code Assignment for Visual Mediators by Question

Codes recheck number by number.	Q1	Q2	Q3	Q4	Q5	Q6A	Q6B	Q6C	Q6D	Q7A	Q7B	Q7C	Q7D	Total
Written Expression														
Positive number		7	5		3	38			15	25			8	101
Zero		2								1				3
Negative number/sign		7	5		3	16			8	47			13	99
Addition		8	3			21			8	20			7	67
Subtraction			4			18			6	19			8	55
Positive sign explicitly			1			2								3
Identification of ‘ - ’														
Subtraction sign				14										14
Negative sign				12										12
Chip model notations														
Drawn						9				6				15
One color						7		25		4		23		59
Separate colors								16		3		24		43
Mixed colors								15				10		25
Number line notations														
Participant drawn	1	1			2	12	1			9				26
Spoken of	2				9	2	1	2		4	2	2		24
Colored chips as number line								2				5		7
Dot-hops-dot		1			1	6	1		44	9			43	105
Left or right arrow						3			7				4	14
Numbered steps taken									4				4	8
Dots along									4				3	7
Total	3	26	18	26	18	134	3	60	96	147	2	64	90	687

Appendix K: Code Assignment for Routines by Participant

Codes recheck number by number.	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae	Total
Actions with numerals															
Portions of numbers with different signs can cancel to give sum or difference		1		3	2				2	3	3	4		1	19
Adding to a negative number as subtraction	5	2			1		2		3	1		1	1		16
Subtracting a negative is adding a positive number	4	5	2		1	1	3	1	1	2		1	4	1	26
Subtraction as adding a negative number	3			2	1		3	1	1	2		5	1	2	21
Actions with the chip model															
Adding as joining	5	7	9	6	7	5	4	1	8	7	6	7	7	8	87
Opposite colored chips stacked		3	5	3	3	2			3	3	4	3		3	32
Subtraction as taking away	4	3	1	1	1	2	2	1	2	1	1	1	3	2	25
Opposite colors chips swapped		1	1		1	3			1			2			9
Action with number line															
Go up	4	5	6	10	4	4	4	9	4	4	3	4	9	7	77
Go down	4	5	5	7	4	4	4	8	5	4	4	4	10	6	74
Total	29	32	29	32	25	21	22	21	30	27	21	32	35	30	386

Appendix L: Code Assignment for Routines by Question

Codes recheck number by number.	Q1	Q2	Q3	Q4	Q5	Q6A	Q6B	Q6C	Q6D	Q7A	Q7B	Q7C	Q7D	Total
Actions with numerals														
Portions of numbers with different signs can cancel to give sum or difference						2		6		4		7		19
Adding to a negative as subtraction		1				5	1	3		2		4		16
Subtracting a negative number is adding a positive number			2			13	2	5	4					26
Subtraction as adding a negative			1					1	1	5	2	11		21
Actions with chip model														
Adding as joining						5		34		6		42		87
Opposite colored chips stacked								9		4		19		32
Subtraction as taking away						4		17				4		25
Opposite colors chips swapped								4		1		4		9
Action with number line														
Go up		1			3	9	2	1	40	4	1	1	15	77
Go down					2	3	1		17	7	1	3	40	74
Total		2	3		5	41	6	80	62	33	4	95	55	386

Appendix M: Code Assignment for Narratives by Participant

Codes	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae	Total
Relationship between positive integers and whole numbers															
Are synonyms	1				1				1						3
Mean different things		1	1	1		1	1	1		1	1	1	1	1	11
'Addition makes larger' remains true with integers															
Yes, it remains true				1	1							1			3
No, not always	1	1	1			1	1	1	1	1	1		1	1	11
Types of numbers generated by integer subtraction															
Positive number	1		1	1		1	1	1	1	1	1	1	1		11
Negative number	1		1	1	1	1	1	1	1	1	1	1	1		12
Integer					1					1					2
Any number											1			1	2
Larger number		1													1
Smaller number		1								1	1				3
Properties of additive inverse or the pair '-7 and 7'															
Are opposites	1		1	2	2	1	2	3		3	1	1	2	1	20
Add to zero	2	3	2				2		1		2	1	2		15
Operation and signifiers both need to be considered															
Adding a negative number and a positive number can be treated as subtraction	3				1		1			1			2		8
Subtracting a negative number can be treated as addition of a positive number		1		1			2	2	1	1		3	3	1	15
Subtracting a positive number can be treated as adding a negative number	1						1	1	1	1			1	1	7
When you have a negative number you automatically subtract							1								1
Symbols provide inconsistent meanings		1													1
Results of subtraction															
Larger number from a smaller number is not proper	2	1					1								4
Negative number cannot be done	1							1	1						3
Similarity of operations on the number line															
Addition		2		2	1			4	1	2		1	2	1	16
Subtraction	1	4		3	1			4	1		1	1	3	1	20
Total	15	16	7	12	9	5	14	19	10	14	10	11	19	8	169

Appendix N: Code Assignment for Narratives by Question

Codes	Q1	Q2	Q3	Q4	Q5	Q6A	Q6B	Q6C	Q6D	Q7A	Q7B	Q7C	Q7D	Total
Relationship between positive integers and whole numbers														
Are synonyms	3													3
Mean different things	11													11
'Addition makes larger' remains true with integers														
Yes, it remains true		3												3
No, not always		11												11
Types of numbers generated by integer subtraction														
Positive number			11											11
Negative number			12											12
Integer			2											2
Any number			2											2
Larger number			1											1
Smaller number			2						1					3
Properties of additive inverse or the pair '-7 and 7'														
Are opposites					20									20
Add to zero					15									15
Operation and signifiers both need to be considered														
Adding a negative number and a positive number can be treated as subtraction		1			1	2	2					1	1	8
Subtracting a negative number can be treated as addition of a positive number			1			6	3	2	3					15
Subtracting a positive number can be treated as adding a negative number										2		5		7
When you have a negative number you automatically subtract							1							1
Symbols provide inconsistent meanings													1	1
Results of subtraction														
Larger number from a smaller number is not proper										1	1	2		4
Negative number cannot be done						1		1	1					3
Similarity of operations on the number line														
Addition						2			6				8	16
Subtraction						2			6			1	11	20
Total	14	15	31	0	36	13	6	3	17	3	1	9	21	169

Appendix O: Participant Code Assignments by Question Prompt

Code assignment for Amanda’s transcript by turn.

			Sets of numbers					Sign of number read or introduced			Role of sign verbalized or introduced									Written Expression including						Identification of ‘- ’		Chip model notations				Number line notations						
	Question number	Participant voiced confusion	‘Positive integer’	‘Negative integer’	‘Whole number’	Integer	Other names for sets of numbers	Positive number	Zero	Negative number	Addition	‘Plus’ as addition sign	‘Minus’ as subtraction sign	Subtraction	‘Take away’ for subtraction	Negative sign	‘Positive sign’	Negative for subtraction sign	‘Minus’ for negative	Positive number	Zero	Negative number/sign	Addition	Subtraction	Positive sign explicitly	Subtraction sign	Negative sign	Drawn	One color	Separate colors	Mixed colors	Participant drawn	Spoke of number line	Chips as colored number line	Dot-hops-dot	Left or right arrow	Numbered steps taken	Dots along number line
01	Q1		*		*																												*					
02	Q2							*		*		*		*						*		*	*															
03	Q2							*	*	*		*								*	*	*	*															
04	Q3							*		*			*	*						*		*	*	*														
05	Q4			*						*																*	*											
06	Q5								*		*																					*						
07	Q5A								*		*																						*					
08	Q5B																																					
09	Q5C							*		*																												
10	Q6																																					
11	Q6A ‘6 + 2’							*				*								*			*															
12	Q6A ‘10 – 2’							*					*							*				*														
13	Q6A ‘-1 + 9’							*		*			*	*		*				*		*	*	*														
14	Q6A ‘6 – -2’							*		*		*	*							*		*																
15	Q6B							*		*	*			*																								
16	Q6C ‘6 + 2’							*			*	*																			*							
17	Q6C ‘10 – 2’							*					*		*														*									
18	Q6C ‘6 – -2’							*		*	*		*																*									
19	Q6C ‘-1 + 9’							*		*																			*									
20	Q6D ‘6 + ‘}							*				*																							*			
21	Q6D ‘10 – 2’							*																											*			
22	Q6D ‘6 – -2’							*		*	*		*																						*			
23	Q6D ‘-1 + 9’							*		*		*																							*			
24	Q7																																					
25	Q7A ‘2 – 8’							*		*		*	*							*		*	*	*														
26	Q7A ‘-7 + 1’							*		*		*	*							*		*	*	*														
27	Q7A ‘-4 – 2’							*		*		*	*									*	*															
28	Q7A ‘-1 + -5’									*	*											*	*															
29	Q7B							*		*	*			*																								
30	Q7C ‘2 – 8’							*		*				*	*																*							
31	Q7C ‘-7 + 1’							*		*	*	*		*															*									
32	Q7C ‘-4 – 2’							*		*	*		*																*									
33	Q7C ‘-1 + -5’									*	*																		*									
34	Q7D ‘2 – 8’							*		*				*																					*			
35	Q7D ‘-7 + 1’							*		*	*																								*			
36	Q7D ‘-4 – 2’							*		*				*																					*			
37	Q7D ‘-1 + -5’									*	*			*																					*			

		Actions with numerals				Actions with the chip model				Action with number line		Relationship between positive integers and whole numbers		'Addition makes larger' remains true with integers		Types of numbers generated by integer subtraction						Properties of additive inverse or the pair '-7 and 7'		Operation and signifiers both need to be considered					Results of subtraction		Similarity of operations on the number line	
	Question number	Portions of numbers with different signs can cancel to give sum or difference	Adding to a negative number as subtraction	Subtracting a negative is adding a positive number	Subtraction as adding a negative number	Adding as joining	Opposite colored chips stacked	Subtraction as taking away	Opposite colors chips swapped	Go up number line	Go down number line	Are synonyms	Mean different things	, it remains true	No, not always	Positive number	Negative number	Integer	Any number	Larger number	Smaller number	Add to zero	Are opposites	Adding a negative number and a positive number can be treated as subtraction	Subtracting a negative number can be treated as addition of a positive number	Subtracting a positive number can be treated as adding a negative number	When you have a negative number you automatically subtract	Symbols provide inconsistent meanings	Larger number from a smaller number is not proper	Negative number cannot be done	Addition	Subtraction
01	Q1											*																				
02	Q2		*												*									*								
03	Q2																															
04	Q3			*	*											*	*															
05	Q4																															
06	Q5																					*	*									
07	Q5A																					*										
08	Q5B																															
09	Q5C																															
10	Q6																															
11	Q6A '6 + 2'																															
12	Q6A '10 – 2'																															
13	Q6A '-1 + 9'		*																					*								
14	Q6A '6 – -2'			*																												
15	Q6B																															
16	Q6C '6 + 2'					*																										
17	Q6C '10 – 2'							*																								
18	Q6C '6 – -2'			*		*																								*		
19	Q6C '-1 + 9'		*					*																								
20	Q6D '6 + '}									*																						
21	Q6D '10 – 2'										*																					
22	Q6D '6 – -2'			*						*																						
23	Q6D '-1 + 9'									*																						
24	Q7																															
25	Q7A '2 – 8'				*																											
26	Q7A '-7 + 1'		*																													
27	Q7A '-4 – 2'																															
28	Q7A '-1 + -5'																															
29	Q7B																												*			
30	Q7C '2 – 8'					*		*																					*			
31	Q7C '-7 + 1'		*					*																								
32	Q7C '-4 – 2'				*	*																				*						
33	Q7C '-1 + -5'					*																										
34	Q7D '2 – 8'										*																					*
35	Q7D '-7 + 1'									*																						
36	Q7D '-4 – 2'										*																					
37	Q7D '-1 + -5'										*													*								

Code assignment for Ashley's transcript by turn.

			Sets of numbers					Sign of number read or introduced			Role of sign verbalized or introduced									Written Expression including						Identification of '-'		Chip model notations				Number line notations						
	Question number	Participant voiced confusion	'Positive integer'	'Negative integer'	'Whole number'	Integer	Other names for sets of numbers	Positive number	Zero	Negative number	Addition	'Plus' as addition sign	'Minus' as subtraction sign	Subtraction	'Take away' for subtraction	Negative sign	'Positive sign'	Negative for subtraction sign	'Minus' for negative	Positive number	Zero	Negative number/sign	Addition	Subtraction	Positive sign explicitly	Subtraction sign	Negative sign	Drawn	One color	Separate colors	Mixed colors	Participant drawn	Spoke of number line	Chips as colored number line	Dot-hops-dot	Left or right arrow	Numbered steps taken	Dots along number line
01	Q1		*	*	*		*	*		*																							*					
02	Q2			*						*	*	*										*	*															
03	Q3										*			*																								
04	Q4							*							*											*												
05	Q5	*							*		*																											
06	Q5A							*	*	*	*																						*					
07	Q5B																																					
08	Q5C																			*		*																
09	Q5A							*	*	*	*																											
10	Q6A '10 – 2'																																*					
11	Q6A '10 – 2'							*					*	*	*					*				*				*	*				*					
12	Q6A '-1 + 9'							*		*		*	*		*	*				*		*	*	*				*										
13	Q6A '6 – -2'																			*		*	*	*				*	*									
14	Q6A '6 + 2'							*			*	*																*	*									
15	Q6A '6 – -2'							*		*		*	*			*																						
16	Q6B							*			*			*	*																							
17	Q6C '10 – 2'							*					*	*	*														*									
18	Q6C '-1 + 9'	*						*		*		*			*															*								
19	Q6C '6 – -2'	*												*																*								
20	Q6C '6 + 2'		*					*			*																		*									
21	Q6D '10 – 2'							*						*																					*			
22	Q6D '-1 + 9'							*		*		*																							*			
23	Q6D '6 – -2'							*		*			*	*																					*			
24	Q6D '6 + 2'							*				*																							*			
25	Q6D '6 – -2'		*					*		*	*		*																									
26	Q6D																																					
27	Q7A																																					
28	Q7A '-1 + -5'			*						*	*	*										*	*						*									
29	Q7A '-4 – 2'	*	*	*				*		*			*	*						*		*		*					*									
30	Q7A '-7 + 1'		*	*				*		*	*	*								*		*	*					*										
31	Q7A '2 – 8'							*		*			*	*						*				*								*			*			
32	Q7A '-4 – 2'	*								*										*		*	*	*									*					
33	Q7B									*	*	*																										
34	Q7C			*																										*								
35	Q7C '-1 + -5'									*	*																		*									
36	Q7C '-4 – 2'	*						*		*			*																	*								
37	Q7C '-7 + 1'							*		*	*																			*								
38	Q7C '2 – 8'	*						*					*																*									
39	Q7D '-1 + -5'									*	*	*																							*			
40	Q7D '-4 – 2'							*		*				*																					*			
41	Q7D '-7 + 1'							*		*		*																							*			
42	Q7D '2 – 8'							*		*				*																					*			
43	Q7D '-4 – 2'		*	*							*			*																								

		Actions with numerals				Actions with the chip model				Action with number line		Relationship between positive integers and whole numbers		'Addition makes larger' remains true with integers		Types of numbers generated by integer subtraction						Properties of additive inverse or the pair '-7 and 7'		Operation and signifiers both need to be considered					Results of subtraction		Similarity of operations on the number line	
	Question number	Portions of numbers with different signs can cancel to give sum or difference	Adding to a negative number as subtraction	Subtracting a negative is adding a positive number	Subtraction as adding a negative number	Adding as joining	Opposite colored chips stacked	Subtraction as taking away	Opposite colors chips swapped	Go up number line	Go down number line	Are synonyms	Mean different things	, it remains true	No, not always	Positive number	Negative number	Integer	Any number	Larger number	Smaller number	Add to zero	Are opposites	Adding a negative number and a positive number can be treated as subtraction	Subtracting a negative number can be treated as addition of a positive number	Subtracting a positive number can be treated as adding a negative number	When you have a negative number you automatically subtract	Symbols provide inconsistent meanings	Larger number from a smaller number is not proper	Negative number cannot be done	Addition	Subtraction
01	Q1											*																				
02	Q2													*																		
03	Q3																		*	*											
04	Q4																																
05	Q5																				*											
06	Q5A								*											*											
07	Q5B																																
08	Q5C																																
09	Q5A																				*											
10	Q6A ‘10 – 2’																																
11	Q6A ‘10 – 2’						*																									
12	Q6A ‘-1 + 9’		*					*																									
13	Q6A ‘6 – -2’			*		*																									
14	Q6A ‘6 + 2’				*																									
15	Q6A ‘6 – -2’			*																				*							
16	Q6B		*	*																											
17	Q6C ‘10 – 2’						*																									
18	Q6C ‘-1 + 9’				*	*																								
19	Q6C ‘6 – -2’			*					*																							
20	Q6C ‘6 + 2’				*																									
21	Q6D ‘10 – 2’									*																					*
22	Q6D ‘-1 + 9’								*																					*	
23	Q6D ‘6 – -2’								*																					
24	Q6D ‘6 + 2’								*																					
25	Q6D ‘6 – -2’			*																												*
26	Q6D																																
27	Q7A																																
28	Q7A ‘-1 + -5’																																
29	Q7A ‘-4 – 2’																																
30	Q7A ‘-7 + 1’				*	*																								
31	Q7A ‘2 – 8’									*																		*			
32	Q7A ‘-4 – 2’																																
33	Q7B																																
34	Q7C																																
35	Q7C ‘-1 + -5’				*																									
36	Q7C ‘-4 – 2’																																
37	Q7C ‘-7 + 1’	*				*	*																								
38	Q7C ‘2 – 8’																																
39	Q7D ‘-1 + -5’									*																				*	
40	Q7D ‘-4 – 2’									*																					*
41	Q7D ‘-7 + 1’								*																					
42	Q7D ‘2 – 8’									*																					*
43	Q7D ‘-4 – 2’																											*				

Code assignment for Bailey's transcript by turn.

			Sets of numbers					Sign of number read or introduced			Role of sign verbalized or introduced									Written Expression including						Identification of '- '		Chip model notations				Number line notations						
	Question number	Participant voiced confusion	'Positive integer'	'Negative integer'	'Whole number'	Integer	Other names for sets of numbers	Positive number	Zero	Negative number	Addition	'Plus' as addition sign	'Minus' as subtraction sign	Subtraction	'Take away' for subtraction	Negative sign	'Positive sign'	Negative for subtraction sign	Minus' for negative	Positive number	Zero	Negative number/sign	Addition	Subtraction	Positive sign explicitly	Subtraction sign	Negative sign	Drawn	One color	Separate colors	Mixed colors	Participant drawn	Spoke of number line	Chips as colored number line	Dot-hops-dot	Left or right arrow	Numbered steps taken	Dots along number line
01	Q1		*		*		*																															
02	Q2	*								*										*	*	*	*															
03	Q3							*		*										*		*	*															
04	Q4														*											*	*											
05	Q5	*							*		*																											
06	Q5A							*	*	*	*																						*					
07	Q5B																																					
08	Q5B																																					
09	Q5C							*		*																												
10	Q6A '10 – 2'							*						*	*																	*				*		
11	Q6A '-1 + 9'							*		*	*	*																				*				*		
12	Q6A '6 + 2'							*				*																				*				*		
13	Q6A '6 – -2'	*						*																														
14	Q6B							*			*			*																								
15	Q6C																																					
16	Q6C '6 + 2'							*			*																				*							
17	Q6C '6 – -2'							*		*			*																		*							
18	Q6C																																					
19	Q6C '-1 + 9'	*																											*									
20	Q6C '10 – 2'							*							*														*									
21	Q6D '6 + 2'																																			*		
22	Q6D '10 – 2'							*					*							*				*												*		
23	Q6D '-1 + 9'							*		*		*								*		*	*													*		
24	Q6D '6 – -2'							*		*		*	*							*		*		*												*		
25	Q7A																																					
26	Q7A '2 – 8'							*		*			*	*	*					*		*		*				*		*								
27	Q7A '-4 – 2'							*		*			*	*						*		*		*														
28	Q7A '-7 + 1'																			*		*	*					*		*								
29	Q7A '-1 + -5'									*	*											*	*					*	*									
30	Q7A '-4 – 2'							*					*									*						*		*								
31	Q7B	*						*		*	*	*		*																								
32	Q7C '2 – 8'							*		*																					*							
33	Q7C '-1 + -5'									*		*																	*									
34	Q7C '-7 + 1'							*		*		*																			*							
35	Q7C '-4 – 2'							*		*	*		*																		*							
36	Q7D '2 – 8'							*		*			*							*		*		*												*		
37	Q7D '-7 + 1'					*	*	*		*	*	*								*		*	*													*		
38	Q7D '-4 – 2'	*						*		*			*							*		*		*												*		
39	Q7D '-1 + -5'							*		*		*										*	*															

		Actions with numerals				Actions with the chip model				Action with number line		Relationship between positive integers and whole numbers		‘Addition makes larger’ remains true with integers		Types of numbers generated by integer subtraction						Properties of additive inverse for the pair ‘-7 and 7’		Operation and signifiers both need to be considered					Results of subtraction		Similarity of operations on the number line	
	Question number	Portions of numbers with different signs can cancel to give sum or difference	Adding to a negative number as subtraction	Subtracting a negative is adding a positive number	Subtraction as adding a negative number	Adding as joining	Opposite colored chips stacked	Subtraction as taking away	Opposite colors chips swapped	Go up number line	Go down number line	Are synonyms	Mean different things	, it remains true	No, not always	Positive number	Negative number	Integer	Any number	Larger number	Smaller number	Add to zero	Are opposites	Adding a negative number and a positive number can be treated as subtraction	Subtracting a negative number can be treated as addition of a positive number	Subtracting a positive number can be treated as adding a negative number	When you have a negative number you automatically subtract	Symbols provide inconsistent meanings	Larger number from a smaller number is not proper	Negative number cannot be done	Addition	Subtraction
01	Q1												*																			
02	Q2														*																	
03	Q3															*	*															
04	Q4																															
05	Q5																					*										
06	Q5A																					*	*									
07	Q5B																															
08	Q5B																															
09	Q5C																															
10	Q6A ‘10 – 2’										*																					
11	Q6A ‘-1 + 9’									*																						
12	Q6A ‘6 + 2’																															
13	Q6A ‘6 – -2’			*																												
14	Q6B																															
15	Q6C																															
16	Q6C ‘6 + 2’					*																										
17	Q6C ‘6 – -2’					*																										
18	Q6C																															
19	Q6C ‘-1 + 9’																															
20	Q6C ‘10 – 2’							*																								
21	Q6D ‘6 + 2’									*																						
22	Q6D ‘10 – 2’										*																					
23	Q6D ‘-1 + 9’									*																						
24	Q6D ‘6 – -2’			*						*																						
25	Q7A																															
26	Q7A ‘2 – 8’					*	*																									
27	Q7A ‘-4 – 2’																															
28	Q7A ‘-7 + 1’					*	*																									
29	Q7A ‘-1 + -5’																															
30	Q7A ‘-4 – 2’					*			*																							
31	Q7B																															
32	Q7C ‘2 – 8’					*	*																									
33	Q7C ‘-1 + -5’																															
34	Q7C ‘-7 + 1’					*	*																									
35	Q7C ‘-4 – 2’					*	*																									
36	Q7D ‘2 – 8’										*																					
37	Q7D ‘-7 + 1’									*																						
38	Q7D ‘-4 – 2’										*																					
39	Q7D ‘-1 + -5’										*																					

Code assignment for Brooke’s transcript by turn.

			Sets of numbers					Sign of number read or introduced			Role of sign verbalized or introduced									Written Expression including						Identification of ‘- ’		Chip model notations				Number line notations						
	Question number	Participant voiced confusion	‘Positive integer’	‘Negative integer’	‘Whole number’	Integer	Other names for sets of numbers	Positive number	Zero	Negative number	Addition	'Plus' as addition sign	'Minus' as subtraction sign	Subtraction	'Take away' for subtraction	Negative sign	'Positive sign'	Negative for subtraction sign	Minus' for negative	Positive number	Zero	Negative number/sign	Addition	Subtraction	Positive sign explicitly	Subtraction sign	Negative sign	Drawn	One color	Separate colors	Mixed colors	Participant drawn	Spoke of number line	Chips as colored number line	Dot-hops-dot	Left or right arrow	Numbered steps taken	Dots along number line
01	Q1		*		*		*																															
02	Q2			*	*			*	*	*	*	*								*		*	*									*			*			
03	Q3				*	*	*	*		*																												
04	Q4																									*	*											
05	Q5	*						*		*																												
06	Q5A							*		*																												
07	Q5B																																*					
08	Q5C							*		*																												
09	Q6A																																					
10	Q6A																																					
11	Q6A ‘10 – 2’							*					*	*						*				*														
12	Q6A ‘6 + 2’							*				*																										
13	Q6A ‘6 – -2’	*						*		*			*	*																		*						
14	Q6A ‘-1 + 9’							*		*	*																					*			*			
15	Q6B			*		*		*		*		*	*																									
16	Q6C																																					
17	Q6C ‘10 - 2’							*					*	*	*														*									
18	Q6C ‘6 + 2’							*			*	*																	*									
19	Q6C ‘6 – -2’	*						*		*			*																*									
20	Q6C ‘-1 + 9’							*		*		*																		*								
21	Q6D ‘10 – 2’							*						*																					*			
22	Q6D ‘6 + 2’							*			*	*																										
23	Q6D ‘6 – -2’							*		*	*		*	*						*		*	*	*											*			
24	Q6D ‘-1 + 9’							*		*	*	*																							*			
25	Q7A ‘2 – 8’							*		*				*																		*			*			
26	Q7A ‘-7 + 1’							*		*																							*					
27	Q7A ‘-4 – 2’							*		*																							*					
28	Q7A ‘-1 + -5’							*		*		*	*							*		*	*	*														
29	Q7B									*		*																					*					
30	Q7C ‘2 – 8’	*						*		*			*			*														*			*					
31	Q7C ‘-7 + 1’							*		*		*																		*								
32	Q7C ‘-4 – 2’									*	*																		*									
33	Q7C ‘-1 + -5’									*	*																		*									
34	Q7D ‘2 – 8’							*		*				*																					*			
35	Q7D ‘-7 + 1’									*																									*			
36	Q7D ‘-4 – 2’							*		*																									*			
37	Q7D ‘-1 + -5’									*	*			*						*		*	*	*											*			
38	Q7D ‘-4 – 2’	*						*		*			*							*		*		*												*		
39	Q7D ‘-1 + -5’							*		*		*										*	*															

		Actions with numerals				Actions with the chip model				Action with number line		Relationship between positive integers and whole numbers		‘Addition makes larger’ remains true with integers		Types of numbers generated by integer subtraction						Properties of additive inverse or the pair ‘-7 and 7’		Operation and signifiers both need to be considered					Results of subtraction		Similarity of operations on the number line	
	Question number	Portions of numbers with different signs can cancel to give sum or difference	Adding to a negative number as subtraction	Subtracting a negative is adding a positive number	Subtraction as adding a negative number	Adding as joining	Opposite colored chips stacked	Subtraction as taking away	Opposite colors chips swapped	Go up number line	Go down number line	Are synonyms	Mean different things	, it remains true	No, not always	Positive number	Negative number	Integer	Any number	Larger number	Smaller number	Add to zero	Are opposites	Adding a negative number and a positive number can be treated as subtraction	Subtracting a negative number can be treated as addition of a positive number	Subtracting a positive number can be treated as adding a negative number	When you have a negative number you automatically subtract	Symbols provide inconsistent meanings	Larger number from a smaller number is not proper	Negative number cannot be done	Addition	Subtraction
01	Q1												*																			
02	Q2									*				*																		
03	Q3															*	*															
04	Q4																															
05	Q5																															
06	Q5A																						*									
07	Q5B																						*									
08	Q5C																															
09	Q6A																															
10	Q6A																															
11	Q6A ‘10 – 2’																															
12	Q6A ‘6 + 2’																															
13	Q6A ‘6 – -2’									*	*														*							*
14	Q6A ‘-1 + 9’									*																						
15	Q6B																															
16	Q6C																															
17	Q6C ‘10 - 2’							*																								
18	Q6C ‘6 + 2’					*																										
19	Q6C ‘6 – -2’																															
20	Q6C ‘-1 + 9’	*				*	*																									
21	Q6D ‘10 – 2’										*																					*
22	Q6D ‘6 + 2’									*																					*	
23	Q6D ‘6 – -2’									*																						
24	Q6D ‘-1 + 9’									*																					*	
25	Q7A ‘2 – 8’										*																					
26	Q7A ‘-7 + 1’									*																						
27	Q7A ‘-4 – 2’									*																						
28	Q7A ‘-1 + -5’																															
29	Q7B									*	*																					
30	Q7C ‘2 – 8’	*			*	*	*																									
31	Q7C ‘-7 + 1’	*				*	*																									
32	Q7C ‘-4 – 2’				*	*																										
33	Q7C ‘-1 + -5’					*																										
34	Q7D ‘2 – 8’										*																					*
35	Q7D ‘-7 + 1’									*																						
36	Q7D ‘-4 – 2’										*																					
37	Q7D ‘-1 + -5’										*																					
38	Q7D ‘-4 – 2’										*																					
39	Q7D ‘-1 + -5’										*																					

Code assignment for Christina's transcript by turn.

			Sets of numbers					Sign of number read or introduced			Role of sign verbalized or introduced									Written Expression including						Identification of '-'		Chip model notations				Number line notations						
	Question number	Participant voiced confusion	'Positive integer'	'Negative integer'	'Whole number'	Integer	Other names for sets of numbers	Positive number	Zero	Negative number	Addition	'Plus' as addition sign	'Minus' as subtraction sign	Subtraction	'Take away' for subtraction	Negative sign	'Positive sign'	Negative for subtraction sign	'Minus' for negative	Positive number	Zero	Negative number/sign	Addition	Subtraction	Positive sign explicitly	Subtraction sign	Negative sign	Drawn	One color	Separate colors	Mixed colors	Participant drawn	Spoke of number line	Chips as colored number line	Dot-hops-dot	Left or right arrow	Numbered steps taken	Dots along number line
01	Q1				*	*																																
02	Q2	*						*		*	*	*								*			*															
03	Q3							*		*			*							*		*		*														
04	Q4							*							*	*										*	*											
05	Q5	*																																				
06	Q5A																																*					
07	Q5B																																					
08	Q5C							*		*																												
09	Q6A '-1 + 9'							*		*	*	*		*						*		*	*															
10	Q6A '10 – 2'							*					*							*				*														
11	Q6A '6 – -2'	*						*					*							*		*		*														
12	Q6A '6 – -2'							*												*			*		*													
13	Q6A '6 + 2'							*				*								*			*															
14	Q6B							*		*		*	*																									
15	Q6C '-1 + 9'							*		*	*																			*								
16	Q6C '10 – 2'							*						*	*																*							
17	Q6C '6 – -2'	*						*		*					*															*								
18	Q6C '6 + 2'							*				*																			*							
19	Q6D '-1 + 9'							*		*		*																							*			
20	Q6D '10 – 2'							*							*																				*			
21	Q6D '6 – -2'							*		*				*																					*			
22	Q6D '6 + 2'							*			*																											
23	Q7A '-1 + -5'									*		*										*	*															
24	Q7A '2 – 8'							*		*			*							*		*		*														
25	Q7A '-4 – 2'							*		*			*	*						*		*		*														
26	Q7A '-7 + 1'							*		*	*	*								*		*	*															
27	Q7B							*		*	*	*	*		*																							
28	Q7C '-1 + -5'									*		*																	*									
29	Q7C '2 – 8'							*		*				*																*								
30	Q7C '-4 – 2'							*		*	*			*																*								
31	Q7C '-7 + 1'							*		*		*																		*								
32	Q7D '-1 + -5'									*	*	*																							*			
33	Q7D '2 – 8'							*		*					*																				*			
34	Q7D '-4 – 2'							*		*			*	*																					*			
35	Q7D '-7 + 1'							*		*	*	*																							*			

		Actions with numerals				Actions with the chip model				Action with number line		Relationship between positive integers and whole numbers		‘Addition makes larger’ remains true with integers		Types of numbers generated by integer subtraction						Properties of additive inverse or the pair ‘-7 and 7’		Operation and signifiers both need to be considered					Results of subtraction		Similarity of operations on the number line	
	Question number	Portions of numbers with different signs can cancel to give sum or difference	Adding to a negative number as subtraction	Subtracting a negative is adding a positive number	Subtraction as adding a negative number	Adding as joining	Opposite colored chips stacked	Subtraction as taking away	Opposite colors chips swapped	Go up number line	Go down number line	Are synonyms	Mean different things	, it remains true	No, not always	Positive number	Negative number	Integer	Any number	Larger number	Smaller number	Add to zero	Are opposites	Adding a negative number and a positive number can be treated as subtraction	Subtracting a negative number can be treated as addition of a positive number	Subtracting a positive number can be treated as adding a negative number	When you have a negative number you automatically subtract	Symbols provide inconsistent meanings	Larger number from a smaller number is not proper	Negative number cannot be done	Addition	Subtraction
01	Q1											*																				
02	Q2													*																		
03	Q3																*	*														
04	Q4																															
05	Q5																															
06	Q5A																						*									
07	Q5B																						*									
08	Q5C																															
09	Q6A ‘-1 + 9’		*																					*								
10	Q6A ‘10 – 2’																															
11	Q6A ‘6 – -2’																															
12	Q6A ‘6 – -2’			*																												
13	Q6A ‘6 + 2’																															
14	Q6B																															
15	Q6C ‘-1 + 9’	*				*	*																									
16	Q6C ‘10 – 2’							*																								
17	Q6C ‘6 – -2’					*																										
18	Q6C ‘6 + 2’					*																										
19	Q6D ‘-1 + 9’									*																						
20	Q6D ‘10 – 2’										*																					
21	Q6D ‘6 – -2’									*																						
22	Q6D ‘6 + 2’									*																						
23	Q7A ‘-1 + -5’																															
24	Q7A ‘2 – 8’																															
25	Q7A ‘-4 – 2’																															
26	Q7A ‘-7 + 1’																															
27	Q7B																															
28	Q7C ‘-1 + -5’					*																										
29	Q7C ‘2 – 8’					*	*																									
30	Q7C ‘-4 – 2’				*	*			*																							
31	Q7C ‘-7 + 1’	*				*	*																									
32	Q7D ‘-1 + -5’										*																					*
33	Q7D ‘2 – 8’										*																					
34	Q7D ‘-4 – 2’										*																					
35	Q7D ‘-7 + 1’									*																					*	

Code assignment for Courtney’s transcript by turn.

			Sets of numbers					Sign of number read or introduced			Role of sign verbalized or introduced									Written Expression including						Identification of ‘- ’		Chip model notations				Number line notations						
	Question number	Participant voiced confusion	‘Positive integer’	‘Negative integer’	‘Whole number’	Integer	Other names for sets of numbers	Positive number	Zero	Negative number	Addition	‘Plus’ as addition sign	‘Minus’ as subtraction sign	Subtraction	‘Take away’ for subtraction	Negative sign	‘Positive sign’	Negative for subtraction sign	‘Minus’ for negative	Positive number	Zero	Negative number/sign	Addition	Subtraction	Positive sign explicitly	Subtraction sign	Negative sign	Drawn	One color	Separate colors	Mixed colors	Participant drawn	Spoke of number line	Chips as colored number line	Dot-hops-dot	Left or right arrow	Numbered steps taken	Dots along number line
01	Q1		*		*		*																															
02	Q2								*		*																											
03	Q3						*	*	*		*																											
04	Q4						*		*																*	*												
05	Q5	*						*		*																												
06	Q5A	*																																				
07	Q5B																																					
08	Q5C						*		*																													
09	Q6A ‘6 – -2’							*		*			*																									
10	Q6A ‘10 – 2’							*					*																									
11	Q6A ‘-1 + 9’							*		*		*																										
12	Q6A ‘6 + 2’							*				*																										
13	Q6B							*		*																												
14	Q6C																																					
15	Q6C ‘6 - -2’							*																						*								
16	Q6C ‘10 – 2’							*						*															*									
17	Q6C ‘-1 + 9’																													*								
18	Q6C ‘6 + 2’							*																					*									
19	Q6D ‘6 – -2’							*		*										*			*												*			
20	Q6D ‘10 – 2’																			*				*											*			
21	Q6D ‘-1 + 9’							*		*		*								*		*	*												*			
22	Q6D ‘6 + 2’																			*			*												*			
23	Q6D																																					
24	Q7A																																					
25	Q7A ‘2 – 8’							*		*			*																									
26	Q7A ‘-7 + 1’							*		*		*																										
27	Q7A ‘-1 + -5’									*		*																										
28	Q7A ‘-4 – 2’							*		*			*																									
29	Q7B																																					
30	Q7D ‘2 – 8’							*		*										*				*											*			
31	Q7D ‘-1 + -5’									*												*	*	*											*			
32	Q7D ‘-7 + 1’							*												*		*	*												*			
33	Q7D ‘-4 – 2’																			*		*		*											*			
34	Q7C ‘2 – 8’																														*							
35	Q7C ‘-1 + -5’							*																							*							
36	Q7C ‘-7 + 1’																														*							
37	Q7C ‘-4 – 2’							*		*																					*							

		Actions with numerals				Actions with the chip model				Action with number line		Relationship between positive integers and whole numbers		‘Addition makes larger’ remains true with integers		Types of numbers generated by integer subtraction						Properties of additive inverse or the pair ‘-7 and 7’		Operation and signifiers both need to be considered					Results of subtraction		Similarity of operations on the number line	
	Question number	Portions of numbers with different signs can cancel to give sum or difference	Adding to a negative number as subtraction	Subtracting a negative is adding a positive number	Subtraction as adding a negative number	Adding as joining	Opposite colored chips stacked	Subtraction as taking away	Opposite colors chips swapped	Go up number line	Go down number line	Are synonyms	Mean different things	, it remains true	No, not always	Positive number	Negative number	Integer	Any number	Larger number	Smaller number	Add to zero	Are opposites	Adding a negative number and a positive number can be treated as subtraction	Subtracting a negative number can be treated as addition of a positive number	Subtracting a positive number can be treated as adding a negative number	When you have a negative number you automatically subtract	Symbols provide inconsistent meanings	Larger number from a smaller number is not proper	Negative number cannot be done	Addition	Subtraction
01	Q1												*																			
02	Q2														*																	
03	Q3															*	*															
04	Q4																															
05	Q5																						*									
06	Q5A																															
07	Q5B																															
08	Q5C																															
09	Q6A ‘6 – -2’			*																												
10	Q6A ‘10 – 2’																															
11	Q6A ‘-1 + 9’																															
12	Q6A ‘6 + 2’																															
13	Q6B																															
14	Q6C																															
15	Q6C ‘6 - -2’					*			*																							
16	Q6C ‘10 – 2’							*																								
17	Q6C ‘-1 + 9’					*	*																									
18	Q6C ‘6 + 2’																															
19	Q6D ‘6 – -2’									*																						
20	Q6D ‘10 – 2’										*																					
21	Q6D ‘-1 + 9’										*																					
22	Q6D ‘6 + 2’									*																						
23	Q6D																															
24	Q7A																															
25	Q7A ‘2 – 8’																															
26	Q7A ‘-7 + 1’																															
27	Q7A ‘-1 + -5’																															
28	Q7A ‘-4 – 2’																															
29	Q7B																															
30	Q7D ‘2 – 8’									*																						
31	Q7D ‘-1 + -5’										*																					
32	Q7D ‘-7 + 1’									*																						
33	Q7D ‘-4 – 2’										*																					
34	Q7C ‘2 – 8’								*																							
35	Q7C ‘-1 + -5’					*																										
36	Q7C ‘-7 + 1’					*	*	*	*																							
37	Q7C ‘-4 – 2’					*																										

Code assignment for Jacqueline's transcript by turn.

			Sets of numbers					Sign of number read or introduced			Role of sign verbalized or introduced									Written Expression including						Identification of '- '		Chip model notations				Number line notations						
	Question number	Participant voiced confusion	'Positive integer'	'Negative integer'	'Whole number'	Integer	Other names for sets of numbers	Positive number	Zero	Negative number	Addition	'Plus' as addition sign	'Minus' as subtraction sign	Subtraction	'Take away' for subtraction	Negative sign	'Positive sign'	Negative for subtraction sign	'Minus' for negative	Positive number	Zero	Negative number/sign	Addition	Subtraction	Positive sign explicitly	Subtraction sign	Negative sign	Drawn	One color	Separate colors	Mixed colors	Participant drawn	Spoke of number line	Chips as colored number line	Dot-hops-dot	Left or right arrow	Numbered steps taken	Dots along number line
01	Q1		*		*		*																															
02	Q2									*		*																										
03	Q3			*				*		*																												
04	Q4																									*												
05	Q5							*	*	*		*																										
06	Q5A					*		*		*																							*					
07	Q5B								*		*																											
08	Q5C							*	*	*		*																										
09	Q6A '10 – 2'																			*				*														
10	Q6A '6 + 2'																			*			*															
11	Q6A '6 – -2'																			*			*		*													
12	Q6A '10 – 2'				*			*					*		*																							
13	Q6A '6 + 2'				*			*			*	*																										
14	Q6A '6 – -2'							*				*	*	*		*																						
15	Q6A '-1 + 9'							*		*		*								*																		
16	Q6B							*		*	*		*	*																								
17	Q6C																																					
18	Q6C '10 – 2'							*							*																*							
19	Q6C '6 + 2'							*			*																				*							
20	Q6C '-1 + 9'							*		*		*	*			*															*							
21	Q6C '6 – -2'	*						*		*	*		*																		*							
22	Q6D '10 – 2'							*							*																				*			
23	Q6D '6 + 2'							*			*	*																							*			
24	Q6D '-1 + 9'							*		*	*	*																							*			
25	Q6D '6 – -2'	*						*		*	*		*																						*			
26	Q7A '2 – 8'				*			*		*			*		*	*						*																
27	Q7A '-7 + 1'							*		*	*			*								*																
28	Q7A '-4 – 2'							*		*		*	*									*	*															
29	Q7A '-1 + -5'									*		*										*																
30	Q7B							*		*		*	*																									
31	Q7C '2 – 8'							*		*			*		*														*									
32	Q7C '-7 + 1'							*		*		*	*																*									
33	Q7C '-4 – 2'	*						*		*	*		*																		*							
34	Q7C '-1 + -5'									*		*																			*							
35	Q7D '2 – 8'							*		*					*																				*			
36	Q7D '-7 + 1'				*			*		*	*	*																							*			
37	Q7D '-4 – 2'							*		*			*		*																				*			
38	Q7D '-1 + -5'									*		*																							*			

		Actions with numerals				Actions with the chip model				Action with number line		Relationship between positive integers and whole numbers		‘Addition makes larger’ remains true with integers		Types of numbers generated by integer subtraction						Properties of additive inverse or the pair ‘-7 and 7’		Operation and signifiers both need to be considered					Results of subtraction		Similarity of operations on the number line	
	Question number	Portions of numbers with different signs can cancel to give sum or difference	Adding to a negative number as subtraction	Subtracting a negative is adding a positive number	Subtraction as adding a negative number	Adding as joining	Opposite colored chips stacked	Subtraction as taking away	Opposite colors chips swapped	Go up number line	Go down number line	Are synonyms	Mean different things	, it remains true	No, not always	Positive number	Negative number	Integer	Any number	Larger number	Smaller number	Add to zero	Are opposites	Adding a negative number and a positive number can be treated as subtraction	Subtracting a negative number can be treated as addition of a positive number	Subtracting a positive number can be treated as adding a negative number	When you have a negative number you automatically subtract	Symbols provide inconsistent meanings	Larger number from a smaller number is not proper	Negative number cannot be done	Addition	Subtraction
01	Q1												*																			
02	Q2														*																	
03	Q3															*	*															
04	Q4																															
05	Q5																					*	*									
06	Q5A																						*									
07	Q5B																					*										
08	Q5C																															
09	Q6A ‘10 – 2’																															
10	Q6A ‘6 + 2’																															
11	Q6A ‘6 – -2’			*																												
12	Q6A ‘10 – 2’																															
13	Q6A ‘6 + 2’																															
14	Q6A ‘6 – -2’			*																					*							
15	Q6A ‘-1 + 9’																															
16	Q6B																							*	*		*					
17	Q6C																															
18	Q6C ‘10 – 2’							*																								
19	Q6C ‘6 + 2’					*																										
20	Q6C ‘-1 + 9’		*					*																								
21	Q6C ‘6 – -2’			*		*																										
22	Q6D ‘10 – 2’										*																					
23	Q6D ‘6 + 2’									*																						
24	Q6D ‘-1 + 9’									*																						
25	Q6D ‘6 – -2’									*																						
26	Q7A ‘2 – 8’																															
27	Q7A ‘-7 + 1’																															
28	Q7A ‘-4 – 2’				*																					*						
29	Q7A ‘-1 + -5’																															
30	Q7B				*																											
31	Q7C ‘2 – 8’																												*			
32	Q7C ‘-7 + 1’		*																													
33	Q7C ‘-4 – 2’				*	*																										
34	Q7C ‘-1 + -5’					*																										
35	Q7D ‘2 – 8’										*																					
36	Q7D ‘-7 + 1’									*																						
37	Q7D ‘-4 – 2’										*																					
38	Q7D ‘-1 + -5’										*																					

Code assignment for Jamie's transcript by turn.

			Sets of numbers					Sign of number read or introduced			Role of sign verbalized or introduced									Written Expression including						Identification of '- '		Chip model notations				Number line notations						
	Question number	Participant voiced confusion	'Positive integer'	'Negative integer'	'Whole number'	Integer	Other names for sets of numbers	Positive number	Zero	Negative number	Addition	'Plus' as addition sign	'Minus' as subtraction sign	Subtraction	'Take away' for subtraction	Negative sign	'Positive sign'	Negative for subtraction sign	'Minus' for negative	Positive number	Zero	Negative number/sign	Addition	Subtraction	Positive sign explicitly	Subtraction sign	Negative sign	Drawn	One color	Separate colors	Mixed colors	Participant drawn	Spoke of number line	Chips as colored number line	Dot-hops-dot	Left or right arrow	Numbered steps taken	Dots along number line
01	Q1		*		*			*																														
02	Q2			*				*		*	*	*																										
03	Q3							*		*			*	*																								
04	Q4						*	*																		*	*											
05	Q5							*	*	*	*				*			*																				
06	Q5C							*		*																												
07	Q5B																																*					
08	Q5A							*		*																							*					
09	Q6A '10 – 2'							*					*							*				*								*						
10	Q6A '6 + 2'							*			*	*								*			*									*			*			
11	Q6A '6 – -2'	*						*		*	*	*	*		*					*		*	*	*								*						
12	Q6A '-1 + 9'	*						*		*	*	*								*		*	*									*			*			
13	Q6B	*						*		*	*	*		*																		*			*			
14	Q6C	*																																				
15	Q6C																																					
16	Q6C '10 – 2'		*					*					*	*	*														*									
17	Q6C '6 + 2'							*			*	*																	*									
18	Q6C '6 – -2'	*						*		*	*		*		*															*								
19	Q6C '-1 + 9'							*		*	*																			*			*	*				
20	Q6D '10 – 2'							*					*	*																					*	*		
21	Q6D '6 + 2'							*			*	*								*			*												*	*		
22	Q6D '6 – -2'	*						*		*	*		*		*					*		*		*											*			
23	Q6D '-1 + 9'							*		*	*	*								*		*	*												*	*		
24	Q7																																					
25	Q7A '-1 + -5'									*	*	*										*	*															
26	Q7A '-4 – 2'	*						*		*			*		*					*		*	*	*														
27	Q7A '-7 + 1'							*		*	*	*								*		*	*															
28	Q7A '2 – 8'							*		*			*							*		*		*														
29	Q7B		*	*				*	*	*	*		*		*																							
30	Q7C '-1 + -5'									*	*	*																		*			*	*				
31	Q7C '-4 – 2'	*						*		*			*																	*				*				
32	Q7C '-7 + 1'		*					*		*	*	*																		*				*				
33	Q7C '2 – 8'							*		*			*																	*				*				
34	Q7D '-1 + -5'									*	*		*									*													*			
35	Q7D '-4 – 2'	*						*		*	*	*	*		*					*		*	*	*											*			
36	Q7D '-7 + 1'							*		*	*	*																							*			
37	Q7D '2 – 8'							*					*		*																				*			

		Actions with numerals				Actions with the chip model				Action with number line		Relationship between positive integers and whole numbers		‘Addition makes larger’ remains true with integers		Types of numbers generated by integer subtraction						Properties of additive inverse or the pair ‘-7 and 7’		Operation and signifiers both need to be considered					Results of subtraction		Similarity of operations on the number line	
	Question number	Portions of numbers with different signs can cancel to give sum or difference	Adding to a negative number as subtraction	Subtracting a negative is adding a positive number	Subtraction as adding a negative number	Adding as joining	Opposite colored chips stacked	Subtraction as taking away	Opposite colors chips swapped	Go up number line	Go down number line	Are synonyms	Mean different things	, it remains true	No, not always	Positive number	Negative number	Integer	Any number	Larger number	Smaller number	Add to zero	Are opposites	Adding a negative number and a positive number can be treated as subtraction	Subtracting a negative number can be treated as addition of a positive number	Subtracting a positive number can be treated as adding a negative number	When you have a negative number you automatically subtract	Symbols provide inconsistent meanings	Larger number from a smaller number is not proper	Negative number cannot be done	Addition	Subtraction
01	Q1												*																			
02	Q2														*																	
03	Q3															*	*															
04	Q4																															
05	Q5																						*									
06	Q5C																															
07	Q5B																						*									
08	Q5A																						*									
09	Q6A ‘10 – 2’										*																					
10	Q6A ‘6 + 2’									*																					*	*
11	Q6A ‘6 – -2’			*																					*							
12	Q6A ‘-1 + 9’									*																					*	
13	Q6B									*	*																					
14	Q6C																															
15	Q6C																															
16	Q6C ‘10 – 2’							*																								
17	Q6C ‘6 + 2’					*																										
18	Q6C ‘6 – -2’																															
19	Q6C ‘-1 + 9’									*																						
20	Q6D ‘10 – 2’										*																					*
21	Q6D ‘6 + 2’									*																					*	
22	Q6D ‘6 – -2’									*															*					*		
23	Q6D ‘-1 + 9’									*																					*	
24	Q7																															
25	Q7A ‘-1 + -5’																															
26	Q7A ‘-4 – 2’				*																					*						
27	Q7A ‘-7 + 1’																															
28	Q7A ‘2 – 8’																															
29	Q7B																															
30	Q7C ‘-1 + -5’										*																					
31	Q7C ‘-4 – 2’																															
32	Q7C ‘-7 + 1’									*																						
33	Q7C ‘2 – 8’										*																					*
34	Q7D ‘-1 + -5’										*																					
35	Q7D ‘-4 – 2’										*																					
36	Q7D ‘-7 + 1’									*																						
37	Q7D ‘2 – 8’										*																					*

Code assignment for Jordan’s transcript by turn.

			Sets of numbers					Sign of number read or introduced			Role of sign verbalized or introduced									Written Expression including						Identification of ‘- ’		Chip model notations				Number line notations						
	Question number	Participant voiced confusion	‘Positive integer’	‘Negative integer’	‘Whole number’	Integer	Other names for sets of numbers	Positive number	Zero	Negative number	Addition	‘Plus’ as addition sign	‘Minus’ as subtraction sign	Subtraction	‘Take away’ for subtraction	Negative sign	‘Positive sign’	Negative for subtraction sign	‘Minus’ for negative	Positive number	Zero	Negative number/sign	Addition	Subtraction	Positive sign explicitly	Subtraction sign	Negative sign	Drawn	One color	Separate colors	Mixed colors	Participant drawn	Spoke of number line	Chips as colored number line	Dot-hops-dot	Left or right arrow	Numbered steps taken	Dots along number line
01	Q1	*	*		*	*	*	*	*	*																												
02	Q2	*	*	*	*	*		*		*	*																											
03	Q3							*		*			*	*																								
04	Q4									*																*	*											
05	Q5	*						*		*									*																			
06	Q5A							*	*	*	*			*																								
07	Q5B										*			*																								
08	Q5C							*		*																												
09	Q6A ‘6 + 2’							*				*								*			*															
10	Q6A ‘-1 + 9’							*		*	*	*			*					*		*	*															
11	Q6A ‘10 – 2’							*					*		*					*				*														
12	Q6A ‘6 – -2’	*						*		*		*	*		*					*		*		*														
13	Q6B						*	*		*	*			*																								
14	Q6C ‘6 + 2’							*			*																				*							
15	Q6C ‘-1 + 9’				*			*		*	*				*															*								
16	Q6C ‘10 – 2’							*							*														*									
17	Q6C ‘6 – -2’	*						*		*					*															*								
18	Q6D ‘6 + 2’							*			*	*																										*
19	Q6D ‘-1 + 9’							*		*	*																											*
20	Q6D ‘10 – 2’							*							*																							*
21	Q6D ‘6 – -2’	*						*		*					*							*																*
22	Q7A ‘2 – 8’	*						*		*					*					*		*		*				*										
23	Q7A ‘-7 + 1’				*			*		*		*								*		*	*															
24	Q7A ‘-4 – 2’							*		*			*	*						*		*		*														
25	Q7A ‘-1 + -5’							*		*		*										*	*															
26	Q6A ‘6 – -2’																			*		*																
27	Q7B							*		*	*			*																								
28	Q7C ‘2 – 8’							*		*					*															*								
29	Q7C ‘-7 + 1’							*		*	*	*			*															*								
30	Q7C ‘-4 – 2’							*		*	*			*															*									
31	Q7C ‘-1 + -5’									*	*																		*									
32	Q6D ‘6 – -2’							*		*	*																								*			
33	Q7D ‘2 – 8’							*		*				*																								*
34	Q7D ‘-7 + 1’							*		*	*																									*		
35	Q7D ‘-4 – 2’							*		*				*																								*
36	Q7D ‘-1 + -5’									*	*																											*

		Actions with numerals				Actions with the chip model				Action with number line		Relationship between positive integers and whole numbers		‘Addition makes larger’ remains true with integers		Types of numbers generated by integer subtraction						Properties of additive inverse or the pair ‘-7 and 7’		Operation and signifiers both need to be considered					Results of subtraction		Similarity of operations on the number line	
	Question number	Portions of numbers with different signs can cancel to give sum or difference	Adding to a negative number as subtraction	Subtracting a negative is adding a positive number	Subtraction as adding a negative number	Adding as joining	Opposite colored chips stacked	Subtraction as taking away	Opposite colors chips swapped	Go up number line	Go down number line	Are synonyms	Mean different things	, it remains true	No, not always	Positive number	Negative number	Integer	Any number	Larger number	Smaller number	Add to zero	Are opposites	Adding a negative number and a positive number can be treated as subtraction	Subtracting a negative number can be treated as addition of a positive number	Subtracting a positive number can be treated as adding a negative number	When you have a negative number you automatically subtract	Symbols provide inconsistent meanings	Larger number from a smaller number is not proper	Negative number cannot be done	Addition	Subtraction
01	Q1											*																				
02	Q2														*																	
03	Q3															*	*															
04	Q4																															
05	Q5																															
06	Q5A																					*										
07	Q5B																															
08	Q5C																															
09	Q6A ‘6 + 2’																															
10	Q6A ‘-1 + 9’	*	*																													
11	Q6A ‘10 – 2’																															
12	Q6A ‘6 – -2’																													*		
13	Q6B																															
14	Q6C ‘6 + 2’					*																										
15	Q6C ‘-1 + 9’		*			*		*	*																							
16	Q6C ‘10 – 2’							*																								
17	Q6C ‘6 – -2’																															
18	Q6D ‘6 + 2’									*																						
19	Q6D ‘-1 + 9’									*																						
20	Q6D ‘10 – 2’										*																					
21	Q6D ‘6 – -2’										*																					
22	Q7A ‘2 – 8’					*	*																									
23	Q7A ‘-7 + 1’	*																														
24	Q7A ‘-4 – 2’																															
25	Q7A ‘-1 + -5’																															
26	Q6A ‘6 – -2’			*																					*							
27	Q7B																															
28	Q7C ‘2 – 8’					*	*																									
29	Q7C ‘-7 + 1’		*			*	*																									
30	Q7C ‘-4 – 2’				*	*																				*						
31	Q7C ‘-1 + -5’					*																										
32	Q6D ‘6 – -2’									*																						
33	Q7D ‘2 – 8’										*																					
34	Q7D ‘-7 + 1’									*																						
35	Q7D ‘-4 – 2’										*																					*
36	Q7D ‘-1 + -5’										*																				*	

Code assignment for Kayla’s transcript by turn.

			Sets of numbers					Sign of number read or introduced			Role of sign verbalized or introduced									Written Expression including						Identification of ‘- ’		Chip model notations				Number line notations						
	Question number	Participant voiced confusion	‘Positive integer’	‘Negative integer’	‘Whole number’	Integer	Other names for sets of numbers	Positive number	Zero	Negative number	Addition	‘Plus’ as addition sign	‘Minus’ as subtraction sign	Subtraction	‘Take away’ for subtraction	Negative sign	‘Positive sign’	Negative for subtraction sign	‘Minus’ for negative	Positive number	Zero	Negative number/sign	Addition	Subtraction	Positive sign explicitly	Subtraction sign	Negative sign	Drawn	One color	Separate colors	Mixed colors	Participant drawn	Spoke of number line	Chips as colored number line	Dot-hops-dot	Left or right arrow	Numbered steps taken	Dots along number line
01	Q1		*		*					*																												
02	Q2									*	*	*																										
03	Q3			*		*		*		*				*																								
04	Q4																									*	*											
05	Q5	*	*	*																																		
06	Q5A							*		*																												
07	Q5B					*		*		*																												
08	Q5C							*		*																												
09	Q6A ‘10 – 2’							*					*		*																							
10	Q6A ‘6 + 2’							*			*																											
11	Q6A ‘6 – -2’							*		*	*	*	*	*																								
12	Q6A ‘-1 + 9’							*		*		*	*																									
13	Q6B													*																								
14	Q6B							*		*	*			*																								
15	Q6C ‘10 – 2’							*						*															*									
16	Q6C ‘6 + 2’							*			*																		*									
17	Q6C ‘6 – -2’	*						*		*	*			*																	*							
18	Q6C ‘-1 + 9’							*	*	*	*	*																		*								
19	Q6D ‘10 – 2’							*						*																					*			
20	Q6D ‘6 + 2’							*				*																							*			
21	Q6D ‘6 – -2’	*						*		*			*	*																					*			
22	Q6D ‘-1 + 9’							*		*		*																							*			
23	Q7A ‘2 – 8’							*	*	*			*	*						*	*	*		*														
24	Q7A ‘-7 + 1’							*		*	*	*										*																
25	Q7A ‘-4 – 2’							*		*		*	*									*																
26	Q7A ‘-1 + -5’									*		*										*																
27	Q7B							*		*	*			*																								
28	Q7C ‘2 – 8’							*		*	*			*	*															*								
29	Q7C ‘-7 + 1’							*		*	*	*																		*								
30	Q7C ‘-4 – 2’							*		*	*		*	*															*									
31	Q7C ‘-1 + -5’									*	*	*																	*									
32	Q7D ‘2 – 8’							*		*			*																						*			
33	Q7D ‘-7 + 1’							*		*	*	*																							*			
34	Q7D ‘-4 – 2’							*		*			*	*																					*			
35	Q7D ‘-1 + -5’									*	*	*																							*			

		Actions with numerals				Actions with the chip model				Action with number line		Relationship between positive integers and whole numbers		'Addition makes larger' remains true with integers		Types of numbers generated by integer subtraction						Properties of additive inverse or the pair '-7 and 7'		Operation and signifiers both need to be considered					Results of subtraction		Similarity of operations on the number line	
	Question number	Portions of numbers with different signs can cancel to give sum or difference	Adding to a negative number as subtraction	Subtracting a negative is adding a positive number	Subtraction as adding a negative number	Adding as joining	Opposite colored chips stacked	Subtraction as taking away	Opposite colors chips swapped	Go up number line	Go down number line	Are synonyms	Mean different things	, it remains true	No, not always	Positive number	Negative number	Integer	Any number	Larger number	Smaller number	Add to zero	Are opposites	Adding a negative number and a positive number can be treated as subtraction	Subtracting a negative number can be treated as addition of a positive number	Subtracting a positive number can be treated as adding a negative number	When you have a negative number you automatically subtract	Symbols provide inconsistent meanings	Larger number from a smaller number is not proper	Negative number cannot be done	Addition	Subtraction
01	Q1												*																			
02	Q2														*																	
03	Q3															*	*	*			*											
04	Q4																															
05	Q5																						*									
06	Q5A																						*									
07	Q5B																						*									
08	Q5C																															
09	Q6A '10 – 2'																															
10	Q6A '6 + 2'																															
11	Q6A '6 – -2'			*																												
12	Q6A '-1 + 9'		*																													
13	Q6B																								*							
14	Q6B																							*								
15	Q6C '10 – 2'							*																								
16	Q6C '6 + 2'					*																										
17	Q6C '6 – -2'			*		*																										
18	Q6C '-1 + 9'	*				*	*																									
19	Q6D '10 – 2'										*																					
20	Q6D '6 + 2'									*																						
21	Q6D '6 – -2'									*																						
22	Q6D '-1 + 9'									*																						
23	Q7A '2 – 8'	*																														
24	Q7A '-7 + 1'																															
25	Q7A '-4 – 2'				*																											
26	Q7A '-1 + -5'																															
27	Q7B																															
28	Q7C '2 – 8'					*	*																									
29	Q7C '-7 + 1'	*				*	*																									
30	Q7C '-4 – 2'				*	*																				*						
31	Q7C '-1 + -5'					*																										
32	Q7D '2 – 8'										*																					
33	Q7D '-7 + 1'																														*	
34	Q7D '-4 – 2'										*																					
35	Q7D '-1 + -5'										*																				*	

Code assignment for Monique’s transcript by turn.

			Sets of numbers					Sign of number read or introduced			Role of sign verbalized or introduced									Written Expression including						Identification of ‘- ’		Chip model notations				Number line notations						
	Question number	Participant voiced confusion	‘Positive integer’	‘Negative integer’	‘Whole number’	Integer	Other names for sets of numbers	Positive number	Zero	Negative number	Addition	‘Plus’ as addition sign	‘Minus’ as subtraction sign	Subtraction	‘Take away’ for subtraction	Negative sign	‘Positive sign’	Negative for subtraction sign	‘Minus’ for negative	Positive number	Zero	Negative number/sign	Addition	Subtraction	Positive sign explicitly	Subtraction sign	Negative sign	Drawn	One color	Separate colors	Mixed colors	Participant drawn	Spoke of number line	Chips as colored number line	Dot-hops-dot	Left or right arrow	Numbered steps taken	Dots along number line
01	Q1		*		*		*																															
02	Q2				*	*		*		*	*																											
03	Q3						*	*		*																												
04	Q4			*																						*	*											
05	Q5	*						*	*	*	*																											
06	Q5C							*		*																												
07	Q5B							*		*																												
08	Q5A	*						*	*	*	*																											
09	Q6A																																					
10	Q6A ‘10 – 2’							*					*	*	*					*				*														
11	Q6A ‘6 + 2’							*			*	*								*			*															
12	Q6A ‘-1 + 9’							*		*		*								*		*	*															
13	Q6A ‘6 – -2’							*		*					*					*		*		*														
14	Q6B							*				*				*																						
15	Q6C ‘6 + 2’							*			*	*																			*							
16	Q6C ‘10 – 2’							*					*																*									
17	Q6C ‘-1 + 9’	*																												*								
18	Q6C ‘6 – -2’	*																												*								
19	Q6D ‘6 + 2’							*			*	*																							*			
20	Q6D ‘10 – 2’							*							*																				*			
21	Q6D ‘-1 + 9’							*		*	*	*																							*			
22	Q6D ‘6 – -2’							*		*					*																				*			
23	Q7A																																					
24	Q7A ‘-7 + 1’							*		*		*								*		*	*															
25	Q7A ‘2 – 8’							*							*					*		*		*														
26	Q7A ‘-1 + -5’									*		*										*	*															
27	Q7A ‘-4 – 2’							*		*					*					*		*		*														
28	Q7B			*		*					*		*		*	*	*																					
29	Q7C ‘2 – 8’							*		*					*															*								
30	Q7C ‘-7 + 1’							*		*		*																		*								
31	Q7C ‘-1 + -5’																												*									
32	Q7C ‘-4 – 2’	*						*		*			*																*									
33	Q7D ‘2 – 8’									*					*																				*			
34	Q7D ‘-7 + 1’									*	*																								*			
35	Q7D ‘-1 + -5’									*	*																								*			
36	Q7D ‘-4 – 2’	*						*		*					*																							

		Actions with numerals				Actions with the chip model				Action with number line		Relationship between positive integers and whole numbers		'Addition makes larger' remains true with integers		Types of numbers generated by integer subtraction						Properties of additive inverse for the pair '-7 and 7'		Operation and signifiers both need to be considered					Results of subtraction		Similarity of operations on the number line	
	Question number	Portions of numbers with different signs can cancel to give sum or difference	Adding to a negative number as subtraction	Subtracting a negative is adding a positive number	Subtraction as adding a negative number	Adding as joining	Opposite colored chips stacked	Subtraction as taking away	Opposite colors chips swapped	Go up number line	Go down number line	Are synonyms	Mean different things	, it remains true	No, not always	Positive number	Negative number	Integer	Any number	Larger number	Smaller number	Add to zero	Are opposites	Adding a negative number and a positive number can be treated as subtraction	Subtracting a negative number can be treated as addition of a positive number	Subtracting a positive number can be treated as adding a negative number	When you have a negative number you automatically subtract	Symbols provide inconsistent meanings	Larger number from a smaller number is not proper	Negative number cannot be done	Addition	Subtraction
01	Q1												*																			
02	Q2														*																	
03	Q3															*	*		*													
04	Q4																															
05	Q5																					*										
06	Q5C																															
07	Q5B																						*									
08	Q5A																					*										
09	Q6A																															
10	Q6A '10 – 2'																															
11	Q6A '6 + 2'																															
12	Q6A '-1 + 9'																															
13	Q6A '6 – -2'																															
14	Q6B																															
15	Q6C '6 + 2'					*																										
16	Q6C '10 – 2'							*																								
17	Q6C '-1 + 9'	*				*	*																									
18	Q6C '6 – -2'					*	*																									
19	Q6D '6 + 2'									*																						
20	Q6D '10 – 2'										*										*											*
21	Q6D '-1 + 9'									*																						
22	Q6D '6 – -2'										*																					
23	Q7A																															
24	Q7A '-7 + 1'	*																														
25	Q7A '2 – 8'																															
26	Q7A '-1 + -5'																															
27	Q7A '-4 – 2'																															
28	Q7B																															
29	Q7C '2 – 8'					*	*																									
30	Q7C '-7 + 1'	*				*	*																									
31	Q7C '-1 + -5'					*																										
32	Q7C '-4 – 2'																															
33	Q7D '2 – 8'										*																					
34	Q7D '-7 + 1'									*																						
35	Q7D '-1 + -5'										*																					
36	Q7D '-4 – 2'																															

Code assignment for Nicole's transcript by turn.

			Sets of numbers					Sign of number read or introduced			Role of sign verbalized or introduced									Written Expression including						Identification of '-'		Chip model notations				Number line notations						
	Question number	Participant voiced confusion	'Positive integer'	'Negative integer'	'Whole number'	Integer	Other names for sets of numbers	Positive number	Zero	Negative number	Addition	'Plus' as addition sign	'Minus' as subtraction sign	Subtraction	'Take away' for subtraction	Negative sign	'Positive sign'	Negative for subtraction sign	'Minus' for negative	Positive number	Zero	Negative number/sign	Addition	Subtraction	Positive sign explicitly	Subtraction sign	Negative sign	Drawn	One color	Separate colors	Mixed colors	Participant drawn	Spoke of number line	Chips as colored number line	Dot-hops-dot	Left or right arrow	Numbered steps taken	Dots along number line
01	Q1		*		*		*	*		*																												
02	Q2		*	*			*	*	*	*	*																											
03	Q3							*	*	*			*	*																								
04	Q4																									*	*											
05	Q5	*						*	*	*	*			*		*	*																					
06	Q5B																																					
07	Q5C							*		*																												
08	Q6A '-1 + 9'							*	*	*	*	*																										
09	Q6A '6 + 2'							*				*																										
10	Q6A '10 – 2'							*					*																									
11	Q6A '6 – -2'										*		*			*	*	*																				
12	Q6B							*	*	*	*			*																								
13	Q6C '-1 + 9'							*		*	*																			*								
14	Q6C '6 + 2'							*			*	*																	*									
15	Q6C '10 – 2'							*		*			*	*															*									
16	Q6C '6 – -2'							*		*	*			*																	*							
17	Q6D '-1 + 9'							*		*	*	*																									*	
18	Q6D '6 + 2'							*			*	*																									*	
19	Q6D '10 – 2'							*		*			*																								*	
20	Q6D '6 – -2'							*		*	*		*							*		*	*	*													*	
21	Q7A '2 – 8'							*		*			*																				*					
22	Q7A '-4 – 2'							*		*		*	*	*																								
23	Q7A '-7 + 1'							*		*	*	*		*																								
24	Q7A '-1 + -5'							*		*	*	*																										
25	Q7B							*		*	*																											
26	Q7C '2 – 8'							*		*				*																*								
27	Q7C '-4 – 2'							*		*			*	*																*								
28	Q7C '-7 + 1'							*		*		*																		*								
29	Q7C '-1 + -5'									*	*	*																	*									
30	Q7D '2 – 8'							*		*			*																								*	
31	Q7D '-4 – 2'							*		*			*	*																							*	
32	Q7D '-7 + 1'							*		*	*	*																									*	
33	Q7D '-1 + -5'									*	*	*		*				*				*	*	*													*	

		Actions with numerals				Actions with the chip model				Action with number line		Relationship between positive integers and whole numbers		‘Addition makes larger’ remains true with integers		Types of numbers generated by integer subtraction						Properties of additive inverse or the pair ‘-7 and 7’		Operation and signifiers both need to be considered					Results of subtraction		Similarity of operations on the number line	
	Question number	Portions of numbers with different signs can cancel to give sum or difference	Adding to a negative number as subtraction	Subtracting a negative is adding a positive number	Subtraction as adding a negative number	Adding as joining	Opposite colored chips stacked	Subtraction as taking away	Opposite colors chips swapped	Go up number line	Go down number line	Are synonyms	Mean different things	, it remains true	No, not always	Positive number	Negative number	Integer	Any number	Larger number	Smaller number	Add to zero	Are opposites	Adding a negative number and a positive number can be treated as subtraction	Subtracting a negative number can be treated as addition of a positive number	Subtracting a positive number can be treated as adding a negative number	When you have a negative number you automatically subtract	Symbols provide inconsistent meanings	Larger number from a smaller number is not proper	Negative number cannot be done	Addition	Subtraction
01	Q1												*																			
02	Q2													*																		
03	Q3															*	*															
04	Q4																															
05	Q5																					*										
06	Q5B																						*									
07	Q5C																															
08	Q6A ‘-1 + 9’	*																														
09	Q6A ‘6 + 2’																															
10	Q6A ‘10 – 2’																															
11	Q6A ‘6 – -2’			*																					*							
12	Q6B																															
13	Q6C ‘-1 + 9’	*				*	*																									
14	Q6C ‘6 + 2’					*																										
15	Q6C ‘10 – 2’							*	*																							
16	Q6C ‘6 – -2’				*	*																			*							
17	Q6D ‘-1 + 9’									*																						
18	Q6D ‘6 + 2’									*																						
19	Q6D ‘10 – 2’				*						*																					
20	Q6D ‘6 – -2’									*															*							
21	Q7A ‘2 – 8’																															
22	Q7A ‘-4 – 2’				*																											
23	Q7A ‘-7 + 1’	*	*																													
24	Q7A ‘-1 + -5’																															
25	Q7B				*																											
26	Q7C ‘2 – 8’				*	*	*																									
27	Q7C ‘-4 – 2’					*			*																							
28	Q7C ‘-7 + 1’	*				*	*																									
29	Q7C ‘-1 + -5’					*																										
30	Q7D ‘2 – 8’										*																					
31	Q7D ‘-4 – 2’										*																					*
32	Q7D ‘-7 + 1’									*																					*	
33	Q7D ‘-1 + -5’										*																					

Code assignment for Rebecca's transcript by turn.

			Sets of numbers					Sign of number read or introduced			Role of sign verbalized or introduced									Written Expression including						Identification of '- '		Chip model notations				Number line notations						
	Question number	Participant voiced confusion	'Positive integer'	'Negative integer'	'Whole number'	Integer	Other names for sets of numbers	Positive number	Zero	Negative number	Addition	'Plus' as addition sign	'Minus' as subtraction sign	Subtraction	'Take away' for subtraction	Negative sign	'Positive sign'	Negative for subtraction sign	'Minus' for negative	Positive number	Zero	Negative number/sign	Addition	Subtraction	Positive sign explicitly	Subtraction sign	Negative sign	Drawn	One color	Separate colors	Mixed colors	Participant drawn	Spoke of number line	Chips as colored number line	Dot-hops-dot	Left or right arrow	Numbered steps taken	Dots along number line
01	Q1		*		*		*	*		*																												
02	Q2				*	*		*		*	*	*								*		*	*															
03	Q3		*	*		*		*		*			*	*						*		*	*	*	*													
04	Q4				*	*																				*	*											
05	Q5							*	*	*	*			*																		*			*			
06	Q5A		*	*				*								*																						
07	Q5B							*		*						*	*			*		*																
08	Q5C							*		*																												
09	Q6A																																					
10	Q6A '10 – 2'							*						*														*	*									
11	Q6A '6 + 2'							*			*	*								*								*	*									
12	Q6A '6 – -2'			*				*		*			*	*		*				*		*	*	*				*										
13	Q6A '-1 + 9'							*		*		*								*		*	*									*			*			
14	Q6B			*				*		*			*			*		*															*					
15	Q6C '10 – 2'							*						*															*									
16	Q6C '6 + 2'																												*									
17	Q6C '6 – -2'							*		*	*		*	*	*														*									
18	Q6C '-1 + 9'										*																				*		*	*				
19	Q6D '10 – 2'							*					*							*															*			
20	Q6D '6 + 2'		*					*				*								*															*			
21	Q6D '6 – -2'							*		*			*							*															*			
22	Q6D '-1 + 9'							*		*																									*			
23	Q7A '2 – 8'					*		*		*			*	*						*		*		*								*			*			
24	Q7A '-7 + 1'		*					*		*		*										*										*			*			
25	Q7A '-4 – 2'							*		*			*									*										*			*			
26	Q7A '-1 + -5'									*	*	*	*	*						*		*		*								*			*			
27	Q7B		*	*					*	*	*			*		*																						
28	Q7C '2 – 8'																													*				*				
29	Q7C '-7 + 1'							*		*		*																	*									
30	Q7C '-4 – 2'							*		*	*		*																*									
31	Q7C '-1 + -5'							*		*	*	*	*																*									
32	Q7D '2 – 8'		*					*		*			*																						*			
33	Q7D '-7 + 1'		*					*		*	*	*										*													*			
34	Q7D '-4 – 2'		*					*		*			*																						*			
35	Q7D '-1 + -5'			*				*		*	*	*	*									*													*			

	Question number	Actions with numerals				Actions with the chip model				Action with number line		Relationship between positive integers and whole numbers		‘Addition makes larger’ remains true with integers		Types of numbers generated by integer subtraction						Properties of additive inverse or the pair ‘-7 and 7’		Operation and signifiers both need to be considered					Results of subtraction		Similarity of operations on the number line	
		Portions of numbers with different signs can cancel to give sum or difference	Adding to a negative number as subtraction	Subtracting a negative is adding a positive number	Subtraction as adding a negative number	Adding as joining	Opposite colored chips stacked	Subtraction as taking away	Opposite colors chips swapped	Go up number line	Go down number line	Are synonyms	Mean different things	, it remains true	No, not always	Positive number	Negative number	Integer	Any number	Larger number	Smaller number	Add to zero	Are opposites	Adding a negative number and a positive number can be treated as subtraction	Subtracting a negative number can be treated as addition of a positive number	Subtracting a positive number can be treated as adding a negative number	When you have a negative number you automatically subtract	Symbols provide inconsistent meanings	Larger number from a smaller number is not proper	Negative number cannot be done	Addition	Subtraction
01	Q1												*																			
02	Q2														*																	
03	Q3			*												*	*								*							
04	Q4																															
05	Q5									*	*											*	*	*								
06	Q5A																					*										
07	Q5B									*	*												*									
08	Q5C																															
09	Q6A																															
10	Q6A ‘10 – 2’							*																								
11	Q6A ‘6 + 2’					*																										
12	Q6A ‘6 – -2’			*		*																										
13	Q6A ‘-1 + 9’									*																						
14	Q6B									*																						
15	Q6C ‘10 – 2’							*																								
16	Q6C ‘6 + 2’					*																										
17	Q6C ‘6 – -2’			*		*																			*							
18	Q6C ‘-1 + 9’					*																										
19	Q6D ‘10 – 2’										*																					
20	Q6D ‘6 + 2’									*																					*	
21	Q6D ‘6 – -2’			*						*															*							
22	Q6D ‘-1 + 9’									*																						
23	Q7A ‘2 – 8’										*																					
24	Q7A ‘-7 + 1’									*																						
25	Q7A ‘-4 – 2’										*																					
26	Q7A ‘-1 + -5’										*																					
27	Q7B																															
28	Q7C ‘2 – 8’										*																					
29	Q7C ‘-7 + 1’		*					*																*								
30	Q7C ‘-4 – 2’				*	*																				*						
31	Q7C ‘-1 + -5’					*																										
32	Q7D ‘2 – 8’										*																					*
33	Q7D ‘-7 + 1’									*																					*	
34	Q7D ‘-4 – 2’										*																					*
35	Q7D ‘-1 + -5’										*																					*

Code assignment for Tae’s transcript by turn.

			Sets of numbers					Sign of number read or introduced			Role of sign verbalized or introduced									Written Expression including						Identification of ‘- ’		Chip model notations				Number line notations						
	Question number	Participant voiced confusion	‘Positive integer’	‘Negative integer’	‘Whole number’	Integer	Other names for sets of numbers	Positive number	Zero	Negative number	Addition	‘Plus’ as addition sign	‘Minus’ as subtraction sign	Subtraction	‘Take away’ for subtraction	Negative sign	‘Positive sign’	Negative for subtraction sign	‘Minus’ for negative	Positive number	Zero	Negative number/sign	Addition	Subtraction	Positive sign explicitly	Subtraction sign	Negative sign	Drawn	One color	Separate colors	Mixed colors	Participant drawn	Spoke of number line	Chips as colored number line	Dot-hops-dot	Left or right arrow	Numbered steps taken	Dots along number line
01	Q1		*	*	*																											*						
02	Q2						*	*		*	*	*								*		*	*															
03	Q3			*		*	*	*		*			*	*						*		*		*														
04	Q1						*	*																														
05	Q4																									*	*											
06	Q5	*						*			*																											
07	Q5C																			*		*																
08	Q5																																					
09	Q5B																																*					
10	Q6A							*						*						*																		
11	Q6A ‘10 – 2’							*						*														*	*									
12	Q6A ‘6 + 2’																											*	*									
13	Q6A ‘6 – -2’							*		*																						*			*			
14	Q6A ‘-1 + 9’																															*			*			
15	Q6B							*		*	*	*	*	*																								
16	Q6C ‘10 – 2’							*					*	*															*									
17	Q6C ‘6 + 2’							*			*	*																	*									
18	Q6C ‘-1 + 9’		*	*				*		*	*	*																		*								
19	Q6C ‘6 – -2’	*						*		*																					*							
20	Q6D ‘10 – 2’							*					*	*																					*			
21	Q6D ‘6 + 2’							*			*	*																							*			
22	Q6D ‘6 – -2’							*		*			*																						*			
23	Q6D ‘-1 + 9’							*		*																												
24	Q7A ‘2 – 8’							*					*									*										*			*			
25	Q7A ‘-7 + 1’							*		*		*										*										*						
26	Q7A ‘-4 – 2’																					*										*			*			
27	Q7A ‘-1 + -5’																					*							*									
28	Q7B									*	*																						*					
29	Q7C ‘2 – 8’			*				*		*	*	*	*																	*								
30	Q7C ‘-7 + 1’			*				*		*	*	*																		*								
31	Q7C ‘-4 – 2’									*			*																*									
32	Q7C ‘-1 + -5’									*	*																		*									
33	Q7D ’2 – 8’							*					*																						*			
34	Q7D ‘-7 + 1’							*		*	*	*																							*			
35	Q7D ‘-4 – 2’							*		*			*																						*			
36	Q7D ‘-1 + -5’									*																									*			

		Actions with numerals				Actions with the chip model				Action with number line		Relationship between positive integers and whole numbers		'Addition makes larger' remains true with integers		Types of numbers generated by integer subtraction						Properties of additive inverse or the pair '-7 and 7'		Operation and signifiers both need to be considered					Results of subtraction		Similarity of operations on the number line	
	Question number	Portions of numbers with different signs can cancel to give sum or difference	Adding to a negative number as subtraction	Subtracting a negative is adding a positive number	Subtraction as adding a negative number	Adding as joining	Opposite colored chips stacked	Subtraction as taking away	Opposite colors chips swapped	Go up number line	Go down number line	Are synonyms	Mean different things	, it remains true	No, not always	Positive number	Negative number	Integer	Any number	Larger number	Smaller number	Add to zero	Are opposites	Adding a negative number and a positive number can be treated as subtraction	Subtracting a negative number can be treated as addition of a positive number	Subtracting a positive number can be treated as adding a negative number	When you have a negative number you automatically subtract	Symbols provide inconsistent meanings	Larger number from a smaller number is not proper	Negative number cannot be done	Addition	Subtraction
01	Q1												*																			
02	Q2														*																	
03	Q3																		*													
04	Q1																															
05	Q4																															
06	Q5																															
07	Q5C																															
08	Q5																						*									
09	Q5B																															
10	Q6A																															
11	Q6A '10 – 2'							*																								
12	Q6A '6 + 2'					*																										
13	Q6A '6 – -2'									*																						
14	Q6A '-1 + 9'									*																						
15	Q6B			*																					*							
16	Q6C '10 – 2'							*																								
17	Q6C '6 + 2'					*																										
18	Q6C '-1 + 9'	*				*	*																									
19	Q6C '6 – -2'					*																										
20	Q6D '10 – 2'										*																					*
21	Q6D '6 + 2'									*																						
22	Q6D '6 – -2'									*																						
23	Q6D '-1 + 9'									*																						
24	Q7A '2 – 8'										*																					
25	Q7A '-7 + 1'									*																						
26	Q7A '-4 – 2'										*																					
27	Q7A '-1 + -5'																															
28	Q7B																															
29	Q7C '2 – 8'				*	*	*																			*						
30	Q7C '-7 + 1'					*	*																									
31	Q7C '-4 – 2'				*	*																										
32	Q7C '-1 + -5'					*																										
33	Q7D '2 – 8'										*																					
34	Q7D '-7 + 1'									*																					*	
35	Q7D '-4 – 2'										*																					
36	Q7D '-1 + -5'										*																					

Appendix P: Code assignment by PST per Question for Questions 1 - 5

Codes associated with the difference between 'whole numbers' and 'positive integers'

Code	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae	Total
Word Use															
Sets of numbers															
'Whole number'	*	*	*	*	*	*	*	*	*	*	*	*	*	*	14
'Positive integer'	*	*	*	*		*	*	*	*	*	*	*	*	*	13
'Negative Integer'		*												*	2
Other names for sets of numbers		*	*	*		*	*		*		*	*	*	*	10
Integer					*				*						2
Sign of number read or introduced															
Positive number		*						*	*			*	*	*	6
Negative number		*							*	*		*	*		5
Zero									*						1
Visual Mediator															
Number line notation															
Spoke of number line	*	*													2
Drawn number line														*	1
Narrative															
Positive integers and whole numbers															
Mean different things		*	*	*		*	*	*		*	*	*	*	*	11
Synonyms	*				*				*						3
Participant voiced confusion									*						1
Total	4	8	4	4	3	4	4	4	9	4	4	6	6	7	71

Codes associated with results of addition of integers

Code	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae	Total
Word Use															
Sets of numbers															
'Positive integer'									*			*			2
'Negative integer'		*		*				*	*			*			5
'Whole number'				*					*		*		*		4
Integer									*		*		*		3
Other names for sets of numbers												*		*	2
Sign of number read or introduced															
Negative number	**	*	*	*	*	*	*	*	*	*	*	*	*	*	15
Positive number	**			*	*			*	*		*	*	*	*	10
Zero	*			*								*			3
Role of sign read or introduced															
'Plus' as addition sign	**	*		*	*	*	*	*		*			*	*	11
Addition		*		*	*			*	*	*	*	*	*	*	10
Subtraction	*														1
Visual Mediator															
Written expression															
Addition	**	*	*	*	*								*	*	8
Positive number	**		*	*	*								*	*	7
Negative number/sign	**	*	*	*									*	*	7
Zero	*		*												2
Number line notations															
Participant drawn				*											1
Dot-hops-dot				*											1
Routine															
Addition with numerals															
Adding to a negative as subtraction	*														1
Action with number line															
Go up number line				*											1
Narrative															
'Addition makes larger' remains true with integers															
Yes, it remains true				*	*							*			3
No, not always	*	*	*			*	*	*	*	*	*		*	*	11
Operation and signifiers both need to be considered															
Adding a negative and a positive can be treated as subtraction	*														1
Participant voiced confusion			*		*				*						3
Total	18	7	7	14	8	3	3	6	8	5	6	8	10	9	112

Codes associated with results of integer subtraction

Code	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae	Total
Word Use															
Sets of numbers															
'Positive integer'													*		1
'Negative integer'							*			*			*	*	4
'Whole number'				*											1
Integer				*						*			*	*	4
Other names for sets of numbers				*							*			*	3
Sign of number read or introduced															
Negative number	*		*	*	*	*	*	*	*	*	*	*	*	*	13
Positive number	*		*	*	*	*	*	*	*	*	*	*	*	*	13
Zero						*						*			2
Role of sign read or introduced															
'Minus' as subtraction sign	*				*			*	*			*	*	*	7
'Plus' as addition sign						*									1
Addition		*													1
Subtraction	*	*						*	*	*		*	*	*	8
Visual Mediator															
Written Expression including…															
Positive number	*		*		*								*	*	5
Negative number/sign	*		*		*								*	*	5
Subtraction	*				*								*	*	4
Addition	*		*										*		3
Positive sign explicitly													*		1
Routine															
Subtraction with numerals	*												*		2
Subtracting a negative is adding a positive	*														1
Narrative															
Results of subtraction															
Positive number	*		*	*		*	*	*	*	*	*	*	*		11
Negative number	*		*	*	*	*	*	*	*	*	*	*	*		12
Integer					*					*					2
Any number											*			*	2
Smaller number		*								*					2
Larger number		*													1
Operation and signifiers both need to be considered															
Negative can be treated as addition of a positive													*		1
Total	12	4	7	7	8	6	5	6	6	9	6	7	16	11	112

Codes associated with roles of ‘ - ’

Speaker	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae	Total
Word Use															
Sets of numbers															
‘Negative integer’	*										*				2
‘Whole number’												*			1
Integer													*		1
Other names for sets of numbers					*										1
Sign of number read or introduced															
Positive number	*			*	*		*								4
Negative number	*					*			*						3
Role of sign read or introduced															
'Take away' for subtraction	*	*		*											3
Negative sign				*											1
Visual Mediator															
Identification of ‘ - ’															
Subtraction sign	*	*	*	*	*	*	*	*	*	*	*	*	*	*	14
Negative sign	*		*	*	*	*		*	*	*	*	*	*	*	12
Total	6	2	2	5	4	3	2	2	3	2	3	3	3	2	42

Codes in response to "What does the phrase additive inverse mean?"

Code	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae	Total
Word Use															
Sets of numbers															
‘Positive integer’										*					1
‘Negative integer’										*					1
Sign of number read or introduced															
Positive number				*		*	*	*	*		*	*	*	*	9
Negative number				*		*	*	*	*		*	*	*		8
Zero	*	*	*				*	*			*	*	*		8
Role of sign read or introduced															
Addition	*	*	*					*			*	*	*	*	8
‘Plus’ as addition sign							*								1
Subtraction												*	*		2
‘Take away’ for subtraction								*							1
Negative sign												*			1
‘Positive sign’												*			1
‘Minus’ for negative									*						1
Negative for subtraction sign								*							1
Relationship between numbers															
‘Opposites’	*			*		*	*								4
‘Same number’								*		*				*	3
Visual Mediator															
Number line notations															
Drawn number line	*												*		2
Number line dot hops dot													*		1
Routine															
Action with number line															
Go up number line													*		1
Go down number line													*		1
Narrative															
Properties of additive inverse or the pair ‘-7 and 7’															
Add to zero	*	*	*				*				*	*	*		7
Are opposites, or one is positive the other is negative	*					*	*	*		*			*	*	7
Operation and signifier both need to be considered															
Adding a negative and a positive can be treated as subtraction													*		1
Participant voiced confusion		*	*	*	*	*			*	*	*	*		*	10
Total	6	4	4	4	1	5	7	8	4	5	6	9	12	5	80

Codes associated with "why are 7 and -7 additive inverses?"

Code	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae	Total
Word Use															
Sets of numbers															
'Positive integer'													*		1
'Negative integer'													*		1
Integer							*								1
Sign of number read or introduced															
Positive number		**	*	*			*	*	*	*	*		*		10
Negative number		**	*	*			*	*	*	*	*				9
Zero	*	**	*						*		*				6
Role of sign read or introduced															
Addition	*	**	*						*		*				6
Subtraction									*						1
Negative sign													*		1
Relationship between numbers															
'Opposites'			*	*	*		*				*		*		6
'Same number'								*		*	*		*		4
Visual Mediator															
Number line notations															
Spoke of number line	*	*	*		*		*	*							6
Routine															
Action with number line															
Go up number line		*													1
Narrative															
Properties of additive inverse or the pair '-7 and 7'															
Add to zero	*	**	*						*		*		*		7
Are opposites, or one is positive the other is negative			*	*	*		*	*		*					6
Participant voiced confusion						*					*				2
Total	4	12	8	4	3	1	6	5	6	4	8	0	7	0	68

Codes for responses to "How else would you describe two numbers that are additive inverses?"

Code	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae	Total
Word Use															
Sets of numbers															
Integer										*					1
Sign of number read or introduced															
Positive number										*	*		*		3
Negative number										*	*		*		3
Zero							*								1
Role of sign read or introduced															
Addition							*		*						2
Subtraction									*						1
Negative sign													*		1
'Positive sign'													*		1
Relationship between numbers															
'Opposites'	*		*	*								*			4
'Same number'									*	*	*		*		4
Visual Mediator															
Written Expression including…															
Positive number													*		1
Negative number/sign													*		1
Number line notations															
Spoke of number line				*				*						*	3
Routine															
Action with number line															
Go up number line													*		1
Go down number line													*		1
Narrative															
Properties of additive inverse or the pair '-7 and 7'															
Add to zero							*								1
Are opposites, or one is positive the other is negative				*	*			*		*	*	*	*		7
Total	1	0	1	3	1	0	3	2	3	5	4	2	10	1	34

Codes used in response to the prompt about a pair of numbers similar to '-7 and 7'

Code	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae	Total
Word Use															
Sign of number read or introduced															
Positive number	*		*	*	*	*	*	*	*	*	*	*	*		12
Negative number	*		*	*	*	*	*	*	*	*	*	*	*		12
Zero							*								1
Role of sign read or introduced															
'Plus' as addition sign							*								1
Visual Mediator															
Written Expression including…															
Positive number		*												*	2
Negative number/sign		*												*	2
Total	2	2	2	2	2	2	4	2	2	2	2	2	2	2	30

Appendix Q: Code assignment by PST per Question by modeling approach

'6 + 2' initial exposure.

	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Participant voiced confusion															
Sets of numbers															
'Whole number'								*							
Sign of number read or introduced															
Positive number	*	*	*	*	*	*		*	*	*	*	*	*	*	
Role of sign verbalized or introduced															
Addition		*						*	*		*	*		*	
'Plus' as addition sign	*	*	*	*	*	*		*	*	*		*	*	*	
Written Expression including															
Positive number	*		*		*		*		*	*		*		*	*
Addition	*		*		*		*		*	*		*			
Chip model notations															
Drawn		*												*	*
One color		*												*	*
Number line notations															
Participant drawn			*						*						
Dot-hops-dot									*						
Left or right arrow			*												
Actions with the chip model															
Adding as joining		*												*	*
Action with number line															
Go up number line			*						*						
Similarity of operations on the number line															
Addition									*						
Subtraction									*						

'6 + 2' chip model.

	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Participant voiced confusion														
Sets of numbers														
'Positive integer'		*												
Sign of number read or introduced														
Positive number	*	*	*	*	*	*	*	*	*	*	*	*		*
Role of sign verbalized or introduced														
Addition	*	*	*	*			*	*	*	*	*	*		*
'Plus' as addition sign	*			*	*			*			*	*		*
Chip model notations														
One color		*		*		*		*		*		*	*	*
Mixed colors	*		*		*		*		*		*			
Actions with the chip model														
Adding as joining	*	*	*	*	*		*	*	*	*	*	*	*	*

'6 + 2' number line.

	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Participant voiced confusion														
Sets of numbers														
'Positive integer'													*	
Sign of number read or introduced														
Positive number	*	*		*	*		*	*	*	*	*	*	*	*
Role of sign verbalized or introduced														
Addition				*	*		*	*	*		*	*		*
'Plus' as addition sign	*	*		*			*	*	*	*	*	*	*	*
Written Expression including														
Positive number						*		*					*	
Addition						*		*						
Number line notations														
Dot-hops-dot	*	*		*	*	*	*	*		*	*		*	*
Left or right arrow			*					*						
Numbered steps taken												*		
Dots along number line									*					
Action with number line														
Go up number line	*	*	*	*	*	*	*	*	*	*	*	*	*	*
Similarity of operations on the number line														
Addition				*				*					*	

‘-1 + 9’ initial interaction.

	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Participant voiced confusion								*						
Sign of number read or introduced														
Positive number	*	*	*	*	*	*	*	*	*	*	*	*	*	
Zero												*		
Negative number	*	*	*	*	*	*	*	*	*	*	*	*	*	
Role of sign verbalized or introduced														
Addition			*	*	*			*	*			*		
'Plus' as addition sign		*	*		*	*	*	*	*	*	*	*	*	
'Minus' as subtraction sign	*	*								*				
Subtraction	*				*									
'Take away' for subtraction		*							*					
Negative sign	*	*												
Written Expression including														
Positive number	*	*	*		*		*	*	*		*		*	
Zero														
Negative number/sign	*	*	*		*			*	*		*		*	
Addition	*	*	*		*			*	*		*		*	
Subtraction	*	*												
Chip model notations														
Drawn		*												
Number line notations														
Participant drawn			*	*				*					*	*
Dot-hops-dot				*				*					*	*
Left or right arrow			*											
Actions with numerals														
Portions of numbers with different signs can cancel to give sum or difference									*			*		
Adding to a negative number as subtraction	*	*			*				*	*				
Actions with the chip model														
Subtraction as taking away		*												
Actions with number line														
Go up number line			*	*				*					*	*
Properties of additive inverse or the pair ‘-7 and 7’														
Adding a negative number and a positive number can be treated as subtraction	*				*									
Similarity of operation on the number line														
Addition								*						

‘-1 + 9’ chip model.

	Amanda	Ashley	Bailey	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Participant voiced confusion		*	*									*			
Sets of numbers															
‘Positive integer’															*
‘Negative integer’															*
‘Whole number’										*					
Sign of number read or introduced															
Positive number	*	*			*	*		*	*	*	*		*		*
Zero											*				
Negative number	*	*			*	*		*	*	*	*		*		*
Role of sign verbalized or introduced															
Addition						*			*	*	*		*	*	*
'Plus' as addition sign		*			*			*			*				*
'Minus' as subtraction sign								*							
'Take away' for subtraction		*								*					
Negative sign								*							
Chip model notations															
One color	*		*												
Separate colors		*			*	*	*		*	*	*	*	*		*
Mixed colors								*						*	
Number line notations															
Spoke of number line									*					*	
Chips as colored number line									*					*	
Actions with numerals															
Portions of numbers with different signs can cancel to give sum or difference					*	*					*	*	*		*
Adding to a negative number as subtraction	*							*		*					
Actions with the chip model															
Adding as joining		*			*	*	*			*	*	*	*	*	*
Opposite colored chips stacked		*			*	*	*				*	*	*		*
Subtraction as taking away	*							*		*					
Opposite colors chips swapped										*					
Action with number line															
Go up number line									*						

'-1 + 9' number line.

	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Mon	Nicole	Rebecca	Tae
Participant voiced confusion														
Sign of number read or introduced														
Positive number	*	*	*	*	*	*	*	*	*	*	*	*	*	*
Negative number	*	*	*	*	*	*	*	*	*	*	*	*	*	*
Role of sign verbalized or introduced														
Addition				*			*	*	*		*	*		
'Plus' as addition sign	*	*	*	*	*	*	*	*		*	*	*		
Written Expression including														
Positive number			*			*		*						
Negative number/sign			*			*		*						
Addition			*			*		*						
Number line notations														
Dot-hops-dot	*	*		*	*	*	*	*		*	*		*	
Left or right arrow			*					*						
Numbered steps taken												*		
Dots along number line									*					
Action with number line														
Go up number line	*	*	*	*	*		*	*	*	*	*	*	*	*
Go down number line						*								
Similarity of operations on the number line														
Addition		*		*				*						

'-7 + 1' initial interaction.

	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Participant voiced confusion														
Sets of numbers														
'Positive integer'		*											*	
'Negative integer'		*												
'Whole number'									*					
Sign of number read or introduced														
Positive number	*	*		*	*	*	*	*	*	*	*	*	*	*
Negative number	*	*		*	*	*	*	*	*	*	*	*	*	*
Role of sign verbalized or introduced														
Addition		*			*		*	*		*		*		
'Plus' as addition sign	*	*			*	*		*	*	*	*	*	*	*
'Minus' as subtraction sign	*													
Subtraction							*					*		
Written Expression including														
Positive number	*	*	*		*			*	*		*			
Negative number/sign	*	*	*		*		*	*	*	*	*		*	*
Addition	*	*	*		*			*	*		*			
Subtraction	*													
Chip model notations														
Drawn		*	*											
Separate colors			*											
Number line notations														
Participant drawn													*	*
Spoke of number line				*										
Dot-hops-dot													*	*
Actions with numerals														
Portions of numbers with different signs can cancel to give sum or difference									*		*	*		
Adding to a negative number as subtraction	*											*		
Actions with the chip model														
Adding as joining		*	*											
Opposite colored chips stacked		*	*											
Action with number line														
Go up number line				*									*	*

‘-7 + 1’ chip model.

	Amanda	Ashley	Bailey	Brooke	Christine	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Participant voiced confusion														
Sets of numbers														
‘Positive integer’								*						
‘Negative integer’														*
Sign of number read or introduced														
Positive number	*	*	*	*	*		*	*	*	*	*	*	*	*
Negative number	*	*	*	*	*		*	*	*	*	*	*	*	*
Role of sign verbalized or introduced														
Addition	*	*						*	*	*				*
'Plus' as addition sign	*		*	*	*		*	*	*	*	*	*	*	*
'Minus' as subtraction sign							*							
Subtraction	*													
'Take away' for subtraction									*					
Chip model notations														
One color	*						*						*	
Separate colors		*		*	*			*	*	*	*	*		*
Mixed colors			*			*								
Number line notations														
Chips as colored number line								*						
Actions with numerals														
Portions of numbers with different signs can cancel to give sum or difference		*		*	*					*	*	*		
Adding to a negative number as subtraction	*						*		*				*	
Actions with the chip model														
Adding as joining		*	*	*	*	*			*	*	*	*		*
Opposite colored chips stacked		*	*	*	*	*			*	*	*	*		*
Subtraction as taking away	*					*							*	
Opposite colors chips swapped						*								
Action with number line														
Go up number line								*						
Operation and signifiers both need to be considered														
Adding a negative number and a positive number can be treated as subtraction													*	

'-7 + 1' number line.

	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Participant voiced confusion														
Sets of numbers														
'Positive integer'													*	
'Whole number'							*							
Integer			*											
Other names for sets of numbers			*											
Sign of number read or introduced														
Positive number	*	*	*		*	*	*	*	*	*		*	*	*
Negative number	*	*	*	*	*		*	*	*	*	*	*	*	*
Role of sign verbalized or introduced														
Addition	*		*		*		*	*	*	*	*	*	*	*
'Plus' as addition sign		*	*		*		*	*		*		*	*	*
Written Expression including														
Positive number			*			*								
Negative number/sign			*			*							*	
Addition			*			*								
Number line notations														
Dot-hops-dot	*	*		*	*	*	*	*		*	*		*	*
Left or right arrow			*						*					
Numbered steps taken												*		
Action with number line														
Go up number line	*	*	*	*	*	*	*	*	*	*	*	*	*	*
Similarity of operations on the number line														
Addition					*					*		*	*	*

'6 – -2' initial interaction.

	Amanda	Ashley	Ashley	Bailey	Brooke	Christina	Christina	Courtney	Jacqueline	Jacqueline	Jamie	Jordan	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Participant voiced confusion				*	*	*					*	*						
Sets of numbers																		
'Negative integer'																	*	
Sign of number read or introduced																		
Positive number	*		*	*	*	*	*	*		*	*	*		*	*		*	*
Negative number	*		*		*			*			*	*		*	*		*	*
Role of sign verbalized or introduced																		
Addition											*			*		*		
'Plus' as addition sign	*		*							*	*	*		*				
'Minus' as subtraction sign	*		*		*	*		*		*	*	*		*		*	*	
Subtraction					*					*				*			*	
'Take away' for subtraction											*	*			*			
Negative sign			*							*						*	*	
'Positive sign'																*		
Negative for subtraction sign																*		
Written Expression including																		
Positive number	*	*				*	*		*		*	*	*		*		*	
Negative number/sign	*	*				*					*	*	*		*		*	
Addition	*	*					*		*		*						*	
Subtraction	*	*				*					*	*			*		*	
Positive sign explicitly	*						*		*									
Chip model notations																		
Drawn		*															*	
One color		*																
Number line notations																		
Participant drawn					*						*							*
Dot-hops-dot																		*
Actions with numerals																		
Subtracting a negative is adding a positive number	*	*	*	*			*	*	*	*	*		*	*		*	*	
Actions with the chip model																		
Adding as joining		*															*	
Action with number line																		
Go up number line					*													*
Go down number line					*													
Operation and signifiers both need to be considered																		
Subtracting a negative number can be treated as addition of a positive number			*		*					*	*		*			*		
Results of subtraction																		
Negative number cannot be done												*						
Similarity of operations on the number line																		
Subtraction					*													

'6 – -2' chip model.

	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Mon	Nicole	Rebecca	Tae
Participant voiced confusion		*		*	*		*	*	*	*	*			*
Sign of number read or introduced														
Positive number	*		*	*	*	*	*	*	*	*		*	*	*
Negative number	*		*	*	*		*	*	*	*		*	*	*
Role of sign verbalized or introduced														
Addition	*						*	*		*		*	*	
'Minus' as subtraction sign	*		*	*			*	*					*	
Subtraction		*								*		*	*	
'Take away' for subtraction					*			*	*				*	
Chip model notations														
One color	*			*									*	
Separate colors		*			*	*		*	*		*			
Mixed colors			*				*			*		*		*
Actions with numerals														
Subtracting a negative is adding a positive number	*	*					*			*			*	
Actions with the chip model												*		
Adding as joining	*		*		*	*	*		*	*	*	*	*	*
Opposite colored chips stacked											*			
Opposite colors chips swapped		*				*								
Operation and signifiers both need to be considered														
Subtracting a negative number can be treated as addition of a positive number												*	*	
Results of subtraction														
Negative number cannot be done	*													

'6 – -2' number line.

	Amanda	Ashley	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Participant voiced confusion								*	*	*		*				
Sets of numbers																
'Positive integer'			*													
Sign of number read or introduced																
Positive number	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
Negative number	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
Role of sign verbalized or introduced																
Addition	*		*		*			*	*		*			*		
'Plus' as addition sign				*												
'Minus' as subtraction sign	*	*	*	*	*			*	*			*		*	*	*
Subtraction		*			*	*						*				
'Take away' for subtraction									*	*			*			
Written Expression including																
Positive number				*	*		*		*					*	*	
Negative number/sign				*	*				*	*				*		
Addition					*		*							*		
Subtraction				*	*				*					*		
Number line notations																
Dot-hops-dot	*	*			*	*	*	*	*		*	*	*		*	*
Left or right arrow				*												
Numbered steps taken														*		
Dots along number line										*						
Actions with numerals																
Subtracting a negative is adding a positive number	*		*	*											*	
Action with number line																
Go up number line	*	*		*	*	*	*	*	*		*	*		*	*	*
Go down number line										*			*			
Operation and signifiers both need to be considered																
Subtracting a negative number can be treated as addition of a positive number									*					*	*	
Results of subtraction																
Negative number cannot be done									*							
Similarity of operations on the number line																
Subtraction			*													

'10 - 2' initial interaction.

	Amanda	Ashley	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Participant voiced confusion																
Sets of numbers																
'Whole number'									*							
Sign of number read or introduced																
Positive number	*		*	*	*	*	*		*	*	*	*	*	*	*	*
Role of sign verbalized or introduced																
'Minus' as subtraction sign	*		*		*	*	*		*	*	*	*	*	*		
Subtraction			*	*	*								*		*	*
'Take away' for subtraction			*	*					*		*	*	*			
Written Expression including																
Positive number	*		*	*	*	*		*		*	*		*			
Subtraction	*		*	*	*	*		*		*	*		*			
Chip model notations																
Drawn			*												*	*
One color			*												*	*
Number line notations																
Participant drawn				*						*						
Spoke of number line		*	*													
Left or right arrow				*												
Actions with the chip model																
Subtraction as taking away			*												*	*
Action with number line																
Go down number line				*						*						

'10 - 2' chip model.

	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Participant voiced confusion														
Sets of numbers														
'Positive integer'								*						
Sign of number read or introduced														
Positive number	*	*	*	*	*	*	*	*	*	*	*	*	*	*
Negative number												*		
Role of sign verbalized or introduced														
'Minus' as subtraction sign	*	*		*				*			*	*		*
Subtraction		*		*	*	*		*		*		*	*	*
'Take away' for subtraction	*	*	*	*	*		*	*	*					
Chip model notations														
One color	*	*	*	*		*		*	*	*	*	*	*	*
Mixed colors					*		*							
Actions with the chip model														
Subtraction as taking away	*	*	*	*	*	*	*	*	*	*	*	*	*	*
Opposite colors chips swapped												*		

'10 - 2' number line.

	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Participant voiced confusion														
Sign of number read or introduced														
Positive number	*	*	*	*	*		*	*	*	*	*	*	*	*
Negative number												*		
Role of sign verbalized or introduced														
'Minus' as subtraction sign			*					*				*	*	*
Subtraction		*		*				*		*				*
'Take away' for subtraction					*		*		*		*			
Written Expression including														
Positive number			*			*							*	
Subtraction			*			*								
Number line notations														
Dot-hops-dot	*	*		*	*	*	*	*		*	*		*	*
Left or right arrow			*					*						
Numbered steps taken												*		
Dots along number line									*					
Actions with numerals														
Subtraction as adding a negative number												*		
Action with number line														
Go down number line	*	*	*	*	*	*	*	*	*	*	*	*	*	*
Types of numbers generated by integer subtraction														
Smaller number											*			
Similarity of operations on the number line														
Subtraction		*		*				*			*			*

‘2 - 8’ initial interaction.

	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Participant voiced confusion									*					
Sets of numbers														
‘Whole number’							*							
Integer													*	
Sign of number read or introduced														
Positive number	*	*	*	*	*	*	*	*	*	*	*	*	*	*
Zero										*				
Negative number	*	*	*	*	*	*	*	*	*	*		*	*	
Role of sign verbalized or introduced														
'Plus' as addition sign	*													
'Minus' as subtraction sign	*	*	*		*	*	*	*		*		*	*	*
Subtraction		*	*	*						*			*	
'Take away' for subtraction			*				*		*		*			
Negative sign							*							
Written Expression including														
Positive number	*	*	*		*			*	*	*	*		*	
Zero										*				
Negative number/sign	*	*	*		*		*	*	*	*	*		*	*
Addition	*													
Subtraction	*	*	*		*			*	*	*	*		*	
Chip model notations														
Drawn			*						*					
Separate colors			*											
Number line notations														
Participant drawn		*		*									*	*
Spoke of number line												*		
Dot-hops-dot		*		*									*	*
Actions with numerals														
Portions of numbers with different signs can cancel to give sum or difference										*				
Subtraction as adding a negative number	*													
Actions with the chip model														
Adding as joining			*						*					
Opposite colored chips stacked			*						*					
Action with number line														
Go down number line		*		*									*	*
Results of subtraction														
Larger number from a smaller number is not proper		*												

'2 - 8' chip model.

	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Participant voiced confusion		*		*										
Sets of numbers														
'Negative integer'														*
Sign of number read or introduced														
Positive number	*	*	*	*	*		*	*	*	*	*	*		*
Negative number	*		*	*	*		*	*	*	*	*	*		*
Role of sign verbalized or introduced														
Addition										*				*
'Plus' as addition sign														*
'Minus' as subtraction sign		*		*			*	*						*
Subtraction	*				*					*		*		
'Take away' for subtraction	*						*		*	*	*			
Negative sign				*										
Chip model notations														
One color		*					*							
Separate colors				*	*			*	*	*	*	*	*	*
Mixed colors	*		*			*								
Number line notations														
Spoke of number line				*										
Chips as colored number line								*					*	
Actions with numerals														
Portions of numbers with different signs can cancel to give sum or difference				*										
Subtraction as adding a negative number				*								*		*
Actions with the chip model														
Adding as joining	*		*	*	*				*	*	*	*		*
Opposite colored chips stacked			*	*	*				*	*	*	*		*
Subtraction as taking away	*													
Opposite colors chips swapped						*								
Action with number line														
Go down number line								*					*	
Operation and signifiers both need to be considered														
Subtracting a positive number can be treated as adding a negative number														*
Results of subtraction														
Larger number from a smaller number is not proper	*						*							
Similarity of operations on the number line														
Subtraction								*						

'2 - 8' number line

	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Participant voiced confusion														
Sets of numbers														
'Positive integer'													*	
Sign of number read or introduced														
Positive number	*	*	*	*	*	*	*	*	*	*		*	*	*
Negative number	*	*	*	*	*	*	*		*	*	*	*	*	
Role of sign verbalized or introduced														
'Minus' as subtraction sign			*					*		*		*	*	*
Subtraction	*	*		*					*					
'Take away' for subtraction					*		*	*			*			
Written Expression including														
Positive number			*			*								
Negative number/sign			*											
Subtraction			*			*								
Number line notations														
Dot-hops-dot	*	*		*	*	*	*	*		*	*		*	*
Left or right arrow			*											
Numbered steps taken												*		
Dots along number line									*					
Action with number line														
Go up number line						*								
Go down number line	*	*	*	*	*		*	*	*	*	*	*	*	*
Similarity of operations on the number line														
Subtraction	*	*		*				*					*	

'-4 – 2' initial interaction.

	Amanda	Ashley	Ashley	Bailey	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Participant voiced confusion		*	*							*						
Sets of numbers																
'Positive integer'		*														
'Negative integer'		*														
Sign of number read or introduced																
Positive number	*	*		*	*	*	*	*	*	*	*	*	*	*	*	
Negative number	*	*	*	*		*	*	*	*	*	*	*	*	*	*	
Role of sign verbalized or introduced																
'Plus' as addition sign	*								*			*		*		
'Minus' as subtraction sign	*	*		*	*		*	*	*	*	*	*		*	*	
Subtraction		*		*			*				*			*		
Written Expression including																
Positive number		*	*	*			*			*	*		*			
Negative number/sign	*	*	*	*	*		*		*	*	*	*	*		*	*
Addition	*		*						*	*						
Subtraction		*	*	*			*			*	*		*			
Chip model notations																
Drawn					*											
One color		*														
Separate colors					*											
Number line notations																
Participant drawn															*	*
Spoke of number line			*			*										
Dot-hops-dot															*	*
Actions with numerals																
Subtraction as adding a negative number									*	*		*		*		
Actions with the chip model																
Adding as joining					*											
Opposite colors chips swapped					*											
Action with number line																
Go up number line						*										
Go down number line															*	*
Operation and signifiers both need to be considered																
Subtracting a positive number can be treated as adding a negative number									*	*						

'-4 – 2' chip model.

	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Participant voiced confusion		*					*	*			*			
Sign of number read or introduced														
Positive number	*	*	*		*	*	*	*	*	*	*	*	*	
Negative number	*	*	*	*	*	*	*	*	*	*	*	*	*	*
Role of sign verbalized or introduced														
Addition	*		*	*	*		*		*	*			*	
'Minus' as subtraction sign	*	*	*				*	*		*	*	*	*	*
Subtraction					*				*	*		*		
Chip model notations														
One color	*			*					*	*	*		*	*
Separate colors		*			*			*				*		
Mixed colors			*			*	*							
Number line notations														
Chips as colored number line								*						
Actions with numerals														
Subtraction as adding a negative number	*			*	*		*		*	*			*	*
Actions with the chip model														
Adding as joining	*		*	*	*	*	*		*	*		*	*	*
Opposite colored chips stacked			*											
Opposite colors chips swapped					*							*		
Operation and signifiers both need to be considered														
Subtracting a positive number can be treated as adding a negative number	*								*	*			*	

'-4 – 2' number line.

	Amanda	Ashley	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Participant voiced confusion				*					*			*			
Sets of numbers															
'Positive integer'			*											*	
'Negative integer'			*												
Sign of number read or introduced															
Positive number	*	*		*	*	*		*	*	*	*	*	*	*	*
Negative number	*	*		*	*	*		*	*	*	*	*	*	*	*
Role of sign verbalized or introduced															
Addition			*						*						
'Plus' as addition sign									*						
'Minus' as subtraction sign				*		*		*	*		*		*	*	*
Subtraction	*	*	*			*				*	*		*		
'Take away' for subtraction								*	*			*			
Written Expression including															
Positive number				*			*		*						
Negative number/sign				*			*		*						
Addition									*						
Subtraction				*			*		*						
Number line notations															
Dot-hops-dot	*	*			*	*	*	*	*		*			*	*
Left or right arrow				*											
Numbered steps taken													*		
Dots along number line										*					
Action with number line															
Go down number line	*	*		*	*	*	*	*	*	*	*		*	*	*
Operation and signifiers both need to be considered															
Symbols provide inconsistent meanings			*												
Similarity of operations on the number line															
Subtraction		*				*				*			*	*	

'-1 + -5' initial interaction.

	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Participant voiced confusion														
Sets of numbers														
'Negative integer'		*												
Sign of number read or introduced														
Positive number				*					*			*		
Negative number	*	*	*	*	*	*	*	*	*	*	*	*	*	
Role of sign verbalized or introduced														
Addition	*	*	*					*				*	*	
'Plus' as addition sign		*		*	*	*	*	*	*	*	*	*	*	
'Minus' as subtraction sign				*									*	
Subtraction													*	
Written Expression including														
Positive number				*									*	
Negative number/sign	*	*	*	*	*		*	*	*	*	*		*	*
Addition	*	*	*	*	*			*	*		*			
Subtraction				*									*	
Chip model notations														
Drawn			*											
One color		*	*											*
Number line notations														
Participant drawn													*	
Dot-hops-dot													*	
Action with number line														
Go down number line													*	

'-1 + -5' chip model.

	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Participant voiced confusion														
Sign of number read or introduced														
Positive number						*							*	
Negative number	*	*	*	*	*		*	*	*	*		*	*	*
Role of sign verbalized or introduced														
Addition	*	*		*				*	*	*		*	*	*
'Plus' as addition sign			*		*		*	*		*		*	*	
'Minus' as subtraction sign													*	
Chip model notations														
One color	*	*	*	*	*				*	*	*	*	*	*
Separate colors								*						
Mixed colors						*	*							
Number line notations														
Spoke of number line								*						
Chips as colored number line								*						
Actions with the chip model														
Adding as joining	*	*		*	*	*	*		*	*	*	*	*	*
Action with number line														
Go down number line								*						

'-1 + -5' number line.

	Amanda	Ashley	Bailey	Brooke	Christina	Courtney	Jacqueline	Jamie	Jordan	Kayla	Monique	Nicole	Rebecca	Tae
Participant voiced confusion														
Sets of numbers														
'Negative integer'													*	
Sign of number read or introduced														
Positive number			*										*	
Negative number	*	*	*	*	*	*	*	*	*	*	*	*	*	*
Role of sign verbalized or introduced														
Addition	*	*		*	*			*	*	*	*	*	*	
'Plus' as addition sign		*	*		*		*			*		*	*	
'Minus' as subtraction sign								*					*	
Subtraction	*			*								*		
Negative for subtraction sign												*		
Written Expression including														
Positive number				*										
Negative number/sign			*	*		*		*				*	*	
Addition			*	*		*						*		
Subtraction				*		*						*		
Number line notations														
Dot-hops-dot	*	*		*	*	*	*	*		*	*		*	*
Numbered steps taken												*		
Dots along number line									*					
Action with number line														
Go down number line	*	*	*	*	*	*	*	*	*	*	*	*	*	*
Operation and signifiers both need to be considered														
Adding a negative number and a positive number can be treated as subtraction	*													
Results of subtraction														
Negative number cannot be done														
Similarity of operations on the number line														
Addition		*							*	*				
Subtraction													*	

Appendix R: Participant Interview Transcripts Arranged by Interview Date

Interview Participant Pseudonym: Amanda
Date of Interview: 11/7/17
Duration of Interview: about 16 Minutes

(Doug Prompt 1) "Let me re-ask you that first question now that I have everything up and running. If a seventh-grade student were to ask you "what is the difference between whole numbers and positive integers?", how would you respond? I think you had just said they are the same thing."

(Amanda Turn 01 Prompt 1) "Yeah, a positive integer's anything like, on a number line, it's anything after zero and a whole number is the same thing, it's anything greater than zero. I think, so I think there the same thing."

Act: Moving her hands from together at a central position and moving her right hand out, while referencing both positive integers and whole numbers

(Doug Prompt 2) "Ok, it is a familiar notion that addition makes larger. This is true when non-zero whole numbers are added together. Is this also true when integers are added? And please provide some examples."

(Amanda Turn 02 Prompt 2) "No, so if you do negative ten plus three, you get negative seven. So, it isn't so, it doesn't make it greater because your actually, if you do this"

Act: Picking up the pen

(Amanda Turn 02 Prompt 2) "What did I say, three right? [yes] you actually subtract it, so you end, because the ten is negative and it is greater you get negative seven, so you are actually subtracting, so even though there is an addition sign, it's not always, you do not always get a bigger number."

Act: Writing the expression and solution -10 + 3 = -7

(Doug Prompt 2) "Ok, can you think of any other examples where"

(Amanda Turn 03 Prompt 2) "If you do. I am trying to think. If you do this you get zero, it you do negative two plus positive two, you get zero, so that's not greater. What else. If you do two negatives you do get greater, and if you do two positives you do get greater, but if you do, if you have a negative and a positive then you don't."

Act: Writing the expression and solution -2 + 2 = 0

(Doug Pr3) "Ok, what types of numbers can you get when one integer is subtracted from another integer?"

(Amanda Turn 04 Prompt 3) "What type of numbers [yup]. Well you can get subtracted right? [yes] [so you have ten minus three equals seven] and if you do a negative, if you just do negative ten minus three you have to change that to addition, and you get negative thirteen. [ok] If you did, if you did ten minus negative three then that cancel and then that becomes positive thirteen. That's an x [ok] I think that's it. So, you can get a negative or a positive number."

Act: Writing these three expressions, 10 – 3 = 7, -10 – 3 changed to -10 + -3 = -13,10 - -3 = 13 changed to 10 + 3 = 13

(Doug Prompt 4) "What roles can this symbol play in a mathematics problem?"

Act: Placing "-" paper on the table.

(Amanda Turn 05 Prompt 4) "It can be a subtraction sign, or it can be, it can be in front of a number and it can make the number negative. So, it can be a negative integer. Either or."

(Doug) "So far so good. This is the first multipart question." [ok]

(Doug Prompt 5) "The numbers negative seven and seven are called additive inverses of each other. What does the Prompt additive inverse mean?"

Act: Placing "-7, 7" paper on the table.

(Amanda Turn 06 Prompt 5) "Well if you add, if you add them together you get zero. So, I guess from zero they are both seven away, but they are opposite of each other cause this one's a negative, if you want I can do this on a number line, and this was zero they are both seven away, but one's negative and one's positive. So, they are on different sides of a number line. And if you add them together you get zero."

Act: Moving the tip of the pen about the same distance left and right of the comma to identify distance from zero. Drawing a number line and marking -7, 0, and 7. Spreading her arms wide to show different directions.

(Doug Prompt 5A) "I think you answered the next question, but why are seven and negative seven additive inverses?"

(Amanda Turn 07 Prompt 5A) "Like I said cause they're both seven away they are both the same distance away from zero so if you add them together you get zero on the number line."

(Doug Prompt 5B) "How else can you describe two numbers that are additive inverses?"

(Amanda Turn 08 Prompt 5B) "I guess that they're opposites of each other [ok] does that work."

(Doug Prompt 5C) "I can't tell you yes or no, at this point. Can you give me another pair of additive inverses?"

(Amanda Turn 09 Prompt 5C) "Negative two and two. Negative four and four."

(Doug) "Ok, anything you want to add to anything we have looked at so far?"

(Amanda) "I don't think so."

(Doug Prompt 6) "Ok, so the next question, actually the next two questions both have multiple parts as well and you are going to be asked to look at four different mathematical expressions and describe how they work in a couple different ways. So, we have, I would like you to consider, these four expressions. And you can take all the scrap paper you'd like."

Act: Placing '6 + 2', '10 – 2', '6 – 2' and '-1 + 9' papers on the table.

(Amanda) "I will just take a piece"

(Doug Prompt 6A) "And can you show and explain how you would evaluate each expression."

(Amanda Prompt 6A '6 + 2') "For six plus two you would just do six it is just basic, so it would just be eight."

Act: Wrote the expression "6 + 2 = 8" down and simplified it.

(Amanda Turn 11 Prompt 6A '10 – 2') "And then ten minus two, same thing."

Act: Wrote the expression "10 – 2 = 8" down and simplified it

(Amanda Turn 12 Prompt 6A '-1 + 9') "But then this one negative one minus {nine}. Since it has a negative in it you have to subtract so it would actually be nine minus one equals eight, but since the one is negative, it is the smaller number so is stays, it is still a positive number, so it still would be positive eight."
Act: Pointing to '-1 + 9' then writing it down. Then writing the 9 -1 vertical and recording that it is eight, then points to -1 and 9

(Amanda Turn 13 Prompt 6A '6 – -2') "And if you had six minus negative two these cancel, this becomes a plus and this cancels so this also becomes a positive eight."
Act: Wrote 6 – (-2) then putting verticals to make the minus and negative add and positive.

(Doug Prompt 6B) "In what ways are these expressions similar and it what ways are they different?"

(Amanda Turn 14Prompt 6B) "They are similar because they all have, they all come up with the same answer so they all equal positive eight. But they're different, well these two are both addition, but it's different because this one ('6 + 2') is just a regular addition it is just basic, but this ('-1 + 9') is actually you have to do subtraction to find the answer."
Act: Pointing to '6 + 2' and '-1 + 9'

(Amanda Turn 14 Prompt 6B) "And the same thing with this these are both subtraction but this ('10 - 2') is just basic subtraction but this ('6 - -2') one you have to actually do addition in order to find the answer cause of the negative number."
Act: Pointing to '6- -2' and '10 - 2'

(Doug Prompt 6C) "And different manipulative, or different models are often used to help students understand mathematics concepts. One of the key ones used in integers is called the chip model, and would you be willing to try to show me the chip model [sure] Ok, go ahead when you're ready."

(Amanda Turn 15 Prompt 6C '6 + 2') "For each of these? [yup for each of those four] alright, so if you had six, we are doing six plus two, so you have six right there, and you have two, add them together and you have eight."
Act: Sorted out six brown and two beige chips in two groups, then slid them together to make a mixed color group of eight

(Amanda Turn 16 Prompt 6C '10 – 2') "And then if you were doing ten minus two, so [hold on a minute just one second] one, two, three, four, five, six, seven, eight, nine, ten. One, two. Oh, no, that would be ten take away two and you have eight."
Act: Sorted out ten brown chips and two beige chips. Then as a correction go rid of the two beige colored chips and removed two brown chips to show the subtraction.

(Amanda Turn 17 Prompt 6C '6 – -2') "Alright and if you did, six, so if they, ah, so if it is six minus negative two, but you really, you can't take away negative two from here, so you end up adding two, so you have eight."
Act: Sorted out six brown chips then slid in two more brown chips

(Amanda Turn 18 Prompt 6C '-1 + 9') "And one, two, three, four, five, six, and you have negative one plus nine, so you have nine, and because it is a negative one, take away, you are taking away one. And you are left with eight."

Act: Counting out nine brown chips then slid away one of the brown chips.

(Doug Prompt 6D) "And would you be willing to show me how you might approach these using the number line." [sure]

(Amanda Turn 19 Prompt 6D '6 + 2') "Ok, so we will do six plus two first. So, if you have six, and then you go one, two, then you get eight."

Act: Pointing at 6 then placing a dot, then doing two hops to the right and placing a dot at 8 this was in blue

(Amanda Turn 20 Prompt 6D '10 – 2') "If you have, if you're at ten, and you have to go back two, you go one, two and you're at eight."

Act: Placed a red dot at ten, making two hops to the left and placing a dot at 8

(Amanda Turn 21 Prompt 6D '6 – -2') "And then, six minus, so you're at six which is here. So, if you do six minus negative two, you are really adding two once you do, once you do the cancelation so there the same thing, so you do seven, eight."

Act: Placing a green dot at 6 making two hops to the right and placing a dot at 8

(Amanda Turn 22 Prompt 6D '-1 + 9') "Then if you're at negative one and you do plus nine, you go to one, two, three, four, five, six, seven, eight, nine, and you get to eight, and you find your answers eight."

Act: Placing a black dot at -1 then making 9 hops to the right and a dot at 8

(Doug Prompt 7) "And we're going to do the same process with these four expressions."

(Amanda Prompt 7) "On the number line or are we going to do all of it."

(Doug Prompt 7) "We are going to, you can set the number line aside. I will have you work through symbolically, initially first,"

(Amanda Turn 23 Prompt 7A '2 – 8') "Two minus eight equals, so to me it is like you are doing plus a negative, so it becomes negative six."

Act: Wrote 2 – 8, then converted it to 2 + -8 and found -6.

(Amanda Turn 24 Prompt 7A '-7 + 1') "Negative seven plus one, really it is just seven minus one, we just, because the seven's bigger it is a negative."

Act: Wrote -7 + 1 horizontally then 7 – 1 vertically and found 6 then attached the sign.

(Amanda Turn 25 Prompt 7A '-4 – 2') "Negative four plus, oh wait, so if it is minus two then you are really going to do plus a negative two so that's a negative six."

Act: Wrote -4 + -2 directly and found -6

(Amanda Turn 26 Prompt 7A '-1 + -5') "So, if you have two negatives you just add them together, so you have negative six. So, they all are negative six."

Act: Wrote -1 + -5 and found = -6

(Doug Prompt 7B) "In what ways are these expressions similar, and in what ways are they different?"

(Amanda Turn 27 Prompt 7B) "Well they all got the same answer negative six, but they're all different because, like this one ('-1+ -5') is just normal addition because they're both, they have the same symbol. So, they're both negative. That's how you get negative six, but then for this one ('-4 – 2') you have to do, you have to change the subtraction to addition and then make this a negative two to get negative six. And for this one ('-7 + 1') you actually have to subtract, because the negative seven is bigger than the one. So, you subtract to get negative six. And for this one ('2 – 8'), you can't subtract eight from, so you get negative six because it not a proper subtraction."

Act: Pointing to the printed '-1 + -5' then at the written -6. Then she shifted to '-4 – 2' followed by '-7 + 1'. This was followed by pointing out '2 – 8'.

(Doug Prompt 7C) "Ok, as before, I would like you to use the chip model if you feel comfortable doing so."

(Amanda Turn 28 Prompt 7C '2 – 8') "Ok, so if you have two you can't subtract eight from two, so you have to give, technically, so these are negative, technically you have to take away negative six. But they don't really exist, when you get, negative six right?"

Act: Sorted out two brown chips, then selected six beige chips grouped all eight chips together then slid the six beige chips to the side.

(Amanda Turn 29 Prompt 7C '-7 + 1') "And if you are doing negative seven plus one. One, two, three, four, five, six, seven. So, if you have negative seven and you are adding one, you are really subtracting one so there you get negative six right there. You just take one away."

Act: Counting out 7 brown chips, then slid 1 brown chip away to leave six brown chips in the center.

(Amanda Turn 30 Prompt 7C '-4 – 2') "And if you have negative four minus two, because it is a negative you actually add a negative two, so if these were all negatives and you add two you get negative six."

Act: Sorted four brown chips then slid two more brown chips to join the other four.

(Amanda Turn 31 Prompt 7C '-1 + -5') "And for this, if you have a negative one and you add five negatives, you get negative six."

Act: Started with 1 brown chip and counted out 5 more brown chips, which she slid to join the original.

(Doug Prompt 7D) "Ok, next, can you display each of these using the number line?"

(Amanda Turn 32 Prompt 7D '2 – 8') "Ok, so if you have two and you are subtracting, so you are going backwards eight, you are going to go, one, two, three, four, five, six, seven, eight. And you get to negative six."

Act: Placing a blue dot at 2 then making 8 hops to the left and placing a blue dot at -6

(Amanda Turn 33 Prompt 7D '-7 + 1') "If you have negative seven and you add one on the number line, you get to negative six."

Act: Placing a red dot at -7 then making a hop to the right and placing a red dot at -6

(Amanda Turn 34 Prompt 7D '-4 – 2') "If you have negative four and you subtract two which means you go this way, you would get to negative six too."

Act: Pointing left. Placing a green dot at -4 the pointed to the left, then drew to hops to the left and a dot at -6

(Amanda Turn 35 Prompt 7D '-1 + -5') "And then, you're at negative one, and you are adding negative five which means you are subtracting, so you got this way one, two, three, four, five. And you get to a negative six."
Act: Placed a black dot at -1 then drew 5 hops to the left and placed a dot at -6.

Interview Participant Pseudonym: Ashley
Date of Interview: November 3rd, 2017
Duration of Interview: 48 minutes
(Doug Prompt 1) "If a seventh-grade student were to ask you, what is the difference between whole numbers and positive integers, how would you respond?"
(Ashley Turn 01 Prompt 1) "Whole numbers and positive integers I'd probably say something like whole numbers are encompass negative and positive numbers, integers, so I'd say something like a whole number can be a positive number or a negative number. I would show them like a number line and then I would say that all of these numbers on here are whole numbers and they're not a part of. They're not a fraction or a decimal, they're a number, or, or, I'm trying to think about how I would Prompt it."
(Doug Prompt 1) "Ok, take a moment and"
(Ashley Turn 01 Prompt 1) "Like I said before, I show them, I'd show them a number line and I'd say umm numbers on the right side of the zero are positive integers, and a number on the left side of the zero would be negative integers and then all together, they're all whole numbers and then there are numbers in between whole numbers that aren't whole numbers."
Act: Pointing to the left and right of a common center point as if on a number line
(Doug Prompt 1) "So we have whole numbers are everywhere, [yes] positive and negative integers are split by zero, and there are numbers that fall between the whole numbers and integers [yes] Anything else you want to add to that question? [no]"
(Doug Prompt 2) "Ok, it is a familiar notion that addition makes larger, [um huh] this is true for non-zero whole numbers, when non-zero whole numbers are added together, is this also true when integers are added? And please provide some examples. Addition makes larger is this true when integers are added?"
(Ashley Turn 02 Prompt 2) "Not all the time because when you add two negative integers you get a number that's smaller than what, then both of them, those numbers, so like if you were to do negative seven plus negative one, that would be negative eight which is something which is smaller than what you started with negative seven, so not addition is not always associated with getting a larger number."
Act: Wrote the expression -7 + -1 and solved it as -8
(Doug Prompt 2) "Ok umm, can you provide me with a second example"
(Ashley Turn 02 Prompt 2) "Second example, of an example of when it is when the number is smaller?"
(Doug Prompt 2) "Yes, an example that would contrast with addition would make larger."
(Ashley Turn 02 Prompt 2) "I guess negative six plus negative six negative twelve."
Act: Wrote the expression -6 + -6 and solved it as -12.

(Doug Prompt 3) "What types of numbers can you get when one integer is subtracted from another integer? [Wait can you say that again] what types of numbers can you get when one integer is subtracted from another integer?"

(Ashely Turn 03 Prompt 3) "When one integer is subtracted from another integer [yes, so] you can get a number that is bigger than what you started with, or a number that is smaller than you started with or what I'm, the number that. I keep saying the number that you started with I'm like referring to these [ok] I guess, the number that you are adding or the number that you are subtracting in this case."

Act: Circling the first numbers in the addition problems from earlier.

(Doug Prompt 4) "Ok, so far so good. What roles can this symbol take within a mathematics problem? What roles can that symbol take within a mathematics problem?

Act: Placing ' - ' on the table.

(Ashley Turn 04 Prompt 4) "Umm, well you can this is take away you would use this to refer to taking away something, so if you had the number eight, like eight of these things, like little coin things, and then you would want to portray take away two, you would take away two, so that is how you would umm portray that."

Act: Points to the symbols and the chips.

(Doug Prompt 4) "Ok, any other roles you can think of for that?"

(Ashley Turn 04 Prompt 4) "Umm, not at the moment [ok]"

(Doug Prompt 5) "The numbers negative seven and seven are called additive inverses of each other, what does the Prompt additive inverse mean?"

Act: Placing the -7, 7 cards on the table.

(Ashley Turn 05 Prompt 5) "Added inverse?"

(Doug Prompt 5) "Additive inverse"

(Ashley Turn 05 Prompt 5) "Additive inverse, so if you were to add these two together, you would get the number zero."

(Doug Prompt 5A) "Ok, why are they additive inverses?"

(Ashley Turn 06 Prompt 5A) "They are added inverses because I mean if you were to look at it on a number line, if you found where negative seven was on a number line and you added seven to it, you'd get zero."

(Doug Prompt 5B) "Ok, how else would you describe two numbers that are additive inverses?"

(Ashley Turn 07 Prompt 5B) "How else would I describe two numbers that are additive inverses? [yes] I'm not sure"

(Doug Prompt 5C) "Ok, that's fine, ok, can you give me another pair of additive inverses?"

(Ashley Turn 08 Prompt 5C) "Sure"

Act: Writing the numbers -6 and 6.

(Doug Prompt 5C) "Ok, so you have negative six and six"

(Doug Prompt 5A) "Ok and why are they additive inverses?"

(Ashley Turn 09 Prompt 5A) "Because if you added six to negative six they would be zero or they would kind of, I don't want to say cancel each other out, I mean they do yeah, they just equal to zero. [ok] if you added them."

(Doug Prompt 6) Ok, so questions six and seven, are structured similarly and in each case, I am going to present you with four expressions to consider, and we are going to look at how you might display solving them in a couple different ways, so here we have the numbers, here we have these, four operations, four expressions. How would you, show and explain how you would evaluate such expressions."

(Ashley Turn 10 Prompt 6) "How I would evaluate [yup] these expressions. I would probably use the number line for all of them. Just because it, would be more s, it would be more clear. So, I would actually have to do it, [well yes]"

(Doug Prompt 6) "We'll take a look at the number line in a moment but for now can you work them through just symbolically?"

(Ashley Prompt 6) "What do you mean by symbolically?"

(Doug Prompt 6) "Just with the numbers as they're written."

(Ashley Turn 11 Prompt 6A '10 – 2') "So, I'd say something without using the number line, I would probably say something like ten minus two, I am thinking about what learn in mthm 201 about how to explain ten minus two without using just mental math ok, so you'd have just lets same if I were to have ten ones and then two ones you'd be subtracting them taking, so you'd take away two of these ten ones to get eight ones."

<u>Act:</u> Wrote the expression 10 - 2 and then the line of ten circles and a pair of circles, then crossed off two of the ten. Circled the eight that remain.

(Doug Prompt 6A '-1 + 9') "Ok, how would you approach negative one plus nine"

(Ashley Turn 12 Prompt 6A '-1 + 9') "Negative one plus nine. I'd probably see what I was taught in was to change the negative and the one to just minus, So, I mean you can just do it like what I did here, nine ones and then take away one so it would be eight yeah, so I'd probably change the negative sign, I'd probably yeah, yeah."

<u>Act:</u> Wrote the expression as -1 + 9 then 9 – 1. Then made a line of nine circle and crossed one out.

(Doug Prompt 6A '6 – -2') "Ok, how about the next one?"

(Ashley Turn 13 Prompt 6A '6 – -2') "The next one. I was taught to just I don't know if this would be even but like in school I was taught to just make this a plus sign, so I guess, six ones and two ones and I would just add them together to make eight ones. Yeah"

<u>Act:</u> Miswrote an expression then crossed it out, then wrote 6 - -2 changed - - into a +. Then drew 6 circles and 2 circles then the arrow to join them.

(Doug Prompt 6A '6 + 2') "Ok and the next one."

(Ashley Turn 14 Prompt 6A '6 + 2') "Six plus two I would draw six ones and two ones and then add them together to make eight ones."

<u>Act:</u> Drew a line of 6 circles and then 2 circles then a set of 8 circles

(Doug Prompt 6A) "Ok, what would be"

(Ashley Turn 15 Prompt 6A) "Sorry you see like hum, I kind of made these two the same I am trying to think of a better way to explain six minus negative two without using the number line.

Act: Pointing at '6 – -2' and '6 + 2'

(Doug) "Once the interview is over I can help you work through that [ok]"

(Ashley Prompt 6A) "Because I was just taught in school change the minus sign subtraction sign and the negative sign and just turn them into an addition sign."

(Doug Prompt 6B) "In what ways are these expressions similar and in what ways are they different?"

(Ashley Turn 16 Prompt 6B) "I mean essentially with the with this one ('-1 + 9') you are taking away one from the you are taking away one ones from nine ones, so this is essentially subtraction because you are taking away one from nine. This ('10 – 2") is subtraction, you are taking away two ones from ten ones and this ('6 – -2') is addition so you'd add two ones to six ones, and this ('6 + 2') is also addition so you add two ones to the six ones again. So, they're similar, these two ('6 - -2' and '6 + 2') cause you're addition two to six ones and these two ('10 – 2' and '-1 + 9') are similar cause they are both subtraction. I mean, you're adding negative one ('-1 + 9') it would just be different phrasing you, you're, because you don't have one to begin with, ok, sorry."

Act: Pointing to '-1 + 9'. Pointing to '10 – 2'. Pointing to '6 – -2'. Pointing to '6 + 2'. Pointing to '6 – -2' and '6 + 2'. Pointing to '10 – 2' and '-1 + 9'.

(Doug) "That's alright."

(Ashley) "I am just rambling, I am rambling my thoughts, I don't know if it's helping but."

(Doug) "I think that is a, the more you talk the better, the more you express your thinking the better cause I can't read minds."

(Doug Prompt 6C) "The chip model is a model that is often used with integers, are you willing to try to display how you would solve, model solving these with the chip model? [Sure] How would you show each of the four problems using the chip model?"

(Ashley Turn 17 Prompt 6C '10 – 2') "Ok, so let's say I guess I will split them up and say that the brown chips would umm, represent positive integers and the white chips represent negative integers. So, I would for ten minus two I would take one, two, three, four, five, six, seven, eight, nine, ten. Do I have ten one, two, three, four, five, six, seven, eight, nine, ten and then I would umm since we are doing subtraction I would take away two and then I am left with eight. Positive integers."

Act: Sorted the chips by color. Counted out 10 brown chips and made a blob took 2 away from the blob of 10.

(Ashley Turn 18 Prompt 6C '-1 + 9') "For negative one plus nine, I would use the white chip for the negative one and because, like I said before the white chips would represent negative integers and the brown chips would represent positive integers, and so I'd do one, two, three, four, five, six, seven, eight, nine, and then I would, maybe I'd say that one of these, means you are taking away, so removing one of these would equal this. I mean, I know I'd be left with eight, but I don't know, I mean, I'm not sure how I would go about explaining how this white chip kind of represents me taking away one of these brown chips. Maybe I can say ok yeah."

Act: Selected a white chip, recounted the brown chips and added 1 to get up to 9. Paired up a brown with the beige and removed both.

(Ashley Turn 19 Prompt 6C '6 – -2') "And then I mean I'll move on, if I think of anything better I'll come back to it [sounds good] of for the next question, I would do, six positive integers one, two, three, four, five, six, wait, one, two, three, four, five, six, and then two negative integers and then I would oh man, so my explanation for this one ('-1 + 9') didn't even work out because I can't even say that I would take away. I would probably have to think of it some other way to explain oh [keep in mind the operation between the two problems] true, I missed that umm, true so then I can say something like if I am subtracting these I would do the opposite and add two positive integers, so I would be doing the opposite of what it is asking umm, I mean let me just to the last one real quick and I will figure it out."

Act: Counted out 6 brown chips and 2 beige chips swapped to brown chips in for the two beige chips

(Ashley Turn 20 Prompt 6C '6 + 2') "One, two, three, four, five, six, the last one is easy because that is easy to explain, I would just add two of the brown chips to get eight brown chips or to eight positive integers. But then the middle two ('-1 + 9' and '6 – -2") are hard to explain."

Act: Counting out six brown chips and 2 brown chips put the two sets of brown chips together.

(Doug Prompt 6D) "Ok, let's move from those on to representing with the number line which sounded like the one you would have wanted to do in the first place, [I yeah, yup] So on the white board we have the number line yup you can pull it towards you, don't worry nothings fragile. Whenever you are ready, one thing I do ask it that you try not to mark right on the white stickers."

(Ashley Turn 21 Prompt 6D '10 – 2') "Ok, so if we have ten make a little dot here to represent that I have ten and then in order to subtract two I just umm, move to the left two times. To get eight. Because a subtraction sign tells me that I am going backwards so that's what's telling me to go left twice."

Act: Making a blue dot at 10 made 2 hops to the left made a dot at 8.

(Ashley Turn 22 Prompt 6D '-1 + 9') "With ok, negative one plus nine, so I start with negative one, I find negative one on the number line and then I move to the right nine times cause the addition symbol is telling me to move to the right. So, one, two, three, four, five, six, seven, eight, nine. So, it would be I ended up at eight on the number line. And then positive eight, and then."

Act: Made a blue dot at -1 made 9 hops to the right and a dot at 8.

(Ashley Turn 23 Prompt 6D '6 – -2') "Six minus negative two so I start with six minus negative two, so the minus would tell me to go left but then the negative two would kind of cancel that out so then I would end up going right instead of left because if I subtract negative two I would actually I'll go back to that."

Act: Made a blue dot at 6 made 2 hops to the right and a dot at 8.

(Ashley Turn 24 Prompt 6D '6 + 2' "Six plus two the addition symbol is telling me to go right, move two places right on the number line one two and I would get eight."

Act: Made a blue dot at 6 made 2 hops to the right and a dot at 8

(Ashley Turn 25 Prompt 6D '6 – -2') "And going back to six minus negative two, I mean I kind of know what I am trying to say, in that the subtraction symbol is telling me to go left but since I'm subtracting a negative integer it cancels it out, so I am essentially adding two to six. I'm turning it into I mean you can't just turn it into a positive integer. Sorry I am sort of just going through the thoughts in my head and thinking about how to adequately explain I mean if it comes to me I'll say something later on.

(Doug) "Yeah, I don't mean to cut you off, but I don't want you to get bogged down in that problem either."

(Ashley) "Sorry if I get stuck on something I tend to keep trying."

(Doug) "In a few minutes I'll get you unstuck and we'll get you much more comfortable with these then some of them are making you at this point."

(Doug Prompt 7A) "Ok, so we have these four expressions, and we want to evaluate them symbolically, then with the chip model, and the number line again, so we will go through one piece at a time. So, using the numbers as they're written, how would you model the solution to each of those four expressions?"

Act: Dealing out '-1 + -5', '-4 – 2', '-7 + 1' and '2 – 8'

(Ashley Turn 26 Prompt 7A '-1 + -5') "Ok, symbolically I would negative one plus negative five. I have one negative integer and then I would be adding five negative integers. One, two, three, four, five. Oh. essentially drew the chip model on paper, I just realized. I mean, one, two, three, four, five, I kind of realized what I was doing. Well we would be left with negative six."

Act: Wrote the expression -1 + -5 then drew the 1 and 5 open circles then wrote the number -6.

(Ashley Turn 27 Prompt 7A '-4 – 2') "Negative four minus two. I mean I guess, I'll just, should I be doing mental math, or would I have to show it."

Act: Wrote the expression -4 -2.

(Doug) "Mental math is an acceptable option. Whatever you, however you would approach it."

(Ashley Turn 28 Prompt 7A '-4 – 2') "Ok, so four negative integers. One, two, three, four. And I would be subtracting two, and two, yeah, two is a positive integer. So, then I'd be hum, I'll go back to that [ok] Cause that one is hard to explain too.

Act: Added the 4 circles to the expression '-4 – 2'.

(Ashley Turn 29 Prompt 7A '-7 + 1') "Seven, negative seven plus one, I'd have seven negative integers, one, two, three, four, five, six, seven, and then I would be adding one positive integer. Which takes away one of the negative integers. And I would be left with one, two, three, six negative integers. I mean, yeah, no, what am I saying, I'd be left with negative six, which is a negative integer."

Act: Wrote the expression -7 + 1 and drew 7 circles then crossed off a circle and counted the remaining circles.

(Ashley Turn 30 Prompt 7A '2 – 8') "And then two minus eight I don't have two, I don't have enough to subtract two minus eight so then I'll. I'm gunna cheat. One, two, minus eight, one, wait, two, wait, one, two, three, four, five, six, seven, eight. So, I would be left with negative six. But I am not sure who I'd explain that on paper and without using a number line."

Act: Wrote the expression 2 – 8 and drew the 2 circles. Drew the number line, labeled it and added the hops.

(Doug Prompt 7A '-4 – 2') "Ok and what number do you think this one would give you."

Act: Pointing to '-4 -2'

(Ashley Turn 31 Prompt 7A '-4 – 2') "This should give me negative six. But one again, I am not sure how I would explain that on paper. Without using a number line."

Act: Wrote -6 and changed the initial expression -4 – 2 to -4 + -2.

(Doug) "Ok, we'll talk about that more [ok] in a few minutes."

(Doug Prompt 7B) "And in what ways are the expressions similar, and in what ways are they different?"

(Ashley Turn 32 Prompt 7B) "In what ways are they similar. I mean, negative one plus negative five adding the negative. These two ('-1 + -5' and '-4 – 2') are similar because you end up with the same answer, you end up with the same answer. And then these two ('-7 + 1' and '2 – 8') I mean obviously these two are similar because they're these two {"-1 + -5" and "-7 + 1"} have the addition symbol, and these two {"-4 – 2" and "2 – 8"} have the subtraction symbol."

Act: Pointing to '-1 + -5' and '-4 – 2'. Pointing to '-7 + 1' and '2 – 8'. Pointing to '-7 + 1' and '2 – 8'. Pointing to '-1 + -5' and '-7 + 1'. Pointing to "-4 – 2" and "2 – 8".

(Doug Prompt 7C) "Ok, alright so, using the chip model can you model some or all of them?"

(Ashley Turn 33 Prompt 7C '-1 + -5') "Sure oh, what did I say before these are the, these are the negative integers [yeah, yes the whites as negative] one, two, one, two, three, four, five. So, I am adding a one negative integer to five negative integers. Yeah, I am addition one negative to negative five. To get negative six."

Act: Sorting the chips by color, with white as negative. Set out a single beige chip and a group of 5 beige chips, slid the single chip to join the other 5 and repeated the process.

(Ashley Turn 34 Prompt 7C -4 – 2') "And then for the second one I'd do. I'd have, this represents a negative four because this. One represents negative one. Minus two. I'll go back to that one."

Act: Counted out 4 beige chips pulled out 2 brown chips then set them aside.

(Ashley Turn 35 Prompt 7C '-7 + 1') "So this represents negative seven adding one which is positive. So, I would, hum. So, I would cancel these out. To get negative six."

Act: Set out 7 beige colored chips and pulled out 1 brown chip, paired a beige and brown chip and moved them away.

(Ashley Turn 36 Prompt 7C '2 – 8') "And then, two, one, two, three, four, five, six, seven, eight. Two minus eight. I'd say. See I'm not sure how to explain these two {'-4 – 2' and '2 – 8'} using the chip model."

Act: Pulled out a group of 2 brown chips and a group of 8 brown chips and counted the second set.

(Doug Prompt 7D) "Ok, then why don't we start with those or shift over to the number line model."

(Ashley Turn 37 Prompt 7D '-1 + -5') "So then negative [you can start with any one you would like] oh, ok. Negative one, I find negative one on the number line and then I, the addition symbol tells me to go, right, but I am adding a negative number, so instead I go to the left. So, it is telling me to do the opposite. One, two, three, four, five. To get negative six."

Act: Made a blue dot at -1 made 5 hops to the left and made a dot at -6.

(Ashley Turn 38 Prompt 7D '-4 – 2') "I'd find negative four on the number line and it is telling me to subtract two, and the subtraction symbol would tell me to go to the left, and it is a positive number. I'll go back to that one again."

Act: Made a blue dot at -4 made 2 hops to the left and made a dot at -6.

(Ashley Turn 39 Prompt 7D '-7 + 1') "Negative seven plus one. So, it would tell me to go to the right one. So that would be negative six. I'd end up with negative six."

Act: Made a blue dot at -7 and made 1 hop to the right and made a dot at -6.

(Ashley Turn 40 Prompt 7D '2 – 8') "And then I find two on the number line and then I subtract eight. I would, yeah, I find two on the number line and then I subtract eight, and the subtraction symbol tells me to go left on the number line, so I'd go, one, two, three, four, five, six, seven, eight, and I am left with negative six."

Act: Made a blue dot at 2 then made 8 hops to the left and made a dot at -6.

(Ashley Turn 41 Prompt 7D) "I guess, I mean, what I am noticing, I guess, I mean, I'm trying to say. See I don't know what I am trying to say. I am trying to say that sometimes you would have to do the opposite of what the symbol is asking you to do depending on if, depending on if the number you are adding, or subtracting is a negative or a positive integer."

Act: Pointing at "-4 -2"

Interview Participant Pseudonym: Bailey

Date of Interview: November 21st, 2017

Duration of Interview: 41 minutes

(Doug Prompt 1) "If a seventh grader were to ask you "what is the difference between whole numbers and positive integers?' how would you respond?"

(Bailey Turn 01 Prompt 1) "Do you want me to just [yes]"

(Doug Prompt 1) "What is the difference between whole numbers and positive integers?"

(Bailey Turn 01 Prompt 1) "Well I kind of remember whole numbers as being different because positive numbers are, positive integers are like rational numbers I think, that word, irrational numbers I guess you could say."

(Doug Prompt 2) "Ok It, is a familiar notion that addition makes larger. This is true when non-zero whole numbers are added together. Is this also always true when integers are added? And please provide some examples. When whole numbers are added, addition makes larger, does addition of integers also always produce a larger number?"

(Bailey Turn 02 Prompt 2) "non-zero whole numbers, what?"

(Doug Prompt 2) "An example of that would be like three plus four [ok] would equal seven. Where seven is larger than either the three or the four. So just when in general when whole numbers are added together it makes a result that is larger than either of them started out to be. So, when integers are added do we end with a result that is larger than?"

(Bailey Turn 02 Prompt 2) "No, not always, sometimes there's negative numbers [ok, can you show me an example that would produce that] I think that's the answer."

Act: Wrote this expression '-1 +1=0'

(Doug Prompt 2) "Ok so, you have negative one plus one equals zero."

(Bailey Turn 02 Prompt 2) "Yeah, I think that's the answer. I don't really know."

(Doug Prompt 3) "Alright, what types of numbers can you get when one integer is subtracted from another integer?"

(Bailey Turn 03 Prompt 3) "When one integer [is subtracted from another integer]. What?"

(Doug Prompt 3) "What types of number can you get, can you get positive numbers, negative numbers, zero, and can you give me an example of any of those cases? Where that is possible. [sorry] what types of numbers can you get when one integer is subtracted from another integer? So, can you get positive number when you subtract two integers?"

(Bailey Turn 03 Prompt 3) "Yeah, you can get positive and negative numbers."

(Doug Prompt 3) "Ok, can you show me an example where the difference of a positive or two integers is a positive?"

(Bailey Turn 03 Prompt 3) "So this is negative, and this is positive."

Act: She wrote these two expressions '-1 + -1 = -2' and '-2 + 1 = 1'

(Doug Prompt 3) "Ok, so you have negative one plus negative one equals negative two, and negative two plus one is one."

(Doug) "Ok, anything you want to add to these three questions so far?"

(Bailey) "Umm, no."

(Doug Prompt 4) "Ok, when this symbol appears in a mathematics problem what roles does it convey? Or what roles can that symbol take within a mathematics problem."

Act: Placing ' - ' on the table.

(Bailey Turn 04 Prompt 4) "It could tell you to take away a certain number or it can tell you that a number is negative"

(Doug Prompt 5) "Ok, the numbers negative seven and seven are additive inverses of each other. What does the Prompt additive inverse mean?"

Act: Placing '-7, 7' on the table.

(Bailey Turn 05 Prompt 5) "Sorry, I haven't done negative numbers in such a long time."

(Doug) "I just want your natural reaction if you were being asked to teach it today, so just continue on."

(Bailey Turn 05 Prompt 5) "Ok, so if you, I'm going to take a guess, so their additive inverses of each other, that means that if you add these numbers together you'd get zero."

(Doug Prompt 5A) "Ok, I think you might have answered it but why are negative seven and seven additive inverses?"

(Bailey Turn 06 Prompt 5A) "Why [yes] because, I guess if you add them together you get a number, you get zero, their also opposite from each other on the number line so from zero to seven and from zero to negative seven.

Act: Used her fingers to simulate distances from zero on the number line.

(Doug Prompt 5B) "Ok, I think you might have beat me to the next question, how else would you describe two numbers that are additive inverses, you just described them as opposites. [yeah] anything else you want to add to that?"

(Doug Prompt 5C) "Can you give me another pair of additive inverses?"

(Bailey Turn 08 Prompt 5C) "Negative nine and nine."

(Doug Prompt 6A) "Ok, the next couple questions have several parts and involve a group of several different expressions. That I would like you to show how you compute and evaluate each of the four expressions. So, for these four expressions, how would you show and explain to a student how you would get their results."

Act: Placing the expressions on the table with four on a page and as one expression per card.

(Bailey Prompt 6A) "I would probably draw a number line. Do a zero here and then I'd do."

Act: She drew a number line the full width of the paper.

(Bailey Turn 09 Prompt 6A '10 – 2') "So I would mark the ten right here and draw another line and put them to go back to because negative, in subtraction we take away two, so they would go, one, two. So, then their answer would be eight for neg, ten minus two. [ok]

Act: She put a mark at the ten put an arrow from zero to ten then put a shorter arrow back two from ten.

(Doug Prompt 6A) "continue on with the other three"

(Bailey Turn 10 Prompt 6A '-1 + 9') "So for negative one plus nine, you start at negative one and you add nine, so you go one, two, three, four, five, six, seven, eight, nine. So, for that one, you would also get eight. [ok, if you have six plus two]"

Act: Put a dot at the -1 on the number line drew an arrow to the right nine units long.

(Bailey Turn 11 Prompt 6A '6 + 2') "For six plus two, the six and then you one, two.

Act: She put a dot at 6 and drew and arrow two to the right.

(Doug Prompt 6A '6 – -2') "And how would you model, six minus negative two, or how would you compute six minus negative two."

(Bailey Turn 12 Prompt 6A '6 – -2') "I know this turns into like a positive but, I don't know how to show it to them.

Act: Points to the card

(Doug Prompt 6A '6 – -2') "Ok, what result would you get from this computation doing it the way that you remember?"

(Bailey Turn 12 Prompt 6A '6 – -2') "It would be I think, eight."

(Doug Prompt 6B) "Ok, looking at these four expressions at the same time, what's similar and what's different between the four."

(Bailey Turn 13 Prompt 6B) "What's similar, you all I guess get to the answer eight. [ok] and then what's different some ('-1 +9' and '6 – -2') are negatives and others ('6 + 2' and '10 – 2') are basic adding and subtracting.
Act: Pointing to '-1 +9' and '6 - -2'. Pointing to '6 + 2' and '10 – 2'.
(Doug Prompt 6C) "Ok, one way that's often used to represent operations with integers is what's called the chip model. Are you willing to give a try to model these using the chip model?"
(Bailey Prompt 6C) "I would give it a try. [ok]"
(Doug Prompt 6C) "We are going to approach the same four problems. There are our chips. Give me just a second to get something to write notes on. And then, all right, go ahead. Which problem are you going to model first?"
(Bailey Turn 14 Prompt 6C '6 + 2') "This one [ok, so six plus two] So you'd have six, one, two, three, four, five, six. And then you are going to add two, I'm going to represent these {2} as the darker color. So, six plus two which is eight."
Act: Pointing to 6 + 2. Counting out 6 beige chips and 2 brown chips creating a set of 8 chips with two different colors.
(Doug) "Let me write the down, give me a second. Alright, go ahead."
(Bailey Turn 15 Prompt 6C '6 – -2') "OK, and then I am going to do this again six minus two and then then and then these {brown chips} are going to be negative two. I don't know if that's like showing them."
Act: Counting out 6 beige chips and 2 brown chips creating a group of 8 chips made of two different colors.
(Doug) "At the end I will show you a couple ways that I might approach it. But just continue on."
(Bailey) "Ok, so."
(Doug) "Which expression are you thinking about right now? [this one] Ok, negative one plus nine."
(Bailey Turn 16 Prompt 6C '-1 + 9') "I am going to do nine. I have no idea."
Act: Points to "-1 + 9" Counting out 9 brown chips.
(Doug Prompt 6C '10 – 2') "Ok, we can move on to ten minus two if you'd like."
(Bailey Turn 17 Prompt 6C '10 – 2') "Ok, I mean ten and then you take away two."
Act: Expanded the 9 to 10 brown chips the slid 2 chips away.
(Doug Prompt 6D) "And it seems like you've done many of them already, but I would like you to if you are willing to try to show and explain to me again how you might approach these on the number line. We'll move the chip model out of the way, and I have the number line white board over there. If you could give that a shot. [you want me to do these] yup the same four."
(Bailey Turn 18 Prompt 6D '6 + 2') "[ok so you did six plus two and got eight]"
Act: Put a red dot at 6 then drew and arrow to the right two units long and wrote 6 + 2 = 8.
(Bailey Turn 19 Prompt 6D '10 -2') "Then I will do this one. So, for ten minus two we have ten and then we're are going to take, go down two and that would be eight. [ok]"

Act: Pointing to '10 – 2'. Made a black dot at 10 then drew an arrow 2 units long to the left and wrote 10 – 2 = 8.

(Bailey Turn 20 Prompt 6D '-1 + 9') "For negative one plus nine, you start at negative one, and you go up nine. And that is eight. [ok]"

Act: Made a green dot at -1 then drew 1 an arrow 9 units long to the right and wrote -1 + 9 = 8

(Bailey Turn 21 Prompt 6D '6 – -2') "And then six negative, six minus negative two since it is minus negative two. Well it turns to six plus eight. I don't know why, but I guess to show that you can start at two because two becomes positive and go up six and so then it would be six minus negative two which equals eight."

Act: Pointed at the number line with her finger and counted, in the negative region starting for -6. Counted from 2 up 6. Made a blue dot at 2 then drew an arrow the right 6 units long and wrote 6 – (-2) = 8.

(Doug Prompt 7) "Ok, we just have one more question. Seven is similar to six. We have a couple expressions that go to take a look at. So, we have these four expressions and I would like to know what solutions you would get when you simplify or evaluate these expressions."

Act: Placing the second set of expressions on the table.

(Bailey Prompt 7) "Ok, do you want me to just?"

(Doug Prompt 7A) "You can just point to it, or just write it out on the paper. We will come back to the number line later. Can you hand me the white board?"

(Bailey Turn 22 Prompt 7A '2 – 8') "So two minus eight, and this is how I would show."

Act: Wrote the expression 2- 8 down.

(Doug Prompt 7A '2 – 8') "Yes, how would you show a seventh grader [ok] what these simplified formulas would be."

(Bailey Turn 22 Prompt 7A '2 – 8') "So you have two and if you subtract eight, it's not there, can I have another pen. So, then this would be gone, and you still need to take away six more. One, two, three, four, five, six. So, then you would have One, two, three, four, five, six. [Ok] Negative six. [Alright so] cause they're not actually there. They are just below zero."

Act: Drew 2 circles in red cross out the 2 red circles and drew 6 circles in blue and counted the blue circles wrote 6 then corrected to -6 immediately.

(Doug Prompt 7A '-4 – 2') "Alright, so which problem would you like to look at next?"

(Bailey Turn 23 Prompt 7A '-4 – 2') "Negative four minus two [ok] this one, I think would be to me, since it is bigger the answer is going to be negative, so if you actually subtract it, it would be negative two.

Act: Wrote the expression -4 – 2 in red, wrote -2

(Doug Prompt 7A '-7 + 1') "Ok, how about negative seven plus one?"

(Bailey Turn 24 Prompt 7A '-7 + 1') "How do you teach this stuff?"

Act: Wrote down the expression -7 + 1 then drew 7 blue circles.

(Doug) "I will show you a couple of ways in a few minutes. Continue if you'd like, if you're getting frustrated we can take a break or stop."

(Bailey Turn 24 Prompt 7A '-7 + 1') "Can you go up one? You would actually go down. You'd get less. You would get negative six."

Act: Continuing to consider "-7 + 1" crossing out a blue circle and writing -6.

(Doug Prompt 7A '-1 + -5') "Alright, and do you want to take a look at negative one plus negative five. And if you want you can use another piece of paper."

(Bailey Turn 25 Prompt 7A '-1 + -5') "So you have negative one, you add negative five and then you have negative six. Oh, so this ('-4 – 2') one's wrong. Minus two."

Act: She drew the blue and red circles and then combined them. She then changed the -2 to a -6.

(Bailey Turn 25 Prompt 7A '-1 + -5')

Act: Wrote the expression -1 + -5 and drew 1 then 5 blue circles then drew 6 new blue circles.

(Bailey Turn 26 Prompt 7A '-4 – 2')

Act: Drew 4 blue circles minus 2 red circles then 6 blue circles and changes -2 to -6.

(Doug Prompt 7B) "Ok and in what ways are these four expressions similar and in what ways are they different?"

(Bailey Turn 27 Prompt 7B) "So in negative seven plus one you are actually subtracting. This one, ('-1 + -5') you're adding. This one ('-4 – 2') I don't know what you're doing it just happened, and then this one '2 – 8') you're also kind of adding on to the first number."

(Doug Prompt 7C) "Ok, and you have sort of done so on the paper, but can you try to show this with the chip model."

(Bailey Turn 28 Prompt 7C '2 – 8') "I am going to start with two, and these would go over, two, three, four.

Act: Placed 2 beige chips on the table, then placed 2 brown chips on top of the beige chips then counted out 4 more brown chips.

(Doug Prompt 7C '2 – 8') "Ok, so what are you left with?"

(Bailey Turn 28 Prompt 7C '2 – 8') "negative six. These are negative numbers."

Act: Pointing to a brown chip.

(Doug Prompt 7C '-1 + -5') "Ok, then what expression do you want to look at next?"

(Bailey Turn 29 Prompt 7C '-1 + -5') "I'll do negative one plus negative five. [ok]. That's negative one plus five, plus negative five. One, two, three, four, five. So, we have negative numbers as the dark."

Act: Placed 1 brown chip on the table then set out 5 more brown chips. Identified brown as negative numbers.

(Bailey Turn 30 Prompt 7C '-7 + 1') "I'm doing negative seven plus one. Two, three, four, five, six, seven and this one. [ok, what?] You get negative six. [negative six]"

Act: Counted out 7 brown chips in a line then placed a beige chip on top of one of the brown chips and identified the line as showing -6

(Bailey Turn 31 Prompt 7C '-4 – 2') "And then negative four, minus two. I am just going to add two. I get negative six. So negative six."

Act: Counted out 4 brown chips then added 2 beige chips which she identified as -6 then placed 2 brown chips on the 2 beige chips.

(Doug Prompt 7D) "Are you willing to give a shot to solving these on a number line? [uh huh]"

(Bailey Turn 32 Prompt 7D '2 – 8') "So with the number line I am going to do two minus eight. [ok] so I'll start at two and go down eight. [ok] so, one, two, three, four, five, six, seven, eight. So, two minus eight equals negative six. [ok]"

Act: Drew a black dot at 2 then drew an arrow to the left 8 units. Placed a dot at -6 and wrote 2 – 8 = -6.

(Bailey Turn 33 Prompt 7D '-7 + 1') "Then negative seven plus one. So, you start at negative seven, whole integers would just over one. Negative seven adding one. [ok]"

Act: Drew a black dot at -7 then a line to the right one unit and a dot at -6 wrote -7 + 1

(Bailey Turn 34 Prompt 7D '-4 – 2') "So positive is this way negative numbers are this way."

Act: Drawing an arrow to her right {for positive}. Drawing an arrow to the left {for negative}

(Doug Prompt 7D '-4 – 2') "Which one's next?"

(Bailey Turn 34 Prompt 7D '-4 – 2') "Negative four minus two, so you have negative, I am going to use a different color. So negative four minus two, you would go back, this way. This one have me stumped as to why it works or what it does, so."

Act: Drew a green dot at -4 drew a line to the left 2 units and made a dot a -6 wrote -4 – 2. Pointing to ('-4 – 2')

(Bailey Turn 35 Prompt 7D '-1 + -5') "Negative one, plus five, start here and go up negative five numbers.

Act: Drew a red dot at -1 and line 5 units to the left and drew a dot at -6 wrote -1 + -5 = -6

Interview Participant Pseudonym: Brooke
Date of Interview: November 16th, 2017
Duration of Interview: 30 minutes

(Doug Prompt 1) "So, if a seventh grade student, or a sixth grade student, were to ask you what is the difference between whole numbers and positive integers how would you respond?"

(Brooke Turn 01 Prompt 1) "I would tell them that positive integers can have a decimal or a fraction of a whole number."

(Doug Prompt 2) "Ok, it is a familiar notion that addition makes larger, this is true when non-zero whole numbers are added together is this also always true when integers are added? And please provide some examples."

(Brooke Prompt 2) "Can you repeat the question?"

(Doug Prompt 2) "Is it always true that addition makes larger when we are adding integers?"

(Brooke Turn 02 Prompt 2) "I would say yes, because even if you add together negative integers, umm, and they'd still be a negative number, but it can be larger or closer to zero closer to a whole number then it would be before."

(Doug Prompt 2) "Ok can you write down an example in which that would happen?"

(Brooke Turn 02 Prompt 2) "So, I'm thinking, like negative six plus negative four and then it would help me out having the number line. I'm sorry that wouldn't be plus negative. That would just be plus four, so I'd me moving to the right, so negative six plus four, one, two, three, four, which would give me negative two which is bigger than where I started at negative six, and it is closer to a whole number as well."

Act: Wrote -6 + -4 then drew and labeled a number line. Scribbled out the negative on the four. Pointed to -6, then counted and moved up four jumps put a dot and circled -2.

(Doug) "Anything you would like to add to the questions we have looked at so far? [no]"

(Doug Prompt 3) "Ok, when two integers are subtracted, what kinds of numbers can you produce? So, when you subtract one integer from another what types of numbers can be produced?"

(Brooke Turn 03 Prompt 3) "You can have whole numbers whether it be positive or negative, and then you can also have integers meaning decimals or fractions of a whole number."

(Doug Prompt 4) "Ok. What role or roles can that symbol take within a mathematics problem?"

Act: Placing ' - ' on the table.

(Brooke Turn 04 Prompt 4) "It could indicate either negative or subtraction."

(Doug Prompt 5) "Um, ok. The numbers negative seven and seven are called additive inverses. What does the Prompt, additive inverse mean?"

Act: Placing '-7, 7' on the table.

(Brooke Turn 05 Prompt 5) "When I think about it, I am not familiar with the entire term, but just using context clues, I think inverse means opposite, so here you have a negative seven so the opposite of that would be positive seven."

(Doug Prompt 5A) "And you may have hinted at an answer for this, but I will ask it anyway, and why are seven and negative seven additive inverses?"

(Brooke Turn 06 Prompt 5A) "Because the opposite of the negative seven would be a positive seven."

(Doug Prompt 5B) "Ok, how else would you describe two numbers that are additive inverses?"

(Brooke Turn 07 Prompt 5B) "I guess you could say they are opposite, I don't know if that would confuse students, but I guess you could also describe it as, they are the same distance on the number line from zero just in different directions."

(Doug Prompt 5C) "And can you give me another pair of additive inverses?"

(Brooke Turn 08 Prompt 5C) "You can also say negative five and positive five."

(Doug Prompt 6) "The next couple problems we are going to be looking at are slightly longer than what we've looked at so far, and they'll contain four parts and we'll be looking at four mathematical expressions in each. So, with those four expressions show and explain how you might evaluate each of those expressions."

Act: Clearing the table and sliding over the first four expressions.

(Brooke Prompt 6) "By evaluate do you just mean like solve the equation."

(Doug Prompt 6) "Yes"

(Brooke Turn 09 Prompt 6A '10 – 2') "And you said this is for like sixth, seventh graders? [yes] ok. So, I guess for ten subtracted by two, I would think maybe at that age they could just do mental math a do it, otherwise just lining the numbers up and regrouping from the tens place over to the one's place to get eight."
Act: Wrote '10 – 2' vertically, then crossed out the 1 and wrote a 0 and the o and wrote a 10, then wrote 8.

(Brooke Turn 10 Prompt 6A '6 + 2') "[ok] same with six plus two, I think that's also mental math, and then the student would just get they're answer as eight."

(Brooke Turn 11 Prompt 6A '6 – -2") "Whereas six subtracted negative two I know when I was taught in school that when I say two negatives it always meant a positive. I am not exactly sure why so I'm not, I guess, I would have to brush up on who to teach that to students. I guess maybe if I were to draw a number line. So, if we start at six, and we are going to subtract negative two, Yeah, I am not quite sure how I would explain to a student how, even though its subtracting a negative you are still going to the right of the number line cause when I see subtraction I always think you have to go to the left of the number line."
Act: Drawing and labeling a number line. Places a dot at 6. Moving the pen to the right along the number line.

(Doug) "Ok, at the end we'll take a look at how we can maybe make sense out of some of that. [ok]"

(Brooke Turn 12 Prompt 6A '-1 + 9') "So for this one ('-1 + 9') it kind of makes more sense for me cause when I see addition I always think you move to the right of the number line. So again, that would be. So, you start at negative one and then you would move to the right of the number line nine spaces. Which would be one, two, three, four, five, six, seven, eight, nine. Which would give you positive eight."
Act: Drawing and labeling a number line. Placing a dot on -1 counting to the right, then putting a dot at and circling 8.

(Doug Prompt 6B) "Ok, in what ways are these expressions similar and in what ways are they different?"

(Brooke Turn 13 Prompt 6B) "So they're all integers. These, ten minus two and then six plus two are similar because they don't involve any negative numbers in the sense that six minus negative two or negative one plus nine have negative numbers. And then six minus negative two and negative one plus nine have negative integers in them."
Act: Pointing to '10 – 2' and '6 + 2'. Pointing to '6 – - 2' and '-1 + 9'. Pointing to '6 – -2' and '-1 + 9'

(Doug Prompt 6c) "One of the regular ways we use to show integers is the chip model. are you willing to try modeling some of these with the chip model?"

(Brooke) "I can try."

(Doug Prompt 6C) "Ok, here are our chips, working on the same four expressions."

(Brooke Turn 14 Prompt 6C '10 – 2') "Ok, so for ten minus two you would start with ten chips. Two, four, six, eight, ten. And then since it is subtraction it means you take away, so you take away two and then you're left with two, four, six, eight chips."

Act: Counted out 10 beige chips by pairs, lined up in two columns. Slide 2 beige chips away from the others and counted up to 8 on the remaining chips.

(Brooke Turn 15 Prompt 6C '6 + 2') "So now we have six plus two. So, we start with six chips, and then we're adding two more, so now we have eight chips in total."

Act: Starting with 6 beige chips in two columns of three then slides in a pair of beige chips to make two columns of four, shows the 8 chips.

(Brooke Turn 16 Prompt 6C '6 – -2') "Six minus negative two, like I said before when I was solving it, I am not sure how I would show a student how to do that, do the problem using chips as well."

Act: She left 6 beige chips on the table.

(Brooke Turn 17 Prompt 6C '-1 + 9') "And then, negative one plus nine. I'm not sure if this would make sense to a student, but the way I see it as I can represent negative one with the dark brown chip, and positive nine with the light brown ones, and then in order to solve the problem, since I have negative one and I have a positive one they would cancel out each other since negative one plus positive one would give me zero, so now I am left with eight positive chips."

Act: Pulled all the chips back to one mixed pile, counted out 9 beige chips in pairs and formed an angled pair of rows. Then reangles the beige chips. And took out a single brown chip. Pointed to the beige chips while referring to them. Picked up one of each color chips and placed them together. Set the mixed color pair aside and rearranged the beige chips to show 8.

(Doug Prompt 6D) "Ok, number lines as you used a little bit before are also used to represent mathematical computations. And are you willing to try to model using the number line? [yes] get the chips out of the way and just reach over and grab the number line."

Act: Getting the chips out of the way and the number line out.

(Brooke Turn 18 Prompt 6D '10 – 2') "So again we always start at the first number of the equation, so you'd start at ten and then your subtracting two, which means you are moving left on the number line. So, one two hopes gets you to eight, so the answer would be eight."

Act: Drew a dot at 10, made 2 hops to the left and placed a dot at 8.

(Brooke Turn 19 Prompt 6D '6 + 2') "Six plus two, you start at six, since you're adding you're moving to the right of the number line, so two hops would bring you to eight."

Act: Drew a dot at 6, made 2 hops to the right and placed a dot at 8.

(Brooke Turn 20 Prompt 6D '6 – -2') "Negative six. Oh, no I'm sorry positive six, I guess now that I am thinking about it, thinking about our conversations, before, the, I can also explain this as an inverse of the equation, so instead of six minus negative two, I would take the inverse of negative two which would be a positive two which would also mean I would take the inverse of subtraction, which would mean addition. Perhaps, that is how I could explain it, and that way at least I know the answer to this would be positive eight, so if you move over two spots, it gets you positive eight. [Ok]"

Act: Drew a dot at -6, erased that dot and drew one at 6. Points to the expression on the board. Writes 6 - -2 on the white board circles -2 and writes 2 points to then circles the subtraction sign. Writes a plus and the six made 2 hops to the right from 6 and puts a dot at 8.

(Brooke Turn 21 Prompt 6D '- 1 + 9') "And now with negative one plus nine, you'd start at negative one and you're adding so you're moving to the right of the number line nine spaces. One, two, three, four, five, six, seven, eight, nine. That gets you positive eight."

Act: Drew a dot at -1, while counting made 9 hops to the right from -1 to 8. Makes a dot at 8.

(Doug Prompt 7) "We are going to be looking at these four expressions, you can look at the individuals, or you can look at the full sheet it is up to you and I would like you to show and explain how you would evaluate each expression. [ok, umm] Actually if you want you can move the white board out of the way."

Act: Moving papers and items around.

(Brooke Turn 22 Prompt 7A '2 – 8') "So I think I would continue using the number lines, because I know for me that's what helps me make sense of subtracting a larger number than the number that we begin with. So, you would start with your first number at positive two, since it is subtraction you are moving to the left of the number line eight spaces, which is one, two, three, four, five, six, seven, eight. Which gives you negative six as your answer."

Act: Makes darker marks on the first number line she drew before. Drew and labeled a new number line put a dot at 2, made 8 hops to the left and put a dot at -6.

(Brooke Turn 23 Prompt 7A '-7 + 1') "For this one ('-7 + 1') I wouldn't have to use the number line myself, but I would also use the same logic that I used on the previous one, you start at negative seven and you are moving to the right of the number line one spot which would bring you to negative six."

Act: Points to the numbers '-7 + 1' as she talks about them, then makes a virtual hop to the right as she points to the 1.

(Brooke Turn 24 Prompt 7A '-4 – 2') "Same with this one ('-4 – 2') you start at negative four and you're moving to the left of the number line two spots, so that would be negative six.

Act: Points to the numbers as she talks about them in '-4 – 2'.

(Brooke Turn 25 Prompt 7A '-1 + -5') "And then negative one plus negative five. This problem kind of reminds me of the other one, I think it was six minus negative two, which after I thought that one out helps me solve this one too. So even though it's negative one plus negative five, if we do the inverse of negative five which would just be positive five and also do the inverse of our addition which would be subtraction, just gives us negative one minus five, or negative one minus a positive five, but you don't necessarily need the positive sign there it is

just assumed, so then you would start at the negative one, and you would go to the left of the number line five spaces which would give you negative six."
Act: Writes -1 + -5 on the paper. Circles -5 and writes 5 circles the addition sign and writes a subtraction sign writes a -1. Points to the numbers.

(Doug Prompt 7B) "Ok, in what ways are these four expressions similar and in what ways are they different?"

(Brooke Turn 26 Prompt 7B) "They all gave me negative answers, they're different they're not all moving in the same direction on the number line, some are moving to the left, and some are moving to the right, and say negative one plus negative five also sticks out to me as well because you have two signs which may be confusing as well having a negative number next to an addition symbol."
Act: Does not point to any expressions during most of her talking but points to -1 + -5 when referring to it.

(Doug Prompt 7C) "How might you model these with the chip model?"

(Brooke Turn 27 Prompt 7C '2 – 8') "So like I did for the other problems, I am going to represent my positive two, with two light brown chips and I am going to represent my eight with the dark brown chips. So, the dark brown chips are gonna represent negative eight. I am not sure exactly how I would explain why that is, when at first, when I was solving before, I just describe the subtraction sign as movement on the number line whereas now I am using it as a negative sign, but that's what makes sense to me right now [ok] umm, so like I did before, my positive two and negative two would cancel each other out, so now since now I know my darker chips represent negative, this would be negative six."
Act: Counts out 2 beige chips and 8 brown chips, with the brown chips on her left and the beige chips on the right, then reverses them. Points to the beige and brown chips in turn. The chips are in pairs. Took a pair of brown chips and matched them with a pair of beige chips and removed the cancelled pairs from the display area. Repositioned the 6 brown chips.

(Brooke Turn 28 Prompt 7C '-7 + 1') "For, negative seven plus a positive one, again my negative number represented by the darker chips and the positive by the lighter chips, I know one negative one and a positive one cancel each other out, which leaves me with negative six."
Act: Moved the 6 brown chips and added 1 to make 7 and placed a beige chip next on the table as well. Pointed to the chips as she referenced the colors. Paired up a positive and negative pair and removed it. Repositioned the 6 brown chips.

(Brooke Turn 29 Prompt 7C '-4 – 2') "Now I have negative four, and negative two and since I don't have anything cancelling each other out, they're the same color chips, essentially, I add them together. And I get negative six."
Act: Made a group of 4 brown chips and a pair of brown chips, slid the pair to join the 4 brown chips Forming two rows of three brown chips of 6 brown chips.

(Brooke Turn 30 Prompt 7C '-1 + -5') "And then negative one and I'm adding negative five which gives me negative six."
Act: Positioned 1 brown chip separate from 5 brown chips, then slid the single chip to join the 5 chips. Forming two rows of three brown chips of 6 brown chips.

(Doug Prompt 7D) "Ok, and we are going to take another look at the number line at these expressions."

(Brooke Turn 31 Prompt 7D '2 – 8') "So I start at positive two and since I'm subtracting I am moving to the left of the number line eight spaces. One, two, three, four, five, six, seven, eight, which brings me to negative six."

Act: Drew a dot at 2, made 8 hops to the left to -6 and placed a dot.

(Brooke Turn 32 Prompt 7D '-7 + 1') "Now, I am at negative seven, and I'm moving to the right of the number line one space which brings me to negative six."

Act: Drew a dot at -7 and made a hop to the right and a dot at -6.

(Brooke Turn 33 Prompt 7D '-4 – 2') "And now I start at negative four and I am moving to the left of the number line two spots. So, I move two gives me negative six."

Act: Drew a dot at -4 and made 2 hops to the left and a dot at -6.

(Brooke Turn 34 Prompt 7D '-1 + -5') "Negative one and I am adding a negative, I would also use the same logic I did before when explaining it at first, so I have negative one plus negative five, I am going to take the inverse of negative five which is positive five and the inverse of addition which is subtraction, and negative one minus five, so now I know I am moving to the left, of the number line five spaces that's one, two, three, four, five, gives me negative six.

Act: Placed a dot at -1. Wrote -1 + -5. Circled the -5 and wrote 5. Circled the + and wrote and – then a -1. Made 5 hops to the left and a dot at -6.

(Doug) "Anything you would like to add to anything we have discussed so far? [no]"

Interview Participant Pseudonym: Christina
Date of Interview: November 6th, 2017
Duration of Interview: 25 minutes

(Doug Prompt 1) "If a seventh grader were to ask you "what is the difference between whole numbers and positive integers?", how would you respond?"

(Christina Turn 01 Prompt 1) "I thought they were the same thing. Integers are whole numbers so a pos, a positive. I think I would tell them it's the same thing."

(Doug Prompt 2) "Ok, sounds good. My, one thing I should have mentioned before, is my goal is to be as neutral as I can during a response [I totally get it.] So, I should be reacting the same way to a right, wrong or Chinese answer. [Got it.] It is a familiar notion that addition makes larger [yes] ok, and this is true when non-zero whole numbers are added together, three plus four is seven [right, yes, yes, yes] Is this also true when integers are added together? And if so, please provide some examples."

(Christina Turn 02 Prompt 2) "Ahh, yes, but if its negative then, I don't. Well if you're adding negative numbers, I guess it's, I think it's true. I think, I don't know, I'm trying to like visualize it and I can't think of a way to explain it so, so I guess yeah."

(Doug Prompt 2) "If you would like to write down a problem or two that might confirm or challenge the notion that addition makes larger. [Do you want me to write it on paper?] Yup.

(Christina Turn 02 Prompt 2) "Can you repeat the question?"

(Doug Prompt 2) "Does it remain true that addition makes larger when we are talking about adding integers?"

(Christina Turn 02 Prompt 2) "Right. Yes, so that would be like two plus two equals four and since you're adding it making it a larger number than two."

<u>Act:</u> Wrote '2 + 2 = 4'

(Doug Prompt 3) "OK, what types of numbers can you get when one integer is subtracted from another integer?"

(Christina Turn 03 Prompt 3) "Well you'll still get. You'll still get an integer."

<u>Act:</u> Wrote '4 – 2 = 2'

(Doug Prompt 3) "Is it possible to get a negative number when you subtract?"

(Christina Turn 03 Prompt 3) "Yeah if you did like four minus six, you get negative two."

<u>Act:</u> Wrote '4 – 6 = -2'

(Doug) "Anything else you would like to add to any of the questions that we have looked at so far?"

(Christina) "I don't think so."

(Doug Prompt 4) "What rolls can this symbol take within a mathematics problem?"

<u>Act:</u> Placing the ' - ' on the table.

(Christina Turn 04 Prompt 4) "Ok, that can be used as a subtraction sign so, if you have four cookies and that identified removing, like taking away. So, like if you have four cookies and then you take away two, so you're making the number smaller, but it can also be a negative sign, which indicates that the value is less than zero."

(Doug Prompt 5) "Ok, the numbers negative seven and seven are additive inverses of each other, what does the Prompt additive inverse mean?"

<u>Act:</u> Placing the card with -7, 7 on the table.

(Christina Turn 05 Prompt 5) "No idea."

(Doug Prompt 5C) "Can you give me another pair of numbers that are also additive inverses?"

(Christina Turn 06 Prompt 5C) "Like negative two, positive two, negative ten, positive ten."

(Doug Prompt 5B) "Ok, so what is it about negative seven and seven or negative two and two that make then additive inverses?"

(Christina Turn 07 Prompt 5B) "They are, it's the not the opposite, but like on the number line negative seven would be the same amount away from zero that positive seven is, just negative."

(Doug Prompt 5A) "Ok, I think we got this piece as well, but how else would you describe two numbers that are additive inverses?"

(Christine Turn 08 Prompt 5A) "I guess, the same way, you can describe it as the one is the positive of the other one, they would be the same distance away from zero except one is going negative the other's going positive."

(Doug Prompt 6) "Ok, the next question has a couple different parts, and also a couple different expressions, so I am going to be asking you to look at these four expressions, and for the, their also on these, little cards. Show and explain how you would evaluate each expression in a symbolic form.

Act: Placing the four expressions on a single sheet and on individual sheets on the table.

(Christina Turn 09 Prompt 6A '10 – 2') "So, like answer them. [Yes.] So, ten minus two is eight, just take two from that.

Act: Wrote '10 – 2 = 8'

(Christina Turn 10 Prompt 6A '6 + 2') "Six plus two equals eight."

Act: Wrote '6 + 2 = 8'.

(Christina Turn 11 Prompt 6A '6 – -2') "Six minus negative two, would be, I think you add it. So, I think it becomes four, no. Let me go back to that."

Act: Wrote '6 – -2'.

(Christina Turn 12 Prompt 6A '-1 + 9') "Then negative one plus nine equals eight because that's negative, if you add a negative number to a positive number you're subtracting."

Act: Wrote "-1 + 9 = 8'

(Christina Turn 13 Prompt 6A '6 – -2') "This is confusing cause I am trying to think of algebra like an algebra equation when you have like the double negative, I am almost certain it becomes positive, but I am second guessing myself, so I am just trying to. It does plus positive is eight as well."

Act: Went back to '6 – -2' changed the subtraction and negative signs to plus and positive and wrote down 8.

(Doug Prompt 6B) "Ok, in what ways are those expressions similar and in what ways are they different?"

(Christina Turn 14 Prompt 6B) "Well, they all have the same answer, but I think that these, ('10 – 2' and '6 + 2') the first two here are a little bit more basic, like I feel like if you gave that problem to someone maybe even in kindergarten, they would be able to answer it, whereas these ('6 – -2' and '-1 + 9') even though it's the same answer and its two numbers I think when you get into the negatives it becomes a little bit more complicated for someone to answer, I mean look at me sitting here trying to do it so.

Act: Wrote a bracket grouping '10 – 2 = 8' and '6 + 2 = 8'. Wrote a bracket grouping '6 – -2 = 8' and '-1 + 9 = 8'.

(Christina Turn 14 Prompt 6B) "I think it's interesting as well that these two, the six minus negative two, and the six plus two are the same numbers you're just have like a different symbol which can become tricky. So, I would say that's probably the most significant differences that I can tell.

Act: Pointed to '6 + 2 = 8' and '6 – -2 = 8'

(Doug Prompt 6C) "Ok, in elementary mathematics teaching we often use different tools to show the idea in addition to symbols. And one that is often used with integers is the chip model. Would you be willing to try to explain how you would solve each of these problems or any of these problems using the chip model? [Sure.]"

(Christina Turn 15 Prompt 6C '10 – 2') "Alright, so here's ten, two, four, six, eight, ten. Ok, so ten and then you're going to subtract two, which means you're going to take away two which would give you two, four, six, eight. Just like six plus."

Act: Counted out 10 chips of mixed colors then took away two, leaving 8 still of mixed colors.

(Doug) "Hold on one second [no problem just let me know] ok go ahead."

(Christina Turn 16 Prompt 6C '6 + 2') "So then six and then plus two is eight."

Act: Left 6 chips of mixed colors and slid in two brown chips to make 8.

(Christina Turn 17 Prompt 6C '6 – -2') "Umm, so now what I am thinking here is to differentiate positive and negative with colors. I'm thinking. I am just going to try it. So, let's say that dark brown is positive so that would be two four six, so we have six. Take away negative two, this is where it gets tricky because neg, it's like hard to pr, to represent something as a negative because when you think a negative you think that it is no there, not really because it is a real number but with negatives there is not usually a physical representation of if like I can't think of being in grammar school and umm, doing a problem like this where something represented a negative number. Like, I just can't. I don't know."

Act: Sorted out 6 brown chips and slid in 2 beige chips.

(Doug) "So, do you want to."

(Christina Turn 18 Prompt 6C '-9 + 1') "This one I can, so you have two, four, six, eight, nine. And then we're adding a negative one, so when you add the negative one to the group you're just gonna cancel out one of these and that's gonna become eight."

Act: Counted out 9 brown chips placed a beige chip on top of one of the brown chips and slid the pair away. Leaving 8 brown chips.

(Doug Prompt 6D) "And would you be willing to show and explain how you would evaluate each expression using the number line model [sure]."

(Christina Turn 19 Prompt 6D '10 – 2') "So, ten is here right, and you are taking away two, so you go one two and you land on eight."

Act: Drew a black dot at 10 then made 2 hops to the left and put a circle at 8.

(Christina Turn 20 Prompt 6D '6 + 2') "[ok] So, you start at six. Let me erase this first. So, we have six right here, and you're going to add two, one two and you also land on eight."

Act: Drew a red dot at 6 then made 2 hops to the right and a circle at 8.

(Christina Turn 21 Prompt 6D '-1 + 9') "[Ok.] Negative one which is here, and we are going to go one two three four five six seven eight nine. Also, eight."

Act: Drew a blue dot at -1 and 9 hops to the right and a dot at 8.

(Christina Turn 22 Prompt 6D '6 – -2') "And ok here and then you're gonna subtract negative two which is going to one two which is also going to put you to eight."

Act: Drew a green dot at 6 then two hops to the right and a dot at 8.

(Doug Prompt 7) "We're going to do the same procedure again with these four problems. [do you want me to start with these] no actually just the numbers as numbers, then we'll do the chip and compare and contrast, then the chips, then the number lines."

(Christina Prompt 7) "Do you want me to use the white board or do you want me to use paper? [Do it on the scrap paper.] No problem. [If you are ok with that.]"

(Christina Turn 23 Prompt 7A '2 – 8') "So, two minus eight is negative six."

Act: Wrote '2 – 8 = -6'.

(Christina Turn 24 Prompt 7A '-1 + -5') "One, negative one plus negative five equals negative six, cause you're going backwards."

Act: Wrote '-1 + -5 = -6'

(Christina Turn 25 Prompt 7A '-4 – 2') "Negative four minus two is also negative six because you're subtracting what you already don't have."

Act: Wrote '-4 – 2 = -6'

(Christina Turn 26 Prompt 7A '-7 + 1') "Negative seven plus one, you're adding a positive to negative seven, but you still have negative six."

Act: Wrote '-7 + 1 = -6'

(Doug Prompt 7B) "Ok, in what ways are these expressions similar and how are they different?"

(Christina Turn 27 Prompt 7B) "Umm, again, they all have the same answers, here, umm, I think that like the first one, the two minus eight is a little bit more basic, I think a very young student would be able to answer that. When you umm, do something like negative four minus two I think you need a little bit more, you need time to like process, like you need to like break it down. I think with all of these actually, you, you need to ahh, understand that this is already a negative and that the subtraction sign means to take away. So, when you're taking, you have a negative and now you're are taking away a positive but you're still going in the negative direction because you're removing something that you already don't have. Where this, this one this problem, you're adding to the negative number which is kind of similar, same thought process. Umm, and then, negative seven plus one, you're adding this positive number to the negative value that you have here, but since it's less than the negative number you're still going to have a negative value."

Act: Added the grouping symbol to the set of expressions Drew the arrow by the first expression '2 – 8 = -6'. Circled the third expression '-4 – 2 = -6'. Circled the subtraction sign. Pointing to '-1 + -5 = -6'. Marked up the last expression '-7 + 1=-6'

(Doug 7C) "Ok. And can you show me how you would explain using the chip model each of those four problems."

(Christina Turn 28 Prompt 7C '2 – 8') "Ok, so you have two and you're gonna two four six eight, you're gonna remove eight so, this is, this is the positive {dark chips} value, and, well actually they're both technically, positive values so two eight, these cancel when you subtract the two, so you're going to remove the two from the eight, like this and you're still gonna have negative six."

Act: Set out 2 brown chips and counted out 8 beige chips point to the top pair of beige chips and both brown chips then removes the 2 brown and 2 beige chips

(Christina Turn 29 Prompt 7C '-1 + -5') "Negative one plus negative five, just going to join the negative one with the negative five and have a negative six."

Act: Set out 1 beige and counted out 5 beige chips slid the 1 to join the 5 to make 6 beige chips.

(Christina Turn 30 Prompt 7C '-4 – 2') "Negative four and you're gonna subtract negative two, no, you're just going to subtract regular two, but since you don't have a positive value to remove the two from, you're really just going to end up adding two more to the negative value cause you are trying to remove something that you don't have positive of so you're just going to have a negative six."

<u>Act:</u> Set out 4 beige and 2 browns in separate piles slid in 2 more beige chips and got rid of the 2 brown chips leaving 6 beige chips.

(Christine Turn 31 Prompt 7C '-7 + 1') "Ok, negative seven, plus positive one is just gonna cancel, these two out right here. And you have negative six as well."

<u>Act:</u> Set out 7 beige chips and picked 1 brown chip. Paired the brown with one of the beige slid them together and away leaving 6 beige chips.

(Doug 7D) "And can you show me how you might approach them with the number line."

(Christina Turn 32 Prompt 7D '2 – 8') "So you begin on two and you are going to take away eight, so here you go, one two three four five six seven eight. Right on negative six."

<u>Act:</u> Drew a blue dot at 2 and make 8 hops to the left put a dot at -6.

(Christina Turn 33 Prompt 7D '-1 + -5') "Negative one plus negative five. So, you're adding, but you're still going in the negative direction, so you're one two three four five. Negative six again."

<u>Act:</u> Drew a red dot at -1 and made 5 hops to the left and put a dot at -6.

(Christina Turn 34 Prompt 7D '-4 – 2') "Negative four minus two. Negative four, here, subtracting positive two, so you're still going in the negative direction, since it's two you do one two. Negative six."

<u>Act:</u> Drew a blue dot at -4 then made two hops to the left and put a dot at -6.

(Christina Turn 35 Prompt 7D '-7 + 1') "Negative seven plus one start on negative seven right here and you're adding positive one, so you're going in the positive direction right on to negative six."

<u>Act:</u> Drew a blue dot at -7 and made 1 hop to the right and put a dot at -6.

Interview Participant Pseudonym: Courtney
Date of Interview: November 2nd, 2017
Duration of Interview: 23 minutes

(Doug Prompt 1) "If a seventh grade student were to ask you 'What is the difference between whole numbers and positive integers?' How would you respond?"

(Courtney Turn 01 Prompt 1) "I feel like whole numbers would still be cons, you could still do a fraction, so it would be like twelve out of twelve or like ten out of ten, where like positive integers is just like seven. Or like six it is just whole numbers not an actual fraction. It would be like point one hundred."

(Doug) "Was there anything else you wanted to add to that at this moment?"

(Doug Prompt 2) "It is a familiar notation that addition makes larger, this is true when non-zero whole numbers are added together. Is this also true when integers are added?"

(Courtney Turn 02 Prompt 2) "Well if you do like a negative no."

(Doug Prompt 2) "Ok, can you provide me a specific or a couple of specific examples?"

(Courtney Turn 02 Prompt 2) "You can do like negative seven plus negative six, you are going to get negative thirteen."

(Doug Prompt 3) "Ok, so far so good, ok. What types of numbers can you get when one integer is subtracted from another, and by types of number [like positive and negative] yeah."

(Courtney Turn 03 Prompt 3) "Yeah, you can get positive or negative, depending on the question given."

(Doug Prompt 3) "Can you give me an example of a question that would yield a negative number?"

(Courtney Turn 03 Prompt 3) "If you do zero plus a negative seven. Would be negative seven. If you do seven plus seven you would get fourteen. Positive fourteen."

(Doug Prompt 4) "What roles can that symbol take within a mathematics problem?"

Act: Sliding the ' - ' symbol across the table.

(Courtney Turn 04 Prompt 4) "If can be a subtraction, it can be attached to a number, so let's stay you attach that to six you get negative six, instead of just a positive six."

(Doug Prompt 5) "The numbers, seven and negative seven are additive inverses of each other. What does the Prompt additive inverse mean?"

Act: Placing a paper containing '-7, 7' on the table.

(Courtney Turn 05 Prompt 5) "You, know when I was taught about inverses, it is like the flipped version of seven or negative seven. Negative seven can turn into a positive seven. Adverse I don't remember going over ever."

(Doug Prompt 5A) "Umm, ok, so you said that additive inverses, or like inverses are like flipped versions of each other. [yeah] and why, what's special about seven and negative seven that they have that relationship?"

(Courtney Turn 06 Prompt 5A) "I am not entirely sure. I thought most numbers besides zero have that same relationship."

(Doug Prompt 5B) "Is there any other terms that you can use to describe two numbers that have this relationship?"

(Courtney Turn 07 Prompt 5B) "Not that I know of."

(Doug Prompt 5C) "Can you give me another pair of numbers that behave this way?"

(Courtney Turn 08 Prompt 5C) "Negative eleven and positive eleven."

(Doug Prompt 6) "Please consider the following expressions. Six minus negative two, ten minus two, negative one plus nine and six plus two. Show and explain how you would evaluate each expression."

Act: Placing first four expressions on the table.

(Courtney Turn 09 Prompt 6A '6 – -2') "For six minus negative two you would change the two negatives into a positive, so it would be eight."

Act: She wrote nothing down.

(Courtney Turn 10 Prompt 6A '10 – 2') "Ten minus two is already a positive eight. Those two pair together."

Act: She wrote nothing down.

(Courtney Turn 11 Prompt 6A '-1 + 9') "Negative one plus nine is also eight."

Act: She wrote nothing down.

(Courtney Turn 12 Prompt 6A '6 + 2') "and then six plus two equals eight."

Act: She wrote nothing down.

(Doug Prompt 6B) “Ok, in what ways are these expressions similar?”

(Courtney Turn 13 Prompt 6B) “They all have the same answer. They’re all four different problems where you’re getting that answer.”

(Doug Prompt 6B) “Ok, you may have just answered the second part of this question, in what ways are they different.”

(Courtney Turn 13 Prompt 6B) “They’re all, even though they have similar signs and have the same answer they’re all made up of different ones, two of them have negatives two of them have all positives, yet, they still have different symbols.”

(Doug Prompt 6C) “Sometimes when we try to represent ideas, for the students we use means other than just numbers and symbols. One of them would be a chip model. Are you willing, show and explain how you would use the chip model.”

(Courtney Prompt 6C) “Yeah, sure.”

(Doug) “So whenever you’re ready.”

(Courtney) “Does it matter which one I do.”

(Doug) “I am going to want you to do all four of them.”

(Courtney) “All four of them.”

(Doug) “The order you do them is up to you.”

(Courtney Turn 14 Prompt ‘6 – -2’) “Two four six. So, this will be the positive six, and then you have the two, because this one’s a negative you basically just go like that. Instead of just having the two left alone.”

Act: Sorted the chips into beige and brown. Counted 6 beige and 2 brown, swapped the 2 brown, with 2 beiges.

(Doug) “I am going to try to keep track of what you are doing, ok.”

(Courtney Turn 15 Prompt ’10 – 2’) “Since these are both positive two four six eight ten and then two, so you start with basically twelve and then subtract two.”

Act: Counted out 10 brown chips and 2 more brown chips for a total group of 12 brown chips slid 2 brown chips away leaving 10 brown chips then slid away 2 more leaving 8.

(Courtney) “Negative one.”

(Doug) “I’m sorry I missed can you do that one, one more time.”

(Courtney Turn 16 Prompt 6C ‘-1 + 9’) “Two four six eight nine, negative one. Canceling, eight.”

Act: Counted out 9 brown chips and 1 beige chip paired up a beige and brown leaving a group of 8 which she then counts out quietly into two rows.

(Courtney Turn 17 Prompt 6C ‘6 + 2’) “Six you need six and two.”

Act: Formed a group of 6 brown and 2 brown chips leaves the chips as two separate sets.

(Doug Prompt 6D) “And I am also going to ask you to show and explain how you might evaluate each expression using a number line model. I just ask you to try to not write on the white stickers.”

(Courtney Turn 18 Prompt 6D ‘6 – -2’) “Ok you have negative one, six then you have negative two.”

Act: Two to here. Initially puts a blue dot at -6 erases that and puts a blue dot at 6 writes the expressions as 6 + 2 makes two hops to the right in red and puts a red dot at 8.

(Courtney Turn 19 Prompt 6D '10 – 2')

Act: Makes a red dot at 10 then makes 2 hops to the left in blue writes the expression 10 - 2 then puts a dot in blue at 8.

(Courtney Turn 20 Prompt 6D '-1 + 9') "Nine plus negative one, you go back one gives eight."

Act: Writes the expression -1 + 9 draws a red dot at 9 then makes a hop to the left in blue and a blue dot at 8.

(Courtney Turn 21 Prompt 6D '6 + 2') "Then you go one two. If you want you can really start from zero but, and that's how most kids would do it, but I would hope at this point they know what they're doing."

Act: Writes the expression 6 + 2 puts a dot in red at 6 then makes 2 hops to the right in blue and a blue dot at 8.

(Doug) "Can you repeat what you, the last thing you just said."

(Courtney Turn 21 Prompt 6D '6 + 2') "Usually people start from like you can go like one two three four five six, but if you know what you are doing, and you are like advanced you can start from wherever you want. Like it is not necessary until you get to the college lever. For kids it is usually and all that."

(Doug Prompt 7A) "And we are going to be doing the same series of questions with these four. Ok, you have negative four minus two, two minus eight, negative one plus negative five and negative seven plus one, so I want to see how you would show and explain how you would evaluate these expressions."

(Courtney) "Do you want me to do like the checkers or the board, it doesn't matter."

(Doug) "With the symbolic form first. Then we will do the checkers and the board after."

(Courtney Turn 22 Prompt 7A '2 – 8') "You would do two and negative eight, two minus eight would be negative six."

Act: She wrote nothing down.

(Courtney Turn 23 Prompt 7A '-7 + 1') "Then negative seven plus one is negative six.

Act: She wrote nothing down.

(Courtney Turn 24 Prompt 7A '-1 + -5') "Negative one plus negative five is negative six.

Act: She wrote nothing down.

(Courtney Turn 25 Prompt 7A '-4 – 2') "And then negative four minus two is negative six."

Act: She wrote nothing down.

(Doug Prompt 7B) "Ok, so in what ways are they similar? And in what ways are they different?"

(Courtney Turn 26 Prompt 7B) "The same like before they are all expressed differently but they all get the same answer."

(Doug Prompt 7D) "Since you have the number line model in front of you already can you show me how to do how you would approach that?"

(Courtney Turn 27 Prompt 7D '2 – 8') "I have negative eight go back one two. And that gets a negative six."

Act: Wrote the expression '2 – 8' in green put a dot in green at -8 then made 2 hops in blue and made a dot in blue at -6.

(Courtney Turn 28 Prompt 7D '-1 + -5') "Negative five and this goes over one."

Act: Writing the expression '-1 + -5' in read then put + - in parenthesis and put – above that made a red dot at -5 and changes the – above the parenthesis to a plus and made a hop to the left in red and put a red dot at -6.

(Courtney Turn 29 Prompt 7D '-7 + 1') "One."

Act: "Write the expression -7 + 1 then made a blue dot at -7 made a hop to the right in red and put a red dot at -6."

(Courtney Turn 30 Prompt 7D '-4 – 2')

Act: Wrote the expression -4 – 2 made a red dot at -4 then two hops to the left in green and a green dot at -6.

(Doug Prompt 7C) "And if you can show me how you would approach these with the chip model."

(Courtney Turn 31 Prompt 7C '2 – 8') "Four six eight."

Act: Counting out 8 brown chips and 2 beige chips slid 2 brown chips away and 2 beige chips to be with the brown chips.

(Doug Prompt 7C) "Ok, which one was that?"

(Courtney Turn 32 Prompt 7C '2 – 8') "This one's going to be eight, and this one's two."

Act: Regrouped the 8 brown chips and 2 beige chips repeated sliding the 2 brown chips away and 2 beige chips in.

(Doug) "Ok, so you are looking at two minus eight. Ok."

(Courtney Turn 33 Prompt 7C '-1 + -5') "And this would be six."

Act: Grouped 5 brown chips together then selected 1 beige chip and combined the multicolored chips to give 6.

(Courtney Turn 34 Prompt 7C '-7 + 1') "Two four six seven and six."

Act: Counted out 5 beige chips while saying six, and a 6th beige chip while saying 7 then removed a beige chip and placing a brown chip in its place then removing the brown chip and sliding in beige chips to get 7 then sliding away 1 beige and 1 brown to leave 6 beige chips.

(Courtney Turn 35 Prompt 7C '-4 – 2') "Negative four and two."

Act: Grouping 4 beige chips and 2 brown chips to give a mixed color group of 6 chips.

Interview Participant Pseudonym: Jacqueline
Date of Interview: November 14th, 2017
Duration of Interview: 20 minutes

(Doug Prompt 1) "If a seventh grade student were to ask you 'what is the difference between whole numbers and positive integers?' How would you respond?"

(Jacqueline Turn 01 Prompt 1) "I would say that a whole number is non-decimals and a positive integer would be, a positive integer would be anything that is over, above zero."

(Doug Prompt 2) "It is a familiar notion that 'addition makes larger'. This is true when non-zero whole numbers are added together. Is this also true when integers are added? That addition makes larger."

(Jacqueline Turn 02 Prompt 2) "No, because you did negative five plus negative three you would get negative eight and that would be less than, so that would be a smaller number."

(Doug Turn 02 Prompt 2) "Ok, can you give me any other examples?"

(Jacqueline Turn 02 Prompt 2) "Umm, I would say no."

(Doug Prompt 3) "Ok, what types of numbers can you get when one integer is subtracted from another integer?"

(Jacqueline Turn 03 Prompt 3) "When an integer, you can get. You would only get positive because negative, you can't have a negative integer."

(Doug Prompt 4) "What roles can that symbol take within a mathematics problem?"
Act: Placing ' - ' on the table.

(Jacqueline Turn 04 Prompt 4) "What roles? Subtraction."

(Doug) "Anything you would like to add to any of the problems questions we have looked at so far? [No.]"

(Doug Prompt 5) "The numbers negative seven and seven are called additive inverses of each other. What does the Prompt additive inverse mean to you?"
Act: Placing '-7, 7' on the table.

(Jacqueline Turn 05 Prompt 5) "The opposite of each other, so you would have negative seven and positive seven, so they would equal, negative seven plus seven is zero."

(Doug Prompt 5A) "Ok, and I think you might have gone into it but why are negative seven and seven additive inverses?"

(Jacqueline Turn 06 Prompt 5A) "Because they are opposite, umm, negative and positive, so opposite integers, um, I am trying to think of a word. Like completely different, they're the same distance on the number line but the negatives and then the positives on this side. [ok] the same distance from zero."

(Doug Prompt 5B) "And how else can you describe two numbers that are additive inverses?"

(Jacqueline Turn 07 Prompt 5B) "If you add them together you get zero."

(Doug Prompt 5C) "Ok, and can you give me another an example of another pair of additive inverses?"

(Jacqueline Turn 08 Prompt 5C) "Umm, negative two plus two, you get zero."

(Doug Prompt 6) "The next two questions will ask you to consider a quartet of expressions, [ok] umm, in a couple different ways. And I have done them in the collective form and individual form. And please consider, the, those four expressions can you show and explain in writing how you would evaluate each expression."
Act: Placing the first four problems '10 – 2', '6 + 2', '6 – -2' and '-1 + 9' on the table on both a single sheet and individual slips.

(Jacqueline Turn 09 Prompt 6A '10 – 2') "So you want me to solve them? [yup]"
Act: Wrote the '10 – 2' and 8 on the paper.

(Jacqueline Turn 10 Prompt 6A '6 + 2')

Act: Wrote the 6 + 2 and 8 on the paper

(Jacqueline Turn 11 Prompt 6A '6 – -2')

Act: Wrote the vertical slashes through the minus and negative signs and made them addition and positive. Then wrote the 6 + 2 =8.

(Doug Prompt 6A) "And can you think out loud for those."

(Jacqueline Turn 12 Prompt 6A '10 – 2') "Ok, ten minus two we have two whole numbers so ten we have a total of ten we take away two there would be eight."

Act: Pointing to the expression '10 – 2'

(Jacqueline Turn 12 Prompt 6A '6 + 2') "Same thing, we would be addition the two whole numbers together so six plus two is eight.

Act: Then '6 + 2' and her work for those

(Jacqueline Turn 13 Prompt 6A '6 – -2') "And when we have subtraction and a negative together then combine to make positive, so six plus two."

Act: Pointing to '6 – -2' and her additional marks and 6 + 2 = 8.

(Jacqueline Turn 14 Prompt 6A '-1 + 9') "And then we have negative one plus nine, so the negative is in the smaller number so that would be eight as well, so we'll get the positive part of it."

Act: Wrote the 8.

(Doug Prompt 6B) "Ok, in what ways are these expressions similar? And in what ways are they different?"

(Jacqueline Turn 15 Prompt 6B) "Umm, I would say that the negative is, both of these ['6 – -2' and '-1 + 9'] are different then the two above ['10 – 2' and '6 + 2'] because they include negative numbers, but they're also the same because you umm, once you have a negative number you automatically subtract them, so that would be the same as ten minus two, and then when you're adding them, when you're subtracting and you have a negative number it would be the same.

Act: Pointing initially towards the -1 then to both '6 – -2' and '-1 + 9'. Then pointed to '10 – 2' and '6 + 2'. Pointing to '-1 + 9' then pointing to the pair '10 – 2' and '-1 + 9'. Pointing at the minus and negative signs then pairing up the '6 – -2' and '6 + 2'.

(Doug Prompt 6C) "Ok, one of the ways of modeling mathematical operations, particularly integers, would be something called the chip model. Would you be willing to try to show me how you might attempt to explain these with the chip model? [Sure.] Here are a set of chips that you can use. I am going be trying to mark down and follow what you are doing on paper, so?

(Jacqueline Prompt 6C) "I am showing these with the chips?"

(Doug Prompt 6C) "Yup, the same four questions, with the chips. [ok]"

(Jacqueline Turn 16 Prompt 6C '10 – 2') "So we have ten, we have a total of ten and we take away two. So that would be, we would have eight left over." [Ok, hold on one second.]"

Act: Counting out 10 chips some beige some brown, then removing the bottom 2 chips, leaving 3 brown and 5 beige chips for 8 total.

(Jacqueline Turn 17 Prompt 6C '6 + 2') "[Ok, go ahead.] Then we have, we have six and then we add two more to make eight."

Act: Counted out 6 chips half beige and half brown and slid 2 more beige chips for a total of 8 mixed color chips.

(Jacqueline Turn 18 Prompt 6C '-1 + 9') "[Ok] And now for negative one plus nine we have nine, one two three four. We have nine and it would basically be the same thing as saying nine minus one since we have the negative so then we just take one away. So that would be our minus one. And get eight."

Act: Counted out 9 chips some beige and some brown and removed 1 chip leaving 8 mixed color chips.

(Jacqueline Turn 19 Prompt 6C '6 – -2') "[Ok] And then our six minus negative two, I am putting six and the only way that I can think about doing it is just, six minus negative two so we would be, I don't know how to show that, but we would add two more to make eight. I don't know how to explain that one. With the chips."

Act: Counted our 6 chips half beige and half brown then slid 2 more beige chips to make 8 mixed color chips.

(Doug Prompt 6D) "Ok, we also use number lines a lot to try to explain ideas and on the white board there is a number line would you be willing to show me how you might model those with the number line? [yes]"

(Jacqueline Turn 20 Prompt 6D '10 – 2') "So we have ten which is where we are starting and then we take away two, one two, so we are going to have eight as our answer. [ok]"

Act: Placed a black dash at 10 then made 2 hops to the left and put a dash at 8.

(Jacqueline Turn 21 Prompt 6D '6 + 2') "And then we have six, six plus two and we would add two more, one two so it gets eight. [ok]"

Act: Placed a dash at 6 then made 2 hops to the right and made a dash at 8.

(Jacqueline Turn 22 Prompt 6D '-1 + 9') "And then we have negative one plus nine, we would start at negative one then we would add nine, one, two three four five six seven eight nine. So that's eight."

Act: Placed a dash at -1 and 9 hops to the right and a dash at 8.

(Jacqueline Turn 23 Prompt 6D '6 – -2') "And then for six minus negative two I would start at negative two and then add six, that would give me four. I don't know how to show this. If you did six I don't know how to show this one."

Act: Placed a dash at -2 then made 6 hops to the right and put a dash at 4. Then erased this before placing a dash at 6 the making 2 hops to the right.

(Doug Prompt 7) "Ok we'll I'll show you a possible way when we are done. We are going to look at the same batch of questions with this group of expressions. Considering these four expressions how would you show and explain them to a seventh grade student."

(Jacqueline Turn 24 Prompt 7A '2 – 8') "Ok so we have two whole numbers. Two is a whole number and we take away eight from it, so it, would be in the negatives, so it would technically be eight minus two which would be a negative number with a negative in front of it."

Act: Wrote -6

(Jacqueline Turn 25 Prompt 7A '-7 + 1') "We have negative seven and we add one, so it would be, so it would get the negative number because the higher, the bigger number has the seven the bigger number has negative in front of it, so we have subtract seven minus one to get negative six."

Act: Wrote the negative then 6.

(Jacqueline Turn 26 Prompt 7A '-4 – 2') "Negative four minus two you would do, it is technically negative four plus negative two, so it would be, two negatives still make a negative, so it would be negative four plus negative two is negative six."

Act: Wrote the plus sign between the numbers solved as '-4 + -2'. Then wrote the -6.

(Jacqueline Turn 27 Prompt 7A '-1 + -5') "And the same as well for this negative one plus negative five, negative six."

Act: Wrote the -6.

(Doug Prompt 7B) "Ok, how are these expressions similar and in what ways are they different?"

(Jacqueline Turn 28 Prompt 7B) "So I would say the negative four plus negative two is the same, is very similar to negative one plus negative five the just, basically this is like you saying negative one minus five you would get the same response from that. And then negative seven plus one carries the negative and two minus eight this is the same thing as saying negative eight plus two. So, it would kind of be in the same category as that."

Act: Pointing at '-4 -2' then pointed at '-1 + -5' wrote -1 – 5 under the expression as seen above. Pointing at '-7 + 1' and pointing at '2 – 8' wrote -8 + 2 on paper below the expression.

(Doug Prompt 7C) "Ok and if you were to use the chip model for these how might you approach that?"

(Jacqueline Turn 28 Prompt 7C '2 – 8') "Umm, so two minus eight. Umm if we start with two, if we start with two and know that we can't take away eight so if we have two we need six more that would be negative six that we don't have."

Act: Counted out 2 beige chips.

(Jacqueline Turn 29 Prompt 7C '-7 + 1') "[ok] and our negative seven plus one, so we have a total of one and we know that we need seven, but we don't have seven more, so we only have one so seven minus one would be six. So, we need six more so that would make a negative six."

Act: Have 1 beige chip.

(Jacqueline Turn 30 Prompt 7C '-4 – 2') "Negative four minus two umm, we have two whole chips and we need four, negative four minus two so we already start with negative four that we don't have, and we need to make we're adding two more to that so that would be negative six that we don't have as well. I don't know how to explain these."

Act: Counted out 4 chips of different colors pulled in 2 more chips (beige) ended with 6 chips of mixed color.

(Jacqueline Turn 31 Prompt 7C '-1 + -5') "Negative one plus negative five that would be the same as well, so we start with one and we know that we need an imaginary five more, so it would be negative six that we don't have."

Act: Selected 1 beige chip and counted out 5 more chips of mixed colors and slid them in for 6 chips of mixed colors.
(Doug Prompt 7D) "And if we were to on the number line."
(Jacqueline Turn 32 Prompt 7D '2 – 8') "So we have two and you take away eight, so you one, two, three, four, five, six, seven, eight. To get negative six."
Act: Put a dash at 2 and made 8 hops to the left and put a dash at -6.
(Jacqueline Turn 33 Prompt 7D '-7 + 1') "Negative seven plus one you start at negative seven and you add one whole number. So that would be negative six."
Act: Placed a dash at -7 and made 1 hop to the right.
(Jacqueline Turn 34 Prompt 7D '-4 – 2') "Negative four minus two we would start at negative four and we would take away two more to get a negative six."
Act: Made a dash at -4 and made 2 hops to the left.
(Jacqueline Turn 35 Prompt 7D '-1 + -5') "And then negative one plus negative five, so we would start at negative one and we would know we need to go back because it is negative five so one, two three four five to get to negative six."
Act: Made a dash at -1 and made 5 hops to the left.
(Doug) "Anything you would like to add to anything we have discussed."
(Jacqueline) "No."

Interview Participant Pseudonym: Jamie
Date of Interview: November 9th, 2017
Duration of Interview: 39 minutes
(Doug Prompt 1) "If a seventh grader were to ask you 'What's the difference between whole numbers and positive integers?' How would you respond?"
(Jamie Turn 01 Prompt 1) "Oh no, I guess I would say that whole numbers can be positive or negative, it doesn't have to be. If you were talking about a positive integer specifically, you are talking about a positive number, whereas a whole number could be positive or negative. I would respond like that I guess."
(Doug Prompt 2) "It is a familiar notation that addition makes larger. This is true when non-zero whole numbers are added together, is this also true when integers are added? Please provide some examples.
(Jamie Turn 02 Prompt 2) "It is not true because integers can be negative so if you are adding negative numbers together you are not going to get a larger number, so let's say you were adding negative two plus negative two you get negative four and that's further from zero than you would if you were adding two and two, which aren't negative. So, when you add negative integers together, you go, your numbers are smaller."
Act: Moved hands to the left when referencing the result of adding two negative numbers together.
(Doug Prompt 3) "Ok, what types of numbers can you get when one integer is subtracted from another integer?"
(Jamie Turn 03 Prompt 3) "What kind of number? Positive or negative numbers."
(Doug Prompt 3) "Ok, can you give me an example of when integer subtraction would give a positive number?"

(Jamie Turn 03 Prompt 3) "So, if you were doing, well if you were doing umm, two positive numbers, so you're doing like ten minus three you'd get seven and that's a positive number."

(Doug Prompt 3) "And you mentioned that you can also get negative numbers, can you give me an example that would generate a negative?"

(Jamie Turn 03 Prompt 3) "Yeah, so this is subtraction, right? [yes] ok, so you can do umm, you can do three minus seven and you would get a negative number. You would get a negative four.

(Doug) "Ok, anything you would like to add to the problems we have looked at so far?" [No.]

(Jamie) "[Ok] by the way do you want me to ever use any of this, or only when you give me problems."

(Doug) "You can write things down at any particular point, but at this point I am not sure whether anything needed to be written down, so. Feel free to write down anything you would like to."

(Doug Prompt 4) "What roles can that symbol take in a mathematics problem?"
Act: Sliding the ' - ' symbol across the table.

(Jamie Turn 04 Prompt 4) "So that can be used umm, to give a number a negative identity, it can be used if you are subtracting numbers together, umm, it can be used if you're, basically also if you're showing a relationship between like if you want to give a negative identity to any kind of like if it is a fraction or whatever it would be. That could be, that could actually be the fraction line if you wanted to call it that if you wanted to do a division you could do like twenty-seven there and you could do like three down there. I mean like whatever you want to do for that also."

(Doug) "Ok, the numbers seven and negative seven are additive inverses of each other. What does the Prompt additive inverse mean?"
Act: Placing the '-7, 7' paper on the table.

(Jamie Turn 05 Prompt 5) "Ok, so additive inverse, I think it means. So, let's say you're starting at. First thing I thought of, if you're starting at zero and you're adding seven you're going to get positive seven and then if you're starting at zero and you're taking away seven you get negative seven, so I'm seeing them sort of as they're the same number technically, but they're inverses because one is positive and one is negative, I guess, I guess additive because yeah, I guess I am just going to go with you can add seven and then you can also add I guess like, I don't want to say add negative seven to zero but they are sort of just like on opposite sides of the zero and they're the same distance from zero. So that is what I was thinking."

(Doug Prompt 5C) "Ok, can you give me another pair of additive inverses?"

(Jamie Turn 06 Prompt 5C) "Negative three and positive three."

(Doug Prompt 5A) "How else would you describe two numbers that are additive inverses?"

(Jamie Turn 07 Prompt 5A) "Additive inverses, how else would I describe it? [yes] You can describe it as two numbers that are both the same distance away from zero just on different sides, like one is positive and then one can be negative. So, if

you want, I guess different from what I said I would say the same distance from zero, but one is on the negative side one is one the positive side."
Act: Moving her hands to the left and right of a common point showing positive and negative.

(Doug Prompt 5B) "Ok, and you may have answered it, but why are negative seven and seven additive inverses?"

(Jamie Turn 08 Prompt 5B) "Because they are, because one is positive, one is negative, they're both, they both have the same value if you take the identity away from them, so you can say that they are inverses of each other, this one's positive, one's negative and then because of where they fall on the number line. I would say that's why."

(Doug Prompt 6A) "Ok, each of the two remaining problems are, they actually are a family of problems and they will each involve four expressions with four sub-parts. So, the first four expressions are these as spelled out on the individual cards as well. So, one's a duplicate of the other. And please show and explain how you would evaluate each of those expressions.
Act: Sliding the four expressions '-1 + 9', '10 – 2', '6 + 2' and '6 – -2' onto the table as both a collection of small papers and a large paper with the four expressions.

(Jamie Prompt 6A) "How I would evaluate each one, in terms of what's going on in the expression or in terms of what each number is or like how I would solve it."

(Doug) "How would you show them being solved?"

(Jamie Turn 09 Prompt 6A '10 – 2') "Ok, [and you can please write down your work] yeah. So, let's say we start with ten minus two so, how would I show they the result, so. I guess I would start with, I would write ten, and then minus two, I guess if you wanted to just show it like without doing a, assuming a person didn't know a procedural method I guess I would show it, you can draw assuming that those were equal. So, you can start at ten and then you can go back two, so ten nine eight and then you can say that eight is your answer."
Act: Wrote the expression '10 – 2'. Drew the number line from zero to ten pointed to the ten, counted back 2 while pointing at the numbers in turn, then circled the eight.

(Jamie Turn 10 Prompt 6A '6 + 2') "[ok] Yeah, for six plus two you can do the same type of strategy where you can start at so let me just note that this is six plus two, and you could do the same type of things. So, it is already zero to eight. So, we would start at six and then we could, we could add two to that, so when we move we are moving right on the number line on left, cause when we subtract we go left and when we add we go right so we go one two and then we will land at nine and that would be your answer. Oh my gosh, wow, that was a mistake, that was a mistake scratch that from the record, scratch it, scratch it, so six, so wow, I wish I could work, so six seven eight would be two bumps and you'd get, you'd get eight as your answer for that one [good catch] I was like man, man simple problem like six plus two."

Act: Pointed to the number line from the previous example. Wrote out the expression, the drew a new number line from zero to 8 skipping 7 pointed at 6 then extended the number line to 9 starting at 6 moved two units to the right and circled the 9. Crossed off the 9 on the number line and squeezed in the 7 to make the correction. Starting at 6 going up 2 and circling 8 as the destination.

(Jamie Turn 11 Prompt 6A '6 – -2') "So this one's ['6 – -2'] a little more complicated. So, I learned like that keep change, change procedure, but that really doesn't show any understanding of what you're actually doing, umm, cause if I did that it would be what six plus two because it would be keep change, change. That doesn't really show understanding so let's say we had a number line so let's do this again, zero two three four five six seven eight ok, so it is six minus negative two, so this is like a double negative her so. Saying six minus negative two it actually is the same as saying six plus two but it is because if you start at six and you were going to take away, let's just say it was six minus two if you were going to take away a positive two it would just be six five four cause your just taking away a positive number but since the number is negative it's like your saying six minus man like how do you even like explain that, it's like, it's like six taking away these double negatives just don't. Just doesn't make sense. I wish I would know, to be honest I really did, I learned yeah, my teachers all they ever said was if you're faced with this kind of issue you add and add, you just change them. I guess in my head I thought I knew why, but I just don't. I think, I just don't know how to explain that. In terms of why we are taking it away."

Act: Wrote the original expression '6 – -2' then '6 + 2'. Crossed out the '6 + 2'. Drew a number line from 0 to 8. Pointed to - -. Pointed 6. Wrote '6 – 2' and circled the 2 pointed to 6, 5, 4, on the number line.

(Doug) "In the end you a way that can make that make sense." ["Yeah, Because I am lost with that one."]

(Jamie Turn 12 Prompt 6A '-1 + 9') "I wish I knew, well ok, so negative one plus nine so we got. Let's put zero here this time. That's a negative one so this is one two three four five six seven eight nine I will go to ten. So this one, so you're still this is an addition problem so you can just use the number line the same way you would in the other problems, so let's say you started at negative one we just have to add nine to negative one so you just move right on the number line the same way you would with the other ones so one two three, one two three four five six seven eight nine. And you get eight as your answer."

Act: Wrote the expression '-1 + 9' and drew the number line from -1 to 10. Made hops from -1 to 8 and circled -8.

(Doug Prompt 6B) "In what ways are these expressions similar and in what ways are they different?"

(Jamie Turn 13 Prompt 6B) "Both show adding and subtracting numbers. The only difference is some of the numbers in the problem are positive, some are negative. Umm, some will. Oh, actually, I noticed, they all give positive answers that's something they all leave you with a positive answer, cause even though I didn't know how to explain this one, I know that it would be eight. Oh, they all give you eight. Every problem gives you eight, positive eight. And it is just a bunch of different ways to show how to get the same answer which is cool. Oh, I didn't realize that till now."

Act: Circled the 8 on the third number line. Moving over the expressions without clearly pointing to any in particular.

(Doug Prompt 6B) "Ok, anything else on similarities or differences?"

(Jamie Turn 13 Prompt 6B) "These six plus two, what do you call it oh this one so even though this one's subtraction, I noticed that I would end up going up on the number line, so I would be going to the right, whereas this one is subtraction, yet I had to move to the left. So that is one thing I notice as a difference between these two subtraction problems. That is the only thing I notice."

Act: Pointing to '6 + 2' and '-1 + 9'. Then pointing to '6 – -2' adds 2 hops to the number line for '6 – -2'. Adds 2 hops to the number line for '10 – 2'. Then pointing to '10 – 2' and '-1 + 9'.

(Doug Prompt 6C) "Ok, teachers are getting into the habit of more and more using other manipulatives, including we have the chip model for integers. Are you willing to show and explain how you might evaluate these expressions using the chip model?"

(Jamie) "The chip model for integers? [yes] umm, I've never heard of the, I've never been taught the chip model, for integers but I will do my best to show you what I think it could be."

(Doug Prompt 6C) "Ok, if you are willing to try I would like to see [yeah] if you don't feel comfortable we can skip it."

(Jamie) "I can try, I am willing to try it. I just don't know what it is, but"

(Doug) "We are going to be working on those same problems so don't move those pages to far, here are our chips."

(Jamie)"So, I am going to say, alright, so I am just going to assume right now that these are my positive integers {beige chips} and these are my negative one's {brown chips}. So yeah.

Act: Sorted the chips by color into a stack of beige chips and a stack of brown chips.

(Jamie Turn 14 Prompt 6C '10 – 2') "So, let's say we had ten minus two so both numbers are positive, ten and two are both positive, so, we'll start with ten the first number, so one two three four five six seven eight nine ten, we'll go with that. Now we're subtracting so I am going to take away two from ten, so one two away, and then we're left with one two three four five six seven eight. So that would be eight."

Act: Counted out 10 beige colored chips then slid away 2 beige colored chips made a rough row of the 8 remaining beige chips.

(Doug) "Ok, let me write that down, ok."

(Jamie Turn 15 Prompt 6C '6 + 2') "So, for six plus two, umm, they are both positive, so I am going to keep with the same color chips, so I am going to count out six of them cause the first number one two three four five six. And then we have to add two to that, so we have six here we'll add two because it is a positive we'll add two from our positive pile, so it will be one two and then that will get us one two three four, one two three four five six seven eight. Eight again."

Act: Counted out 6 beige chips lined up those 6 chips slid in 2 chips 1 at a time. To make a line of 8 beige chips.

(Jamie Turn 16 Prompt 6C '6 – -2') "So next problem, I'm just gonna keep my six here this was a positive six minus negative two so we have our six chips here so minus negative two, once again I hit this problem, so you might assume you take away but you don't if it was a positive I would just take two away but they're not, so it's almost like I have to add on. I'm like thinking that I would use, that wouldn't make sense. Minus negative two. I am not sure how to show this one."

Act: Counted out 6 beige chips acts to slide 2 away when referring to subtracting positive 2. Holding a brown chip.

(Doug) "Ok, so then we'll skip it and move to the next one."

(Jamie Turn 17 Prompt 6C '-1 + 9') "So, negative one I kinda, is it ok if I switch around what I want to do with this one, like the way I want to show it. [yup] So I am actually gonna show it kind of like a line, but I am going to use the chips, so I would say this is negative. So, we'll keep negative one there, I am going to put my finger were zero would be because, I don't want to count that, well actually I will count it I don't want it, so negative one then it is going to be one two, I am going to like these up. I am just going to line them up as if this was a number line so let's just put these down. Yeah, let's just line them up. So, I am going to start here and going to say that because I have negative umm actually yeah this is going to represent zero too, so let's say we have our negative, so you can just add a few more if you want or whatever, these are negatives, so we can assume, I am going to put this one a little higher so we know that is zero, so this would be negative one. So, I want to add nine to that, so I would move to the right, my right, so I would just add nine to where negative one is, so one two three four five six seven eight nine and then I would just count a distance that this number is from zero so one two three four five six seven eight would be my answer, positive eight.

Act: Placed a brown chip out, then left a gap held by her finger and moved in a beige chip closed the gap then opened it again and made a long line of beige chips. Pointed to one of the beige chips to represent zero and added a pair of brown chips slide the zero beige chip out of line pointing to -1 brown chip counts 9 chips right and counts the number of beige chips to the right of zero.

(Doug 6D) "Ok, though it might be duplicating what you did on paper I would like to see how you might approach this with a number line model explicitly, [sure, sure] For that you can use the white board."

(Jamie) "Oh, I'm sorry I have ruined the chip model … [yeah we're good] cool."

Act: Knocking some chips to the floor.

(Jamie Turn 18 Prompt 6D '10 – 2') "So just show them all on here, [yup] ok. So, for ten minus two. So, we see our first number's ten, so we start on, on ten right there. So, there's ten, so ten is here so we are going to subtract two from there so in order to do that we, when we're subtracting we move left, so we would go back to, so we'd go one two and then we'd get eight as our answer. So that would be ten minus two is eight."

Act: Drew a vertical line in red at 10 drew and arrow to the left a distance from the number line made 2 left hops and wrote the number 8.

(Jamie Turn 19 Prompt 6D '6 + 2) "Can I erase it, are you ok, [yes] So the next one is six plus two, so start on the first number which is six and we are adding so that's a positive relationship so we're going to move over to the right, cause they are both positive so it is ok to go to the right this time not that it would always make a difference but so we are going to add two so we are going to move over two, so we get eight as our answer again."

Act: Wrote the expression '6 + 2' made a dash in blue at 6 Drew an arrow to the right, then made 2 hops to the right and wrote 8.

(Jamie Turn 20 Prompt 6D '6 - -2') "So here we are again, six minus negative two, I am going to try this time. So, we're at six we want to take away a negative number. So, taking away a negative number is basically like adding on a positive one, it's like when you say, I am not not going to do it, it's like the same principle. When you're taking away a negative they sort of cancel each other out, I just can't. I wish I knew how to explain it better but umm, so we have six taking away a negative, we would do the opp, inverse so take away a negative. It's the same thing as adding a positive. Cause you can't take away a negative number. Ok, so, so I am just going to move it here and then you get eight, I don't know how to explain it [ok] my thoughts like so incomplete, umm, ok."

Act: Wrote the expression '6 – -2' and made a dash at 6 pointing to the - -. Wrote +2 made 2 hops to the right and wrote 8.

(Jamie Turn 21 Prompt 6D '-1 + 9') "So then negative one plus nine. We would start at negative one and we're adding so we go to the right, so we will count up nine from negative one, so we'll go one two three four five six seven eight nine. And then we'll get eight for our answer."

Act: Wrote the expression '-1 + 9' and made a mark at -1 drew an arrow to the right made 9 hops to the right and wrote 8.

(Doug Prompt 7A) "We are going to do the same batch of procedures with a new batch of problems. So, take these and set them aside, [ok] and exchange them for those."

Act: Passing over the second batch of problems.

(Jamie) "So, should we start."

(Doug Prompt 7A) "Let's start with the paper and pencil initially. Show how you would evaluated those to determine they're sums or differences."

(Jamie Turn 22 Prompt 7A '-1 + -5') "Yeah, so negative one plus negative five. So negative one plus negative five, umm, they're both negative numbers and we're adding them together. So, if you add two negative numbers together you are going to get a larger negative number or a smaller negative number, I guess, however you'd word that, but umm so we would take negative one plus negative five and you'd get negative six, now you want me to show my work, right? Like show how I did it, or is my explanation enough? Or"

Act: Wrote the expression '-1 + -5' pointing to both numbers repeatedly wrote -6.

(Doug) "The explanation seems fine."

(Jamie) "Ok, yeah, I just don't want to short you with anything."

(Jamie Turn 23 Prompt 7A '-4 – 2') "So then negative four minus two, umm ok, so we have negative four minus two. Oh no, So, you were taking away a positive number from a negative number so that would be umm. I really, my gut reaction is to just do this the procedural method of keep change, change. I just don't know why I'm doing it, I haven't been able to like learn this yet in my classes like undo this procedural learning yet, so I don't know, [ok, but what answer would that give you] oh, that would give me a negative six."

Act: Wrote the expression '-4 – 2'. The wrote '-4 + -2' then –6 when prompted.

(Jamie Turn 24 Prompt 7A '-7 + 1') "So the next one negative seven plus one umm, so that would be we're adding a positive number to a negative one so it's the same idea where if we had a line where we would be adding a positive, one positive to a negative so it would be negative six because negative seven, if you want to make that number larger by one number you, it would go, the number would get bigger by one. And it would be negative six, cause that's bigger than negative seven."

Act: Wrote the expression '-7 + 1' then wrote the -6.

(Jamie Turn 25 Prompt 7A '2 – 8') "So two minus eight. So, this one, we just have to count down and then go, just count down from two until you get count down from, count down eight from two. So, it would be two one zero negative one negative two negative three negative four negative five negative six and then you get negative six as your answer."

Act: Wrote the expression '2 – 8' Counting out on the fingers to -6 and recording -6.

(Jamie Turn 26 Prompt 7B) "So things I notice I got negative six for all my answers. I noticed that when you add two positive integers I mean when you add two negative one's together you get a negative answer cause umm you're just increasing you're making a number smaller therefore making it a bigger negative number. I notice when you add a positive integer and a negative integer umm it becomes a smaller negative number because it's getting bigger and the smaller the negative number the bigger it is and also with two minus eight no well when you take away a bigger positive integer from a smaller positive integer you are going to get a negative number because it's bigger than the small one so you're going to go past zero. To the negative zone."

Act: Pointing to -1 and -5 pointing to '-4 – 2'. Pointing to '2 – 8'.

(Doug Prompt 7B) "Ok and I think you just answer the next question which would have been in what ways are the expressions similar and in what ways are they different? [oh yeah, I sort of skipped that, I remember you asked that before.] no, that's fine if. Is there anything you'd like to add to that? that's ok. [I think I covered it all.] It seems that way."

(Doug Prompt 7C) "Would you like to attempt these with the chip model? [ah, sure] ok. Let's give it a shot. Again, if you want to skip any of them you can, don't worry about it."

(Jamie) "I keep doing this, I'm ruining the chip model. I'm mad that I don't know it, [they are pretty durable] I am try to, I'm trying to I'm mad I don't know it, so I am throwing them on the ground."

<u>Act:</u> Sliding chips back to in front of her. Some of which fall to the floor as she attempts to move them.

(Jamie Prompt 7C) "So I liked what I did before were I had them lined up I kind of liked that representation I notice I am really gearing towards. I feel like with negative numbers it is like really good to, at least for me, my understanding comes the most when I see it in a number line, so I just feel like having different chips with different colors is a cool way to show positive and negative. Um, I want to figure out. I want this to be zero those two because I don't want to call, well yeah, I'll just make those two zero that's fine. I'll leave it like that, it's fine."

<u>Act:</u> Lined up all the chips with the brown chips on the left and the beige chips on the right and pulled out the middle brown and beige chips as zero.

(Jamie Turn 27 Prompt 7C '-1 + -5') "So negative one plus negative five so we would start at negative one and we are adding another negative number to it. So, because they're both negative it's just going to get smaller. Umm, so it would just be negative one, adding negative five so it's you would go, you'd start at negative one and you'd be add negative five so umm so one two three four five and we count positions, so it would be one two three four five six and you get negative six as the answer."

<u>Act:</u> Pointing at the brown chip closest to "zero" in the line of chips then counted to the left 5 units then count 6 to the left and called it -6.

(Jamie Turn 28 Prompt 7C '-4 – 2') "Negative four minus two, one two three four, umm so again, I am not really sure how to show that one, so I am just going to skip it cause were not going to get anything at it is going to leave me confused."

<u>Act:</u> Using the same long line of chips, pointed to the 4th brown chip.

(Jamie Turn 29 Prompt 7C '-7 + 1') "Negative seven plus one, ok so we would count back so we could get to seven so from zero, so one, two three four five six seven. So were at negative seven right here and were adding a positive integer to it so it's going to make the number bigger and closer to zero this one, so we are going to add a positive one which will move it to the right, so one and we're right there and we're at negative six. [ok]"

<u>Act:</u> Using the same line of chips and counting to the 7th brown chips on the left then moved 1 unit to the right. Then points to the 6th brown chip and called it -6.

(Jamie Turn 30 Prompt 7C '2 – 8') "Two minus eight, so we'll start at positive two, one two and we'll then just count down eight, to the left cause it is going to become smaller. So, one two three four five six seven eight. And if you count, one two three four five six you're at negative six again."

<u>Act:</u> Pointing to the 2nd beige chip to the right of zero counted 8 to the left going 6 brown chips in counting the 6 brown chips and calling it -6.

(Doug) "Ok, and I would like to go back to the number line once more [sorry this time I am going to go kind of like this get them as far away from me as possible.]"

<u>Act:</u> As she slides the chips out of the way.

(Jamie Turn 31 Prompt 7D '-1 + -5') "Ok, for the white board so for negative one minus negative five we would start at negative one and then we're adding a negative number to it, so it is going to get smaller cause it's already negative, so we'll go one two three four five and we'll be left with negative six."

Act: Made a mark at -1 made 5 hops to the left and wrote -6.

(Jamie Turn 32 Prompt 7D '-4 – 2') "So the next one oh yeah this one again I am really just not sure how to show it I don't know how to show it that I would be counting up like this, we'll, I wouldn't be doing that oh my gosh, never mind, I'm not. Oh, wait, I think I actually just had an epiphany oh, I actually understand this one now, ok wow, all I had to see was it hear on the number line so it works the same way that this works, so if you have negative four and you want to take a value of two away from it, it's making the number smaller the same way that if it was, oh, the same way that if it was negative four plus two it would make it ah, umm, on, now I am just going to confuse myself but negative four minus two would give us a smaller value cause were taking a value of two numbers away from it so negative four taking two numbers away from it would be one two cause I am making it a smaller number. Yeah, yeah, if I was adding two if would go this way because I'm making the number bigger. Ok. I completely understand that now, wow that was an epiphany.

Act: Made a mark at -4 then made 2 hops to the right then erased that pointed to '-7 + 1'. Wrote '-4 + 2' then erased that and wrote '-4 – 2'. Then made 2 hops to the left then changed the subtraction to an addition for -4 + 2 and drew 2 hops to the right.

(Jamie Turn 33 Prompt 7D '-7 + 1') "Ok, number lines really help, with that. Ok, cool, feels good, I don't have to use that the procedure, ok, so negative seven plus one, we start at negative seven, again, it's what I said before we're adding one value to it, so we are making the number bigger, so it is negative seven plus one, one value would give us negative six. Ok."

Act: Made a dash at -7 and 1 hop to the right and wrote -6.

(Jamie Turn 34 Prompt 7D '2 – 8') "The last one, two minus eight. We would start at two, and we're taking eight away from it, so we would go to the left, so it would be one two three four five six seven eight because they're both positive. Ok, cool."

Act: Made a mark at 2 then 8 hops to the left after drawing an arrow to the left

Interview Participant Pseudonym: Jordan
Date of Interview: November 9th, 2017
Duration of Interview: 29 minutes

(Doug Prompt 1) "If a seventh grader were to ask you 'What is the difference between whole numbers and positive integers?' How would you respond?"

(Jordan Turn 01 Prompt 1) "Whole numbers and positive integers? [yes] So whole numbers meaning no fractions, no decimals, and positive integers also meaning. This is a tough one, I am not really sure how I would respond. I understand that integers mean all positive and negative numbers also zero, but no fraction decimals, but I also see a whole number as being the same thing, so I would, I don't know I would have to do more research before I answered that child."

(Doug Prompt 2) "Ok, we'll take a quick point at that a little bit later. It is a familiar notation that addition makes larger. This is true when non-zero whole numbers are added together. Is this also true when integers are added? Please provide some examples."

(Jordan Turn 02 Prompt 2) "Well yeah because integers can also be non-zero whole numbers so that's definitely the case, but I don't know if that's always the case. Can you repeat the end of the question?"

(Doug Prompt 2) "Is it, is this also true that addition makes larger when integers are added?"

(Jordan Turn 02 Prompt 2) "When two integers are added together [yes] in some cases, when you add one and one together you're making a larger number, but if you add two negatives, you're making a lesser number, so I guess you're not always making a larger number with integers, since integers include positives and negatives."

(Doug) "Ok, anything you'd like to add on either of those two so far?"

(Doug Prompt 3) "What types of numbers can you get when one integer is subtracted from another integer?"

(Jordan Turn 03 Prompt 3) "Positive numbers, negative numbers, is that what is meant by that?" ["Yes"]

(Doug Prompt 3) "Ok, so can you give me an example of when you would get a positive number when you subtract integers?"

(Jordan Turn 03 Prompt 3) "When you subtract integers umm how would, wait what did you say, positive."

(Doug Prompt 3) "When you subtract two integers can you show me, give me an example of when you'd get a positive answer."

(Jordan Turn 03 Prompt 3) "Well if you do three minus two you're answers one and that's still a positive number."

(Doug Prompt 3) "Ok and you mentioned you can also get negative numbers when integers are subtracted how might, what problem might show that?"

(Jordan Turn 03 Prompt 3) "Negative five minus negative three your answer will still be a negative number."

(Doug Prompt 4) "Ok, what role can that symbol represent within a mathematics problem?"

Act: Placing ' - ' on the table.

(Jordan Turn 04 Prompt 4) "It can represent a negative number, it can represent subtraction. Yeah."

(Doug Prompt 5) "Ok, the numbers seven and negative seven are called additive inverses of each other. [ok] what do you think the Prompt additive inverse means?"

Act: Placing '-7, 7' on the table

(Jordan Turn 05 Prompt 5) "One is basically plus seven while one is minus seven. They're inverses because. I don't know how to describe it. Additive integers. What was it again, additive [additive inverse] inverse, additive inverse."

(Doug Prompt 5A) "And why are negative seven and seven additive inverses?"

(Jordan Turn 06 Prompt 5A) "Maybe if you add seven, maybe when you add seven you get zero if you had minus seven, ok so if you add seven to one you get zero and if you subtract seven from one you also get zero."

(Doug) "When you say one do you mean the number one or one of the."

(Jordan Turn 06 Prompt 5A) "Did I say one or did I say seven, I meant seven [you said when you subtract seven from one] I meant when you subtract seven, when you add seven to negative seven you would get zero and when you subtract seven from seven you would get zero."

(Doug Prompt 5B) "Ok, how else would you describe two numbers that are additive inverses?"

(Jordan Turn 07 Prompt 5B) "So maybe they both can be, one can be they will get the same number when adding to one and subtracting to the other the same number."

(Doug Prompt 5C) "Ok, can you give me an example of another pair of additive inverses?"

(Jordan Turn 08 Prompt 5C) "Negative six and six."

(Doug) "Ok, anything you want to add to what you've looked at in the previous couple questions?"

(Jordan) "I don't think so."

(Doug Prompt 6) "The next two questions are similar and they're each based on a bunch of different numbers and I would like you to consider these four mathematical expressions. They are the same thing on the paper and on the cards, can you show and explain how you would evaluate each expression in its written symbolic form."

Act: Placing the first batch of expressions on the table.

(Jordan Turn 09 Prompt 6A '6 + 2') "So, for six plus two. How do you mean draw it? I would just count it six, seven, eight." [ok]

Act: Wrote the expression '6 + 2' then the 8.

(Jordan Turn 10 Prompt 6A '-1 + 9') "Ok, so then you have negative one plus nine, so you really have eight, because if you that's like saying nine take away one, because you have negative one so you don't even have one you have less than zero and then your adding eight to that, so one covers the negative one and so you'd get eight."

Act: Wrote the expression '-1 + 9' then 8.

(Jordan Turn 11 Prompt 6A '10 – 2') "And then ten minus two, you just have ten and you take away two, and you get eight."

Act: Wrote the expression '10 – 2' then the 8.

(Jordan Turn 12 Prompt 6A '6 – -2') "And then, six minus negative two. Would also be eight. Negative eight, you'd have six and you're taking away. I am getting thrown off, are they all supposed to be the same? This is tricky, because you can't take away a negative there, so it is like saying plus, six plus a negative I don't know, I'm stumped I'm getting frazzled." [ok]

Act: Wrote the expression '6 – -2' and the solution 8. Wrote parentheses around the -2.

(Doug Prompt 6B) "In what ways are the expressions similar and in what ways are they different?"

(Jordan Turn 13 Prompt 6B) "Some have negatives, some have solely positives. It's easier to subtract when you have whole positive numbers. Like for example in this problem ['-1 + 9'] you are adding by you are getting a smaller number than nine because you have a negative here, so that's almost like if you look at the problem you'd think nine you'd think you'd get a bigger number than nine but since it's a negative you're actually going to get a smaller number."

Act: Pointing to '-1 + 9'.

(Doug Prompt 6C) "Ok, one of the key ways that integer operations are often shown is using what's called the chip model. [ok] and would you be willing to try to show me how you would evaluate and explain each of these problems using the chip model?" [Sure.]

(Jordan Turn 14 Prompt 6A '6 – -2') "I'm wrong with this one. Any way's I'll go back to it."

Act: Looking at '6 – -2'

(Jordan Turn 15 Prompt 6C '6 + 2') "Ok, ok so I have six, and I am adding two and that gives me eight." [Give me one, ok whenever you're ready].

Act: Sorting the chips into different colored groups and counts out 6 brown chips then slid over 2 beige chips grouping them together we have a total of 8 chips 6 brown and 2 beiges.

(Jordan Turn 16 Prompt 6C '-1 + 9) "[ok, whenever you're ready] Then I'll say I have nine well you technically have negative one but I'm basically, I look at addition with sub, with a negative as I'm having nine taking away one to get eight. Or you can use this light chip to represent the negative, but since I'm adding when I add this whole number from the nine, well one it takes place of the negative, and then I only have since I used that one I have eight."

Act: Counting out 9 brown chips removed 1 brown chips. Grabbed a beige chip then swapped out the beige chip with a brown chip then slid 7 move brown chips into play to get 8 brown chips.

(Jordan Turn 17 Prompt 6C '10 – 2') "And then for the next one I have ten. I take away two and I have eight."

Act: Counted out 10 brown chips and slid away 2 brown chips.

(Jordan Turn 18 Prompt 6C '6 – -2') "[ok] and then this last one is going to drive me crazy, so I have six and I am taking away negative two. So, I feel like my answer should be negative. I guess these can represent my negatives. I have negative two. So maybe these should all turn to negative eight. That's negative and negative, basically just makes this a plus. I'm confused. I don't know. I want to move on."

Act: Pulls out 6 brown chips pulled out 2 beige chips changed the – (- in "6 – (-2)" into a +.

(Doug Prompt 6D) "Ok, we can do that. I will walk you through how I might approach that a little bit later and clarify. Another way that's often used is number lines and ok we can skip that if you'd like or if you are willing you can show me how you might do them on the number line. [I'll try on the number line] so the number line is on the white board."

(Jordan Turn 19 Prompt 6D '6 + 2') "Ok, so I have six and I am adding two. So, I am jumping to get eight, so you have one two."
Act: Made a black dot at 6 and jumped 2 spaces the right and made a mark at 8 counted out with 2 dashed below 7 and 8.

(Jordan Turn 20 Prompt 6D '-1 + 9') "Next one I have negative one, so I am over here and I'm adding nine, one two three four five six seven eight nine. So, I am going to get eight. This actually might be an easier way. [hold that's ok] I deleted it. So, I am on negative one, [right] and I jumped over one two three four five six seven eight nine cause I'm adding nine, so my answer was eight." [ok]
Act: Made a dot at -1 made 9 dots as she counted to the right getting to 8 then made a mark at 8.

(JordanTurn 21 Prompt 6D '10 – 2') "Next I have ten and I am going to take away two, so I am going to go this way. I am going to take this one away and this one away, and I am going to get eight as my answer."
Act: Made a mark at 10 then made marks as she counts to the left 2 spaces and makes an x at 8.

(Jordan Turn 22 Prompt 6D '6 – -2') "And now for the tricky one, I have negative six and you are taking away two. So, I am going this way, so I would have negative eight. So that is making me believe my first instinct was right, yeah, so I guess it would be negative eight. That is what I am going to go with. [ok] So this is wrong."
Act: Made a mark at -6 then moves two spots to the left. Changes 6 - -2 = 8 to a -8.

(Doug Prompt 7A) "Do you want to take a moment to take a sip of your iced tea or water. [Sure.] Ok, ready to resume? [yes] Ok so, I now have in front of you a new set of four problems and we're going to go through the same activity that we just or set of activities that we just did. So, I would like you to explain how you'd show on paper how you'd evaluate each of these expressions?"
Act: I am placing the next batch of problems on the table.

(Jordan Turn 23 Prompt 7A '2 – 8') "Ok, so I have two I am taking away eight, so I would have negative six, I don't know how I would show that, oh, just on the paper, [just on the paper for now]. So, I have two, I only have two of those, but I am taking away eight, so I am taking away one two and then three four five six seven eight. But they're negatives so I have one two three four five six left but they're negative."
Act: Wrote the expression '2 – 8' then =-6 wrote 2 filled in dots and 6 open dots having crossed off the first 2 dots. The 6 open dots give -6.

(JordanTurn 24 Prompt 7A '-7 + 1') "[ok] And then, negative seven plus one would give me negative six, because I have the whole the one whole, umm, makes up for one of the negative ones.
Act: Wrote the expression and =-6.

(Jordan Turn 25 Prompt 7A '-4 – 2') "and one four minus, negative four minus two so, I'm negative four and I am subtracting two more from that, so I'd have negative six."

Act: Wrote the expression and =-6.

(Jordan Turn 26 Prompt 7A '-1 + -5') "And then negative one plus negative five would give me negative six, which is leading me to believe that this should have been eight. So, I feel like again I was right with the plus now. Any way's moving on from that." [ok]

Act: Wrote the expression and = -6. Changes the answer to '6 – -2' back to 8.

(Jordan Turn 27 Prompt 6A '6 – -2') "So I got negative six for all of them. [alright, so] I want to change my answer for this one ['6 – -2'], I think it's positive eight. [ok] for sure. Cause a negative and a negative is a positive, plus."

Act: Pointing to '6 – -2'.

(Doug Prompt 7B) "In what ways are these, are the current problems similar? And in what ways are they different?"

(Jordan Turn 28 Prompt 7B) "Different from this set or different from each other. [different amongst themselves], well for one they all get the same answer so that's similar. What's different is some of them have negatives, some of them have positives, some are subtraction, some are addition. And yeah.

Act: Pointing to the first set of expressions on her paper. Pointing to all 4 of the second group of expressions points to '-7 + 1' and '2 – 8' then to '-1 + -5' and '-4 – 2'

(Doug Prompt 7C) "Ok, so using the chip model, I would like you to approach each of these."

(Jordan Turn 29 Prompt 7C '2 – 8') "So I have, I will use these for my negatives, oh, no wait, I have two, but now I am going to take away eight, I am going to use these as my negatives. So, I am taking away these two, one two from. I'll just take them away, one two three four five six seven eight and then when you count these up, I get six and since I used the light colors for negatives, I know these are negative six."

Act: Sorted the chips into beige and brown Pulled over 2 brown chips placed beige chips on top of the 2 brown chips to form 0 pairs removed the brown part of these pairs, reused the beige part of the pairs and counted out 4 additional beige chips for a total of 6 beige chips.

(Jordan Turn 30 Prompt 7C '-7 + 1') "[ok] So now I have negative seven, so I have all these negatives, and I'm adding, I'm adding one whole, but by adding this one whole I'm really just taking away one negative, so, I have six negatives still. Rather than seven, negative seven."

Act: Set out 7 beige chips then picks up 1 brown chip then matches that with 1 beige chip and removed the beige and brown chip leaving 6 beige chips.

(Jordan Turn 31 Prompt 7C '-4 – 2') "Next I have negative four and I am subtracting two, So I'm subtracting two so that is just adding two more negatives, since I am subtracting, it is adding more of negatives."

Act: Setting out 4 beige chips and sliding in 2 beige chips for a total of 6 beige chips.

(Jordan Turn 32 Prompt 7C '-1 + -5') "And then lastly, I have negative one and I am just going to add five more negatives, and again that gives me negative six."

Act: Setting out 1 beige and 5 beige chips which are slid together to give 6 beige chips.

(Doug Prompt 7D) "Ok, we're going with the number line if you would."

(Jordan Turn 33 Prompt 6D '6 – -2') "Oh, I just realized what I did before, I went to negative six, when it's positive six, so I should have gone right here and then added two which would give me eight. Now I see what I did there. I went on the wrong number."

Act: Pointing at -6 then marking at 6 made to hops to the right.

(Jordan Turn 34 Prompt 7D '2 – 8') "Ok, so for this one I am at positive two and I have to subtract eight, so I have to go this way and I am going to subtract one two three four five six seven eight. So, I land on negative six."

Act: Made a mark at 2 and made 8 marks as she moved to the left and made an x at -6.

(Jordan Turn 35 Prompt 7D '-7 + 1') "And the next one I have negative seven. So, I am right here, and I am going to add one, so I am going this way which gives me negative six again, I am only going one spot over. To the right."

Act: Draws a dash at -7 then drew an arrow to the right 1 unit long and made a mark at -6.

(Jordan Turn 36 Prompt 7D '-4 -2') "And then I have negative four, so I am right here, and I am going to subtract two, and since I am subtracting I am going to go to the left two over which would give me negative six."

Act: Made a mark at -4 and marks as she is counting to the left 2, and an x at -6.

(Jordan Turn 37 Prompt 7D '-1 + -5') "And then last one I have negative one right here and I'm gonna add negatives, but since I'm adding negatives I'm going this way to the left, one two three four five. Which would give me negative six again."

Act: Makes a mark at -1 then makes an arrow with her finger pointing to the left counts out and makes 5 marks to the left and an x at -6.

Interview Participant Pseudonym: Kayla
Date of Interview: November 14th, 2017
Duration of Interview: 23 minutes

(Doug Prompt 1) "If a seventh grader were to ask you 'What is the difference between whole numbers and positive integers?' How would you respond?"

(Kayla Turn 01 Prompt 1) "I guess a whole number could also be a negative number. Positive integer would only be like above zero. Whole number above zero."

(Doug Prompt 2) "Ok, it is a familiar notion that addition makes larger. This is true when non-zero whole numbers are added together is this always true when integers are added?"

(Kayla Turn 02 Prompt 2) "No, because you can add negative numbers and then if you add two negative numbers it would become less, like higher negative number which would is further below zero."

(Doug Prompt 2) "Ok, can you give, provide me with any additional examples."

(Kayla Turn 02 Prompt 2) "So like if you did negative two plus negative six your answers negative eight so it would be less than those two numbers."

(Doug Prompt 3) "Ok, what types of numbers can you get when one integer is subtracted from another integer?"

(Kayla Turn 03 Prompt 3) "What types of numbers do you want? Other integers, like smaller integers."

(Doug Prompt 3) "Ok, so when integers are subtracted you said smaller, does that mean pos, other smaller positive integers?"

(Kayla Turn 03 Prompt 3) "Well it could be either cause if you subtract a larger integer from, like if you subtracted six from two it would be negative four, so it would be negative, a negative integer."

(Doug Prompt 3) "Ok, can you get a positive integer when you subtract integers?"

(Kayla Turn 03 Prompt 3) "Yeah, like if you subtract a smaller number from a bigger number. It is still positive."

(Doug) "Anything you'd like to add to what we've discussed so far? [no]"

(Doug Prompt 4) "What roles can that symbol take within a mathematics problem?"

Act: Sliding the ' - ' sign across the table.

(Kayla Turn 04 Prompt 4) "It's a subtraction sign, so you'd subtract the numbers that go before and after or it's a negative symbol. So yeah"

(Doug Prompt 5) "Ok the numbers negative seven and seven are additive inverses of each other. What does the Prompt additive inverse mean?"

Act: Sliding '-7, 7' across the table.

(Kayla Turn 05 Prompt 5) "I don't know but I'm gonna guess that it means that it's the same integer but one's positive and one's negative."

(Doug Prompt 5A) "And ok, and I think you answered it at least in part by why are seven and negative seven additive inverses?"

(Kayla Turn 06 Prompt 5A) "Because it is the same like number but one's negative and one's positive."

(Doug Prompt 5B) "Ok, how else would you describe two numbers that are additive inverses?"

(Kayla Turn 07 Prompt 5B) "I guess if it's like they're the same integer but one's just positive and one's negative."

(Doug Prompt 5C) "Ok, and can you give me another pair of additive inverses?"

(Kayla Turn 08 Prompt 5C) "Like negative ten and positive ten, or ten."

(Doug Prompt 6) "Ok, the next two problems or the remaining two problems are multipart problems and they involve four different expressions and I have them both written out there and one the smaller cards. Can you show and explain how you would evaluate each expression?"

Act: Sliding both versions of the positive eight expressions across the table.

(Kayla Turn 09 Prompt 6A '10 – 2') "Well ten minus two [please write your work if you want if you would like to] so it would be eight because you can take two away from ten then you're left with eight."

(Kayla Turn 10 Prompt 6A '6 + 2') "And then if you add six and two that would be eight again."

(Kayla Turn 11 Prompt 6A '6 – -2') "Six minus negative two to me like when you subtract a negative number it's the same thing as adding, like that's what I just think of, so it would be six plus eight. I mean six plus two which is eight."

(Kayla Turn 12 Prompt 6A '-1 + 9') "And then if you have negative one plus nine, well I know the answer would be eight, but you would do like, I don't know to me I think like I would like switch it around in my head, so it would be like positive nine minus one so that's eight. [Ok.] I don't know how to get that."

(Doug Prompt 6B) "In what ways are these expressions similar and in what ways are they different?"

(Kayla Turn 13 Prompt 6B) "Well these ['6 + 2' and '6 – -2'] are similar because it is the same number, like yeah, the same numbers, just different um what's the word, like instead of adding you're subtracting, here, but you're subtracting a negative number here so it's like the same, you come up with the same answer. And then, to me like these ['10 – 2' and '-1 + 9'] are kind of similar because for me like I would subtract one from nine and this is also subtraction ten from two. Or two from ten. So, but also, they're all kind of, this is subtraction ['10 – 2'and '6 – -2'] and this is addition {"6+2" and "-1+9"}, but they're like different cause this has a negative number, and this has a positive number, and this has a negative number, and this has a positive number.
Act: Pointing to '6 + 2' and '6 – -2'. Pointing to '10 – 2' and '-1 + 9' Pointing to '10 – 2' and '6 – -2'. Pointing to '6 + 2' and '-1 + 9'. Pointing to '10 – 2' and '6 – -2'. Pointing to '-1 + 9' and '6 + 2'.

(Doug Prompt 6C) "One of the methods often used in modeling integers is the chip model. And would you be willing to try to model these using the chip model? [sure] these would be our chips. We are looking at the same four expressions. And I am going to be taking notes as you go along so."
Act: Sliding the chips from our chip model around.

(Kayla Turn 14 Prompt 6C '10 – 2') "Alright so, I'll count ten. So, you have ten and then you take subtract two, so you just take two away and then you have eight."
Act: Counting out 10 brown chips into 2 rows of 5 slid 2 brown chips away leaving 8 in uneven lines.

(Kayla Turn 15 Prompt 6C '6 + 2') "And then for the next one you have, you start out with six and then you add two, so you just bring two more in and then you'd have eight."
Act: Set out 6 brown chips in 2 rows of 3, slid in 2 more brown chips.

(Kayla Turn 16 Prompt 6C '6 – -2') "And then, you do have six and you subtract negative two, I don't know how you would show that. I know you would just add two more, but I don't know how you would show like a negative number in chips."
Act: Set out 6 brown chips in 2 rows or 3, slid in 2 beige chips.

(Doug) "Ok, we can take a look at that at the end. How I might model that."

(Kayla Turn 16 Prompt 6C '6 – -2') "Ok, so I am just going to pretend these are like negatives with the lighter pieces and then add two for eight."
<u>Act:</u> Leaving her with 6 brown and 2 beiges for her 8.

(Kayla Turn 17 Prompt 6C '-1 + 9') "For negative one plus nine, so this is going to represent my negative one and add one to get it to zero and then I am going to add eight more and you get eight."
<u>Act:</u> Selected 1 beige colored chip then slid in 1 brown chip and slid away both the beige and the brown chip then slid in 8 brown chips in 2 rows of 4.

(Doug Prompt 6D) "Ok and we also often see a number line used to model in mathematics and on the white board over here I have a white, I have a number line, are you willing to show me how you might attempt to model these using that. [Yeah.]"

(Kayla Turn 18 Prompt 6D '10 – 2') "So if we start at ten and then if you subtract two you'd go back to spots, one two and you're at eight."
<u>Act:</u> Made a dot at 10 and a dot at 0 then made 2 hops to the left and put a dot at 8.

(Kayla Turn 19 Prompt 6D '6 + 2') "And then six plus two, you'd start at six and then go up two spots to get to eight."
<u>Act:</u> Made a dot at 0 and a dot at 6 then made 2 hops to the right and made a dot at 8.

(Kayla Turn 20 Prompt 6D '6 – -2') "And then for six minus negative two start at six and then I don't know this one, it is hard so if you subtracted negative two instead, like if you subtracted two you'd go back to two, but since it's negative two you'd go up to, so it's like you'd go to eight again."
<u>Act:</u> Left the dot at 0 and made a dot at 6 mimed moving to the left then made 2 hops to the right and a dot at 8.

(Kayla Turn 21 Prompt 6D '-1 + 9') "And then, for our negative one plus nine you start at negative one and you would just go up nine spots one two three four five six seven eight nine. And then you're at eight."
<u>Act:</u> Made a dot at -1 then made 9 hops to the right and made a dot at 8.

(Doug Prompt 7A) "Ok and we are going to do the same list of questions for these four expressions. So how would you explain and show to a student how you'd simplify each of those expressions."

(Kayla Turn 22 Prompt 7A '2 – 8') "Ok, so if you have two minus eight you would. Ok, so if you subtract eight from two you'd go down, if you start at two you would so two of the eight that you're subtracting would cancel out to zero. So, then you would have six more to subtract, so it would be like zero minus six and that would be negative six. So, I'm thinking of it like this and then you would do like zero minus six because like two gets canceled out and two gets cancelled out here, so we would get zero and six which is negative six and then.
<u>Act:</u> Wrote the expression 2 – 8 and wrote 0 – 6. Canceled that to make it 0 – 6 and wrote = -6.

(Kayla Turn 23 Prompt 7A '-7 + 1') "Seven, negative seven plus one, I just know that it would if you have negative seven and you just add one it would be negative six. Cause you're counting up by one."
Act: Wrote -6.

(Kayla Turn 24 Prompt 7A '-4 – 2') "And then negative four minus two you would just umm, I don't know, to me I think of it as like just four plus two and then it's negative cause like if you subtract a negative number from a negative number it's kind of like adding two positive numbers so I would just kind of count back like in the negatives, so it would be negative six."
Act: Wrote -6 acted like counting back.

(Kayla Turn 25 Prompt 7A '-1 + -5') "And then negative one plus negative five it's kind of like the same thing. Like you would just count back five from negative one, so it would be a negative six."
Act: Wrote -6 Acted like counting back.

(Doug Prompt 7B) "Alright. In what ways are these expressions similar and in what ways are they different?"

(Kayla Turn 26 Prompt 7B) "Well this one ['2 – 8'] and this one ['-4 – 2'] and similar because you're subtracting but for this one you're starting with a positive number and for this one you're starting with a negative number. And then, these one's ['-7 + 1' and '-1 + -5'] are similar to cause you're starting with a negative number and adding but this one you're adding a negative number and this one you're adding a positive number, and you all get the same answer for all them."
Act: Pointing to '2 – 8' and '-4 – 2' as a pair then each in turn. Pointing to '-7 + 1' and '-1 + -5'. Then pointing to '-1 + -5' then pointing to '-7 + 1'.

(Doug Prompt 7C) "Ok, how might you explain these using the chip model?"

(Kayla Turn 27 Prompt 7C '2 - 8 So, I am going to use these to represent negative numbers the lighter chips and then dark ones are going to be positive. Start with positive two and then you subtract eight, so I am going to take away. I am going to add eight of these but since these are negative and these are positive these would cancel out each other out and then you're left with six, negative six because these are negative ones."
Act: Moved 2 brown chips then slid in 8 beige chips cancelled out pairs of beige and brown chips.

(Kayla Turn 28 Prompt 7C '-7 + 1') "And then negative seven [hold on one second, ok go ahead] So then for negative seven plus one, I am going to start with seven negative chips and add a positive chip so that these two would cancel each other out, cause it's positive one and negative one and you're left with six negative chips."
Act: Counted out 7 beige chips then slid in 1 brown chip cancelled and slid away a beige brown pair leaving 6 beige chips.

(Kayla Turn 29 Prompt 7C '-4 – 2') "So for negative four minus two I would do negative four chips and I would just add two more negative chips cause you're subtracting two so. Then it would be negative six again."
Act: Counted out 4 beige chips then slid in 2 beige chips leaving 6 beige chips.

(Kayla Turn 30 Prompt 7C '-1 + -5') "And then for negative one plus negative five I would have one negative chip and then add five more negative chips, three four five. So, you would get negative six."

Act: Counted out 1 beige chip then slid in 5 beige chips leaving 6 beige chips.

(Doug Prompt 7D) "Ok, and how might you approach this using the number line?"

(Kayla Turn 31 Prompt 7D '2 – 8') "So for two minus eight I would start at two and I would go back eight spots one two three four five six seven eight, so you're at negative six."

Act: Made a dash at 2 then made 8 hops to the left and a dot at -6.

(Kayla Turn 32 Prompt 7D '-7 + 1') "And then for negative seven plus one I would start at negative seven and since we're moving this way's like positive and this way's negative to the right is positive I move up one spot cause your adding one and you're at negative six."

Act: Made a dot at -7 then moved to the right indicating positive made 1 hop to the right. And made a dot at -6.

(Kayla Turn 33 Prompt 7D '-4 – 2') "And then for negative four minus two, I'd start at negative four and then I'd subtract two, so I would move to the left two spots so one two and you get negative six."

Act: Made a dot at -4 then made 2 hops to the left and made a dot at -6.

(Kayla Turn 34 Prompt 7D '-1 + -5') "And negative one plus negative. I'd start at negative one and then since you're adding a negative number, you'd move to the left again, cause it's negative so I'd move to the left five spots. One two three four five. Then you're at negative six again."

Act: Made a dot at -1 and made 5 hops to the left and made a dot at -6.

Interview Participant Pseudonym: Monique
Date of Interview: November 2nd, 2017
Duration of Interview: 42 minutes

(Doug Prompt 1) "If a seventh grade student were to ask you, "what is the difference between whole numbers and positive integers?" How would you respond to that?"

(Monique Turn 01 Prompt 1) "Well I'd probably tell them that positive integers are, numbers that are from zero to a hundred and beyond and that whole numbers can include negative numbers, fractions, decimals."

(Doug) "Ok, alright one thing I should have mentioned before is during the process of the interview I'll be trying to remain as neutral as possible, so I will not be giving direct feedback one way or another on your answers, or I will be attempting to not. Hopefully I will keep myself neutral."

(Monique Turn 01 Prompt 1) "Yeah, I might have confused the whole, the whole numbers with the set of real numbers. [right] right there."

(Doug Prompt 1) "Ok, that's a good addition, so anything else you want to add to question one, the difference between whole numbers and positive integers?"

(Monique Turn 01 Prompt 1) "That it is probably very easier to compute positive integers than it is to do the, whole numbers."

(Doug Prompt 2) "Ok, it is a familiar notion that addition makes larger. This is true when non-zero whole numbers are added together. Is this also true when integers are added? And please provide a few examples. [ok can you repeat that question?] ok, it is a familiar notion that addition makes larger. [yes] this is true when non-zero whole numbers are added together. Is this also true when integers are added?"

(Monique Turn 02 Prompt 2) "Not necessarily, cause integers also include the set of negative numbers as well so, I believe that when you add negative numbers you get further and further away from the whole number from the positive number which is greater."

(Doug Prompt 3) "Ok, what type of numbers can you get when an, when one integer is subtracted from another integer? So, when you perform subtraction with integers what types of numbers can you get?"

(Monique Turn 03 Prompt 3) "Well of course you can get positive numbers you can get negative numbers you can get fractions."

(Doug Prompt 3) "Ok, anything else with that?"

(Monique Turn 03 Prompt 3) "Well repeat the question for me so I make sure I'm focused."

(Doug Prompt 3) "What types of number can you get when one integer is subtracted from another integer? You had mentioned positive integers and negative numbers, and anything."

(Monique Turn 03 Prompt 3) "And I believe I might have said fractions [right] I know you can get like, you can get a decimal. I think, yeah, I think that's about it."

(Doug Prompt 4) "Ok, so what roles can that symbol take within a mathematics problem?"

Sliding the ' - ' sign across the table.

(Monique Turn 04 Prompt 4) "It can mean that it is saying you should subtract one number from another or it could just signify a negative integer."

(Doug Prompt 5) "Ok, the numbers seven and negative seven can be referred to as additive inverses of each other. [yeah] what does the Prompt additive inverse mean?"

Act: placing '-7 and 7' on the table.

(Monique Turn 05 Prompt 5) "I believe it means that if you can, inverse. If you add no not zero I do know they're opposite each other and their additives possible if you add a negative seven here negative seven to seven that would give you zero I'm not sure can we skip this question? I think I'm"

(Doug Prompt 5C) "Yeah we can take a, we come back to it. Well maybe actually let's try moving to a different part of this, then we'll come back to the definition later. Ok. If seven and negative seven are additive inverses can you give me another pair of additive inverses?"

(Monique Turn 06 Prompt 5C) "Well um, positive five and negative five."

(Doug Prompt 5B) "Ok, how else would you describe two numbers that are additive inverses?"

(Monique Turn 07 Prompt 5B) "I guess I would say that their um, they look identical, but one is negative, and one is positive. [ok] or the reciprocal."

(Doug Prompt 5A) "Ok, and we'll skip that because you mentioned it but the, why would seven and negative seven be additive inverses?"

(Monique Turn 08 Prompt 5A) "I'm not sure I can give the proper explanation, but I do know that their opposite to each other and I'm thinking possibly is has something to do with zero with adding if you add the positive to the negative and they're the same number that you'd end up with the zero."

(Doug) "Ok that seems good. Before moving on to six do you want a second to take a sip of water or." [sure]

(Doug Prompt 6A) "So we have four numerical expressions here, six minus negative two, ten minus two, negative one plus nine, and six plus two. How would you evaluate each expression?"

Act: Sliding the first set of expressions across the table.

(Monique Prompt 6A) "Do you mean umm, solve? [yes] like go about solving?"

(Doug Prompt 6A) "Yes, and you can write things down if you'd like."

(Monique Prompt 6A) "Should I, can I do as basic as students would do it in."

(Doug Prompt 6A) "However you would choose to do it, if you were doing it in a classroom for students."

(Monique Turn 09 Prompt 6A '10 – 2') "Ok sure. Ok so for ten minus two I would align the numbers in the correct place value so ten minus two. Umm, so ten take away two we don't have any one's here, so we can't subtract two from that, so I will decompose, this represents ten ones, so I'll decompose that, and it becomes zero here and I bring that ten over here. So, I have ten things take away two and I count back two, I'll land on eight, so ten take away two is eight. Equals eight, ten ones take away two ones equals eight ones." ["Ok"]

Act: Wrote the expression on the paper, points to the 0, crosses out the 1 and writes a 1 above the 0. Then preforms the subtraction in the unit's place. Writes 8.

(Monique Turn 10 Prompt 6A '6 + 2') "So this one is done. Six plus two, so I can align the place values six plus two so six things then I add two more on, I'll just count up by two because this is simple, so six seven and eight, so six plus two is eight."

Act: Moves the '10 – 2' card away and writes '6 + 2'. Nods hand up and down twice while counting. Records the 8 and sets the expression aside.

(Monique Turn 11 Prompt 6A '-1 + 9') "Ok, negative one plus nine, again, I'd align the place value, negative one plus nine. The negative one represents something that is owed so it needs to be repaid and so if you have nine, let's say that you have nine dollars, and this is one dollar that is owed, you'd have to pay off your debts first, so you pay out the one dollar so that would be eight dollars that you're left with and it's positive cause its eight dollars that you actually have in your pocket." ["Ok"]

Act: Grabs the '-1 + 9' card. Writes the expression out. Points to the -1. Writes 8.

(Monique Turn 12 Prompt 6A '6 – -2') "And then six take away negative two and the same concept follows that you. Same concept I believe would apply that this is six things that you have and ah, umm, you have you owe someone two dollars and so you're going to pay them their two dollars and so your six take away two would equal four dollars that you have left."

Act: Grabs the '6 – -2' card. Writes the expression on her paper. Writes 4.

(Doug Prompt 6B) "Ok so it what ways are these expressions similar and in what ways are they different?"

(Monique Turn 13 Prompt 6B) "Well this is very basic but it's similar because their all, their just very simple numbers under, ten and under. Hum, this one ['6 – -2'] is different, because it has a subtraction sign and a negative sign. This one ['10 – 2'] only has one, a subtraction sign. These ['-1 + 9' and '6 + 2'] are similar because they have the plus sign but their different because this one ['-1 + 9'] contains a plus sign plus negative and this one ['6 + 2'] does not and their also similar because of these three ['-1 + 9', '10 – 2', '6 + 2'] equals the answer came up to be eight. This one ['6 – -2'] for some reason it's different so maybe I might have made a mistake here.

Act: She points to or taps the expression she is referring to as is indicated by the expressions in quotes.

(Doug Prompt 6C) "Ok, so, in some of the new or newer teaching techniques manipulatives are finding a significant place and one method that is used with modeling integers is called the chip model. [ok] and are you willing to show and explain how you would evaluate each of these expressions using the chip model? ['I'll try.'] Ok, so these are the chips. So slowly, chose one and slowly work through it using the chip model."

Act: Sliding the chips to her side of the table.

(Monique Turn 14 Prompt 6C '6 + 2') "Ok, so we have one two three four five six things and then two things over here. Six plus two if you put them together you add them you'll end up with one two three four five six seven eight. Eight things." ["Ok"]

Act: Counting out six brown chips and two beige chips into separate piles, then slide them together. This produces a line with two different colors of chips which she counts up to eight.

(Monique Turn 15 Prompt 6C '10 – 2') "['Ok, continue.] So, next one is ten minus two, so one two three four five six seven eight nine ten. Ten things and if we take away two from it we move two aside and then we count to see how many are left so two four six eight. Eight things are left."

Act: Counting out 10 brown chips, forming a long line, then slid away 2 brown chips. Then counted the 8 chips in pairs sliding them into a new line.

(Monique Turn 16 Prompt 6C '-1 + 9') "['Ok, you're good.] This one's going to be a little hard. It is hard to do this one, I'm gonna try, [ok] I'll call this negative one and I'll say I have nine things, one two three four five six seven eight nine things. So, this is negative one, that means I owe someone, let's see if this is nine dollars here and this a dollar I owe someone a dollar, and this can represent that empty pocket that needs to be filled with a dollar, ['Ok'] so I pay my debt and I'm left with one two three four five six seven eight. Eight things." ['Ok']

Act: Pulled out a single brown chip to be the -1 then slides nine beige chips into a line. Points to the negative chip. Moved one of the beige chips on top of the brown chip, then counted out the line of 8 beige chips.

(Monique Turn 17 Prompt 6C '6 – -2') "['Ok, next.] Ok, this is the difficult one ok. I have six things one two three four five six. Call this six dollars and I have, I'm owing someone two dollars, so I have to pay my debt, so I pay two dollars and I'm left with one two three four dollars.

Act: Counted out 6 beige chips counts out 2 brown chips places a pair of beige chips on top of the 2 brown chips then counts the line of 4 beige chips that remain.

(Doug Prompt 6D) "Ok and are you willing to try to show and explain how you would do each expression with the number line? ['Ok, I will try.'] And you do have permission to opt out of attempting any of these if you would like to [ok] but you are doing good so far."

(Monique Turn 18 Prompt 6D '6 + 2') "Ok, so I will start here on six. I am adding six plus two ['ok'] so I am at six and I am going to add two more so six plus one more is seven plus one more is eight." ['Ok']

Act: Makes a red x at 6, then makes 2 hops to the right.

(Monique Turn 19 Prompt 6D '10 – 2') "Ten take away two. So, I'll start out at ten and if I take two things away means the number will become smaller, so I'll have to go back two places. So, from ten I take away one so that lands me on nine, I take away another one and that lands me on eight, so ten take away two is eight.

Act: Makes a red x at 10, then makes 2 hops to the left.

(Monique Turn 20 Prompt 6D '-1 + 9') "Ok so negative one plus nine, so I have start at negative one because I have negative one thing and if I add nine to that, I'll take nine steps forward one two three four five six seven eight nine. So negative one plus nine lands me at eight."

Act: Makes a red x at -1 then makes 9 hops to the right.

(Monique Turn 21 Prompt 6D '6 – -2') "Ok, I will attempt this one, here six take away negative two. I have six things and I am going to take away negative two, if I start at six and I move back to places and that lands me at four."

Act: Makes a red x at 6 then makes 2 hops to the left.

(Doug Prompt 7A) "Ok I want you to consider another four set of equation, of expressions. Ok, negative four minus two, two minus eight, negative one plus negative five, and negative seven plus one. Ok, and I would like you to show and explain how you would evaluate or solve each expression."

Act: Placing the second batch of expression on the table.

(Monique Prompt 7a) "I have a choice in what I use, or do you want me to begin with paper."

(Doug Prompt 7A) "Let's begin with paper, but if you prefer to begin with something else you can."

(Monique Turn 22 Prompt 7A '-7 + 1') "Negative seven plus one. Say if I had, I'm in debt seven dollars and someone gives me one dollar now I would have, if I pay off my debt pay one dollars off then I have six dollars left. Seven plus one is six."

Act: Wrote the expression, '-7 + 1' then points to the numbers then wrote the 6.

(Monique Turn 23 Prompt 7A '2 – 8') "Two things take away eight. So, if I, let's say I only have two dollars and I owe someone eight. I can only afford to pay them the two so now I'm still in debt of six dollars.

Act: Wrote the expression '2 – 8', then points to the numbers, wrote -6.

(Monique Turn 24 Prompt 7A '-1 + -5') "Negative one plus negative five so if I owe someone one dollar and if I owe someone else five dollars that means I end up owing a lot more money which would be when I put them together I owe negative six and negative five dollars, negative six dollars."

Act: Wrote the expression '-1 + -5', pointed to the numbers then wrote -6.

(Monique Turn 25 Prompt 7A '-4 – 2') "Negative four take away two. So, if I have negative four dollars and someone, negative four negative two, two, negative two. Negative four dollars and I take away two of that, so maybe that would be like somebody taking that debt away from me so negative four take away negative two would be negative two."

Act: Wrote the expression '-4 – 2', then pointed to the numbers on the card, wrote -2.

(Doug Prompt 7B) "So of those four expressions, in what ways are they similar and in what ways are they different? These four."

(Monique Turn 26 Prompt 7B) "These two ['-4 – 2' and '-1 + -5'] are similar because they have negative signs as well as, they have negative signs, this one ['-7 + 1'] is similar too because it has a negative sign in it. They are different though because this ['-4 – 2'] here is not necessarily a negative sign but a subtraction sign and this ['2 – 8'] one is different from the rest because there's no negative sign at all there, just the subtraction sign. These two ['-7 + 1' and '-1 + -5'] are similar because they have the positive, positive sign as well as the negative sign, negative integers. Further explanation or." ['No, sounds good.']

Act: Pointing at '-4 – 2' and '-1 + -5'. Pointing to '-7 + 1'. Pointing at the subtraction in the middle of '-4 – 2'. Pointing to '2 – 8'. Pointing to '-7 + 1' and '-1 + -5'.

(Doug Prompt 7B) "You might be asked redundant, but can you specify a little bit more how they're different?"

(Monique Turn 26 Prompt 7B) "Their different because in a lot of cases they have a smaller number well actually this might be similar they have a smaller number that is umm, taking away the, a smaller integer taking a larger one. What other difference is there, umm, for the most part that's what I see the smaller integer is taking away the larger one or adding to it. I can't tell of any more difference right now."

Act: Waving over the set of expressions in a general way.

(Doug Prompt 7C) "Ok, so that's fine. Ok, and would you like to show me how you might represent these using the chip model? ['I'll try.'] That's all I can ask. Which one are you looking at?"

Act: Sliding the chips toward her.

(Monique Turn 27 Prompt 7C '2 – 8') "I am looking at two take away eight. ['Ok.'] So, I have two dollars, and these represent the negative of what I, what I owe, so I owe eight dollars. I'm in the negative here, so two four six eight. So, I have to pay my debt. So, this is one, one dollar that's covered, another one that's covered, and so I only have negative six dollars left to be repaid." ['Ok']

Act: She set 2 beige chips aside. The selects 8 brown chips. Stacks a beige chip on one of the brown chips, then repeats the process. Counts the 6 brown chips that are still lined up.

(Monique Turn 28 Prompt 7C '-7 + 1') "I'll do negative seven plus the one, so this is going to represent the negative, negative two four six and one is seven. Negative seven dollars owed, someone gives me a dollar and because I always have to pay my debt, I pay that, this is removed and now I have one two three four five six. Six dollars still owed so negative six."

Act: Selected 7 brown chips then pulls out 1 beige chip Places the beige chip on one of the brown chips. Counts the row of 6 brown chips that remain.

(Monique Turn 29 Prompt 7C '-1 + -5') "Ok this one is easier. I have negative one dollar owed and then I also have a debt of five dollars so one two three four five, so all together, I'm owing negative five plus one negative, plus negative one. Negative six dollars still owed."

Act: Sets 1 brown chip out then selects 5 more brown chips which she puts in a line. Slides the two groups together into a single line of 6 brown chips.

(Monique Turn 30 Prompt 7C '-4 – 2') ['Ok.'] "So, this one is negative four minus two. So negative four. This one is kind of hard to model because you can't have negative four dollars and then you are going to take away two positives from it so, ['If you want to move on to the next one we can so.'] yes."

Act: Lines up 4 brown chips.

(Doug Prompt 7D) "And like the previous set of number, I want see if you are willing to give it a try on the number line. ['I will give it a shot.'] Ok, that's all I ask."

(Monique Prompt 7D '-4 – 2') "Leave this one for last."

Act: Moving the '-4 – 2' away.

(Monique Turn 31 Prompt 7D '2 – 8') "So, let's say I have two things and I want to take away eight I think that's two. It's almost like I am going to be owing. I have two things and I am going to take away eight things that I don't have so maybe I'm borrowing so I'll take eight steps backwards, so the two is one two three four five six seven eight. So, I'd land on negative six."

Act: Makes a blue x at 2, then makes 8 hops to the left to -6.

(Monique Turn 32 Prompt 7D '-7 + 1') "Negative seven. I'm in the negative by seven and if I'm gonna add a dollar to that, that someone is going to give me, I have to pay my debt, I'll take one step and I'll end with negative six again."

Act: Makes a blue x at -7 then a hop to the right.

(Monique Turn 33 Prompt 7D '-1 + -5') ['Ok.'] "So, I have negative one things, and I have to add negative five to that so that's negative, five negative numbers so I'm gonna add one more negative, that will have me at negative two, then its three, then its four, so five steps from here, so one two three four five. I end up with negative six again."

Act: Makes a blue x at -1 then makes five hops to the left.

(Monique Turn 34 Prompt 7D '-4 – 2') "I saved this one for last and we'll see what happens. [ok] we're at negative four here take away two. It is a possibility that you are taking away negative four and negative. Yeah, this one is hard to do, to model.

Act: Makes a blue x a -4. Looks at the other cards.

Interview Participant Pseudonym: Nicole
Date of Interview: October 30th
Duration of Interview: 29 minutes

(Doug Prompt 1) "If a seventh grader were to ask you, 'What is the difference between whole numbers and positive integers?' How would you respond?"

(Nicole Turn 01 Prompt 1) "So a whole number could be anything between negative infinity and infinity. A positive integer is any whole number that is positive. Rational whole number."

(Doug Prompt 2) "Ok are you, it is a familiar notion that 'addition makes larger'. This is true when non-zero whole numbers are added together. Is it, is this also true when integers are added? Please provide some examples."

(Nicole Prompt 2) "Can you read that question again?"

(Doug Prompt 2) "It is a familiar notation that addition makes larger. ['Uh, huh.'] Ok, is this true when integers are added?"

(Nicole Turn 02 Prompt 2) "Yes, because no matter if you're adding negative integers or positive integers you're always going to a higher number. Whether you start in negative, you add a negative and a negative you will end up going toward the zero mark and then if you're adding a positive and a positive, then you'll still get a higher number that closer to the infinity mark. So, it is that always like are you getting to the zero or are you getting to the positive infinity."

(Doug Prompt 3) "Ok thank you. What types of numbers can you get when one integer is subtracted from another integer? Can you, ok. Can you get positive numbers when you, when two integers are subtracted?"

(Nicole Turn 03 Prompt 3) "Yes and no, when two integers, when two integers are subtraction if the sum is or what's the word the difference is above the zero mark then it would still be positive but if you are subtracting and it goes below the zero mark then it would be negative."

(Doug Prompt 4) "Ok, can you get a positive ok you may have just suggested it, but can you get positive numbers when two integers are subtracted?"

(Nicole Turn 03 Prompt 3) "Yes."

(Doug P3) "Are there any other numbers that you can get when two integers are subtracted?"

(Nicole Turn 03 Prompt 3) "Any other numbers, [Yeah,] you can get negative numbers if two integers are subtracted. Meaning if you subtracted two minus three you'd get negative one. If you get two minus two you'd get zero. {unclear statement}"

(Doug Prompt 4) "Ok, thank you and what role can this symbol play within a mathematics problem?"
Act: Placing the ' - ' card on the table.
(Nicole Turn 04 Prompt 4) "It could be a subtraction sign, it could be a negative symbol. It could be the bar in between fractions, it could be a simple dash, I know some teachers that I have seen like just put dashes in between numbers for no, for no appearance reason if it is not even subtraction, just to separate the numbers in between but there is no mathematical computation that deals with it."
(Doug Prompt 5) "Ok, I just want to check this camera, yeah we're good. So, these two numbers, the numbers seven and negative seven are called additive inverses of each other what does the Prompt additive inverse mean?"
Act: Placing the '-7, 7' card on the table.
(Nicole Turn 05 Prompt 5) "I've never heard of that term before but using those numbers and the way you explained it, I would say that added inverse means that if you add the positive to the negative you get the zero, if you subtracted the negative from the positive you get zero. They're inverses of each other meaning you just switch the sign. Whether it be positive or negative."
(Doug Prompt 5A) "How else can you describe two numbers that are additive inverses?"
(Nicole Turn 06 Prompt 5A) "They're opposite, they are whenever I see the comma in between two numbers I always think of a grid and putting those points on the grid but yeah."
(Doug Prompt 5C) "Can you give me another pair of additive inverses?"
(Nicole Turn 07 Prompt 5C) "Negative six and six, negative five and five, negative twenty and twenty."
(Doug Prompt 6A) "Ok, we have, if we consider these four expressions. Ok can you show and explain how you would evaluate each expression?"
(Nicole Turn 08 Prompt 6A '-1 + 9') "So starting with this one negative one plus nine the way I would learn it is see how many it takes to get to zero and then how many is left is your number. So, you take one away from the nine which is a positive one and you add it to the negative one to create zero and you have eight left."
Act: Pointed to the expression did not write anything down.
(Nicole Turn 09 Prompt 6A '6 + 2') "Six plus two is something that I learned when I was in like kindergarten which is just eight."
Act: Pointed to the expression did not write anything down.
(Nicole Turn 10 Prompt 6A '10 – 2') "Ten minus two again something that I've learned as a kid is eight.
Act: Pointed to the expression did not write anything down.
(Nicole Turn 11 Prompt 6A '6 – -2') "And then the way I learned this is that when you have two negatives next to each other, or a subtraction sign and a negative sign you make it a positive sign and then you add them together."
Act: Pointed to the expression did not write anything down.
(Doug PROMPT 6B) "Ok, staying with these four expressions in what ways are they similar?"

(Nicole Turn 12 Prompt 6B) "They all equal eight, whether you are adding from negatives or subtracting from the positive. Or if you are subtracting a negative to it or adding two positives. What's similar, I mean, beside just the sum or the difference that it creates they're the same in a way that you're using a counting strategy whether you're counting down, counting up. Or you're using what's it called derived fact. Like I know if you add one to negative one you'd get zero and then you add eight to it. But like that's direct fact but I know that negative one plus one is zero. And stuff like that and you can use derived facts throughout all four scenarios."

(Doug Prompt 6B) "In what ways are they different?"

(Nicole Turn 12 Prompt 6B) "I mean these two ['-1 + 9' and '6 + 2'] you're adding, and these two ['10 – 2' and '6 – -2'] you're technically subtracting even though when you subtract a negative like you would just add the two together, but it is still a subtraction problem in and of itself, and then these two are addition problems. Yeah."

Act: Pointing to '-1 + 9' and '6 + 2'. Pointing to '10 – 2' and '6 – -2'. Pointing to '-1 + 9' and '6 + 2'.

(Doug Prompt 6C) "Ok so, are you familiar with chip models for addition and subtraction?"

(Nicole Prompt 6C) "I've used Katie Kubes and I've used like modeling of like different objects, but I have never used chips."

(Doug Prompt 6C) "Ok would you be willing to try to explain how you might use a chip model if you your curriculum asked you to." ['Sure']

(Nicole Prompt 6C) "Is there a certain problem you want me to start with first, or?"

Act: Sorted the chips into beige and brown piles

(Doug Prompt 6C) "Any of these four, and I would like you to show me the chip model through all four of them."

(Nicole Turn 13 Prompt 6C '-1 + 9') "Ok so for this one, because there are two different colors I'd separate for at least this one into what color would be negative and which one would be positive. So, I would say the dark color would be negative, um, then you would start adding positives so one two three four five six seven eight nine and because you have one versus the negative it would cancel each other out and then you would have eight positives left."

Act: Held hands over beige when referring to negative and brown when referring to positive. Slid 1 brown chip over and counted out 9 beige chips which she made it to a square pattern paired up 1 beige and 1 brown chip and removed them leaving 8 beige chips.

(Nicole Turn 14 Prompt 6C '6 + 2') "For six plus two you would start with [if you could hold on one second ok] for six plus two you would start with six positive numbers and then you would add two more positive numbers to equal eight."

Act: Made an array of 6 beige chips then slid in 2 beige chips to give 8 beige chips

(Nicole Turn 15 Prompt 6C '10 – 2') ['ok'] "Now for ten minus two you can do it in two different ways you can start with ten positive numbers and subtract two positive numbers."

Act: Counted out 11 beige chips then removed 2 beige chips leaving 9 beige chips.

(Nicole Turn 16 Prompt 6C '10 – 2') "Or you can replace two positive numbers for two negative numbers and because they would cancel each other out leaving only eight left." ['ok']

Act: Made an array of 11 beige chips then slid away 2 beige chips and slid in 2 brown chips then slid away the 2 brown chips leaving 9 beige chips

(Nicole Turn 17 Prompt 6C '6 – -2') "[Ok] and then for six minus two I'd start with six positive numbers and then when you're subtracting two negatives it would just be this, so you're kind of, you're basically just adding two negatives to your positive like I said earlier the way I just earned it is that, I think I just learned this in sixth grade is that when you are subtracting a negative number when it is a subtraction sign next to a negative sign you would create an addition sign with it, and so you kind of just add the two to it. Um, yeah."

Act: Counted out 6 light chips placed in 2 dark chips leaving 8 chips of mixed color

(Doug Prompt 6D) "Ok, another often used example method in teaching integers is the number line and arrows on the number line are you willing to show and explain how you would evaluate or model each expression using the number line model? ['Sure, is there a certain color'] Nope you can use any of the four pens. ['I am partial to blue'] So, go ahead when you're ready."

(Nicole Turn 18 Prompt 6D '-1 + 9') "So negative one plus nine you would start with negative one and then you'd just kind of go up in adding nine, so that's one two three four five six seven eight nine. And so, you get eight which is nine away from negative one."

Act: Made a dot at -1 and numbered her way up 9 spaces then drew and arrow pointing to 8.

(Nicole Turn 19 Prompt 6D '6 + 2') ['Ok'] "And then for six plus two. You would start at six and just add two, one two. And you'd get eight. I like writing the numbers down that way I can keep track of how many numbers I've gone through [ok] that way I don't get confused."

Act: Made a dot at 6 then numbered her way up 2 spaced.

(Nicole Turn 20 Prompt 6D '10 – 2') "Ten minus two you start at ten and you go backwards one two which is basically like saying ten plus a negative one is nine ten plus a negative two is eight. So, on and so forth." ['Ok.']

Act: Made a dot at 10 then numbered her way down 2 spaces.

(Nicole Turn 21 Prompt 6D '6 – -2') "And then six minus a negative two so like I always said six minus a negative two I was taught that you just make it an addition problem. And then you just go one two."

Act: Made a dot at 6 wrote the expression 6 - -2 replaced - - with + then numbered her way up 2 spaced.

(Doug 7A) "Ok. One last collection of problems. So, we have we're going to do the same thing with another batch of problems. So how would you evaluate each expression?"

Act: Placing out the second batch of expressions.

(Nicole Turn 22 Prompt 7A '2 – 8') "Two minus eight is negative six because you know that eight minus two is six so you because it's um the opposite you just flip the sign and you make it negative six. You could also go down from to and go two one zero negative one so on and so forth to negative eight just like a number line you just kind of work your way down the line.

Act: For '2 – 8' points to 8 minus 2 in that order then as if starting at 2 counted down 8 by taping on the table.

(Nicole Turn 23 Prompt 7A '-4 – 2') "Negative four minus two again negative six and then when you're subtracting a negative from another negative you just work down the number line so away from the zero, so it is just like adding two plus four, or four plus two but again it's flipped four, negative five, negative four, negative five, negative six.

Act: Motioning to show counting down 2 from -4.

(Nicole Turn 24 Prompt 7A '-7 + 1') "Negative seven plus one when you're adding a positive to a negative you move toward zero cause you're not subtracting it, by making it a larger negative number, technically it is a smaller negative number cause the closer you are to zero is bigger again negative six, so you would just add the positive one to the seven or if you're going like seven minus one, is equal would be equal to six you just kind of flip around the equation to what you need it to be and you just flip the signs."

Act: Shifted her hands to the right.

(Nicole Turn 25 Prompt 7A '-1 + -5') "And then negative one plus negative five when you're adding two negative numbers that are separate in their entities you just again move down on the number line away from the zero so you go to negative one, negative two, negative three, negative four, negative five, negative six, or you can do if you ignore the signs then it's one plus five and then you just put the signs back in. and you get six and a negative six."

Act: Counting to the left on the table.

(Doug 7B) "And how are they similar and how are they different? Let's do similar."

(Nicole Turn 26 Prompt 7B) "So these two ['-4 – 2' and '-1 + -5'] are similar in a way that you're moving away from the zero technically this one ['2 – 8'] is too but it is kind of funky cause you start in positives. And this one {"-7 + 1"} you're moving towards the zero cause you're making it a larger negative number.

Act: Pointing to '-4 – 2' and '-1 + -5'. Pointing to '2 – 8'. Pointing to '-7 + 1'

(Doug 7B) "Ok, I'm not sure if we hit it so I'll ask again. In what ways are they different?"

(Nicole Turn 26 Prompt 7B) "A way that they're different is that these ['2 – 8'] start in positive numbers and you're moving not only away from zero, but you hit the zero and move past it kind of like in Monopoly you don't just hit go you pass go. You don't get to collect money. I this one ['-4 – 2'] you're moving away from the zero but you're using two negative numbers because if you think of it as two separate entities it is negative four and negative two so when you combine the two you would get negative six. This one ['-7 + 1'] you have both a negative and a positive, so you know that when you add a positive to a negative you move closer to the zero and you make a larger negative number. And then these ['-1 + -5' are two negative numbers however you're adding them together, so it is kind of like this one ['-4 – 2'] in a way that you're adding the two and they're both technically negative but this one you are moving away from the zero when you add the negatives together."

Act: Pointing to '2 – 8'. Pointing to '-4 – 2'. Pointing to '-7 + 1'. Pointing to '-1 + -5'. Pointing to '-4 – 2'.

(Doug Prompt 7C) "Ok, and I would like to see them with both the chip and the number line model. ['Sure'] Which ever one you would like to do first. ['I'll do the chips first, chips are fun, and it looks like you do.']

(Nicole Turn 27 Prompt 7C '2 – 8') "Ok, let's bring then how I did before into negatives and positives and so because these are both technically positive numbers two and then one two three four five six seven eight. You because you're subtracting you would replace each positive eight with a negative one so. So, then you would replace these two negative eights with two positive and then the only negatives you have left are six. Really negative six."

Act: Counted out 2 beige chips and a second group of 8 beige chips swapped out each of the 8 beige chips with a brown chip the removed a pair of browns and a pair of beige leaving 6 brown chips.

(Nicole Turn 28 Prompt 7C '-4 – 2') "And then negative four minus two, again it's like a negative and technically it's a positive but if you want to ignore the negative the subtraction sign as a negative sign it would be a negative four and then two but because you're subtracting you flip them. And then you have negative six in total because you're subtracting two positives which technically make them negative."

Act: Counted out 4 brown chips and 2 beige chips swapped out the 2 beiges with 2 brown chips leaving 6 brown chips.

(Nicole Turn 29 Prompt 7C '-7 + 1') "And then negative seven plus one, one two three four five six seven and then you have a positive one basically it replaces it cancels out the one extra negative from the negative seven and you make a negative six." ['Ok']

Act: Counted out 7 brown chips and 1 beige chip paired up the beige and 1 brown chip leaving 6 brown chips.

(Nicole Turn 30 Prompt 7C '-1 + -5') "And then negative one plus a negative five you would start with one negative and then you have five negatives and because you're adding to negatives together you don't need to replace any because neither of these numbers are positive so then you just add the negative to the other negative, so you just get a negative six."

Act: Counted out 1 brown chips and 5 brown chips in two groups the slid the 1 brown chips to the 5 brown chips making 6 brown chips.

(Nicole Turn 31 Prompt 7D '2 – 8') "['ok'] Two minus eight you start at two and you work your way down to negative eight so you get one two three four five six seven eight and then you would hit negative six or like I said before eight mins if you just flipped the equation around and you start with eight and you go back two and its seven eight or seven six and then you just flip the sign and because you flip the equation around you just flip it." ['ok']

Act: Started numbering down. Made dot at 7 and 6.

(Nicole Turn 32 Prompt 7D '-4 – 2') "We have negative four minus two. You start with, you start at negative four and then because you're subtracting you move down the number line away from the zero and you would go one two and again you would hit negative six."

Act: Made a dot at -4 the numbered down.

(Nicole Turn 33 Prompt 7D '-7 + 1') "Negative seven plus one you start with negative seven and because it's an addition sign you're moving towards the zero and so you just go to the next one. Which is negative six. So, when you add one you move towards the number line. Or toward zero technically.

Act: Made a dot at -7 made a hop to the right 1 and ended the hop with an arrow pointing to -6.

(Nicole Turn 34 Prompt 7D '-1 + -5') "And then negative one plus a negative five you start here when you have a plus and a negative together you cancel out the positive and you just you the negative of the subtraction sign, or the negative sign as your subtraction sign. So even though it does say adding which means you move towards the number line the negative cancels out the positive and you move away from the number line as a subtraction problem and you go negative two, technically this is one two three four five as in negatives, negative five, negative four, negative three negative two you're subtracting one from each.

Act: Made a dot at -1 wrote + - crossed out the + and circled the – numbered down [5 4 3 2 1] the affixed the negative signs [-5 -4 -3 -2 -1]

Interview Participant Pseudonym: Rebecca
Date of Interview: November 21st, 2017
Duration of Interview: 33 minutes

(Doug Prompt 1) "If a seventh grade student were to ask you 'what is the difference between whole numbers and positive integers?' how would you respond?"

(Rebecca Turn 01 Prompt 1) "So although they may seem very similar whole numbers have to do with the number can be positive or a negative number, positive integers may be whole numbers also, but it can only be above zero and whole numbers cannot be one point five, one point seven five they need to take up an entire number of whole. So, it can't be a fraction of a whole number, it can't be a decimal of a number it needs to represent a whole number."

Act: Moving hands to the right when referring to the positive number.

(Doug Prompt 2) "Ok, it is a familiar notion that addition makes larger. This is true when non-zero whole numbers are added together, such as three plus four equals seven. Is this also true when integers are added? Please provide some examples."

(Rebecca Turn 02 Prompt 2) "So, so addition makes larger is misleading when we're working with integers. For instance, if we're working with simply whole numbers, whole numbers need to be positive so if we're working with four plus three we see that they're both positive numbers and they're going to equal seven. However, when working with integers, integers can be positive or negative so if you're adding for instance negative five plus three the result will be negative two. Actually wait, if you're adding two negatives together the result will be a lower number. So negative five plus negative three equals negative eight. Even though we're adding here we're adding negative numbers, so the result will be a smaller number than the two addends."

Act: Wrote the first expression 4 + 3 and its solution 7. Wrote -5 + 3 then -2 then crossed out -5 + 3 = -2, and wrote -5 + -3 = -8 circled the + then circled the -5 and -3

(Doug Prompt 3) "Ok, what types of numbers can you get when you subtract one integer from another? When you do a subtraction involving integers, what types of numbers can you get?"

(Rebecca Turn 03 Prompt 3) "Ok, so if you're using two positive integers, you can get a positive or a negative number depending on how large the integers are. So, if you're subtracting three minus two you can get a positive answer of one. However, if you're subtracting the positive integers three minus four you will get an answer of negative one. And when subtracting with negative integers, you can subtract, for instance negative four minus one will get you the result of negative five. However, if you're result, if you would subtract a negative from a negative so negative four minus negative one you will get a negative number also, but the signs will change, and the result will be negative three instead. So as a result of combining integers you can get negative or positive values."

Act: Wrote the expression 3 – 2 = 1. Wrote the expression 3 – 4 = -1. Wrote the expression -4 – 1 = -5. Wrote the expression -4 – (-1) then changes the subtraction and negative sign to addition and positive signs before writing -3.

(Doug Prompt 4) "Ok, what roles can that symbol take within a mathematics problem?"

Act: Placing the ' - ' sign on the table.

(Rebecca Turn 04 Prompt 4) "So the roles that, that symbol can take in a mathematics problem are, they can be used for subtraction between two whole numbers or two integers. Or that symbol can be representative of the negativity of a number so if its placed in front of an integer, that represents that that integer is a negative value."

(Doug Prompt 5A) "Alright. Anything you would like to add to anything we have looked at so far?" ['No.'] So, the numbers negative seven and seven are additive inverses of each other. What does the Prompt additive inverse mean?"

Act: Placing the "-7, 7" card on the table.

(Rebecca Turn 05 Prompt 5A) "So the Prompt additive inverse means that if you add these two values together the result would be zero. So, some people might call that canceling out, the values, but technically speaking, if we drew a number line and we had zero in the middle then one two three four five six seven here and one two three four five six seven here seven and negative seven coming into zero is an equivalent value. Seven bumps this way one two three four five six seven and then seven bumps this way. So, these values are able to meet in the middle at zero, which can be considered or seen as the values canceling out, but it is just because it you add for instance the negative seven to the seven you're really subtracting the value of seven from seven, so you're getting to zero. However, if you add the value seven to negative seven, you're adding seven, seven towards zero, to the right, so that gets you to zero when you add seven to negative seven."

Act: Drew a number line then made 7 marks to the right and labeled at 7 and 7 marks to the left and labeled at -7 made large hops from -7 and 7 to zero counting out the 7 bumps

(Doug Prompt 5A) "Ok, I think you probably covered most of these other parts, but I'll go through them anyway. And why are negative seven and seven additive inverses? Do you have anything you want to add about that."

(Rebecca Turn 06 Prompt 5A) "Well, it helps that they're technically opposing positive and negative integers, but the same value so seven. They both have a seven in them. In the one's place, however one of them has a negative sign so it's a negative integer whereas this is a positive integer of seven so when you combine them they even out to zero."

(Doug Prompt 5B) "Ok, how else would you describe two numbers that are additive inverses?"

(Rebecca Turn 07 Prompt 5B) "Hum, so I kind of went through thinking of it as canceling out, if you have a negative seven and then you have a positive seven. I guess, you can just try to think of them as the same, the same value or the same quantity but this sign, so the positive sign indicates that its seven wholes to the right of zero and then this sign indicates left so its seven wholes to the left of zero for negative seven."

Act: Wrote -7 and +7 circled the 7s then circled the signs.

(Doug Prompt 5C) "Ok, and can you give me another pair of additive inverses?"

(Rebecca Turn 08 Prompt 5C) "Four and negative four."

(Doug Prompt 6A) "Ok I think we covered that. The next couple questions involve several parts and several different mathematical expressions. That I am going to ask you to evaluate and show in a couple different ways, so we have these four expressions. You can either look at them all four at the same time or one at a time with the different cards."

Act: Placing the first four expressions on the table on a single sheet and on four individual cards.

(Rebecca Prompt 6A) "Sure can I right on the or use the manipulatives or?"

(Doug Prompt 6A) "We'll come back to the chips and number lines in a little bit but how, just starting with how would you explain to a student how you would evaluate each expression."

(Rebecca Turn 09 Prompt 6A '10 – 2') "Ok, so for the first one that's a simple subtraction problem so I would draw out ten circles and I see that I'm taking two away from that ten or I would begin to express it and apply it to the class saying ten students want an ice cream, ten students want ice cream if two students decide they no longer want ice cream how many students want ice cream now. Or something along those lines to make it more relatable to them so then I'm subtracting two circles, from these, from the original ten circles. I can cross off these two, cross off these two and then I count them all up together at the end and I am left with eight. Or eight people who still want ice cream."

Act: Drew 10 open circles then circled the 2 on the original paper wrote – and drew 2 open circles crossed off the 2 circles and 2 of the original 10 circles. Then put a box around the 8 open circles and wrote and boxed 8

(Rebecca Turn 10 Prompt 6A '6 + 2') ['ok'] "For the next one this ['6 + 2'] is a simple addition problem, six plus two. So again, I could say make it into a word problem because I feel like that's something that's very accessible to kids. So, six people want, six people want to play soccer and two people want to play, two people join and want to play too, how many people are playing soccer now. So, I would again draw out, one two three four five six and then add the extra two and in total I would have eight people who are playing soccer now."

Act: Drew 6 open circles a + and 2 more open circles then wrote = 8 which she circled

(Rebecca Turn 11 Prompt 6A '6 – -2') "Then this one ['6 – -2'] so six minus negative two so again I would try to attach this in a similar fashion, I would draw out the six, one two three four five six, I know that I'm subtracting again, however I'm subtracting a negative integer now, so I would write in here a negative sign also and two dots. However, these two whole, or two dots are from going over two times to the left below zero, so I need to. I can take I guess the hold on this is six minus negative two. I could look at the outside of the parentheses and put a one there and multiply negative one times negative two because I placed that within the parentheses and even though there's no one there, it's implied, so negative one times negative two, negative times a negative is a positive so six plus two is eight."

Act: Wrote parenthesis around the -2 then drew 6 open circles and wrote a – parentheses negative sign and 2 open circles wrote 6 – (-2) wrote a 1 in front of the parentheses wrote 6 + 2 then wrote and circled 8.

(Rebecca Turn 12 Prompt 6A '-1 + 9') "And then negative one plus nine, I could show this one on a number line, I guess I could have shown the other one on a number line too, but I can show this is zero, this is negative one. One two three four five six seven eight nine and if I'm going from negative one plus nine to the right, I could go one two three four five six seven eight nine and I get a result of eight, which I looking back at it I easily could have done for the other problem."

Act: Wrote the expression -1 + 9 and drew a number line which she labeled circled -1 then made 9 hops to the right wrote 8 and circled it.

(Doug Prompt 6B) "Ok in what ways are these expressions similar and in what ways are they different?"

(Rebecca Turn 13 Prompt 6B) "So they're similar in the sense that all the results are eight and technically speaking I could solve all of them using a number line and a similar fashion it's just a matter of where you're starting on a number line. This ['6 – -2'] is the one that's more different because if I had six minus negative two the two negatives together could confuse a student whereas seeing the ['-1 + 9'] negative and just seeing have to go to the right nine is more of a clear-cut problem. They're different in the sense that these ['6 – -2' and '-1 + 9'] one's involve a negative integer whereas these ['10 – 2' and '6 + 2'] are a typical problem that are more accessible to students. And I think this ['6 – -2'] one is the most, will be the most difficult for a student to comprehend even with the assistance of a number line."

Act: Pointing to '6 – -2'. Pointing to '-1 + 9'. Pointing to '6 – -2' and '-1 + 9'. Pointing to '10 – 2' and '6 + 2'. Pointing to '6 – -2'.

(Doug Prompt 6C) "Ok, one of the models that is often used for integers is the chip model which is similar to a lot of what you just drew out, but are you willing to try modeling those four same problems using the chip model. ['I can try to'] ok, at the end I will take the opportunity to show you some examples of how I might approach it. So just do your best now and I will try to clarify anything that needs to be clarified later.

(Rebecca Turn 14 Prompt 6C '10 – 2') "Ok, for the first problem ['10 – 2'] I start with ten so then I am going to make a collection of ten chips, one two three four five six seven eight nine ten. I'm taking way cause that's what the subtraction indicates taking way two of the chips, putting them to the side now how many chips do I have left, one two three four five six seven eight. So that's how you solve that one."

Act: Counted out 10 brown chips in a random order selected and removed 2 brown chips counting off 8 brown chips.

(Rebecca Turn 15 Prompt 6C '6 + 2') "Then for the next one I have six, so one two three four five six, and then I'm bringing in to the six, collecting with the six two more so I take two more and count how many do I have now, one two three four five six seven eight. ['Actually, give me one second.'] sorry."

Act: Counted out 6 brown chips then slid in 2 more brown chips then counted off the 8 chips

(Rebecca Turn 16 Prompt 6C '6 – -2') ['Ok, now proceed.'] "For the six minus negative two, ok so one two three four five six. Umm, so it this were a typical subtraction problem I would be taking away two but because this is the additive inverse I will instead be adding two, so I add two chips to the result."

Act: Counted out 6 brown chips slid 2 more brown chips in resulting in 8 brown chips.

(Rebecca Turn 17 Prompt 6C '-1 + 9') "Then for this one ['-1 + 9'] I'm starting with negative one chips so, we can call this negative one, this is like a marker and then we're adding nine more so negative one, so zero one two three four five six seven eight and that's one two three four five six seven eight nine and I used this and sort of a place value for me to see where negative one is and to use each of these almost like a number line. For what comes afterwards."

Act: Selected 1 beige chip slid in 9 brown chips one at a time counting them as she went leaving a line of 10 chips of mixed colors.

(Doug Prompt 6D) "Ok, and what total did you say that that gave? ['Eight'] Ok, and number lines are also often used, so would you mind doing these on a number line? ['Yeah']

(Rebecca Turn 18 Prompt 6D '10 – 2') "So the first one ['10 – 2'] is ten so that's over here and then minus two so what I usually do is I create like arches to represent jumping between values so I'm going down two so one two and that brings me down to eight and.

Act: Made a mark at 10 made 2 hops to the left and made a mark at 8 and wrote 8.

(Rebecca Turn 19 Prompt 6D '6 + 2') "And then for the next one I am at six and instead of going down two like I did for the last problem I am going to the right two which is what is indicated by the plus sign. The plus sign combined with the positive integer so up two, going one two and I am also at eight. In that case."

Act: Made a mark at 6 made 2 hops to the right and made a mark at 8 and wrote 8.

(Rebecca Turn 20 Prompt 6D '6 – -2') "And then I'm starting with six but it's minus negative two so similarly to how I was discussing the additive inverse before in terms of cancelling out the two negatives combined create a positive and also you see they because it's a negative two you wouldn't be doing the same thing that you would do for a positive two and for a positive two you would go down two so because it is the additive inverse you would need to do the opposite direction then you would for the positive integer so it's minus negative two, so you're going to go up two and you're going to get a result of eight."

Act: Made a mark at 6 and wrote 6 pointed to the left of 6 while referencing subtracting positive 2 made 2 hops to the right and wrote 8.

(Rebecca Turn 21 Prompt 6D '-1 + 9') "And then this one, I kind of already showed that one but negative one is here and again I don't need that negative one, and I go up nine, so nine bumps so one two three four five six seven eight nine, nine bumps which gets me to eight as well."

Act: Made a mark at -1 but wrote -9 then erased that and wrote -1 then made 9 hops to the right then wrote 8.

(Doug Prompt 7A) "And we're going through another set of four problems so … so I would like to know how you would approach solving those four problems."

Act: Removing the white board and placing the second set of expression on the table.

(Rebecca Turn 22 Prompt 7A '2 – 8') "So, as I mentioned before just because the integers are positive that does not necessarily mean you'll be getting a positive result it depends on the distance away from zero and if the first value is smaller than the second value and you're subtracting then the result should be under zero. So, for the first problem ['2 – 8'] it's two minus eight and I'm going to put that on a number line. So, zero one two negative, one negative two negative three negative four negative five negative six negative seven negative eight. So, I'm going from two this is my starting point and then I'm going down eight so eight bumps one two three four five six seven eight. And that gets me to a negative six on my number line. So that's the answer."

Act: Wrote the expression 2 – 8 then drew a number line and labeled it. Marked 2 then made 8 hops to the left and circled -6 wrote and circled -6.

(Rebecca Turn 23 Prompt 7A '-7 + 1') "And then the next one ['-7 + 1'] is negative seven plus one and this I could also do on my number line I'll find negative seven first, so zero, negative one, negative two negative three negative four negative five negative six negative seven, so negative seven because it's a plus I'll be moving and an positive integer I'll be moving to the right one so from negative seven, as my starting point, and one bump over and the result is negative six also, like the last problem.

Act: Drew the number line and labeled it made a mark at -7 then circled the plus on the original problem set made 1 hop to the right and circled -6 and wrote -6.

(Rebecca Turn 24 Prompt 7A '-4 – 2') "And then ['-4 – 2'] my starting point is going to be negative four so I'll draw a number line and zero, negative one negative two negative three negative four negative five negative six negative seven negative eight so if I'm starting at negative four and then minus two and so the minus and the fact that this is a positive integer represents that I'll be going to the left twice so from the starting point here I'll be going down or to the left two points. One two and that also gets me a value of negative six."

Act: Drew a number line and labeled it made a mark at -4 and labeled it with s made 2 hops to the left made a mark at -6 and wrote and circled -6.

(Rebecca Turn 25 Prompt 7A '-1 + -5') "And then for this final one ['-1 + -5'] my starting point is negative one plus negative five so zero, negative one negative two negative three negative four negative five negative six negative seven. So, my starting point is negative one and I'm adding a negative five so if I'm adding a negative that's pretty much the same thing as just subtracting so, it's the same things as saying negative one minus five which means that I'll be going to the left five, so it would be one two three four five and I'll get a result of negative six."

Act: Drew the number line and labeled it made a mark at -1 and labeled it with an x. Wrote -1 – 5. Made 5 hops to the left made a mark at -6 and wrote and circled -6.

(Doug Prompt 6B) "So in what ways are these expressions similar and in what ways are they different?"

(Rebecca Turn 26 Prompt 6B) "Ok, so they're similar in that the result for all of them is negative six they are similar in that they all involve the negative symbol in some way and they all involve, actually they don't all involve that. They're different in the sense that this ['2 - 8'] only has positive integers involved whereas this ['-7 + 1'] is a negative integer, this ['-4 – 2'] is a negative integer, this ['-1 + -5'] is a negative integer and this is a negative integer. This one is ['-1 + -5'] different because it involved two negative integers and adding this ['-7 + 1'] one involved one negative integer and addition. This one ['-4 – 2'] involved one negative integer and subtraction, and this was just two positive integers ['2 – 8']."

Act: Pointing at '2 – 8' then pointing at the negative integers in the other expressions. Then points to both negative integers in '-1 + -5', the '-7 + 1'. and '-4 – 2' then '2 – 8'.

(Doug 7C) "Ok, how would you model these using the chip model?"

(Rebecca Turn 27 Prompt 7C '2 – 8') "So I'll say, I'll represent the two with these brown one's and then going backwards or going to the left the chips, I'll use the beige one's so this is two which means one two this means this is one so this is zero, negative one negative two negative three negative four negative five negative six and to make sure I'm correct I need to make sure I have eight of the beige chips and just count down to, from one zero negative one negative two negative three negative four negative five negative six. And then that's the result it is the last chip that is down."

Act: Counted out 2 brown chips then 8 beige chips then lined up the chips as she was counting down had 7 beige chips counted 1 brown chips and 7 beige chips to get -6. Bounces the last beige chip in the line.

(Rebecca Turn 28 Prompt 7C '-7 + 1') "And then for negative seven plus one. I'll represent the negatives again with beige so one two three four five six seven plus one though means taking away one of the negative pieces so negative seven plus one, I'll take away this, the last piece on the end and so I have one two three four five six pieces but they're negative pieces."

Act: Repositioned the 7 beige chips and counted them slid away 1 beige chip then counted the 6 beige chips.

(Rebecca Turn 29 Prompt 7C '-4 – 2') "Ok and then for negative four minus two. So I'll say that I have these four are negative four cause I'm saying that the beige are representing the negative integers but negative four minus two so because this values already negative if I consider this as almost adding on a negative like how it is in this problem, plus minus five is like minus five so negative four minus two, I can think of that as adding two negative values so then that would be negative five and then negative six after that."

Act: Place 4 beige chips on the table placed 2 more beige chips into the line of the 4 beige chips for a line of 6 beige chip.

(Rebecca Turn 30 Prompt 7C '-1 + -5') "And then this one is negative one, so I'll leave that negative one there, and then plus negative five which again I saw as negative one minus five adding five negatives chips to that, so one two three four five so that's six total, so six negative chips or negative six."

Act: Left 1 beige chip on the table then counted out 5 more beige chips to create a line of 6 beige chips.

(Doug Prompt 7D) "Ok, and if you could show me how you might approach these again with the number line."

(Rebecca Turn 31 Prompt 7D '2 – 8') "Ok, so for the first one my starting point is two and I'm going to the left eight. Which is indicated by the subtraction sign and the positive integer, so one two three four five six seven eight. And that gets me to negative six."

Act: Made a mark at 2 which she labeled as 2 made 8 hops to the left and wrote -6

(Rebecca Turn 32 Prompt 7D '-7 + 1') "And then for this one my starting point is negative seven plus one and I'm adding a positive integer so I'm going to the right one which gets me to negative six also."

Act: Made a mark at -7 and wrote -7 made 1 hop to the right and ended it with an arrow pointing at -6 and wrote -6.

(Rebecca Turn 33 Prompt 7D '-4 – 2') "Then my starting point is negative four minus two so it's minus a positive integer so I'm going to the left two, one two. Negative six is the result."

Act: Made a mark at -4 and wrote -4 made 2 hops to the left and wrote -6.

(Rebecca Turn 34 Prompt 7D '-1 + -5') "And then negative one is my starting point for this final problem and it's plus negative five so, this indicates going to the left five because plus a negative can be translated to just minus five or adding a negative integer so I'm going to the left five one two three four five. And I get another result of negative six.

Act: Made a mark at -1 and wrote -1 made 5 hops to the left and wrote -6

Interview Participant Pseudonym: Tae
Date of Interview: November 30th, 2017
Duration of Interview: 29 minutes

(Doug Prompt 1) "If a seventh grade student were to ask you, 'what is the difference between whole numbers and positive integers?' how would you respond?"

(Tae Turn 01 Prompt 1) "Whole, I would say that whole numbers include all the numbers on the number line, but positive integers are only to the right of zero on the number line. And I would probably like show them the number line."

(Doug Prompt 1) "Ok, actually can you draw what you would show them?"

(Tae Turn 01 Prompt 1) "Can I use anything or just? [Yes] So, I would show them that these are all whole numbers but only this group are the positive integers and these one's are the negative integers."

Act: Drew a number line and labeled it from -4 to 4 then drew an arrow to the right.

(Doug Prompt 2) "Ok, it is a familiar notion that addition makes larger. This is true when non-zero whole numbers are added together. Is this also true when integers are added? If so, please provide some examples."

(Tae Prompt 2) "Would I be able to read it too? I am better at reading things."

(Doug Prompt 2) "It is a familiar notion that addition makes larger, this is true when non-zero whole numbers are added together is that statement also true when integers are added?"

(Tae Turn 02 Prompt 2) "Yes, for examples adding one seventh and three seventh you get four sevenths with is more than both."

Act: Wrote this expression 1/7+3/7=4/7 and its solution.

(Doug Prompt 2) "Ok can you think of any time when you add two numbers together and you get something smaller than you started with?"

(Tae Turn 02 Prompt 2) "Yeah, you could add three sevenths plus negative one seventh, so this would actually be two sevenths."

Act: Wrote this expression 3/7+(-1/7)=2/7 and its solution.

(Doug Prompt 3) "Ok, what types of numbers can you get when one integer is subtracted from another integer? When you perform a subtraction on two integers, what types of numbers can you get?"

(Tae Turn 03 Prompt 3) "You can get any number I think."

(Doug Prompt 3) "Ok, can you give me examples of how subtraction would give us, say a negative number?"

(Tae Turn 03 Prompt 3) "If you subtracted two negative numbers, negative four actually no. Subtracting two integers. Ok so you can do seven minus nine would give you a negative integer."

Act: Wrote -4 then crossed it out and wrote 7 – 9 = -2.

(Doug Prompt 3) "Ok, and can you show were a subtraction can give a positive integer?"

(Tae Turn 03 Prompt 3) "Eleven minus nine and you get two."

Act: Wrote the expression '11 – 9' and its solution '= 2'.

(Doug) "Ok, it there anything you want to add to these three questions so, that we have looked at so far?"

(Tae Turn 04 Prompt 1) "I would probably also add to number one that there are more numbers in between what I wrote like one two three and four, there could be one half, there could be point three or thirty-three hundredths."

Act: Pointing to the number line drawn in problem one.

(Doug Prompt 4) "Ok, what roles can this symbol take within a mathematics problem?"

Act: Placing the ' - ' sign on the table.

(Tae Turn 05 Prompt 4) "That could indicate that you're subtracting. It could indicate that a number is negative." ['Ok']

(Doug Prompt 5) "The numbers negative seven and seven are additive inverses of each other. What does the Prompt, additive inverse mean?"

Act: Placing the '-7, 7' card on the table.

(Tae Turn 06 Prompt 5) "If you, if I were to add seven, that's not true. I know"

(Doug Prompt 5C) "Maybe we'll loop back to that part of the question, can you give me another pair of additive inverses?"

(Tae Turn 07 Prompt 5C)

Act: Wrote this pair -4, 4.

(Doug Prompt 5B) "Ok, what is it about those two pairs of numbers that have them behave the same?"

(Tae Turn 08 Prompt 5B) "I guess in order to be an additive integer, not, I mean additive inverse they both need to be the same number, but one is negative, and one is positive."

(Doug Prompt 5A) "Ok, and how would you describe two numbers that are additive inverses?"

(Tae Turn 09 Prompt 5A) "They, I could say they're the same distance from zero on the number line."

(Doug Prompt 5B) "Ok seemed like you already answered part a why are seven and negative seven additive inverses? Do you have anything to add to them being the same distance from zero on the number line?" ['No.']

(Doug Prompt 6A) "Ok, so the next question, the next couple questions have a number of parts. And they are going to be using the same group of expressions for all four parts, so I would like you to consider those four expressions. You can either have them all out at once or you can arrange them however you like with the cards. And to start out I would like to show, you to show and explain how you would evaluate each of those expressions."

Act: Placing first set of expressions on the table on a single sheet and individual cards.

(Tae Turn 10 Prompt 6A) "As in like this {"10-2"} is subtraction like how I would group them?"

(Doug Prompt 6A) "Not yet, what answer would you get when you perform those operations.)

(Tae Turn 11 Prompt 6A) "Can I write on here? ['Yup.'] I'll get eight for this one {"10 – 2"} eight, eight eight.

Act: Writing on the original problem sheet 8 below '10 – 2', 8 below '6 + 2', 8 below '6 – -2' and 8 below '-1 + 9'.

(Doug Prompt 6A) "How would you show those to a seventh grader?"

(Tae Prompt 6A) "A seventh grader hum, is it ok to kind of go back cause I'm kind of thinking how I would show it to like a younger child."

(Doug) "Ok, you can do that."

(Tae Turn 12 Prompt 6A '10 – 2') "Ok, to I'd probably show them, using like physical units, this one represents ten and two of these units are two, so subtraction just means that you're taking out two from the ten, which would leave you with eight units." ['Ok.']

Act: Drew the long rectangle 10-square and 2 small squares then added the minus sign between them the made 2 small squares at the bottom of the rectangle and crossed them out.

(Tae Turn 13 Prompt 6A '6 + 2') "And for this one I could do the same thing but just with the individual units."

Act: Drew 6 small squares + 2 small squares.

(Tae Turn 14 Prompt 6A '6 – -2') "This one I think it would be easier I think I would show on a number line that they're eight spaces between negative two and six three four five six seven eight."

Act: Drew the number line and labeled it made a dot at -2 and at 6 then made a series of 8 hops to the right from -2.

(Tae Turn 15 Prompt 6A '-1 + 9') "And I think I would do the same for this one too."

Act: Drew a number line and labeled it made a dot at -1 and 9 then made 10 hops to the right from -1.

(Doug Prompt 6B) "Ok so in what ways are these expressions similar and in what ways are they different?"

(Tae Turn 16 Prompt 6B) "Similar you would get eight for all of them [ok] different in how you have to think about it cause some umm. Some student's may think about it as subtraction and some they may think of it as addition."

(Doug Prompt 6B) "Ok in which cases might they think, might some think as subtraction some as addition?"

(Tae Turn 16 Prompt 6B) "Six minus negative two could be confusing my mind automatically thinks of the rules that we learned where you change it to six plus two. But them seeing six minus negative two they may think it's umm, six minus two for example."

Act: Pointing to '6 – -2'

(Doug Prompt 6B) "Ok, anything else with that?"

(Doug Prompt 6C) "Ok, one of the common ways that integers are represented is with what's called the chip model. Are you willing to try presenting those four or any of those with the chip model? ['Ok.'] Give me one second, I take notes as you go alright go ahead."

(Tae Turn 17 Prompt 6C '10 – 2') ['Which problem are you thinking about?'] "Ten minus two ['Ten minus two.'] I am just going to line up the same colored ones, all to make ten and then since I'm subtracting two, I'll just take two of the chips away. And then you have two four six eight."

Act: Counting out 10 beige chips lined up in 2 rows of 5 then slid away 2 of the beige chips then counting pairs of beige chips.

(Tae Turn 18 Prompt 6C '6 + 2') "Ok, I'll do six and then add one two, for six plus two."

Act: Left 6 beige chips on the table then added 2 beige chips one at a time while counting.

(Tae Turn 19 Prompt 6C '-1 + 9') "I'm trying to do negative one plus nine. I am using this to represent negative integers and this to represent positive integers, so I know if I add a negative integer to the positive one's then I have to remove a positive and I can remove the negative and I will have eight."

Act: Selected 1 brown chip and counted out 9 beige chips which were placed in a square pointed to the different colors while stating that brown is representing negative and beige is representing positive slid the brown chip to the beige chips then slid away a beige chip then grabbed the brown chips and paired it with the beige chips and removed the pair leaving 8 beige chips.

(Tae Turn 20 Prompt 6C '6 – -2') "Six minus negative two. I'm not sure how to go about this one"

Act: Left 6 beige chips on the table and slid 2 brown chips over.

(Doug Prompt 6D) "Ok, we can skip it and I'll show you an approach to it later. And number lines are also often used to represent work with integers and on the white board there we have a number line. Would you be willing to show me these four worked out on the number line?

(Tae Turn 21 Prompt 6D '10 – 2') "So for ten minus two. Ten and when I'm subtracting I'm moving to the left. So, I just move one two and have eight."

Act: Made a mark at 10 moved the pen to the left to suggest direction for subtraction made 2 hops to the left and made a dot at 8.

(Tae Turn 22 Prompt 6D '6 + 2') "Six plus two. I have to start at six and add two to get eight."

Act: Made a dot at 6 then 2 hops to the right and a dot at 8.

(Tae Turn 23 Prompt 6D '6 – -2') "Six minus negative two. I would do the same thing that I did here by counting the places to get eight." ['Ok.']

Act: Made a mark at 6 and a mark at -2 made 8 hops to the right to get from -2 to 6 and drew and arrow to the right pointing at 6.

(Tae Turn 24 Prompt 6D '-1 + 9') "Negative one and nine, I would also count these."

Act: Made a mark at -1 and a mark at 9 then made 10 hops to the right from -1 to 9 and drew an arrow to the right towards 9.

(Doug Prompt 7A) "And can I have those four expressions. We have these four other expressions and we are going to do the same thing with them as we did with the other ones. We'll do the number line last. If you want to set that aside that would be good. How would you approach each of these four expressions?"

Act: Placing the second set of expression on the table.

(Tae Turn 25 Prompt 7A '2 – 8') "Two minus eight the easiest way for me is with a number line and then I just count down eight."

Act: Draws number line and labeled the positive side of it then made a mark at 2 and 8 made hops to the left and a dot at -6 then labeled the negative side of the number line then wrote -6.

(Tae Turn 26 Prompt 7A '-7 + 1') "Negative seven plus one."

Act: Drew a number line and labeled it starting at -7 made a dot at -7 made 1 hop to the right and made a -6 then wrote -6.

(Tae Turn 27 Prompt 7A '-4 – 2')

Act: Wrote -6 then drew the number line and labeled it drew a dot at -4 then 2 hops to the left and made a dot at -6.

(Tae Turn 28 Prompt 7A '-1 + -5') "I think I'd show with items instead, I just have these items represent negatives instead of positives."

Act: Wrote -6 drew 1 open circle and 5 open circle.

(Doug Prompt 7B) "Ok, how are those four expressions similar and how are they different?"

(Tae Turn 29 Prompt 7B) "Well similar in their answer, these three ['2 – 8', '-7 + 1' and '-4 – 2'] are similar in that for me I was best able to show with the number line. These ['-7 + 1' and '-1 + -5'] are similar in that they can probably be confusing because you're adding a negative, well this one ['-7 + 1'] you're only adding one negative and then this one ['-1 + -5'] you're adding two negatives. Yeah."

Act: Pointing to '-7 + 1', '2 – 8', and '-4 – 2'. Pointing to '-7 + 1' and '-1 + -5'. Then pointing to individual expressions.

(Doug Prompt 7C) "Ok, so using the chip model how would you present each of these four."

(Tae Turn 30 Prompt 7C '2 – 8') "I think it's easier for me to almost think of minus eight as plus negative eight. So, these would represent negative integers negative eight and this is the two and you add two positives to the negatives you're left with negative six."

Act: Initially selected a pair of beige chips then counted out 8 brown chips then selected 2 beige chips placed the beige chips on the end of the lines of brown chips then removed 2 browns and 2 beiges leaving 6 brown chips in 2 rows of 3.

(Tae Turn 31 Prompt 7C '-7 + 1') ['Ok'] "Negative seven plus one. I'll use these for the negative integers, [ok] and then I'm adding one to it. I know these are negative and this is positive I have to remove and the negative one."

Act: Counted out 7 brown chips then slid 1 beige chip to the end slid 1 brown chip away and then 1 beige chip away leaving 6 brown chips.

(Tae Turn 32 Prompt 7C '-4 – 2') "And then negative four [hold on one second] [and you said negative four] negative four minus negative two cause we're working with the negatives these can be combined."

Act: Arranged 4 brown chips and 2 brown chips in two groups then slid the 2 brown chips in join the 4 brown chips to get 6 brown chips

(Tae Turn 33 Prompt 7C '-1 + -5') "And negative one and negative five we're adding them, so they can combine."

Act: Set out 1 brown chips and 5 brown chips then slid the 1 brown chip to join the 5 beige chips to form 6 brown chips in two rows of 3.

(Doug Prompt 7D) "Ok, and one more time with these on the number line please."

(Tae Turn 34 Prompt 7D '2 – 8') "Two minus eight."

Act: Made a mark at 2 then made 8 hops to the left and made a mark at -6.

(Tae Turn 35 Prompt 7D '-7 + 1') "And negative seven plus one. Adding one, we move to the right."

Act: Made a mark at -7 then made 1 hop to the right and a dot at -6.

(Tae Turn 36 Prompt 7D '-4 – 2') "Negative four minus two."

Act: Made a mark at -4 then made 2 hops to the left and made a dot at -6.

(Tae Turn 37 Prompt 7D '-1 + -5') "And negative one and five I am just going to move over five places."

Act: Made a mark at -1 then made 5 hops to the left and a dot at -6.

Appendix S: Consolidated Codes by Question

	Q1	Q2	Q3	Q4	Q5	Q5A	Q5B	Q5C	Q6A	Q6A ‘-1 + 9’	Q6A ‘10 – 2’	Q6A ‘6 - -2’	Q6A ‘6 + 2’	Q6B	Q6C	Q6C ‘-1 + 9’	Q6C ‘10 – 2’	Q6C ‘6 - -2’	Q6C ‘6 + 2’	Q6D ‘-1 + 9’	Q6D ‘10 – 2’	Q6D ‘6 - -2’	Q6D ‘6 + 2’	Q7A ‘-1 + -5’	Q7A ‘2 – 8’	Q7A ‘-4 – 2’	Q7A ‘-7 + 1’	Q7B	Q7C	Q7C ‘-1 + -5’	Q7C ‘2 – 8’	Q7C ‘-4 – 2’	Q7C ‘-7 + 1’	Q7D ‘-1 + -5’	Q7D ‘2 – 8’	Q7D ‘-4 – 2’	Q7D ‘-7 + 1’
Code for word use																																					
Sets of numbers																																					
‘Positive integer’	13	2	1		1	1										1	1		1			1	1			1	2	2					1		1	2	1
‘Negative integer’	2	5	4	2	1	1						1		2		1								1			1	3	1		1		1	1		1	
‘Whole number’	14	5	1	1							1		1			1									1		1										1
Integer	2	3	7	1		1	1							1	1										1	1		1									1
Other names for sets of numbers	10	3	3	1										1																							1
Sign of number read or introduced																																					
Positive number	6	10	13	4	9	10	3	12	1	13	14	14	14	13		9	13	12	13	14	13	15	12	3	14	13	12	8		2	11	12	13	2	13	13	10
Zero	1	3	2		8	6	1	1		1				1		1									1			2									
Negative number	6	13	12	3	8	9	3	12		13		10		11		9	1	12		14		16		13	12	14	13	12		12	8	14	13	14	12	13	11
Role of sign read or introduced																																					
Addition		10	1		8	6	2			6	1	5	6	9		7		6	11	6		7	8	5			6	11		9	2	7	6	10		2	11
‘Plus’ as addition sign		11	2		1			1		11		6	12	5		5	1		7	11			11	11	1	2	11	4		6	1		12	7		1	9
‘Minus’ as subtraction sign			7							3	11	11		6		1	7	6			4	11		2	11	12	1	5		1	5	10	1	2	6	8	
Subtraction				3	1					2	5	3		1		2	8	3			4	3			4	2		3			5		1		4	3	
‘Take away’ for subtraction		1	8	1	2	1	1		1	2	5	4		9			9	4			5	4		1	5	5	2	5			4	4	1	3	4	7	
Negative sign				1	1	1	1			2		4		2		1									1			2			1						
‘Positive sign’					1		1					1																1		1							
Negative for subtraction sign					1							1		1																				1			
‘Minus’ for negative					1																1																
Code for visual mediators																																					
Written Expression																																					
Positive number		7	5				1	2	1	8	8	7	5							2	1	4	1	2	7	7	7							1	1	3	
Zero		2																																			
Negative number/sign		7	5				1	2		7		6								2		3		12	8	10	9							3		3	
Addition		8	3							7		3	6							2		3	1	8	1	4	7							3		1	
Subtraction			4							2	8	4									1	3		2	7	7	1							2	2	3	
Positive sign explicitly			1									1																									
Identification of ‘ - ’																																					
Subtraction sign				14																																	
Negative sign				12																																	
Chip model notations																																					
Drawn										1	2	3	3											1	2	1	2										
One color										1	3	2	3											1	2	1	2										
Separate colors																								2	1	1											
Mixed colors											3	1	3			3	11	3	8					3		1				11	2	7	3				
Number line notations																																					
Participant drawn	2	1			2		1			5	2	3	2	1										1	4	2	2										
Spoken of	2					6	3				2			1		2									1	2	1	2		1	1						
Colored chips as number line																2														1	2	1	1				
Dot-hops-dot		1			1					2		1	1	1						10	12	12	9	1	3	1	1							11	11	10	11
Left or right arrow										1	1		1							3		2	4				1										2
Numbered steps taken																				1	1	1	1											1	1	1	1
Dots along number line																				1	1	1	1											1	1	1	
Code for Routines																																					
Actions with numerals																																					
Portions of numbers with different signs can cancel to give sum or difference										2						6											3				1		6				
Adding to a negative number as subtraction		1								5				1		3											2						4				
Adding a negative number as subtraction of a positive number																								3						1				3			

	Q1	Q2	Q3	Q4	Q5	Q5A	Q5B	Q5C	Q6A	Q6A ‘-1 + 9’	Q6A ‘10 – 2’	Q6A ‘6 - -2’	Q6A ‘6 + 2’	Q6B	Q6C	Q6C ‘-1 + 9’	Q6C ‘10 – 2’	Q6C ‘6 - -2’	Q6C ‘6 + 2’	Q6D ‘-1 + 9’	Q6D ‘10 – 2’	Q6D ‘6 - -2’	Q6D ‘6 + 2’	Q7A ‘-1 + -5’	Q7A ‘2 – 8’	Q7A ‘-4 – 2’	Q7A ‘-7 + 1’	Q7B	Q7C	Q7C ‘-1 + -5’	Q7C ‘2 – 8’	Q7C ‘-4 – 2’	Q7C ‘-7 + 1’	Q7D ‘-1 + -5’	Q7D ‘2 – 8’	Q7D ‘-4 – 2’	Q7D ‘-7 + 1’
Add negative numbers by adding absolute values than attach sign																								1		1											
Subtracting a negative is adding a positive number			2									13		2				5				4															
Subtraction as adding a negative number			1															1			1				1	4		2			3	8					
Magnitude of terms is used to determine sign of equivalent value										2																1	1						1				
Subtract smaller value from larger value and attach sign of larger value																									2										1		
When subtracting numbers match a portion of the larger number with the smaller to cancel the remaining amount is the difference																									1												
Actions with the chip model																																					
Adding as joining												2	3			10		10	13						2	1	2			12	9	10	10				
Opposite colored chips stacked																8		1							2		2				8	1	10				
Subtraction as taking away										1	3					3	14														1		3				
Opposite colors chips swapped																1	1	2								1					1	2	1				
Action with number line																																					
Go up number line		1			1	1	1			5		2	1	2		1				13		13	14			1	3	1					1		1		13
Go down number line					1		1				2	1		1						1	14	2		1	4	2		1		1	2			14	13	13	
Code for narratives																																					
Relationship between positive integers and whole numbers																																					
Are synonyms	3																																				
Mean different things	11																																				
‘Addition makes larger’ remains true with integers																																					
Yes, it remains true		4																																			
No, not always		10																																			
Types of numbers generated by integer subtraction																																					
Positive number			11																																		
Negative number			12																																		
Integer			2																																		
Any number			2																																		
Larger number			1																																		
Smaller number			2																		1																
Properties of additive inverse or the pair ‘-7 and 7’																																					
Are opposites					7	7	1																														
Add to zero					7	6	7																														
Operation and signifiers both need to be considered																																					
Adding a negative number and a positive number can be treated as subtraction		1			1					2				2																			1	1			
Subtracting a negative number can be treated as addition of a positive number			1									6		3				2				3															
Subtracting a positive number can be treated as adding a negative number																										2					1	4					

	Q1	Q2	Q3	Q4	Q5	Q5A	Q5B	Q5C	Q6A	Q6A ‘-1 + 9’	Q6A ‘10 – 2’	Q6A ‘6 - -2’	Q6A ‘6 + 2’	Q6B	Q6C	Q6C ‘-1 + 9’	Q6C ‘10 – 2’	Q6C ‘6 - -2’	Q6C ‘6 + 2’	Q6D ‘-1 + 9’	Q6D ‘10 – 2’	Q6D ‘6 - -2’	Q6D ‘6 + 2’	Q7A ‘-1 + -5’	Q7A ‘2 – 8’	Q7A ‘-4 – 2’	Q7A ‘-7 + 1’	Q7B	Q7C	Q7C ‘-1 + -5’	Q7C ‘2 – 8’	Q7C ‘-4 – 2’	Q7C ‘-7 + 1’	Q7D ‘-1 + -5’	Q7D ‘2 – 8’	Q7D ‘-4 – 2’	Q7D ‘-7 + 1’
When you have a negative number you automatically subtract														1																							
Symbols provide inconsistent meanings																																				1	
Results of subtraction																																					
Larger number from a smaller number is not proper																									1			1			2						
Negative number cannot be done												1						1				1															
Similarities of operations on the number line																																					
Addition										1			1							3			3											3			5
Subtraction												1	1								5	1									1			1	5	5	

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