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One Teacher's Transformation of Practice Through the Development of Covariational Thinking and Reasoning in Algebra : A Self-Study

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ONE TEACHER'S TRANSFORMATION OF PRACTICE THROUGH THE DEVELOPMENT OF COVARIATIONAL THINKING AND REASONING IN ALGEBRA: A SELF-STUDY

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ABSTRACT

ONE TEACHER'S TRANSFORMATION OF PRACTICE THROUGH THE DEVELOPMENT OF COVARIATIONAL THINKING AND REASONING IN ALGEBRA: A SELF-STUDY

By Jacqueline Dauplaise

CCSSM (2010) describes quantitative reasoning as expertise that mathematics educators should seek to develop in their students. Researchers must then understand how to develop covariational reasoning. The problem is that researchers draw from students' dialogue as the data for understanding quantitative relationships. As a result, the researcher can only conceive the students' reasoning. The objective of using the self-study research methodology is to examine and improve existing teaching practices. To improve my practice, I reflected upon the implementation of my algebra curriculum through a hermeneutics cycle of my personal history and living educational theory. The critical friend provoked through dialogues and narratives the reconceptualization of my smooth covariational reasoning from a "transformational perspective" to a "solving algebraic equations" perspective. This study showed that by creating images in motion, graphs, or algebraic representation, I recognized the importance of students' cognitive development in the conceptual embodied and proceptual symbolic worlds. The results presented the transformation of my teaching practices by building new algebraic connections. By using these findings, researchers can gain additional understanding as to how they can transform their teaching practices.

Keywords: self-study, smooth and chunky covariational reasoning, conceptual embodied, proceptual symbolic, meaningful learning, building algebraic connection

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DEDICATION

This dissertation is dedicated to my daughters: Nicole and Stephanie

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INTRODUCTION

CCSSM (2010) described quantitative reasoning as expertise that mathematics educators sought to develop in their students. To better understand students' cognitive development of quantitative reasoning, the review of literature establishes a relationship between chunky and smooth covariational thinking. The problem is that researchers draw from students' dialogue as the data for understanding this relationship. I suggest that if the researcher reflected on their covariational reasoning, then a more comprehensive analysis would ensue. This reflected knowledge might provide the groundwork for developing a stronger relationship between chunky and smooth reasoning, in an attempt, to facilitate meaningful learning. Therefore, this paper examines, through self-study, the development of my smooth and chunky covariational reasoning for the improvement and transformation of practice.

Background of the Problem

Uppermost on Algebra teachers' minds in the 21st century is not meaningful learning; rather it is their students' Algebra I EOC scores, which reflect directly on their VAM (*value added model*) score. These scores play a part in determining whether teachers are rated highly effective. Amrein-Beardsley (2014) argued administrators made high stake decisions (e.g., merit pay, teacher tenure, teacher termination) based on negative or positive value-added scores, whether appropriate or accurate. However, Cochran-Smith and Boston College Evidence Team (2009) believed there was a need for balance in the evidence-based education movement because there was a difference between a culture where evidence *drove* and a culture where evidence *informed* decisions. They suggested that creating a culture of evidence and inquiry in teacher education that *informed* had the potential to be transformative and add new vitality to teacher curriculum, policy, and practice. However, while teachers focus on these high stake decisions and "strive to improve performance on tests, there is a growing realization that practicing procedures to be able to perform fluently is not sufficient to develop powerful mathematical thinking" (Tall, 2006a, p. 1). Accordingly, teachers are called upon to engage in education reform through collective praxis, first through reflection on their content and pedagogical knowledge, followed by action to transform students into meaningful learners. The transformation of meaningful mathematical learners begins with teachers understanding mathematical cognitive development.

Statement of the Problem

Although, nurturing students' mathematical cognitive development is essential for student learning, another aspect of learning is facilitating meaningful learning. This paper reviews the literature on the diverse ways in which teachers facilitate meaningful learning in their classrooms (e.g., through mathematical dialogue, anchoring, advance organizers, and concept maps). In contrast to rote and procedural learning, meaningful learning requires potentially meaningful material and a meaningful learning set (Ausubel, 2000).

One method of facilitating meaningful learning is developing a broader sense of quantitative or non-numeric reasoning skills in early mathematics prior to introducing algebraic symbols. A relatively new area of quantitative reasoning research is quantitative covariational reasoning. Covariational reasoning is a theoretical construct that conceives two quantities or variables changing simultaneously. Therefore, covariational reasoning, although not explicitly stated in the Algebra curriculum, provides Algebra I students with the opportunity to develop diverse ways to foster meaningful learning in hopes to promote advanced mathematical thinking.

A review of the literature reveals a new perspective on covariational reasoning that conceives of reasoning as either chunky (interval-driven) or smooth (process-driven). By understanding how students think about quantities and variables, we can develop a powerful foundation for covariational thinking. Moreover, if smooth thinking turns out to be the cognitive root of chunky thinking, then we will have identified “a foundation for a powerful form of chunky thinking” (Castillo-Garsow, Johnson, & Moore, 2013, p. 35). Therefore, if researchers understood how someone thinks smoothly, this understanding could provide insight into how to nurture a powerful anchor for chunky reasoning.

A problem arises when a researcher attempts to perceive another’s thinking. The researcher could only draw on what they observed or what the participant explained (Thompson & Carlson, 2017). Thus far, researchers have based their understanding of covariational reasoning on what the researcher perceives students’ thinking to be through dialogue. I argue that the researcher as participant knows their emergent mathematical thinking. To date, there is limited research on how the researcher develops smooth and chunky thinking and then connects covariational reasoning to practice.

Purpose of the Study

To connect how the researcher (myself) thought about covariational reasoning and its effect on practice, I used the self-study methodology. Self-study methodology was specifically designed for educators to “explore ideas, theories, concepts and practice . . .” (Guilfoyle, Hamilton, Pinnegar, & Placier, 2004, p. 1111) as to reframe their thinking and improve their practice. Therefore, a key component of this self-study research was reframing my covariational thinking process for the purpose of improving my practice.

My research drew on connections between the work of Ausubel (1967; 2000), Tall (1989), and Thompson (2012). Ausubel (1967; 2000) recognized the importance of advance organizers as a tool to support students in developing meaningful learning. Similarly, Tall (1989) and Tall and Vinner (1981) used concept maps as tools for students to illustrate their thinking process. Tall (1991) established the Three Worlds of Mathematics, a framework for mathematical cognitive development. Thompson (2012) referred to meaningful learning, the Three Worlds of Mathematics, and connections to Castillo-Garsow's (2012) research on *how* the student thinks covariationally. Finally, each researcher discussed the importance of Algebra as a gateway to quantitative reasoning and advanced mathematical thinking.

Nevertheless, for me, Thompson and Carlson (2017) connected covariational reasoning and self-study when they argued that, thus far, researchers had only been able to infer what the students thought because the thinking occurred in the mind of the student not the researcher. Although there were methodologies that accommodate for this circumstance, the self-study methodology was the most attractive means for me to be the focus of the investigation and explain my thinking. Therefore, this paper explored how I provided students with meaningful learning opportunities through self-study on my own practice to nurture their advanced mathematical thinking. Furthermore, this paper investigated how I supported the development of covariational smooth and chunky quantitative reasoning in the context of algebra.

Theoretical framework. To understand mathematical cognitive development, this paper establishes the Three Worlds of Mathematics as a theoretical framework. Tall's (1995; 2005; 2006b; 2007; 2013) Three Worlds of Mathematics described students' mathematical cognitive development as consisting of conceptual embodied (from the physical world), proceptual symbolic (symbolic process), and axiomatic formalism (generated by axioms). Axiomatic formalism requires advanced mathematical thinking. Teachers who understand the world within the students resides encourages a meaningful learning experience and ideally the students transition within the worlds. Additionally, as teachers nurture the cognitive development of conceptual-embodied and proceptual-symbolic at an early age, they foster skills necessary for students' advanced levels of mathematical thinking expected in higher-level mathematics (e.g., Calculus, Advanced Algebra, and Euclidean Proofs).

This framework provides for the use of prior knowledge or met-befores. Tall (2013) stated met-befores are "mental structured we have now as a result of experiences we have met before . . ." (p. 84). These structures could be advantageous or impede the understanding of new knowledge. A met-before is advantageous when built upon stable prior knowledge; however, the anchor of new knowledge to unstable prior knowledge impedes the retention and learning of new material. There by suggesting, teachers who have awareness to the importance of stable prior knowledge can create a more meaningfully learned experience.

The Study

This self-study presents the reframing of my smooth and chunky covariational reasoning. The dialogue of my thinking provides an articulate form of data to base future understanding to develop a powerful form of smooth and chunky quantitative reasoning; a skill teachers should foster in students to develop advanced mathematical thinking. The following is the research question that guided this study:

> How does analyzing the development of my acquisition and implementation of the lens of smooth and chunky covariational reasoning through reconceptualizing algebraic relationships promote the transformation of my teaching practice?

The primary sources of data in this self-study were videos recordings and daily research logs. The research logs consisted of transcribed video-recorded lesson, provoking thoughts to implement the following day, and a critical friend memo from our dialogue about the lesson.

During my daily teaching practices, I revised the lesson as necessary to create a more perfected lesson and assessed students' understanding either through student discussion, student work, or common assessments; however, student-level data were not included.

A characteristic of this self-study methodology to ensure trustworthiness is working with a critical friend who asks thought provoking questions (Costa & Kallick, 1993), provides an alternate lens to view data (Samaras, 2011), and validates assumptions and interpretations (Samaras, 2011). In this self-study, the critical memos provided the data for the critical friend portfolio and demonstrated an overall progression of dialogue of my thinking about the lessons.

The final data sources were generated during the analysis process through narratives with my critical friend. I kept a Critical Friend Portfolio (CFP) made up of dialogues and narratives between my critical friend and myself. During the critical friend dialogue, my thoughts about the data began to take on an additional depth of understanding. The hermeneutic (back and forth) interpretive process allowed me to reflect on my personal history (Samaras, 2011) and living educational theory (Whitehead, 1989) to reframe my algebraic thinking. The narratives revealed contiguity-based relationships that connected or influenced one and other (Maxwell & Miller, 2008). However, it was during the inter-rater reliability discussions about the concept map that theoretical themes surfaced.

Significance of the study. This self-study examines the reconceptualization of my covariational reasoning to transform my teaching practice such as collective praxis and mathematical dialogue in the facilitation of advanced mathematical thinking. To better understand transformation in practice, we consider Freire's (2000) interpretation that transformation in the educational system requires educators to take action.

Praxis. Paulo Freire (2000) defined praxis as "reflection and action directed at the structures to be transformed" (p. 126). The educational structure is one of the structures that need transformation as demonstrated by the recent adoption of the CCSS of mathematics (Common Core State Standards Initiative, 2010). The stakeholders (e.g., state legislators, the community, administration, and teachers) created the CCMSS (2010) that laid out the vision for the role of teachers to prepare students for college and careers. However, many teachers believe that their teaching methods worked well in the past and that change is unnecessary. Freire (2000) proposed that those who embraced transformational thinking perceived reality as a process rather than a static entity. Therefore, teachers cannot continue to teach as they did in the past. They need to develop new methods of teaching that prepare students for higher-level mathematics.

One approach for teacher educators to transform the educational system is becoming the subject of their research, reflecting on their thinking, taking action, and teaching to enact a vision for that praxis. Lovin, Sanchez, Leatham, Chauvot, Kastberg, and Norton (2012) explained that through a collaborative self-study, they changed their individual practices and also "moved beyond the individual by exemplifying the power of examining and reflecting on other teacher educators' beliefs and practices" (p. 65). Rather than focusing the investigation on their student's thinking, these mathematics teacher educators wanted to investigate their own practice through reflection and ideally transformed their teaching as a result. However, this approach does not imply that as the teacher/researcher examines their thinking that they avoid examining their students' thinking. It simply means that the focus of the study should be directed inward instead of outward. Furthermore, by reflecting inward, the teacher/researcher provides a voice to alter the collective praxis by taking action against static thinking. In other words, as I reflect on my practice and make changes, I share these new practices with others and transform the existing educational system.

Mathematical dialogue. When teachers reflect inward, they confront and challenge their current ways of thinking. Still, it often requires others to provoke the kind of thinking that might otherwise remain unexplored. Often, student dialogue provides rich insight into how the students are thinking about and understanding a concept. In turn, that dialogue serves to alter teachers' and students' perceptions of a concept. In *Principles to Action*, Leinwand (2014) explained that mathematical discourse consisted of dialogue among classmates in the form of verbal, visual, and written communication with a purposeful exchange of ideas. He described the actions necessary to ensure that all students learn to become mathematical thinkers. This book provides guidance to teachers, principals, policymakers, and families on what to expect from the educational system for a successful implementation of the CCMSS.

One reason for a purposeful exchange of ideas is for teachers to provide students with a forum to explain and compare multiple versions of their quantitative thinking processes. For example, in a study of third grade students, Tall (2008b) investigated how students calculated the number of black circles in an array with three rows, each with the same number of elements. Students used various ways to subdivide the rows into smaller parts based on their previous knowledge. The study showed that students developed a broader sense of quantitative reasoning skills when encouraged to share their methods of answering questions.

A second reason for a purposeful exchange of ideas is for teachers to determine whether or not students understand the ideas they share. Carpenter, Franke, and Levi (2003) found sharing different methods of finding an answer helped students to understand the importance of articulating and justifying mathematical claims, reflect and interpret their ideas by comparing them to the ideas of other students, and consider and evaluate the consequences of these ideas.

There are students who offer the correct solutions; however, when they are pressed to share their thinking process, “students have limited ability to reflect on their thinking” (Carpenter & Lehrer, 1999, p. 22). Carpenter and Lehrer (1999) explained that this difficulty might be the result of a lack of understanding. This insight—whether it is the students’ inability to reflect on their thinking or a lack of understanding—should inform the teacher’s subsequent teaching practices.

Another reason for a purposeful exchange of ideas is the verification process proposed by Mason, Burton, and Stacey (1982) in *Thinking Mathematically:* convince yourself, convince a friend, and convince an enemy. Tall (1991) explained the importance of this process in nurturing advanced mathematical thinking further, “ . . . to convince yourself you have to understand why, to convince a friend the argument must be organized and coherent, but to convince an enemy means that the argument must now be analyzed and refined so that it will stand the test of criticism” (p. 20). An individual who can convince an enemy demonstrates an advanced level of understanding. What might an advanced level of mathematical thinking look like in Algebra I? Algebraic proofs provide teachers with an opportunity to challenge student thinking and prepare students at a rudimentary level to begin the construction of formal proofs, which is expected in advanced mathematics studies.

Chapter Summary

In chapter one, I discussed the challenge balancing evidence-based education that drives and informs decisions. Then, I considered how the transformation of practice begins by understanding students’ cognitive development. A method of facilitating meaningful learning is promoting quantitative covariational reasoning. Next, I described the methodology of self-study and the importance of a critical friend to challenge my thinking process asking provoking questions for a different lens during the analysis. This chapter discussed collective praxis as an approach to transform the educational system by taking action to foster students' meaningful learning experiences. Finally, I shared the significance of purposeful exchange of ideas in mathematical dialogue.

Chapter II

Theoretical Framework

In this chapter, I present the Three Worlds of Mathematics framework. First, I explain the students' mathematical development from conceptual embodiment to operational symbolism and then axiomatic formalism (as seen in Figure 1). Next, I connect the Three Worlds of Mathematics to met-befores, met-afters, and obstacles. Finally, I explain the process of the compression of knowledge. The compression of knowledge affects teaching practices because it explains the necessity of connecting new knowledge to prior knowledge into understandable chunks. Taken together, these components provide the teacher with an overall understanding of mathematical cognitive development, its effect on the meaningfulness of the students' learning experience, and the effectiveness of the teachers' practice.

Three Worlds of Mathematics

Although this paper examines the teacher-researcher's covariational quantitative reasoning development within my practice, I must also reflect on the students' covariational quantitative reasoning development to alter my practice throughout the study. Therefore, the theoretical framework used in this study was Tall's (2007) Three Worlds of Mathematics, a framework for analyzing the "long-term" cognitive development of students' mathematical thinking. The students' overall mathematical cognitive development transitions from the physical senses (conceptual embodiment) to a more symbolized process (operational symbolism), and finally, to formalizing concept definitions and logical deductions inspired by advanced mathematical thinking (axiomatic formalism) necessary in higher-level mathematics.

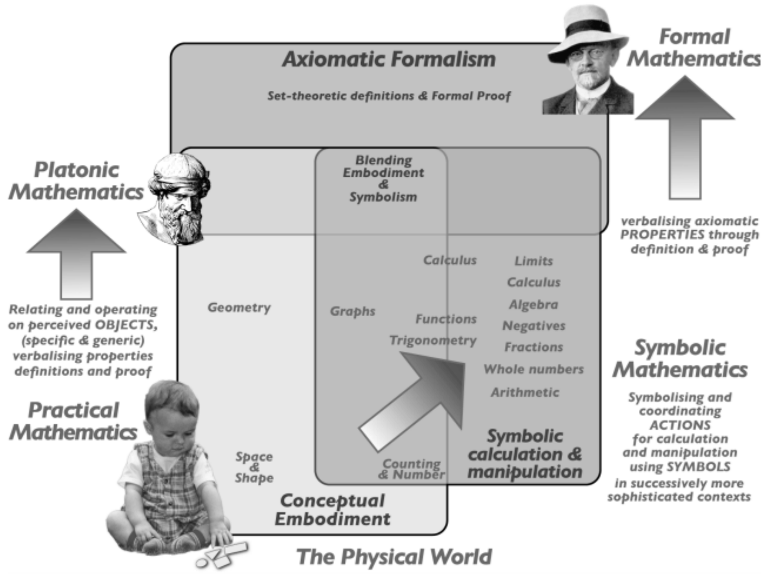


Figure 1. The Three Worlds of Mathematics in Tall, 2007, p. 5

Conceptual embodiment. Conceptual embodiment is the stage at which children begin to use shapes to represent a mathematical mental image. According to Tall (2013), *conceptual embodiment* was the earliest form of cognitive development. Conceptual embodiment built on human perception and actions by developing mental images that were verbalized in increasingly sophisticated ways and became mental entities in our imagination (Tall, 2013). The embodied world was the foundational knowledge the individual developed through sensory perceptions (sensorimotor knowledge) in the physical world. Tall referred to the “object-based conceptual embodied world as reflecting on the senses to observe, describe, define and deduce properties developing from thought” (Tall, 2005, p. 24). A child in the embodiment world may use concrete objects such as shapes, base ten blocks, or algebra tiles to represent abstract concepts. An example of algebra, in this world, can be seen in simplifying the expression, $3a + 4b + 2a$. First, the child *collects* like terms to give $3a + 2a + 4b$ and then *combines* like terms for $5a + 4b$, which is an example of the embodied (physical) technique of “moving around.”

Operational symbolism. The next phase of mathematical cognitive development is operational symbolism and occurs in the *proceptual symbolic world*. Tall (2013) explained that in the world of operational symbolism, mathematical procedures grew out of physical actions. Actions (such as pointing and counting) switch back and forth from *process* to do mathematics to *concepts* to think about mathematics. For example, 3 + 2 in arithmetic has a dual connotation as process (addition) and concept (sum). The amalgamation of a process and concept is called a *procept*. Gray and Tall (1994) defined a procept as "a mental structure, which is an amalgam of process and concept" (p. 115). Therefore, the second action-based world was also referred to as the *proceptual symbolic world or proceptual world*.

There is fluidity between embodiment and symbolism. When given an algebraic word problem, a student may depend on the algebra (symbolism) to answer the problem, but may reflect back to embodiment to interpret whether or not the answer makes sense in the real world. Students who move back and forth between embodiment (concrete objects) and operational symbolism develop a more complete understanding that allows students to think in the realm of complex numbers (e.g., $a^3 - b^3, a^4 - b^4,$ or $a^n - b^n$). These students developed what some mathematicians interpreted as *elementary advanced mathematical thinking,* which referred to advanced mathematical thinking for children in the early years of education (de Beer, Gravemeijer, & van Eijck, 2015; Harel & Sowder, 2005).

Axiomatic formalism. One goal of mathematical cognitive development should be axiomatic formalism or advanced mathematical thinking. *Axiomatic formalism* "builds formal knowledge in axiomatic systems specified by set-theoretic definition, whose properties are deduced by mathematical proof" (Tall, 2013, p. 133). The following examples compared the three worlds, "In the embodied world 3 + 2 is the same as 2 + 3 because I can *see* it is true. In the symbolic world 3 + 2 is the same as 2 + 3 because I can calculate it. In the formal world, $x + y = y + x$ in a specific mathematical structure because it is an axiom" (Tall, 2006b, p. 203).

An individual transitioning from elementary mathematical thinking of embodiment and conceptual symbolism to the advanced mathematical thinking of axiomatic formalism depends less on physical constructs, such as visuo-spatial imagery and *concept images,* and more on *concept definitions*. The *concept definition* was the body of words that described or defined the concept (Tall & Vinner, 1981).

To clarify, "elementary mathematics *describes* objects, whereas advanced mathematics *defines* objects. In both cases, language is used to formulate the properties of objects, but in elementary mathematics, the description is constructed from the experience of the object . . . in advanced mathematics, the properties of the object are constructed from [concept] definitions" (Tall, 1995, p. 7). For example, continuity can be explained in terms of concept images and concept definitions. For instance, students might have an elementary mathematics understanding of the difference between discontinuous and continuous curves as illustrated on a graph (*concept image*), but an advanced mathematics proof of continuity is grounded in the formal definition (*concept definition*).

Facilitating the Three Worlds of Mathematics in Practice

The Three Worlds of Mathematics provides teachers with a framework to recognize their students' cognitive development for specific concepts. Teachers who recognize the cognitive world in which students reside provide the teacher with information to individualize the lesson for the student. Therefore, a teacher who recognizes that a student's cognitive development resides in the world of conceptual embodiment might demonstrate with positive and negative chips to show a visual representation of equality. The lesson may differ for students who think in the *proceptual world;* the teacher might start with fact families and then allow students to calculate equivalent values. The teacher may also nurture the student's cognitive development by applying equivalence to the commutative property (axiomatic formalism). Essentially, the reason teachers need to understand their students' mathematical cognitive development is to provide students with a meaningful learning experience by nurturing the student's transition from one world to the next.

Met-before and met-after. The foundation of a student's mathematical cognitive development is their prior knowledge. Prior knowledge is the knowledge a student met-before. Met-befores are "mental structures we have now as a result of experiences we have met before . . ." (Tall, 2013, p. 84). Analogous to prior knowledge, a met-before could be accurate or erroneous. The accuracy of the met-before determines the fragility of the new learned knowledge.

When the foundational information was fragile, Ausubel and Robinson (1969) claimed the learners would find themselves in the position of being unable to comprehend what follows in the sequence and fall back on a memorized process. For example, a student may have difficulty simplifying the expression $2 + 3x$. The students know 2 + 3 = 5, but do not have an answer for $2 + 3x$. The student interprets this new situation incorrectly by drawing on a met-before to add 2 + 3 to get 5, and then attaching the x to yield $5x$. A learner who resorts to rote (memorized) learning at some point will be unable to perform even the simplest application because they do not understand why they are performing the process. In this case, the student may lack the foundation (met-before) with which to build new learner knowledge, or they may have a fragile understanding of the met-before; either way the new knowledge built will demonstrate fragility at some future point.

Teachers also have concerns with met-afters. Met-afters "refer to the effect of an experience that was met later in development that profoundly changes the way in which earlier ideas are considered" (Tall, 2013, p. 115). A met-after may be accurate or inaccurate knowledge; more precisely, they represent incomplete knowledge. A mathematician might have a met-after on a topic that they learned to solve using advanced mathematical thinking in the world of axiomatic formalism. The challenge for mathematics teachers is to reconceptualize their axiomatic knowledge to explain a concept to middle or high school students because teachers and students think in different cognitive developmental worlds. Tall (2013) explained the trigonometric identities of $sin\ (A + B)$ and $cos\ (A + B)$ was an example of a met-after because it was commonly used in advanced mathematics course, but too challenging for beginning geometry students.

Whether an inaccurate met-before or met-after, both present challenges for teachers. The teacher may be able to reconceptualize how they can teach a concept differently (using accurate met-befores and/or met-afters) merely through discussions with colleagues, but the students' mistaken met-befores present a greater challenge to teachers. Teachers can construct models of their students' thinking, but these are second-order models (constructed by someone other than the model holder); thus, they are not certain. Teachers need to determine if the student's challenge is a result of an inaccurate met-before or caused by an *obstacle*. Tall (1991) deferred to Cornu's thesis (1983) for the definition of an obstacle:

> An obstacle is a piece of knowledge; it is part of the knowledge of the student. This knowledge was at one time generally satisfactory in solving certain problems. It is precisely this satisfactory aspect, which has anchored the concept in the mind. (Cornu,

1983, as cited in Tall, 1991, p. 9 (original in French))

As stated, a met-before is a concept that the students learned consistently before and applied to a new situation. If the met-before is accurate, then the new learner knowledge will be stable. However, if an obstacle exists, this creates an inaccurate met-before resulting in inconsistently. Some obstacles are more challenging to alter than others: "Obstacles [that] arise from deeply held convictions about mathematics [and] are rarely easy to erase from the mind" (Tall, 1991, p. 11). Obstacles may go unnoticed, until someone or something challenges their interpretation on the concept. For example, when students graph the inequality $x > 4$, they mark an open hole at four and a shaded line going right. When the students graph and inequality $x < 4$, they mark an open hole at four and a shaded line goes left. However, when asked to graph $4 < x$, the student may respond by saying the shaded lines go to the left. This obstacle develops because teachers tend to place the variable on the right, so the student soon looks at the inequality sign as a direction arrow. Therefore, teachers should consider accurate or inaccurate met-befores and obstacles in planning lessons to address them during lessons. Because inaccurate met-befores and obstacles are ingrained in a student's concept structure, they may not appear in class without prompting.

Compression of knowledge and thinkable concepts. In the algebra classroom, students learn new material every day, more so than they did in prior years that tend to continually revisit the same concepts. Students need to connect new algebraic knowledge to prior knowledge learned. An effective process for the compression of knowledge is chunking similar concepts. Thurston (1990) remarked that mathematics was *compressible,* which meant that you could replace many ideas and then think of them as one complete idea. For example, the student recognizes whole numbers and their opposites listed on a paper. Compression of knowledge means thinking of this list as integers. Another example of compression of the knowledge is a procept, as discussed earlier. Tall (2013) explained that language facilitated the compression process because only after we thought and named it (thinkable object), could we speak about it. Three methods are used to compress concepts: *categorization* based on recognition (grouping ideas), *encapsulation* based on repeating actions (do it over and over), and *definition* that uses language to formulate a specific concept in a given text (understanding the definition of something).

Meaningful Learning Theory

One way to internalize thinkable ideas is through anchoring (relating new ideas to existing prior knowledge). Ausubel (2000) believed that the "cognitive structure of relevant anchoring ideas, their stability, clarity, and discriminability from related internalized ideas [thinkable concepts] . . . [and] are the most prominent factors that influence meaningful learnability, the degree of learning, and retention of new potentially meaningful instructional material" (p. xi). Those whose cognitive development is based on stable prior knowledge became meaningful learners.

Ausubel's (1967) Meaningful Learning Theory explained the difference between meaningful and rote learning. With respect to rote or meaningless learning, the learner internalizes the concept arbitrarily and verbatim. *Arbitrary* referred to "a presentation of unrelated facts without organizing or explanatory principles" (Ausubel, 2000, p. 7). A concept is rendered meaningless if not anchored to the prior knowledge that gives it meaning. In contrast, meaningful learning anchors new concepts to relevant prior knowledge. It is worth noting that the result is not so much a function of the teaching, but of the learning.

Consider a lesson on graphing a quadratic equation. A teacher uses graphing calculators to enable students to explore transformational shifts of a parabola. The teacher perceives this activity as being meaningful; however, the students may learn it arbitrarily and verbatim (as procedures not tied to meanings). For example, a teacher asks students to explore the transformation of $f(x) = x^2 + 7$. Students realize that the parent function of the parabola shifts up 7 on the y-axis and predicts that for $f(x) = x^2 - 7$ the parabola will shift down to -7 on the y-axis. The students gain confidence because their prediction is correct. When asked to predict the shift of $f(x) = (x - 3)^2$, the students predict a horizontal shift to the left and $f(x) = (x + 3)^2$ as a horizontal shift to the right, when in actuality it is the reverse. Although, many teachers try to explain the shifts based on the formula $f(x) = a(x - h)^2 + k$ when (h, k) is the vertex, students find understanding difficult. Students inevitably remember that the vertex as the opposite of h, k. This memorization process creates rote and meaningless knowledge.

Ausubel (1967) explained that learning new concepts such as function transformations was meaningful when it was nonarbitrarily (plausible, sensible, and nonrandom) and substantively learned. The concept is potentially meaningful to the learner; that is, the learned concept is a cognitive structure or substructure of the student's knowledge. As a result, in the above example, the student may learn to graph parabolas meaningfully. Although the teacher provides opportunities, the learner is ultimately in charge of their learning and depends on whether the student anchors the new knowledge to a stable concept image.

Anchoring, scaffolding, and concept maps. Whether using the term *anchor* (Ausubel, 2000; Tall, 1989) or *scaffold* (Vinner, 1991), the meaning is the same: relating new ideas to existing concept images. A *concept image* refers to the cognitive structure associated with a concept, which includes all of the mental pictures as well as associated properties and processes. When a learner relates new concepts to an existing concept image, the learner has attached the new concept to a stable anchor; thus, the individual has meaningful retention (memory) of the concept. Although Ausubel's conception of assimilation did not quite match Piaget's (1953), his assimilation process interpreted the retention of memory in terms of existing (anchoring) ideas in the cognitive structure (Ausubel, 2000, p. 7). Moreover, if the new information attaches to an unstable concept image then the learner would forget the new anchoring idea.

A tool used to anchor new ideas to existing ideas in the cognitive structure is an "advance organizer" (Ausubel, 2000, p. 11). An advance organizer is a "pedagogic device that bridges the gap between what the learner already knows and what he needs to know if he is to learn new material most actively and expeditiously" (p. 11). Ausubel used the adjective, *advance*, because "an *advance* [emphasis added] organizer is introduced to the learner prior to confronting him with the learning material itself" (p. 12).

Other researchers referred to advance organizers as *graphic organizers* (Baxendall, 2003; Owolabi & Adaramati, 2015; Zollman, 2009) or *concept maps* (McGowan & Tall, 1999; Novak, 2011) because the organizers were used to facilitate the learning of new material. In the following studies, researchers used the organizers to facilitate student learning by connecting the new knowledge with the students' existing knowledge.

Zollman (2009) used Gould and Gould's (1999) four-square and diamond type of graphic organizers for "short-answer, open responses to mathematical assessment problems" (p. 224). The teacher gave students a drawing of a small kitchen. The teacher asked them to find both the length of the border around the kitchen walls and the area of the tiling needed to cover the floor (as seen in Figure 2). He found that "the tool provides multiple starting points for low-ability students to begin solving a problem, helps average-ability students organize their thinking strategies, and encourages high-ability students to improve their problem-solving communication skills" (p. 226). Furthermore, "graphing organizers offer a quick, efficient diagnosis of individual students' [sic] problem-solving abilities, skills, strengths, and weaknesses in a comfortable, familiar, problem-solving instructional setting" (p. 228).

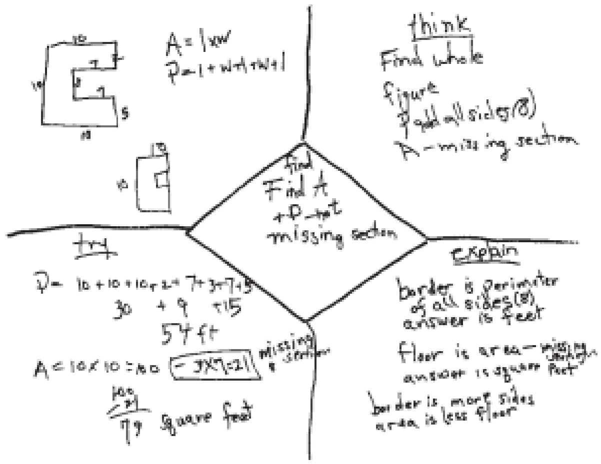


Figure 2. Student work on a perimeter-area problem in Zollman, 2009, p. 226

In a study by McGowan and Tall (1999), the students created three function concept organizers over a 15-week period to document their "cognitive growth over time" (p. 282). The most successful students built the concept maps based on the previous map, whereas the least successful student replaced the previous map and "started almost anew each time" (p. 286). That is, the more "successful [students] had processes of constructing, organizing, and restructuring knowledge that facilitate the building of increasingly complex cognitive structures" (p. 287). The successful students built upon the basic anchoring structure as shown in the initial map (as seen in Figure 3, left) and the second map (as seen in Figure 3, right). These students became flexible thinkers. Tools used to anchor new knowledge to prior knowledge provide (met-befores) students with opportunities to develop reasoning skills.

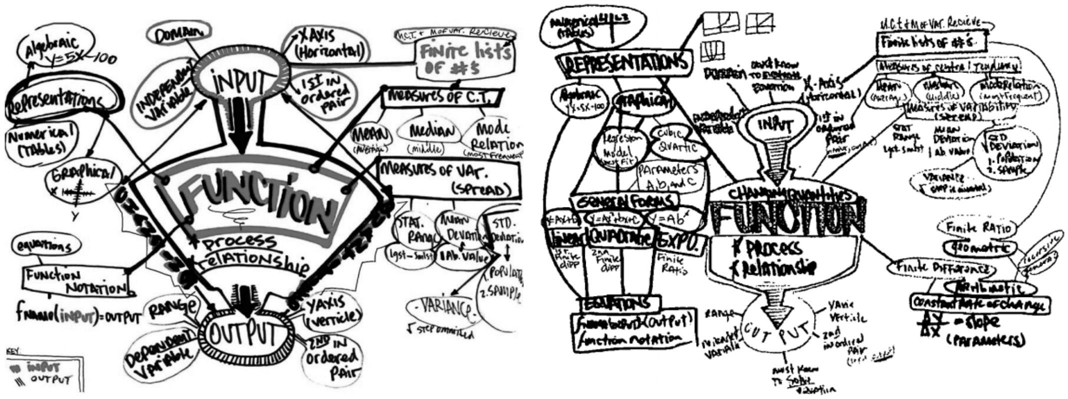


Figure 3. Student Created Concept Maps: On the left: MC’s concept map after Week 4 and on the right: MC’s concept map after 9 weeks in McGowan and Tall, 1999, pp. 282

Chapter Summary

This chapter discussed Tall’s (2007) Three Worlds of Mathematics, a framework for cognitive development (e.g., conceptual embodiment, operational symbolism, and axiomatic formalism). Next, I described the facilitation of these three worlds in practice by explaining the difference between a met-before and met-after, then using compression of knowledge to chunk similar concepts. Finally, I explained the difference between rote and meaningful learning based on Ausubel’s (1967) Meaningful Learning Theory. This chapter concluded with the use of anchors, scaffolding, and concept maps for the retention of knowledge.

Chapter III

Literature Review

The review of literature begins with the significance of promoting quantitative reasoning with school-aged children. Next I provide the description of two types of quantitative reasoning, correspondence relationships with emphasis on explicit rules when thinking about variables (Confrey & Smith 1995) and covariational reasoning expressed in terms of thinking about variables changing asynchronously or simultaneously (Thompson & Carlson, 2017). Covariational reasoning can be thought more specifically as smoothly (process-driven) or chunkily (interval-driven). The conclusions drawn from the review of literature are that researchers described their perception of the students' chunky and smooth reasoning. The final section of this chapter introduces the self-study methodology section to explain the characteristics of self-study that affords the researcher insight about smooth and chunky covariational reasoning to reframe thinking and transform practice, followed by a review of mathematical self-study research.

Emphasizing Quantitative Reasoning

This section begins by emphasizing the reasons for promoting quantitative reasoning at an early age. I then provide an explanation of the difference between variational or correspondence approach and covariational reasoning. Next, this section provides the major levels of covariational reasoning. Finally, I describe the relationship that exists between chunky and smooth covariational reasoning. To understand chunky and smooth covariational reasoning, I begin with understanding quantitative reasoning.

The current mathematics educational curriculum in place across the United States emphasizes the importance of oral or written explanations to demonstrate reasoning skills.

Quantitative reasoning has evolved since Dewey (1933) with more recent quantitative reasoning focusing on the areas of expertise that teachers should develop in their students (e.g., NCTM, 2000; CCSSM, 2010); however, as quantitative reasoning is a standard in the domain for mathematical practices in the CCSSM, teachers interpret quantitative reasoning as a standard being assessed rather than an abstract thinking process to develop. The CCSSM (2010) stated that "quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just computing them; and knowing and flexibly using different properties of operations and objects" (p. 6). Considered with the overall purpose of the CCSSM, this standard of practice suggests that students who develop reasoning skills may become flexible problem solvers, a skill that will be necessary in higher-level mathematics.

Recent studies by Smith and Thompson (2007) and Ellis (2007; 2011) found that introducing quantitative reasoning in elementary school nurtures meaningful and productive arithmetic and algebraic knowledge. Smith and Thompson (2007) suggested that for teachers of first and second grade students to support the development of students' quantitative reasoning, they should focus students' attention away from thinking about numbers and numerical operations (operational embodiment) and focus instead on looking for inferences and representing general relationships (operational symbolism). For example: "Sharon lost 6 marbles to Philip in a game. What can we say about the number of Sharon's marbles before and after the game?" (Smith & Thompson, 2007, p. 38). By not mentioning the total number of Sharon's marbles, the teacher hopes to encourage the student to look for a relationship between quantities. Providing support for students' quantitative reasoning in elementary and middle school allows them to focus on the reasoning (learning meaningfully) rather than on memorized processes or procedures, and then eventually to shift their thinking to algebra that is more symbolic. Smith and Thompson (2007) believed that algebraic knowledge that developed from earlier experiences with quantitative reasoning might serve as the basis for moving toward increasing abstraction (i.e., multi-modal meaning as process, property, or concept).

In a study with seventh and eighth grade students, Ellis (2011) considered that the teacher's role was to shape the discussion by posing appropriate questions and encouraging students to generalize patterns or relationships that are "meaningfully grounded" (p. 235). Meaningfully grounded reasoning with quantities and their relationships "constitutes a powerful way to help students build beginning conceptions of functions at the middle school level" (p. 215). This idea suggested that students with a foundation in quantitative reasoning had a foundation upon which to connect or anchor new algebraic concepts. For example, a group of seventh-grade students explored constant rates of change by investigating gear ratios and constant speed (Ellis, 2007). The students made generalizations and justifications while attempting to simultaneously keep track of the rotation of two gears using masking tape on one of the teeth. Students created a table to pair up the rotations. Eventually, the students found the ways to coordinate the rotations and also developed a covariational language for discussing the coordinated quantities. The covariational approach supported the students' abilities to express the rotation of gears algebraically.

Variational and covariational reasoning. Two types of quantitative reasoning are the variational or correspondence approach and covariational reasoning. Thompson and Carlson's (2017) quantitative covariational reasoning "conceptualiz[ed] individual quantities' values as varying and then conceptualiz[ed] two or more quantities varying simultaneously" (p. 424). In contrast, variational reasoning using a correspondence approach differs because "correspondence is based on an abstract and rather narrow definition of function as a relation between two sets, the domain (A) and the range (B), such that for each *x* in A, there is exactly one *y* in B" (Confrey & Smith, 1995, p. 79). The correspondence approach still predominates. They continue, "This approach places a heavy emphasis on stating the rule explicitly (usually algebraically) and on directionality from x to $f(x)$" (p. 79). An individual's conception of a situation as variational or covariational informs their interpretation of the situation. For example, "a person conceives of a runner's distance from a reference point as varying and of the elapsed time measured on a stopwatch as varying. Uniting the two in thought so that they vary simultaneously constitutes covariational reasoning" (p. 426).

In another example, Moore and Carlson (2012) analyzed the predictions of nine undergraduates' images of the volume of a box. Travis (the student) initially imagined the lengths and widths as being fixed. Travis labeled the dimensions of the paper and denoted the cutout by writing *x* next to the length of the square cutout, then states the formula as $V = 13 \cdot 11x$ as the volume of the box. The interviewer asks, "What does the 13, 11, and x represent?" Travis said, "The 13 is the length of the paper, 11 inches is the width of the paper, *x* is the height of the cutout from a corner." As a result, "his [Travis's] response suggests that he imagined a fixed length and width of the box; these dimensions did not depend on or vary with the length of the side of the square cutout" (p. 52). Throughout a prompted discussion by the interviewer, Travis's image evolved and ultimately he saw the height and length as varying directly. Moore and Carlson (2012) concluded Travis's dialogue suggested he imagined the height of the box varying directly with the length of the side of the cutout. Building on the work of Moore and Carlson (2012), Thompson and Carlson (2017) interpreted Travis's dialogue as conceptualization of the box as a quantitative structure supported by his image of covariation of quantities' values.

Thompson and Carlson (2017) stated, "Covariation happens in the minds —students, teachers, and researchers. When a researcher reports that a situation involves covariation, he is saying how he [the researcher] conceives the situation. The question remains as to how the student or teacher conceives the information" (p. 461). This means that the researcher may ask the student to explain their thinking, but only the students have access to the knowledge in their own minds. As stated earlier, Carpenter and Lehrer (1999) found that "students have limited ability to reflect on their thinking" (p. 22). Therefore, the researcher reports in their investigation the feedback provided to the researcher by the students who may or may not accurately or thoroughly explain their thinking.

Figure 4 presents Thompson and Carlson's (2017) major levels of covariational reasoning. These levels describe a class of behaviors or the characteristics of a person's capacity to reason. In this study, such levels are used as descriptors of a class of behaviors. They demonstrate that an individual could initially think using the precoordination of values on one problem and chunky continuous covariationally on another. That is, an individual may go back and forth between different levels of reasoning depending on the examples presented. Smooth and chunky covariational reasoning is more sophisticated than the precoordination or coordination of values because students must simultaneously conceptualize multiple variables as opposed to conceptualizing the values of the variable varying asynchronously. That is, whereas the student may exhibit behaviors in the *precoordination* of values, e.g. conceptualizing one variable as changing followed by the other, the individual does not anticipate creating an ordered pair as compared to the *coordination* of values in which the individual anticipates creating a discrete ordered pair. In Thompson and Carlson's view, when considering "smooth" and "chunky" reasoning (e. g., Castillo-Garsow, 2014), the characteristics of covariational reasoning were developmental in sophistication. Thompson and Carlson (2017) suspected that these characteristics were developmental, but they left room for the findings of future research.

Level	Description
Smooth continuous covariation	The person envisions increases or decreases in the value of one quantity's or variable's value as happening simultaneously with changes in the value of another variable's value, and the person envisions both variables as varying smoothly and continuously.
Chunky continuous covariation	The person envisions changes in the value of one variable's value as happening simultaneously with changes in the value of another variable's value, and they envision both variables varying with chunky continuous variation.
Coordination of values	The person coordinates the value of one variable (x) with values of another variable (y) with the anticipation of creating a discrete collection of pairs (x, y).
Gross coordination of values	The person forms gross image of quantities' values varying together (e.g., this quantity increases while that value decreases).
Precoordination of values	The person envisions two variables' values varying, but asynchronously—one variable changes, then the second variable changes, and then the first again.
No coordination	The person has no image of variables varying together. The person focuses on one variable or another variable's variation with no coordination of values.

Figure 4. Levels of *covariational* reasoning in Thompson and Carlson, 2017, p. 441

Visualizing chunky and smooth reasoning. Castillo-Garsow et al. (2013) proposed *chunky* (discrete/interval-driven) and *smooth* (continuous/process-driven) has two forms of covariational reasoning. Castillo-Garsow (2014) explained: "*Chunky reasoning* is forming an image of completed change, analogous to [Piaget & Inhelder's (1956)] image as state [action performed on object] . . . *Chunky variation* is forming an image of completed change of a numerical object such as a quantity or variable" (p. 158). Consider chunky reasoning as a completed thought as to have a beginning and end. Smooth reasoning is a continuous thought process and the process does not end. Castillo-Garsow (2014) explained: "*Smooth reasoning* is forming an image of a dynamic change in progress, analogous to [Piaget & Inhelder's (1956)] the image as transformations [dynamic and mobile images in which the transformation is the focus] . . . S*mooth variation* is forming an image of change in progress of a numerical object such as a quantity or variable" (p. 158).

To better understand the difference between smooth and chunky reasoning, Castillo-Garsow (2014) used the metaphor of an animated film. *Chunky* referred to the animator's perception, whereas *smooth* referred to the movie viewer's perception. To the animator, a film was a series of individual frames. To the movie viewer, however, the film ran continuously without interruption. There was an illusion of motion. *Chunky* and *smooth reasoning* accounts for the construction of the image in a dynamic situation such as the animation of a bungee jumper jumping off a bridge. Castillo-Garsow (2014) described the jump as a series of frames (chunky) compared to the movie in progress changing continuously (smooth).

A second visual of the two forms of reasoning was found in Johnson (2015). Using chunky reasoning, an individual envisioned a bottle being filled in two-ounce increments and then assessed the change in the height of the liquid over time in each of those increments. In contrast, using smooth covariational reasoning, or assessing change in progress, the individual envisioned a bottle being filled from a soda dispenser and coordinated the change in the height of the liquid with the continuous change in volume of the liquid.

Reasoning smoothly and chunkily is equally important. It seems that the hybrid or mix of both types of reasoning may be the most desirable because by thinking both smoothly and chunkily, the students can envision the situation going on continuously and also what happens at any given moment. Castillo-Garsow et al. (2013) questioned whether chunky thinking served as a *cognitive root* (a thought from which we can build new thoughts) for smooth thinking. This research considered that a student who initially thought chunkily would find it difficult to transition to smooth thinking. However, if the reverse were true, Castillo-Garsow et al. (2013)

suggested that if smooth thinking could be a cognitive root of chunky thinking, then we might have identified “a foundation for a powerful form of chunky thinking” (p. 35).

Literature on Students’ Chunky and Smooth Reasoning

In this section, I report on studies that provide evidence for how students reason chunkily and/or smoothly. A common covariational reasoning task first developed by Carlson (1998) had students “imagining a bottle filling with water and sketching the graph of the height as a function of the amount of water that’s in the bottle” (p. 123). The bottle problem (as seen in Figure 5) required the student to think of volume simultaneously with the height. In the following studies, the researchers re-invent similar tasks based on Carlson’s (1998) bottle problem.

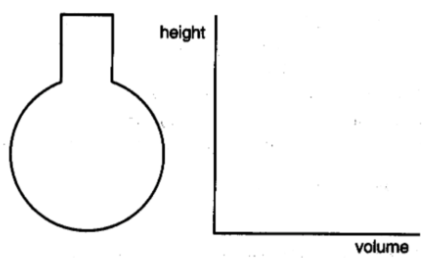


Figure 5. Image of the bottle-filling water task in Carlson, 1998, p. 123

Carlson, Jacobs, Coe, Larsen, and Hsu (2002) had students sketch a graph to illustrate the height of water as a function of the amount of water that is in the bottle to represent the increasing and decreasing average rate of change and instantaneous rate of change of a dynamic situation. In this task, prior to receiving formal teaching, the students demonstrated knowledge of the average and instantaneous rates of change.

Generally speaking, the challenge for students was to represent the changing volume and height of a three-dimensional object (bottle) in a coordinate plane in terms of height and volume. The evidence from this task provided researchers with information on students’ reasoning about the average rate of change and the instantaneous rate of change prior to formal teaching on these topics, suggesting dynamic representations might support students' smooth and chunky covariational reasoning.

In a study of fifth grade students, de Beer et al. (2015) used a modeling-based learning approach and interactive computer simulation to have students interpret the Cartesian graph by describing and reasoning about filling glassware from a discrete to a continuous perspective (operational symbolism). To encourage the students to think smoothly, the researcher asked students to create a realistic depiction of filling the glass without asking specifics about measurement in order to study instantaneous speed (chunky). In the following excerpts, the researchers provided examples of students' (Susan, Paul, and Larry) interpretations of the relationship of the change in volume to height or the change in volume to time while filling glasses (e.g. cocktail, highball, wine, and cognac glasses) with water. The teacher asked the students to create a snapshot or an image to represent moments in time while filling the glass. Susan (the student) commented on the process of pouring the water into a cocktail glass.

> Susan: Here, it is below the water line, and air bubbles appeared everywhere, like, for example, if you pour in water, you often see air bubbles appear (p. 988).

Although, many of the students said they drew bubbles to create a realistic image, Susan's use of the phrase "if you pour water" suggests continuous smooth thinking. In another depiction, Paul (the student) explained the process of filling the cocktail glass.

> Paul: It goes very fast in the beginning and the higher it gets, the slower it gets.
>
> Unknown student: Always more area, it takes longer and longer to fill it (p. 989).

Paul and the unknown student drew pictures of a clock with time, but they represented the process without specific data points, suggesting smooth thinking. Although the researchers tried to encourage smooth thinking by asking students to look at intervals of time, the researcher encouraged chunky thinking.

Each of the students described the process using smooth thinking. However, when asked to create a minimalistic (table or graph) model, the students created a discrete model (conceptual embodiment). This suggests that the students found creating a pictorial image of a familiar dynamic process easier to represent than a graphic representation of an irregular shape (the glass). Larry (the student) described the bend in a cocktail glass, “I think it should go a bit bent” (p. 991), provided confirming evidence of this claim.

In fact, de Beer et al. (2015) found that “students came to the classroom with a continuous conception of speed” (p. 992). The students reverted to discrete (chunky) reasoning, because they lacked a better way to illustrate average speed (smooth). These difficulties in representing a graph-like image does not seem surprising, since the students may not have had the experience of drawing coordinate planes. Therefore, it is understandable that students will struggle when attempting to draw a graph to represent the relationship of volume to time while pouring liquid in a cocktail glass. Nevertheless, the students were able to orally explain instantaneous speed and average speed demonstrating the importance of purposeful dialogue.

Drawing on their (2010) work, Castillo-Garsow (2012) provides exemplars of students’ chunky variational and smooth thinking while explaining what it means to travel 65 miles per hour. The following teaching episode took place between Pat (the observer) and Tiffany (the student). The observer asked the following question and the subsequent exchange between Pat and Tiffany ensued:

Pat: If I’m going 65 miles per hour, what does that mean?

Tiffany: That in one hour you’ve gone, you should have gone 65 miles (p. 60).

Tiffany imagined that Pat *has traveled* 65 miles in one hour. "Tiffany" interpreted one hour as a chunk of time. She imagined the chunk of time in the past. Tiffany's interpretations of chunks of time as having occurred in the past are representative of chunky variational thinking. Castillo-Garsow (2010) claimed that a student engaged in "chunky thinking (variational)" imagines completed change (p. 197). In other words, "chunky variation is forming an image of completed change of a numerical object such as a quantity or variable" (Castillo-Garsow, 2014, p. 158). The excerpt of the episode, in Castillo-Garsow (2012), continued as follows:

Pat: Can I travel for just one second at 65 miles per hour?

Tiffany: No, you have to do . . . You would have to do, um . . . Well, yeah, you could.

Tiffany "reconceptualized that hour as being composed of smaller 'one-second' chunk" (p. 61) when Castillo-Garsow asked her about what was happening during that time period. Even though her chunk was smaller, Tiffany still thought in chunks of time—*chunky variational*. Even with further prompting, Tiffany still did not seem to think smoothly.

In a second teaching episode, the observer asked "Derek" a simple interest question. In Castillo-Garsow (2012), the researcher read the following statement: "*Jordan's bank uses a simple interest policy for their EZ8 investment accounts. The value of an EZ8 account grows at a rate of eight percent of the initial investment per year"* (p. 59). Then, the following exchange between Derek and the researcher ensued.

Derek: It's growing constantly, but it gets one year; it's a total of eight percent higher. And it grows by still eight percent higher than the 500, but just takes that value and it gets up to there each year.

Researcher: Ok, so, what do you mean by "it's growing constantly?"

Derek: "It's always more money is being put in, because . . . and keeps going" (p. 61).

In this case, Derek described a process. He imagined the account *is growing* in the present tense. He is imagining the money growing at this moment in time (present tense). Derek's images illustrated the act of imagining a dynamic change in a process (*smooth reasoning)*. The exchange continues with the researcher asking Derek to describe the situation with a graph over time.

> Researcher: So, can you show me how the money in your . . . in your account is growing, umm.
>
> Derek: On that axis?
>
> Researcher: By moving your finger along this axis, yeah.
>
> Derek: Like starts slow and then just keeps getting faster and faster.
>
> Researcher: Ok, umm, and what about the rate of change?
>
> Derek: "It would also start slow, keep getting faster and faster" (p. 65).

Again, Derek imagined time passing. He also imagined the value of the account growing faster and faster. Derek illustrated the overall behavior of the graph, although he was not calculating the amounts. His reasoning was smooth continuous reasoning as demonstrated by dynamic change in progress. Although not explicitly expressed, it also seemed to be chunky reasoning. Castillo-Garsow (2012) stated, ". . . time was also passing for the account (although much slower, in that he only imagined the first two seconds)" implying chunky reasoning (Castillo-Garsow, 2012, p. 65). Derek's reasoning—both chunky and smooth—suggests a hybrid blend. This means that Derek thought smoothly and chunkily as a result of the interviewer's questions. In other words, Castillo-Garsow (2012) maintained that smooth reasoning was a continuous process, but you can pause (chunky reasoning) and then go back to smooth reasoning. The use of both creates flexibility as expected when using quantitative reasoning. For example, Derek's pause, although short lived, challenged Derek to describe what happened between the months or intervals demonstrating the importance of dialogue in identifying students' smooth and chunky thinking.

In a study that reported on a clinical interview methodology from 2010, Johnson (2015) found evidence of chunky and smooth continuous reasoning during an investigation of how secondary students thought covariationally during multiple tasks. The interviewer asked the students (Mason and Jacob) to "describe how the volume of liquid being poured into the bottle would change as the height of the liquid in the bottle increased" (p. 97).

Mason (the student) began by creating an image of one bottle from the original four choices (as seen in Figure 6, left). His image appears to resemble bottle C (as seen in Figure 6, right).

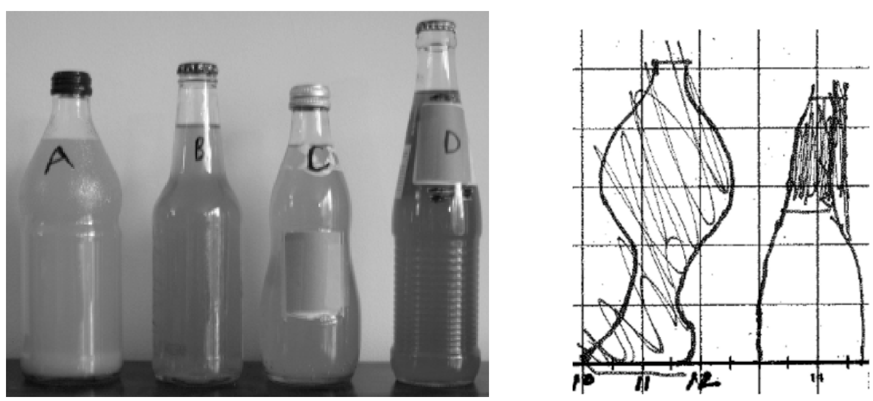


Figure 6. On the Left: Original Bottle choices and on the right: Mason's Work in Johnson, 2015, p. 97 and p.101

In the first example, Mason drew an image of a bottle. After the interviewer asked Mason to "explain his drawing," Mason decided to change the image. In other words, his perception did not match his image. Mason created a new image to the right of the first image and then explains the reason for changing the sketch to the interviewer.

Mason said, ". . . It gets smaller at the top because it's taking up more inches, and it's less amount of volume getting into it. So, I think it is going to start out wide . . .” (p. 101).

By comparing amounts of change in associated quantities, Mason interpreted the given graph as a relationship between quantities (covariational reasoning) rather than as a picture of a bottle. He also envisioned change as occurring “putting in less volume and more inches” (p.101) (continuous reasoning). Finally, Mason envisioned the changes as occurring in intervals of height and volume.

When the interviewer asked, “And so for the new bottle that you drew could you show me where the height would be filling up faster than putting in ounces” (p. 102)?

Mason said, “Right here. This area” (p. 102). He envisioned the volume continuously increasing in height and chunkily when he interpreted the volume at specific heights.

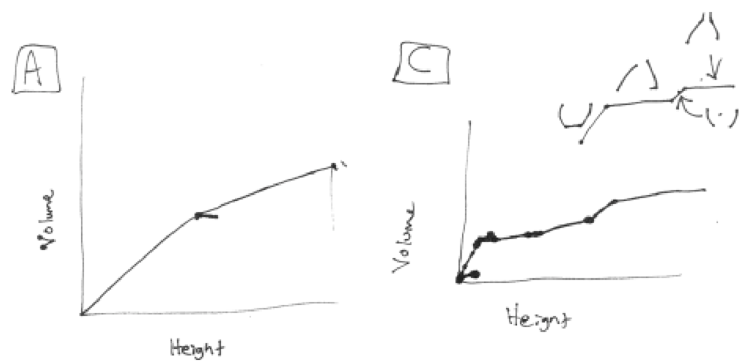


Figure 7. On the left: Height and Volume of Bottles A and on the right: Height and Volume of Bottles C (right) Jacob’s Work in Johnson, 2015, p. 104

In this next example, Jacob (the student) sketched a graph of bottle A (as seen in Figure 7, left) and Bottle C (as seen in Figure 7, right) that compares height and volume of the liquid. For clarification purposes, in the case of Bottle C, the upper image is a detailed image of the lower graph. There are four distinct intervals for this graph (as seen in Figure 7, right) that reflect where the height is moving faster compared to the volume. These intervals are representative of a piecewise function (smooth continuous reasoning) rather than thinking in terms of chunks. Initially, Jacob did not label the axes, but when asked he said, "We'll just leave them the same on the sides so they match" (p. 105). The labels were not important to Jacob. He seemed to believe that neither the horizontal nor the vertical aspect mattered. Jacob also envisioned the change as occurring throughout the intervals (e.g., "like the skinny parts you're not getting as much soda in there but it's still getting higher" (p. 105)).

When the interviewer asks, "And can you tell me why that shows that of the graph" (p. 105)? Jacob started to mark off the height and volume. It appears that the questioning subsequent to the students' initial thinking created an opportunity for thinking chunkily. Jacob initially envisioned the situation using smooth continuous reasoning, but then through responding to perturbing questions, he reasons chunkily.

Johnson (2015) maintained that Mason and Jacob envisioned the change in height and volume as occurring in a continuing process throughout an interval as well as the amount of change that results at the end of each interval (p. 107). She suggested that "students who continue to make comparisons could remain focused on average [chunky] rather than instantaneous rate of change [smooth]; it may be that a root for reasoning about instantaneous rate of change is something other than envisioning changes as having occurred in completed chunks" (p. 108).

Each of these studies demonstrates the importance of productive dialogue between interviewer and student to challenge the students to explain their quantitative reasoning. Through the filling-the-bottle tasks, students envisioned a familiar dynamic phenomenon of average and instantaneous rates of change (conceptual symbolism). During the interview process, the interviewer guided students in explaining their smooth thinking and then through further questioning guided students into thinking within the intervals (chunky thinking).

However, if Tiffany envisioned the dynamic phenomena of a car in motion, she may have more easily visualized the car traveling at 65 mph chunkily, as if moving back and forth between conceptual embodiment and operational symbolism. Presenting students with an authentic dynamic motion model for word problems in a specific context may help their transition between smooth and chunky thinking. Therefore, the construction of a model in a dynamic situation may aid students in developing a more powerful *chunky* and *smooth reasoning.*

In each of the above studies, students explained their reasoning for a given answer; the researcher determined whether the student reasoned chunkily or smoothly. However, the researcher interpreted the reasoning based on how the researcher conceived the situation. To better understand the development of smooth and chunky quantitative reasoning, this paper uses the self-study methodology that affords the researcher additional insight into the chunky and smooth reasoning relationship.

Methodology

This section begins with an explanation of my rationale for choosing the self-study methodology. Next, I briefly describe self-study followed by the review of mathematics and self-study in literature.

Rationale for choosing self-study. The intention of this self-study is to examine my teaching practices in order to understand *how* my role as *self* (e.g., teacher, learner, inquirer, and researcher) affects my thinking about my covariational reasoning that requires thinking of variables varying simultaneously. In general, I tend to use variational reasoning using the correspondence approach that is deeply rooted in my past. As I am a product of education in the 1970s, the message to students was that answering questions did not demand understanding the process, but rather only required the correct answer. Russell (2004) referred to the educational system prior to 1980 as the process-product paradigm characterized by identifying teacher behaviors that correlated to higher test scores (p. 1192). Since it is well known that teachers teach the way they were taught (Lortie, 1975), I chose the self-study methodology because my focus is on improving my practice by reframing my thinking of past teaching and transforming my practice in the future.

Self-study. The overall aim of the self-study research methodology is to examine and improve existing teaching practices. Loughran (2004) further explained that self-study is "used in relation to teaching and researching practice in order to better understand: oneself; teaching; and, the development of knowledge about these" (p. 9). This methodology provides researchers, primarily teacher educators, with opportunities to use multiple methods (e.g., narrative and dialogue), through a recursive practice and collaboration, to determine whether or not their intentions materialize into practice.

The self-study methodology is recursive in nature. Samaras and Freese (2006) referred to this recursive approach as having a hermeneutic quality. When considering the difference between recursion in the thinking process compared to the analysis process, in this study, the intention of recursion aligned with Hunt (1987), Russell (2004), and Samaras and Freese (2006). Russell (2004) and Hunt (1987) considered the *self* to be the starting point (p. 1196) suggesting that teachers look inside (within) before they look outside when initially thinking about this process. In the opinion of Samaras and Freese (2006), the hermeneutic quality was a "research process whereby the researcher shifts forward and backward through the data with no predetermined assumptions" (p. 12) and challenged the teacher educator to build on their reflections of their practice. Whether they begin within or with a shift outward, recursion is essential to the reflection and implementation process.

To allow for the emergence of “living contradictions” (Whitehead, 1989) during the reflection process, Loughran (2004) recommended stepping back or removing yourself as being central to the work. This de-personalization (stepping back) allowed the teacher-researcher to examine the emergent contradictions objectively. The recursive nature of this study allows me (self) to reflect inward and outward, thereby providing unique insight into the development of my thinking about my practice and mathematics, more specifically, covariational reasoning.

Mathematics and Self-Study in the Literature

Although there is a broad collection of self-studies used as a vehicle to transform educational practices, research for mathematics self-studies is limited. Many studies (e.g., Goodell, 2011; Hjalmarson, 2017; Lovin et al., 2012; Marin, 2014; Schuck, 1999) explored mathematics pedagogy whereas others (Guðjónsdóttir & Kristinsdóttir, 2011; Alderton, 2008; Goodell, 2006) balanced mathematics content knowledge and mathematics pedagogy. Because the majority of mathematics self-study researchers were teacher educators of mathematics methods courses, many of these mathematics self-studies represented pedagogical research. The distinctions between mathematics pedagogy and mathematics content knowledge was “viewing prospective teachers as learners of *teaching mathematics*, and not just learners of *mathematics*” (Lovin et al., 2012, p. 59).

Mathematics pedagogy self-studies included research that investigated how teacher educators taught mathematics teachers, developed habits using reflective thinking and writing (Goodell, 2011), developed a learning community for mathematics specialists (Hjalmarson, 2017), found commonalities with co-researchers’ fundamental beliefs about mathematics (Lovin et al., 2012), explored beliefs about mathematical thinking while transitioning from teacher to teacher educator (Marin, 2014), or reflected on views about teaching and learning mathematics and views about the student (Schuck, 1999). The purpose of such studies was to reflect and collaborate to promote future change by transforming the researchers' practice "as learners of *teaching mathematics"* (Lovin et al., 2012, p. 59).

Examining these studies, I found common themes. The first theme in each of these studies was the intent to shift thinking due to internal struggles. For example, Marin (2014) shared her feeling of being "in-between" because of her uncomfortable position that shifted from teacher to teacher educator. The second common theme in each of these studies was that researchers allowed for or made room for emergent findings. Although self-study research has an underlying structure, the researchers did not anticipate the findings prior to the study. In a study of online classes, Hjalmarson (2017) found the themes that emerged from self-study research were student autonomy and engagement, authenticity and practicality, and fostering community. A third common theme found, in Goodell (2011) and Lovin et al. (2012), was that the nature of mathematical knowledge was either fixed (perception-based) or an active construction of mathematics (conception-based). In both studies, the researchers struggled with beliefs about developing knowledge (conception-based) rather than teaching as they were taught (perception-based), as well as the pushback from teachers (students). Finally, Lovin et al. (2012) realized their "initial pedagogical training prepared [them] to teach mathematics to children" (p. 59). However, they realized that they needed to shift their teaching practices and develop new skills for teaching doctorial students.

In the studies that balanced self-studies mathematics content knowledge and mathematics pedagogy to some extent, researchers discussed topics such as a linear system (Guðjónsdóttir & Kristinsdóttir, 2011) or organized fractions (Alderton, 2008). Another study analyzed and discussed critical incidents in a mathematics methods class (Goodell, 2006). Finally, Grandau (2005) investigated the relational and conjectures of algebraic concepts.

I found a common theme throughout these studies. These studies were concerned with transforming the teacher educator's practice and also focused on the development of their students' mathematics knowledge. Alderton (2008) discussed the teacher educator's conflict between beliefs and actions. The teacher educator did not comment on an observation that lacked a constructivist approach during an observation. The teacher educator missed the opportunity for discussion and upon reflection questioned why this was the case. During the next observation, the teacher educator observed a science lesson that addressed a constructivist approach. In another study, Goodell (2006) investigated the use of critical incidents to develop the teachers' (students') reflective practices. The teachers found it more useful to discuss critical incidents in other students' classes than to write up their own critical incidents. In the Guðjónsdóttir and Kristinsdóttir study (2011), teachers with various mathematics backgrounds solved a linear systems question. To solve the problem, the preschool teachers used cubes, whereas the secondary teachers used two equations. This study discussed the most efficient way to solve questions. One could argue that the method used did not matter; however, teachers preferred to learn the most efficient method. Finally, Grandau (2005) came the closest to a true balance of both "*teaching mathematics*" and being a "*learner of mathematics*" (Lovin et al., 2012, p. 59). The study gave numerous examples of mathematics content between a teacher/researcher and students and a teacher/researcher and a critical friend. The reason for the balance may be the fact that the study was conducted in a fourth-grade classroom rather than between teacher educators and teachers as students.

As stated above, there are only a limited number of mathematics self-studies and even fewer that include mathematics content. The only study that truly balanced pedagogy and mathematics content was that of Grandau (2005). Therefore, the lack of mathematical content self-studies demonstrates the need for further research to add to the areas of both the mathematical content knowledge of covariational reasoning and the self-study methodology.

Chapter Summary

This chapter emphasized the importance of quantitative reasoning as a skill necessary to develop flexible problem solvers. Of interest was covariational reasoning that envisioned variables varying simultaneously, more specifically, smooth (process-drive) and chunky (interval-driven) covariational reasoning. An extensive review of literature ensued to demonstrate students' smooth and chunky thinking. Additionally, this chapter briefly discussed the self-study methodology. Followed by a review of mathematics and self-study in literature, the review considered mathematics pedagogy and mathematics content knowledge. In Chapter 4, I discuss the self-study methodology in greater depth.

Chapter IV

Methodology

In this chapter, I provide a brief description of the five characteristics of a self-study: it should be self-initiated and focused, aimed at improvement and interacting with others, employing multiple qualitative methods, and exemplar-based validation (LaBoskey, 2004). I follow this with a description of the hermeneutics interpretive process for understanding the reconceptualizing of my practice and covariational reasoning. I discuss my rationale for choosing the self-study methodology, as well as the methods of inquiry of living educational theory (Whitehead, 1989), personal history (Samaras, 2011), and the developmental portfolio (Samaras & Freese, 2006). Next, I describe the design of this study beginning with a description of the research setting, the mathematics curriculum, and the participants and our positionality as a teacher, learner, and inquirer. I define the categorical terms. Then, I explain the data collection process and data collection tools and sources. This chapter concludes with the data analysis (which includes a description of similarity-based and contiguity-based strategies for coding), the coding analysis procedures, and the validation criteria.

As discussed in Chapter 2, a focus on the teacher's thinking adds a valuable new perspective of evidence within the mathematical (covariational) knowledge base. Therefore, understanding knowledge leads to the theoretical underpinning of self-study that draws from epistemological knowledge, pedagogical beliefs, and moral/ethic/political values.

Characteristics of Self-Study

In this section, I explain the characteristics of the self-study methodology in more detail. The self-study design is self-initiated and focused, aims at improvement, interacts with others and texts, employs multiple qualitative methods, and possesses exemplar-based validation

(LaBoskey, 2004). In other words, to reframe one's thinking and transform one's practice, self-study researchers expect evidence built upon these characteristics to demonstrate validation and trustworthiness.

Self-initiated and focused. The purpose of self-study is to examine one's practice; implicit in the concept is the study of *self*. LaBoskey (2004) identified "the question of 'Who'—both who is doing the research and who is being studied" (p. 842) as being essential to self-study research. This suggests that the "Who" encompasses different positions and perceptions. For example, I may position myself as a classroom teacher (mathematics knower), as a graduate student (learner), or self-study researcher (knower, learner, student, and as a person who reflects on her practices in the classroom). In essence, one self can potentially affect the *other* selves. For example, my own children struggled with their homework. As a result, my views on grading my students' homework have evolved. In this case, my self-as-mother informs my self-as-teacher. As teachers, our memories and our own experiences affect our practice. An essential goal of self-study is the intention to better understand and then, to improve one's practice.

Improvement-aimed. Another characteristic of this methodology is the intention of the researcher to improve their practice. It may be that teachers regularly reflect on practice, however, self-study moves the reflective practice forward to implementation and action within the situated setting. The researcher must aim at making improvements focused on altering existing (deeply held) practices with the ultimate goal of improving, reframing, and transforming practice. To improve our practice, LaBoskey (2004) suggested that it was necessary for "a careful and thorough understanding of our settings, which in turn results in an enhanced understanding of that practice" (p. 845). The ultimate goal of self-study is to improve the researcher's practice, which will eventually lead to making research findings public in order to restructure the body of knowledge within the self-study community and possibly lead to institutional reform.

Interactive. *Interactive* refers to collaboration that allows for reframing *what* is being examined either practice or text. Researchers may collaborate with colleagues both in their situated setting and at a distance to improve their practice. Teacher educators may also interact with students. However, Munby and Russell (1994) warned that in the teacher educators'/pre-service teachers' collaborative relationship, there may be a perception of the *authority of position*. The student (pre-service teacher) may perceive the teacher educator as possessing the *authority of experience*, which may influence the student. Therefore, the positions among the collaborative group should be negotiated before collaboration begins.

Another type of interaction is with text (e.g., dialogues and correspondence). For example, Guilfoyle et al. (2004) referred to conversation as talk, whereas, dialogue referred to critiques and reflections that "endure questions, analysis, alternative interpretations, evaluation, and synthesis" (p. 1158). It is through dialogue that text becomes a form of collaboration and interaction. However, dialogue is not the only method used in self-study research; other methods include narrative and artistic mode, to name but a few. The researcher's choice of methods depends on the evidence that the researcher plans to evaluate.

Use of a variety of quantitative methods. Self-study methodology does not dictate a specific method of collecting evidence. Berry and Taylor (2017) explained that "self-study does not have its own methods, but instead researchers 'borrow' or 'blend' from other methodologies and approaches" (p. 596). Some of these qualitative methods are narrative and dialogue. The researcher chooses the method based on the evidence they require. Because the purpose of self-study is to improve one's practice, typically, the measure is qualitative, lends itself to a conceptual framework that seeks to understand (how), and is also collaborative.

The narrative self-study research method is (e.g., documenting teachers' stories or life experiences, autobiographies, and tall tales) based on their experience. Mishler (1990) pointed out that narrative inquiry was neither structural nor sequential. The narrative is *how* the individual (participant) interprets events and experiences, rather than *assessing* whether or not their interpretation corresponds to the researcher's interpretation. The question to ask is whether the interpretation makes sense to the individual teacher.

A related self-study research method is dialogue that differs depending who is participating in the dialogue (e.g., personal reflections, storytelling, and as a social construct). Therefore, Guilfoyle et al. (2004) clarified that ultimately the person who is responsible for the dialogue determines the type of dialogue it will be. In *dialogue as a pedagogical method,* the teacher leads students toward understanding an idea being taught, whereas in *dialogue as an inquiry method* the *authoritative position* is negotiated. Dialogue, as an inquiry method, requires others such as colleagues and collaborators, referred to as "critical friends" to participate. Costa and Kallick (1993) explained that the purpose of a critical friend is "to ask provocative questions, provide data to be examined through another lens" (p. 49). The dialogue with my critical friend provides a means of validation, since the interactive relationship allows for checks and balances.

Exemplar-based validation. To determine if findings are trustworthy, the reader expects the researcher's analysis to exemplify validity. In general, validation refers to the expectation that if a researcher interpreted another person's data; they would come to a similar conclusion. Mishler (1990) defined validation as "the process (es) through which we make claims for and evaluate the 'trustworthiness' of the reported observations, interpretations, and generalizations" (p. 419). However, the interpretation process may change over time in the same way that norms change; therefore, the "trustworthiness" of the findings may also change. It is possible that if a researcher uses the same data from twenty years ago, the analysis might differ. Nevertheless, the change in societal norms does not change the validity of the data set, only the analysis.

An important characteristic of self-study research is to demonstrate validity through collaboration. Pinnegar, Hamilton, and Fitzgerald (2010) considered collaboration to be a means of validation that supported researchers against claims of misinterpretation. Validation is also providing research data to the reader to confirm that the researcher's analysis is "trustworthy." It may be that self-study research demonstrates a process for validation and trustworthiness; however, the findings are not necessarily generalizable because the narrative is specific to the study, and one's personal narrative is not replicable. There is an expectation that the data analysis researchers' report provides valid evidence to form the basis of future research.

Interpretive Process

The hermeneutics interpretive process is recursive in that it looks both backward and forward to understand and interpret how past experiences influence future experiences. Laverty (2003) explained that the purpose of reflection was to become aware of biases and assumptions. Laverty (2003) described hermeneutic qualitative research as interpretive in that it concentrated on historical meanings of experiences. Therefore, "we are constructing the world from our own background and experiences" (p. 24). I looked backwards into the past in order to determine how I interpreted how those experiences inform the present.

As I analyzed the data collected, I expected my past and my present experiences to influence the interpretation of my analysis. I struggled with understanding how my past experiences affected the reconceptualize my thinking about mathematics. My concern was the extent that my deeply held beliefs about mathematics — such as not believing that learning mathematics can be effective when it is taught using procedures — affected my interpretation of variational and covariational reasoning. Nevertheless, the interpretive process was not concerned about some detrimental affect due to biases. Researchers should not set these positions aside because they are rooted in and essential to the interpretive process. Furthermore, as a researcher’s background is critical to the data analysis, they should give considerable thought to experiences and then make explicit claims about the ways these experiences or positions relate to the issues being researched. My critical friend was again useful in this regard in that they play an important role in helping illuminate experiences so that they do not go unnoticed during the interpretive process.

In regards to this study, I used the hermeneutics interpretive process for understanding the reconceptualizing of my practice and covariational reasoning. I developed meaning from the text of the transcriptions, dialogues, and narratives. The hermeneutic interpretive process involved a circle of understanding among reading, reflective writing, and interpretation. The hermeneutic circle moved from the parts of the experience, to the whole of the experience, and back and forth again to increase the depth of engagement with an understanding of texts (Laverty, 2003). From Koch’s (1999) perspective, understanding occurred while engaging the whole self, including what was real to you and important in your life. When asking a participant to tell a story, it was understood that this story was a construction of their reality. Koch (1999) clarified, “Text is not only the written words and/or our observations of the world but stories told to us” (p. 26). We develop meaning as we read these stories. He suggested thinking about dialogue as interacting with the text, as you read, reflect and reread, and “question what the writer says—developing, connecting what is said and how it is said, dropping behind, going back and running ahead of the writer" (p. 31).

The process of interpretation offered me an opportunity to reflect among the transcriptions, narratives, and dialogues based on my experiences inwardly and outwardly, thereby providing me with unique insights into the development of my thinking about my practice and the teaching of mathematics. The transcriptions were a mirror of my teaching of a lesson and reflected my lessons as I taught a concept. The interpretive process of hermeneutics allowed me to interpret my thinking based on my past and present experiences, while providing an additional perspective from my critical friend about my teaching practices that proved valuable and serve to move my practice forward.

Rationale for Choosing Self-Study

The ultimate objective of using the self-study research methodology is "to provoke, challenge, and illuminate rather than confirm and settle" (Bullough & Pinnegar, 2001, p. 20). As such, teacher educators have used self-study most often to transform their practice. Self-study methodology provides researchers with opportunities to borrow methods. In this study, I chose living educational theory and personal history inquiry methods because they "take place at the intersection of biography and history" (p. 15). If I am going to change my future teaching practices, change must go through the self or one's own experiences. My intention was to examine my past and present mathematical experiences so that I could transform my future practice. Furthermore, through recursive practice and collaboration, self-study researchers determine whether or not their intentions materialize in practice. It is the collaboration between the self and other that may encourage a lasting change in practice over time. However, Guilfoyle et al. (2004) questioned whether we actually executed changes in practice when the collaborators are no longer together, they referred to implementing the changes in practice as "walking one's talk" (p. 1151). That is, it is not through words, but rather through prolong actions, that teachers change these deeply held ideas about teaching. Therefore, until a teacher begins "recognizing, accepting, and learning to deal with failures in one's teaching" (Russell, 2004, p. 1199), I question whether teacher educators can take the steps or actions to reframe their thinking.

Methods

In this section, I provide a concept map in Figure 8 to illustrate the connection between the self-study methodology and its associated methods, collection tools, and data sources (described in the Data Collection section). The connecting lines trace the various lenses I employed within the study along with brief descriptions. My use of the self-study methodology borrowed the living educational theory, personal history, and developmental portfolio methods of inquiry. To provide a brief overview, I employed the living educational theory (Whitehead, 1989) to determine whether my beliefs align with my actions, and the personal history method (Samaras, 2011) to consider how my past experiences affect my teaching practices. I used the developmental portfolio method from Samaras and Freese (2006) to reflect upon the development of my mathematical and professional growth by analyzing a critical friend portfolio, a data collection tool that represents the progression of my covariational thinking in the form of dialogues and narratives. As explained by Guilfoyle et al (2004), "conversation moves beyond mere talk to become dialogue when it contains both critique and reflection — when ideas are not simply stated but ensure intense questioning, analysis, alternative interpretations, evaluation, synthesis" (p. 1159). Narratives, as defined by Samaras (2011), are "stories [or] journaling of your ongoing record, essays, or other reflections about your study" (p. 283). I describe each method in detail below.

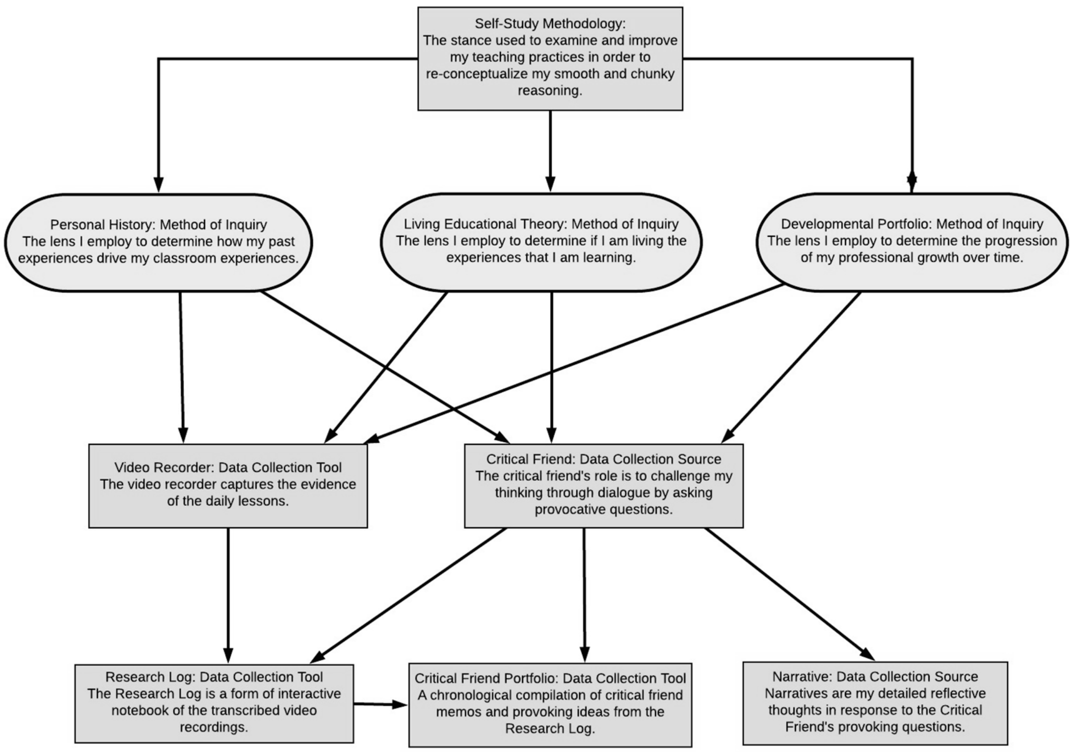


Figure 8. Concept Map of Research Design

Living Educational Theory. The living educational theory (Whitehead, 1989) pertains to the authenticity of beliefs and practices. This method focuses on answering the questions, “How does what I learned in the reflections or narratives change the way I am teaching? How am I going to feel about these changes?” I needed to step back and become the observer rather than a participant to determine if my teaching aligned with my teaching expectations. I asked myself the difficult question of whether I implemented the changes I wanted to see in my classroom. I considered that if my intent really was to provide opportunities to support advanced levels of mathematical thinking in my classroom, I needed to be sure I was asking the students advanced levels of mathematical thinking questions. I also needed to be sure I was teaching and thinking covariationally. In other words, I needed to learning as I was living the experiences.

Personal History Method. Samaras, Hicks, and Berger (2004) referred to personal history as “those formative, contextualized experiences that have influenced teachers’ thinking about teaching and their own practice" (p. 909). By understanding the influences of our thinking, educators can understand how our past affects students' learning. Reflecting upon narratives and dialogues, this method is the lens I employed to re-count how personal experiences, beliefs, and/or background drove my classroom experiences. Although this study was not about my beliefs, per se, it was essential for me to identify and leverage my past experiences as a teacher and a nurturer into this investigation. Educators need to consider that "past experiences create hidden personal narratives about education, school, and schooling that have a profound and sometimes intractable impact on the way teachers teach their students" (p. 998). By interrogating past experiences, Samaras et al., (2004) suggested educators could reform their professional growth, test effective reflecting, and push the boundaries of teaching.

I repeatedly wondered about how my past experiences affected my teaching practices. This disposition to reflect was fundamental to this study in understanding how I developed my covariational reasoning skills to reconceptualize my algebraic thinking, for example. However, I also became aware that I thought in a chunky variational manner. (I explain these forms of mathematical thinking in the Variational and Covariational Reasoning section of the literature review above). In general, I consider that I think in a variational manner using the correspondence approach (Confrey & Smith, 1995) when I am solving algebraic equations and use the precoordination or coordination of values (Thompson & Carlson, 2017) when I am graphing a function on a coordinate plane. Although Thompson and Carlson (2017) refer to the precoordination or coordination of values in their levels of covariational reasoning, I still perceive them to be variational thinking.

As I employed the personal history method (Samaras, 2011), I examined who I was as a teacher and the constraints that precluded me from reaching my goals as a teacher who is intent on reconceptualizing my thinking about algebraic relationships in order to transform my practice.

Developmental Portfolio Method. As I was concerned with my development as a teacher, the developmental portfolio method “presents an opportunity to store, catalog, and study one’s professional growth over a certain time period” (Samaras & Freese 2006, p. 68). To identify and capture growth over time, the developmental portfolio method I employed took the form of a Critical Friend Portfolio (CFP) made up of dialogues and narratives. The dialogues referred to the reflective discussions about my lessons that took place between my critical friend and myself at the end of the day. The narratives reflected the development of my covariational reasoning between my critical friend and myself that were written after the completion of the 29 lessons. I expected my critical friend to ask a provoking question, I then wrote my narrative response. These accumulated documents in the CFP provided a progression of my covariational reasoning in writing. I read the dialogues and narratives in a sequential order to identify the progression of my covariational reasoning. The nature of the developmental portfolio is its capacity to capture the *process* of coming to know, not the *product* that is eventually known. The developmental portfolio, living educational theory, and personal history methods helped me answer my research question using the data collection tools and sources that I used in this study.

Research Design

The purpose of this self-study was to improve my practice by examining my current practice, so that I could better understand how my various roles as self (e.g., teacher, learner, inquirer, and researcher) affected the way I thought about my practice. In this study, I reflected on my daily lessons and the development of my practice so that I transformed and reframed my thinking about content that I associated with the teaching of algebra. In particular, I examined the transformation of my covariational reasoning to reconceptualize my algebraic thinking and develop new practices that I based new understandings of algebraic reasoning. Covariational reasoning is a skill that teachers should foster in their students in order to develop advanced mathematical thinking. To develop covariational reasoning, two variables must be conceived as varying simultaneously. However, this reasoning skill challenges my thinking, since my thinking has been uni-variational (correspondence approach), a way of thinking that is deeply rooted in my childhood learning.

Accordingly, I chose the self-study methodology as it provided me with opportunities to reconceptualize my algebraic thinking and transform my thinking about my practice. The evidence I sought were exemplars of lessons that demonstrated a change in smooth (process-driven) and chunky (interval-driven) thinking using covariational reasoning based on the hermeneutic interpretive process. This was the research question that guided this study:

> *How does analyzing the development of my acquisition and implementation of the lens of smooth and chunky covariational reasoning through reconceptualizing algebraic relationships promote the transformation of my teaching practice?*

Setting. The setting of this study was five algebra honors classes with 7th and 8th grade students, located in a large district in the southeastern United States. Sunshine Middle School (pseudonym) has a population of 1,065 students in grades 6-8. There is a student/teacher ratio of 18:1. The student population is 27% Hispanic and 12% Black, which is below the state average of 61% for minority enrollment. The *diversity score* describes the chance that two students selected at random would be a member of different ethnic group and scored on a scale from 0 (not diverse) to 1 (a more diverse student body). Sunshine Middle School’s diversity score is .65, which is less than the state average of .70. While our diversity score suggests a culturally diverse population, this diversity did not come about as a result of being a “choice” school.

Choice schools choose their students based on specific criteria, such as schools that cater to the arts or music. The Sunshine Middle School's student selection process is random. Therefore, the school does not have control over the student diversity. The students who attend the school live in the school zone or choose to participate in a lottery and are then selected.

Curricular focus. The study took place during the *Summary of Functions* section of the course (over nineteen days) and also during the Post-assessment Algebra II topics (over ten days), for a total of 29 lessons. As these topics took place after the midpoint of the school year, I had taught the topics of solving algebraic linear and quadratic equations prior to the initiation of this study. I had anticipated that the covariational reasoning and the opportunity for advanced mathematical thinking in the *Summary of Functions* section of the Algebra Nation Workbook (2017) would leverage my thinking about transformations, end behaviors, and odd and even functions in the context of a line, quadratic, cubic, exponential, square root, and cube root graphs. Furthermore, because I was using graphic representations to create transformational shifts of functions, I expected that my smooth covariational thinking and teaching would develop through the teaching of the aforementioned functions. After the end of the year Algebra I test, my curriculum set by the district that reflects the correspondence approach delves into the Algebra II topics that I conceive as advanced mathematical thinking types of concepts. These post-assessment concepts were long division, synthetic division, identifying inverse functions, systems of quadratics, graphing inverse functions, simplifying rational expressions, solving radical equations, and creating programs on a graphing calculator to move Rover (a robotic car). In accordance with the goals of the study in relation to the development of my algebraic thinking, my plan was to represent these topics with an emphasis on covariational reasoning and provide opportunities for students to engage in advanced mathematical thinking through productive classroom discussion.

Participants and our positionality.

Teacher/learner/ inquirer. As this research is a self-study, I begin by discussing my concept of self. As I am a product of the feminist movement of the 1970's through the present, I struggle with my role as a woman. As stated in Belenky, Clinchy, Goldberger, and Tarule (1997) "women derive a sense of "who I am" from the definitions others supply and the roles they fill" (p. 81). Accordingly, for this study, my identities or various roles I portray are algebra teacher and a self-study researcher. In addition to my various identities, I also consider my positionality. To me, positionality is how I see myself with respect to others. For example, I see myself as an algebra teacher (the person who is good at algebra) and I also see myself as one of two algebra teachers in my school (the one who is performing a research study). Therefore, I can identify myself as an algebra teacher and also position myself as an algebra teacher with respect to others.

I now elaborate on my positionality and its potential to influence my teaching practices. Herr and Anderson (2005) suggested that researchers have an influence on their study as they bring with them their past lived experiences whether they position themselves as insiders or outsiders. It is through the researcher's lens that interpretation takes place. As Cochran-Smith and Lytle (1993) explained, researchers who are outside of the day-to-day practices of the school are positioned as the outsider/in. In contrast, I positioned myself as the insider/in in that my day-to-day practices as both researcher and teacher took place within the school. It was my critical friend/collaborator who moderated this influence on my study by enhancing validity through narratives, dialogues, critical friend portfolios, member checking, and an inter-reliability discussion. In addition to considering myself as the insider/in, I also positioned myself as a teacher, an algebraic learner, and a pedagogical inquirer. I introduce these positionalities next.

My teaching experience began with four years of teaching 7th grade general mathematics at a middle school and seven years teaching Algebra at a high school in the Northeast. For the past two years, I taught 7th and 8th grade students algebra I in a large district in the southeastern United States. As one of two algebra teachers, the other a novice algebra teacher, I am perceived as an Algebra knower both among my students and my colleagues. Although I position myself as insider/in, I could argue that I also see myself as an outsider/in as I collaborate with my colleagues in middle school mathematics, but rarely in algebra. When I sought collaboration in the teaching of algebra, I looked to my critical friend for support for both mathematics content and lesson implementation.

As a teacher-of-algebra, my responsibility is to facilitate algebraic learning. I am also a nurturer, one who teaches the child holistically, not just their academic identity. Students bring their personal life experiences into the classroom, not just their prior experiences learning mathematics in school, but also all of their out-of-school, home- and community-based experiences. I see all of these forms of knowledge as resources for their learning of algebra.

Finally, I see myself as an inquirer, a reflective practitioner (Schön, 1983). I place myself in the center of my own inquiries and seek to answer the question of how to transform my practice by reflecting on my mathematical knowledge and my methods for teaching algebra. In relation to this study, I have been able to complement these efforts through dialogue with a critical friend.

Critical friend. Samaras (2011) explained that a *critical friend* was a trusted colleague who sought to support and validate research to gain new perspectives in understanding and reframing the researcher's interpretations. The critical friend asks, "provocative questions and provides data to be examined through another lens" (Costa & Kallick, 1993, p. 49). In this study, the critical friend has dual roles. First, the *collaborator* supports and challenges my algebraic thinking by asking provoking questions during the dialogue and narrative writings and second, randomly reviews my coding of the concept map by asking me provoking questions challenging me to justify my findings during the post-data collection phase of the study as a validator or an inter-rater reliability check.

My critical friend describes her positionality as a teacher and learner. Consistent with her constructivist stance on learning, she believes students must have opportunities to construct their own knowledge of mathematical concepts. This stance positions her as a learner as it forces her to extend her specialized mathematical knowledge (Ball, Thames, & Phelps, 2008) in order to provide those opportunities.

In general, I expect my positionality as well as my critical friend's to influence the end of the day dialogue. LaBoskey (2004) explained that our memories and beliefs shape and influence "the construction of our identities, current thinking, and our future behaviors" (p. 843), since former teachers, memories, and experiences affect our practice. As my goal in this self-study was to better understand in order to subsequently improve my practice, I took into account my positionality as teacher, learner, and inquirer.

Definitions of Categorical Terms

Organizational coding categories are broad and established prior to conducting interviews or observations (Maxwell, 2009). I used them to refer to the categories established before the transcription of the video recordings of the classroom lessons. In relation to the mathematics, the organizational categories I employed include chunky and smooth covariational reasoning (Castillo-Garsow, Johnson, & Moore, 2013) and the advanced mathematics opportunities (Tall, 1991), which I described in Chapter 2. In relation to the Critical Friend Portfolio, I employed the organizational categories of narratives of provoking ideas (Connelly & Clandinin, 1990) and critical friend memos (Samaras, 2011), which I described in the data collection section. I referred to provoking ideas as the ideas that provoked my thinking, either after a discussion with my critical friend or upon self-reflection on practice. I now provide five additional organizational categorical terms that are unrelated to the mathematics or the Critical Friend Portfolio. These are brief-but-vivid descriptive accounts, meanings, noticings, decision points, and awareness.

Brief-but-vivid descriptive accounts (BBVDs) portray the situation, concentrating on words that provide vivid images and describe behaviors that could be seen, heard, or otherwise sensed (Mason, 2002). The intent is to capture quotes and/or narrative summaries that capture the crux of the lesson (e.g., mathematical reconceptualization in smooth or chunky reasoning or my own mathematical learning *in situ*) as well as descriptive words that illuminate salient features (e.g., to add emphasis). In contrast, as advised by Mason, words that represent speculation, judgment and evaluation, attempts to justify or explain, actions or opinions, and emotion are excluded. This is because these words are subjective. When describing a situation, in a BBVD, it should be understood that the one who describes the text attaches words that have a personal image; the goal is to describe text in a way that has a possible shared image. Therefore, the word choices used in a BBVD hold a shared meaning, as I discuss below.

A second categorical term, *meanings*, refers to mathematical meanings (Thompson, 2013) that reside within the individual. The teacher's mathematical meanings, for example, inevitably affect how and what students learn. If the teacher perceives the idea they are teaching as being easily understood, the teacher may offer a broad overview of the concept and move on, expecting that students will understand the topic just as the teacher understands it. On the other hand, the teacher may consider the knowledge needed to learn the concept and realize that the students need further explanation. Thompson (2013) stated, “Piaget placed great emphasis on the idea of decentering or attempting to adopt a viewpoint that differs from your own” (p. 63). In this study, I participated in the lessons and then transcribed them. I looked for evidence of shared and unshared meanings by attempting to interpret the students’ conceived mathematical understanding. For example, I attempted to develop second order models of my students’ thinking through an analysis of their classroom discussions. Although, understanding is not recordable, words or text can imply a shared or unshared meaning through clarifying questions or affirmative responses (e.g., raised hands that they agree, a nod, or a thumbs up).

A third categorical term, *noticings*, “is used in everyday language refer to general observations that one makes” (Sherin, Jacobs, & Phillip, 2011, p. 4). Noticing “requires attention to something” (Mason, 2011) and is the natural observation we make daily. For example, the student is happy because they are smiling. Noticings are “a natural part of human sense making” (Ball, 2011, p. xx). I consider these noticing observational. However, in teaching, the noticings can often be less obvious because the teacher may need to notice the student’s words and actions do not align. For example, a student refers to a zero slope verbally, but their hand is moving vertically. Therefore, in teaching, the noticing can often be considered unnatural [a skill a teacher needs to learn] (Ball, 2011) and require sense making (Sherin, Jacobs, & Phillip, 2011) because teacher noticing requires a conscious effort (Mason, 2002) to interpret what the teacher observes. As explained in Ball (2011), “Figuring out what students think, and what they mean, is complicated not only by these ‘gulfs’ of human experience, but also the influences of contexts as well as teachers' desires and assumptions regarding their students' learning" (p. xxi). Teachers may see boredom in their students' faces or glazed eyes and take those to mean that the student does not understand. A teacher who is intently focusing on teaching a mathematical topic may not notice students' behaviors.

Teachers "have to notice the domain they are teaching, with eyes and ears trained to perceive the content both from the perspective of the expert (to know what there is to know and learn to do) and from the fresh perspective of the learner (to see the familiar as strange)" (Ball, 2011, p. xxi). As this is a self-study, my intent was to focus my noticings inward on my behaviors and responses as I replayed the video recordings and transcribed my lessons that affect my students' learning. Additionally, as I also knew what I was thinking during a transcribed lesson, a noticing could also include my reflective thoughts such as the next time I teach the concept I might use a different representation. In general, "teachers' decision making [discussed next] is shaped by what the teachers notice" (Schoenfeld, 2011, p. 233).

Decision points constitute the fourth categorical term. Decision points are the "highlighted opportunities to make a decision that occur in a lecture" (Meehan, O'Shea, & Breen, 2017, p. 3). There are points, either during a lesson or while watching a video recording of it, when a teacher consciously decides whether to react to a noticing. The "decisions that teachers make regarding whether and how to follow up on what they notice are shaped by the teachers' knowledge (more broadly, resources), orientations, and goals (Schoenfeld, 2011, p. 233). In this study, my goal was to reconceptualize my smooth and chunky thinking. Accordingly, I hoped to *notice* opportunities for covariational reasoning while teaching or upon reflection during the transcription process. At a decision point, we can, however, decide that teaching the concept adds an additional depth to the students' covariational reasoning such as expecting students to describe transformations verbally using phrases as shifting the parabola up, down, right, or left.

Lastly, the final categorical term I employ is *awareness*. Mason (1998) explained that awareness is in the subconscious until we become specifically aware of the foci or what is being focused upon. At the point we become aware of the foci, I suggest that we could not become un-aware of them. Simply put, "awareness is what enables action" (Mason, 2011, p. 45). For example, prior to this study I was not aware that there was covariational reasoning. I only knew of the correspondence approach to variational reasoning. Now, that I am aware of covariational reasoning, I think of transformations covariationally. Once brought to my attention, I am now aware and action can take place. Furthermore, awareness is different from noticing. To me, noticings are observable, although may only come to mind upon reflection as compared to awareness that resides in the mind until it surfaces.

Data Collection

In this section, I begin by describing the data I collected followed by my rationale for collecting it in relation to my research question. Then, I present the data collection tools and sources. Finally, I briefly describe how I analyzed the collected data.

The data I collected originated from my daily instruction. For those lessons, I chose the algebraic topics of transformation, end behavior, and odd and even functions in the context of linear, quadratic, cubic, and cube root functions, because these topics presented opportunities for covariational reasoning, which allowed me to reconceptualize my smooth and chunky thinking. I also collected data on topics I perceived as advanced levels of mathematical thinking such as long division, synthetic division, identifying inverse functions, solving radical equations, and creating programs on a graphing calculator to move Rover (a robotic car). These topics furthered my development of covariational reasoning.

During the data collection process, I employed my usual classroom norms, such as expecting students to participate by asking questions and raising their hands. Assessment of my students' understanding was ongoing throughout discussion, written work, and formative (e.g., thumbs up or body motions) and summative (end of unit tests) assessments. It is important to note, however, that student-level data were not included in my data collection, nor did they appear in any research articles that I hoped to publish based on this research.

Data Collection Tools and Sources

In addition to the set of 29 video recordings I described above, I maintained a research log, which I produced from the transcripts of the video recordings, my provoking ideas that reflected my intentions for future lessons, and the critical friend memo, a synopsis of the end of the day dialogue. I also maintained a collection of narratives I wrote to portray my developing thinking as a result of my engagement with the provoking questions posed by the source of that data, my critical friend. Another data collection tool was a compilation of the critical friend memos and provoking ideas from the Research Logs that I refer to as the Critical Friend Portfolio. The final collection tool was the concept map. Each of these data collection tools and sources are described in detail below.

Video data. "Video material catches the non-verbal data that audio recordings cannot catch, which may be particularly useful" (Cohen, Manion, & Morrison, 2011, p. 530) when collecting data. Video recordings also allow for repeated viewing and checking. Klein and Taylor (2017) considered video as a mirror of teaching, where we made sense of how we reflected on the videos and the meanings we made from them.

I used the video recorder laptop as a tool for data collection. The video camera/recorder pointed at the front of the classroom where I stood when I wrote on the board. It was stationary and did not follow me around the classroom as I taught. The video recorder provided documented data of the content taught in the lessons. I could hear and see myself clearly during the replay.

I used the laptop's video recorder to collect 29 video recordings of the 29 lessons I implemented during data collection. Each night following a lesson, I transcribed the video data and described all teaching moves that were relevant to the lesson. As the daily schedule rotated, I video recorded the first hour as a back-up video of teaching (50-minute class) and supplemented the video data when unforeseen situations occurred, such as fire drills, code reds, or testing. Additionally, I video recorded the last hour of class.

Daily research log. Samaras (2011) referred to a teacher research log as a notebook that documented the teacher researcher's meta-conversation to herself and to a critical friend that include "notes, reflections, and primarily ideas unfolding, enactment, and assessment of pedagogical strategies" (p. 175). I used a Daily Research Log as a second data organizational tool that I considered to function as an interactive notebook in which I documented the transcriptions of the video recordings, critical friend memos, and provoking ideas.

Each night following the implementation of a lesson, I transcribed the video-recorded lesson into a daily research log and wrote provoking ideas to plan the actions for the next class. The critical friend memos consisted of an end-of-the-day reflective dialogue about the lesson. As we dialogued, my critical friend asked questions such as, "Do you think the students made a connection between the representation and the graph?" and "How do you think you might change the lesson in the future?" Figure 9 provides an example of a Research Log excerpt. I chose to include the following transcription of a brief but vivid account, because it represents a discussion between the students and myself. This exemplar embodies my attempt to give a visual to words such as when I ask the students, "Do you remember when we were drawing a parabola [I drew a parabola and its reflection]."

Code	Research Log Date: 2/27/18
	Brief but vividly descriptive account: "What I want you to notice is look at these numbers? I am going to graph these [two equations] on the same graph. One half, one, two. Notice anything? S1: It is the reflection. Nice. What allows you to do a reflection? Do you remember when we were drawing a parabola [I drew a parabola and its reflection] x squared, what would negative x squared be? S2: It would be a reflection over the x-axis? Right? Cool right, I just thought this would be a nice point to show you right now."
	Reflection on the brief but vividly descriptive account
	Noticings:
	Decision Points:
	Meanings:
	Awareness-in:
	Advanced mathematical thinking opportunities:
	Smooth and Chunky Reasoning:
	Provoking ideas (discussion with the text)
	Critical Friend Memo:

Figure 9. Excerpt from a Daily Research Log

As I considered the specific data necessary to answer my research question about my covariational reasoning, I included these six organizational categories, which I defined in the preceding section, in the Research Log (see Appendix A for an example of a completed Research

Log): BBVDs (brief-but-vivid descriptive accounts), BBVD reflections (Mason, 2002), meanings (Thompson, 2013), noticings (Mason, 2002), decision points (Meehan, O'Shea, & Breen, 2017), and awareness (Mason, 1998). The categorical terms related to chunky and smooth covariational reasoning (Castillo-Garsow, Johnson, & Moore, 2013) and advanced mathematics opportunities (Tall, 1991) were discussed in Chapter 2. Finally, the last two categories were the provoking ideas (Connelly & Clandinin, 1990) and critical friend memos (Samaras, 2011). Although the management of ten organizational categories may strike the reader as potentially unwieldy, actually they organized and focused my transcriptions, the provoking ideas, and the critical friend memos. The Research Logs provide both the reader and myself with a detailed rich transcription of a salient moment in my teaching.

Narratives. Narratives detail evidence of teachers' stories and classroom experiences. They can include "stories, journaling of your ongoing record, essays, other reflections about your study; can include education-related life history; interpretations of visual data and story of your research process; and can include narratives by participants" (Samaras, 2011, p. 175). In this study, I composed detailed narratives after the completion of the transcription process. I called on my critical friend to ask thought-provoking questions in order to provoke and elicit reflective thinking about my study (see Appendix B for an example of a completed Critical Friend Narrative). For example, the question, "Do you think the connections you're making are changing the way you're teaching the particular concept?" provoked me to reflect on my teaching, which I then captured in a story-like description that I refer to as a "narrative." My intention was for the narratives to convey the story of the transformation of my thinking about covariational reasoning.

Critical friend portfolio. The Critical Friend Portfolio (CFP) was intended to provide a trail of documentation of the progression of the reconceptualization of my thinking about my teaching practice. I used the CFP (see Appendix C for a completed example) to organize the critical friend memos and provoking ideas from my Research Logs. The critical friend memos were a synopsis of verbal dialogues between the critical friend and me. As dialogues are fundamentally social constructions, colleagues and collaborators — or “critical friends” — are called on to participate in their construction (Guilfoyle et al., 2004). In this regard, I conceived the critical friend as a source of data. During the critical friend dialogues, which are meant to include critiques and reflections (Guilfoyle et al., 2004), the critical friend asked provoking questions as we discussed my lessons at the end of the day. I transformed these dialogues into the text of critical friend memos that are compiled into the documents that compose a Critical Friend Portfolio (CFP).

The headings in the CFP are: How do I analyze the development of my acquisition and implementation of the lens of smooth and chunky covariational reasoning?, Reconceptualizing Algebraic Relationships, Promote the transformation of my existing teaching practices, Living Educational Theory, and Personal History Method. I chose these headings as they aligned with my research question and methods. The CFP was a chronological compilation of a rich dialogue that provided evidence of the transformation of my thinking. In Figure 10, I provide an excerpt from the CFP that features the heading, “How do I analyze the development of my acquisition and implementation of the lens of smooth and chunky covariational reasoning?” The CFP can take the form of memos or “letters [notes] you write to your critical friend as a way to more naturally talk about your research while also deeply thinking about it” (Samaras, 2011, p. 179). They could also include “narratives” or an activity or lesson along with a purpose explicitly aligned to the research component, and additional comments could include informal notes about data collection, data analysis, critique/validation, self-assessment, writing/presenting, discussion/professional development, or the collaborative process.

Critical Friend Portfolio
How do I analyze the development of my acquisition and implementation of the lens of smooth and chunky covariational reasoning?
2/26: Critical friend memo: We also discussed my fear that I would not having anything exciting in my research about smooth and chunky thinking. What if I can't think smoothly? She always said, that I couldn't plan my findings. If nothing happens, nothing happens, that is what I will write about.

Figure 10. Excerpt from Critical Friend Portfolio Log; based on Samaras, 2011, p. 91

The critical friend narratives and the critical friend portfolio provided the primary evidence of the evolution of my smooth and chunky covariational reasoning. The purposeful shared dialogue between my critical friend and myself generated an environment in which I could learn about algebra content and practices associated with teaching it.

Concept map. Novak (2011) explained that concept maps helped researchers design and interpret data more effectively. The concept map provided me with a visual representation to analyze the large amount of data. After all the data were collected, I illustrated the organizational and substantive categorized coded data, discussed in the following section, into a concept map, see Figure 20 (p. 149) for a completed example. My intent was to determine if an unexpected theoretical category emerged from the Research Log transcriptions. In the following section, I connect the data collection and the data analysis process.

Data Analysis

This section begins with a discussion on the forms of coding I used in this study. Next, I discuss Maxwell's (2009) three types of categories (organizational, substantive, and theoretical categories) used for categorizing analysis. Then, I relate three strategies for developing substantive and theoretical categories. The strategies I used during the coding process are similarity-based categorizing strategies (coding and thematic analysis), contiguity-based connecting strategies (narrative analysis), and displays (concept maps) and memos. I conclude with a discussion of how these coded categories were applied in the present study.

Coding. The mechanisms of the analysis begin with an introduction of the various forms of coding. Cohen, Minion, and Morrison (2011) referred to “a code simply as a name or label that the researcher gave to a piece of text that contained an idea or a piece of information” (p. 559). In this study, I used organizational, substantive, and theoretical, as in Maxwell (2009), for coding categories. Maxwell (2009) suggested that *organizational* categories were broad and established prior to interviews or observations. In this study, the organizational categories were the categories introduced in the Daily Research Log such as BBVDs, BBVD reflections, meanings, noticings, decision points, awareness, provoking ideas, critical friend memos, chunky and smooth covariational reasoning, and advanced mathematics opportunities.

The *substantive* categories “are primarily *descriptive,* in a broad sense that include description of the participants’ concepts and beliefs; they stay close to the data categorized and don’t inherently imply a more abstract theory” (Maxwell, 2009, p. 237). The descriptions refer to the explicit relationships found. These relationships do not represent a global perspective of the findings in their totality. The *theoretical* categories are a comprehensive perspective of the data collections. Maxwell (2009) stated, “These may be derived either from prior theory or from an inductively developed theory” (p. 238).

As stated, the organizational categories were established prior to the interview or observation, and the size of the data collected was not a factor. However, there was a need for strategies when developing substantive and theoretical categories because there was a large data collection set. This is because “you can’t hold all the relevant data to particular substantive or theoretical points in your mind” (Maxwell, 2009, p. 238). The strategies fall into three main groups: similarity-based categorizing strategies (coding and thematic analysis), contiguity-based connecting strategies (narrative analysis), and displays (concept maps) and memos.

In this study, I used fracturing or connecting strategies to analyze the text. When fracturing, the "organizational categories function primarily as 'bins' for sorting the data for further analysis" (Maxwell, 2009, p. 237). On the other hand, "connecting strategies, instead of fracturing the initial text into discrete elements and re-sorting it into categories, attempt to understand the data [text] in context, using various methods to identify the relationships among the different elements of the text" (p. 238). Expressed in a different way, Mittapalli and Samaras (2008) stated that connecting strategies identified key relationships that bind the data together into a narrative or a sequence" (p. 249).

I used a *similarity-based* categorizing strategy and a *contiguity-based* connecting strategy. A similarity-based strategy helps to address a research question that seeks to understand the way events in a specific context are connected. Maxwell and Miller (2008) explained similarity-based relationships were based on comparison and used coding as a typical categorizing strategy. Similarity-based (categorizing) strategies aim to find common features in the text and are based on comparisons of the text. In the context of this study, the analysis of my transcriptions allowed me to identify organizational codes and then find the emerging themes that surfaced. These themes provided evidence of the reconceptualization of my smooth and chunky covariational reasoning and my teaching practices.

A contiguity-based connecting strategy helps to address a research question that is concerned with similarities and differences across settings, individuals, or general themes. In the context of this study, I used contiguity-based connecting strategies during the analysis of the CFP narratives to determine how I reconceptualized my smooth and chunky covariational reasoning and promoted the transformation of my teaching practice. Contiguity-based (connecting) strategies concern the influence of one item on another, or relations among parts of a text, such as how I interpreted my past and present mathematical connections. The contiguity-based strategies helped me find emergent themes. However, to answer my research question, there was a back and forth movement between the two strategies. I used the similarity-based strategies to find the evidence of the coded research logs to understand *what* and the contiguity-based strategies to find the evidence to understand *how* I reconceptualized my smooth and chunky covariational thinking and the transformation of my practice. Finally, I organized the data into a concept map that provided me an image to compare the coded transcriptions and narratives to gain insight into the comprehensive perspective of the data collections.

Coding Analysis Procedures. In this section, I describe the coding analysis process that includes organizational, substantial, and theoretical categories, and similarity-based (categorization) and contiguity-based (connecting) strategies. The coding process is provided in iterations in Figure 11. These iterations include the initial reading, additional readings, and the review of the transcriptions and coding, followed by the organization of the codes, identifying the themes, and then creating a concept map.

After I transcribed the 29 video-recorded lessons, my first iteration in the analysis was an initial read-through of the transcribed pages to get the totality of the data (Bogdan & Biklen, 2007), while simultaneously labeling the preliminary organizational codes. As I read, I added additional codes to the preliminary list using similarity-based strategies to identify codes used in the Research Log (see Appendix A for a completed coded sample).

Iteration 1: Initial Reading of Transcriptions and Coding
After I transcribed the 29 videotapes using the organizational codes, I read through the transcriptions in one sitting to get the totality of the data (Bogdan & Biklen, 1982) while simultaneously labeling the preliminary organizational codes. I began by using similarity-based strategies to identify additional codes to add to the preliminary list of organizational codes to create the codebook chart. As I read through the transcribed data, I either placed an alphanumeric code or a large x beside a transcription that did not pertain to my research question. I gave each code two- or three- letter category code followed by a numeral 1, in the case of the initial reading.
Iteration 2: Additional Reading of Transcriptions and Coding
I read through the transcriptions two more times in their entirety and continued to use the similarity-based ordering of data to identify additional codes. Each reading took about two days. I updated the sequential numerals upon each new day of analysis to denote the number of times I've performed a read-through. The numeral provided for a verification of the coding process as an audit trail.
Iteration 3: Organized Similar Codes
I double-checked each code as I typed the codes into the computer and underline any corresponding text to code. At the same time, I removed the text that did not pertain to my research question.
Iteration 4: Organized Similar Codes
I printed the coded text and dated each of the codes to preserve the timeline. Next, I cut the codes, text, and corresponding dates into slips of paper and separated the slips into baskets based on the codes. I read through the slips of coded text looking for themes using contiguity-based strategies. Then, I wrote the organizational and substantive categories on sticky notes and stapled the corresponding slips onto the sticky notes placing them on a large poster paper by categories. As I considered the emergent themes, I used reflective practices to identify critical incidents to understand how I developed my thinking about my teaching practices.
Iteration 5: Created a Concept Map
I re-wrote all the themes from the sticky notes on a concept map (McGowan & Tall, 1999). I used the structures of Living Educational Theory, Personal History, Thinking (Smooth, Chunky, Covariational, and Variational), Thinking Process, Tall (Conceptual Embodied and Proceptual Symbolic), and Connections to identify emergent overarching themes to answer my research question of how I promoted the transformation of my teaching practice.

Figure 11. Coding Process Analysis

I recorded each of the codes in the codebook chart as in Figure 12. Per the recommendation of Bernard and Ryan (2010), I maintained a codebook that kept track of the codes in one column along with the abbreviation to which the code corresponds in a second column, and a brief description of the code in a third column. Assigning a code to a piece of data enabled me to discern emerging themes (highlighted connections) in segments of data assigned the same code

Code	Abbreviation	Brief Description
Anchor	ACH	Anchor to prior knowledge
Advance Organizer	ADV	This is an organizer given in advance to learning the material.
Advanced Mathematical Thinking Opportunities	AMT	These are opportunities that I perceive as post Algebra I topics
Met-Befores	MB	These are experiences of past students struggles used in my teaching practices
Chunking Thinking	CH	Thinking in terms of intervals
Conceptual-Embodied	CE	Tall's mathematical cognitive development using sensorimotor including visuals, motion, and movement
Covariational Reasoning	COV	Thinking of the variables simultaneously
Learning Educational Theory	LET	A methodology that states beliefs do not align with the implementation
Learning from Discussion	LFD	I reconceptualized my thinking as a result of classroom discussion or critical friend discussion
New Connections	NCN	I developed new connections
Past Connections	PCN	I taught connections based on past experiences
My Past Personal History	PHM	These are teaching practices based on my past personal experiences
Process Thinking	PT	Unlike steps, I am thinking in terms of a continuous fluid process.
Proceptual-symbolic	OS	Tall's mathematical cognitive development using algebraic symbols
Reconceptualized thinking	RC	Reconceptualized algebraic thinking
Smooth Thinking	SM	Thinking in terms of a process
Steps	STP	Teaching practices based on solving steps, not a fluid thought process.
Variational Reasoning	VAR	Thinking of the variables in terms of input and output (correspondence approach) and precoordination and coordination of values

Figure 12. Codebook Chart

Coded transcriptions of the research logs were used to conduct the analysis. An example appears in Figure 13. As I read through the transcripts, if an excerpt was relevant to my research question, I assigned an alphanumeric code to it. Each alphanumeric code was comprised of a two- or three-letter category code followed by a single-digit number that specifies which read-through of the transcript the code was assigned. For example, COV1 indicated that a segment reflected my covariational thinking and was assigned during the first read-through of the transcript. If an excerpt of the transcript did not pertain to my research question, I assigned it a code of X.

	Date: 3/10/18
COV1 CH1 SM1	Brief but vividly descriptive account: The thing that I have been noticing in each of these equations is that they are starting from the y-intercept when writing the equation. Because of this, the students are looking at the graphs, saying where do the situations start and what trend do I see? They are not focusing on the numbers when graphing. This means they are focusing on the quantitative reasoning. Once they have found where the situation starts, they are looking at the trend, and then how the situation is changing. Next, they determine if it is discrete or continuous.
COV1 RC2 SM1 RC2	Reflection on the brief but vividly descriptive account: The students are thinking in a covariational way. The students barely look at the tables of values except to see the y-intercept. Is it possible that because I have not been focusing on the tables of values when graphing the transformations that the students are using that logic to graph lines. They are finding a key point in this case the y-intercept and following the thought process we have been using, during transformational graphing. [It seems I am modeling covariational reasoning.]

Figure 13. Excerpt from a coded transcription

In the second iteration of the analysis, I read through the transcriptions two more times in their entirety while continuing to use similarity-based ordering of data to identify additional codes. These are codes that were not part of the initial organizational codes. These new codes were emerging themes, like conceptual embodied, operational symbolic, advance organizers, and anchors. Each reading took two days. Each day that I sat down to code, I appended the code with the next sequential number to identify the read-through iteration. The numeral provided for verification of the coding process as an audit trail. For example, I was able to identify if the code was from the initial organizational category or based on subsequent similarity or continuity strategies.

In the third iteration, I double-checked each of the codes as I typed them into the computer and underlined corresponding text to code. At the same time, I removed the text that was not coded, as it did not pertain to my research question.

In the fourth iteration, I printed the text and dated each with the codes to preserve the time line. Then, I cut the slips of coded data with dates and place the slips of coded text in baskets by the codes from the code chart. I read through the coded text looking for themes by using the contiguity connecting strategies from the group of slips of text from the baskets. Then, I wrote the organizational and substantive categories on sticky notes and stapled the corresponding slips onto the sticky notes placing them on a large poster paper by categories.

Although in this iteration I organized the coded transcriptions and summaries, it was the reflective practices used when considering the critical incidents that created a moment of reflection or pause to identify unexpected substantive categories. Tripp (1993) described critical incidents as "essentially cognitive responses of 'surprise' and 'perplexity'" (p. xiii). An outcome of these critical incidents was the "chipping away" of existing teaching practices to make room for a new awareness about my teaching practice.

Through reflection, I deliberated about the effect the critical incidents had on my teaching practice. I drew conclusions from my reflections on these critical incidents, which informed my understanding of how I developed or reframed my thinking about my practice.

In the final iteration, I created a concept map to organize the categorical and substantive categories of Living Educational Theory, Personal History, Thinking (Smooth, Chunky, Covariational, and Variational), Thinking Process, Tall's Conceptual Embodied and Proceptual Symbolic worlds, and Connections. My objective was to create a concept map (Novak, 2011) to dialectically reveal and represent the restructuring of my thinking over time. My plan was for the final iteration of the map to illuminate a global or theoretical categorical perspective of interactions between substantive categories. This means of analyzing the map proved to be much more insightful than leafing through pages of coding.

Validation

In qualitative self-study research, researchers use multiple methods and/or data collection techniques to provide the researcher with opportunities for diverse analytic perspectives. The purpose of multiple data sources is to make validation claims. Mishler (1990) defined validation as “the processes through which we make claims for and evaluate the ‘trustworthiness’ of the reported observations, interpretations, and generalizations” (p. 419). Validity attaches to accounts, rather than to data or methods. It refers to the meaning that subjects give to data and inferences drawn from the data that are important. Moreover, Cohen, Manion, and Morrison (2011) suggested that trustworthiness as an alternative to quantitative notions of reliability and validity. In qualitative studies, the trustworthiness “depends on the credibility of the researcher” (Merriam, 2009, p. 234). It is the researcher who must ensure the study’s findings. In the next section, I discuss the criteria I used to assess credibility (internal validity), transferability (external validity), reliability, and confirmability of the qualitative research and the data sources that inferences were made in the present study (as seen in Figure 14).

Criterion	Data Sources
Credibility (internal validity)	Member check, Peer Debriefing
Transferability (external validity)	In-depth Descriptions, Transcriptions
Reliability	Inter-rater Reliability Discussion during Final Coding Analysis, CF Narratives during Analysis
Confirmability	Research Log, CF Portfolio

Figure 14. Table with the criterion and data sources; based on Lincoln and Guba (1985)

Credibility seeks to accurately describe findings of the phenomena researched. There are several ways to address credibility, such as the authenticity of the data or the ability of the researcher to report the situation in the eyes of the participant. As I am both the researcher and the participant, I authenticated the meaning of the data. Furthermore, I authenticated the data (e.g., dialogues, narratives, and the transcriptions of the video recorder) by the depth of detail in the reflections and transcriptions. Ultimately, however, it was through member checking and peer debriefing that the credibility of the analysis became evident. My critical friend had the role of member checking to ensure that my inferences were reasonable throughout the study. Debriefing occurred during collaboration with my critical friend and me. We dialogued about my past experiences in mathematics, challenged my interpretations of inferences through provoking questions, and then, expected me to justify my analysis. Finally, triangulation through multiple data sources established credibility. Schwandt (2007) explained, " triangulation is both possible and necessary because research is a process of discovery in which the genuine meaning residing within an action or event can be best uncovered by viewing it from different vantage points" (p. 298). Therefore, strategies to promote triangulation are "using multiple investigators, sources of data, or data collection methods to confirm emerging findings" (Merriam, 2009, p.

229). In this study, the assertions emerged from the data of the Critical Friend Portfolio, the dialogues, narratives, and transcriptions of videotapes from which credible inferences were drawn.

Transferability concerns "itself with the extent to which the findings of one study can be applied to another situation" (Merriam, 2009, p. 223). Although this self-study was not generalizable to a wider population, this self-study did provide a rich form of data necessary for transferability that would allow other researchers to draw from, such as exemplars to make connections to their personal experiences and backgrounds. Lincoln and Guba (1985) maintained that researchers should provide sufficiently rich data and thick with exemplars or stories (Mishler, 1990) for the readers and users of the research to determine whether transferability was possible. Merriam (2009) explained, "The researcher has an obligation to provide enough detailed description of the study's context to enable the readers to compare the 'fit' with their situation" (p. 226). There were clear, detailed, and in-depth descriptions so that others could decide if the findings from one piece of research were transferable. These descriptions emerged from the video transcriptions in which I was the participant, which facilitated accurate interpretations and explanation.

Reliability, as defined by Denzin and Lincoln (1994), is the stability of observations, parallel forms, and inter-rater reliability**.** The stability of observations refers to whether the researcher would come to the same conclusion if the observations had been made at a different time. Parallel form refers to whether the researcher would have made the same observation or interpretation if they had paid attention to other phenomena during the observation. In this study, replaying the audio recordings made it possible to watch different phenomena using the *pause* button. During the replay, I was able to observe the video recording multiple times allowing for an accurate account of the lesson including a verbal transcription, actions (pointing), body language (moving my arms to represent an action in mathematics), and facial expressions (frustration). In addition, I replayed the video at a different time, which allowed me to process the lesson before replaying, providing a time delay, adding to the reliability of the claims because the observation took place many times. These video recording were crucial during the narrative analysis process with my critical friend, I replayed specific segments to elaborate on the analysis, when necessary. The critical friend narrative also was a form of reliability as the critical friend checked my analysis findings. *Inter-rater reliability* refers to another individual's interpretation of the finding in the same way. The use of inter-rater reliability occurred during the coding process, as the critical friend randomly checked the coding.

*Confirmability***,** as Schwandt (2007) described it, referred to "establishing the fact that the data and interpretations of an inquiry were not merely figments of the inquirer's imagination" (p. 299). It requires linking assertion, findings, and interpretations of the data in distinct ways, such as the process of auditing. Furthermore, Samaras (2011) explained that transparent research clearly and accurately documents the research process through dialogue and critiques. Therefore, in this study, I used the Research Log as the audit trail (documentation of the raw data analysis and data reduction, process notes of my research process) and the critical friend portfolio as the professional accountability to document evidence of the process. Additionally, my critical friend asked provoking questions during the analysis of the concept map that added a new lens to the analysis.

The validation process demonstrates to the reader that the data and analysis in this study are trustworthy. By meeting the criteria of validity, such as, credibility, transferability, reliability, and confirmability, I was able to make claims, assertions, and inferences based on the data analysis and then determined possible implications that contributes knowledge to mathematics and self-study research.

Chapter V

Results

The purpose of this self-study was to examine how my teaching practice transformed as I developed my smooth and chunky covariational thinking. As I transcribed my lessons each night, I reflected upon my practice. I identified examples of my smooth and chunky thinking. However, I was not able to limit my reflections to my smooth and chunky thinking. I asked myself: How could I improve my lessons? I then decided the lesson adjustments for the future. Therefore, the results from the Critical Friend Portfolio dialogues, narratives, and research logs went beyond the transformation of my smooth and chunky reasoning to encompass the transformation of my practice in general as a result of studying the development of my smooth and chunky thinking.

As stated in Chapter 3, I triangulated multiple resources as a form of validation. The three resources were: narratives, Critical Friend Portfolio dialogues, and the transcriptions as supporting evidence from the Research Logs. Additionally, my critical friend played a pivotal role of member checking to ensure that my inferences were reasonable throughout the study. During the debriefing between my critical friend and me, my responsibility was to justify and argue my conclusions adding to the internal validity about the transformation of my practice.

As described in Chapter 3, my intent was to analyze the transcriptions from the research logs to identify emerging themes that illustrate the transformation of my teaching practices and provide supporting evidence for the narratives and the critical friend portfolio. The organizational categories described in this chapter are the result of the initial coded transcription process from the research log based on classroom lessons. These vignettes provide a basis for answering the research question. In the next chapter, I connect the literature and the results stated in the present chapter.

The themes that emerged from the initial organizational categories are the substantive categories. After each organizational category, I offer a synopsis of the knowledge gleaned from the vignettes that I interpret as the substantive categories. Although the organizational categories are explicitly stated because they can be simplified to a word or phrase such as smooth and chunky covariational reasoning, the substantive categorical themes are not explicitly stated because the themes are my interpretation of my thinking is in real-time and therefore, ever changing such as there is a moving in and out between smooth and chunky reasoning.

To demonstrate the progression of my smooth and chunky reasoning, I present a collection of dialogues and narratives of the reconceptualization of my covariational reasoning. After each narrative, I provide the supporting evidence from the Research Log transcriptions from my classroom lessons in the form of vignettes. In many vignettes, I interacted with students during classroom discussions that illustrate the substantive themes that surfaced. I used the letter T to represent myself/the teacher speaking and S followed by a number to represent the order in which the students spoke during the lesson (e.g., S1 referred to the first student who spoke and S2 referred to the second student who spoke). I referenced the same alpha numeral when the same student responded consecutively. I used the code S Choral to represent multiple students speaking simultaneously. My intent is for the reader to have an insight into the classroom discussion to add to the context of the lesson. Because this is a self-study, I elaborate on the transcriptions when additional explanation might add richness to the evidence.

Results from Provoking Thoughts and Dialogues

As I present my results, I discuss the changes in the development of my thinking by referencing either provoking thoughts or dialogue with the critical friend. I follow the provoking thoughts or dialogue with a vignette from the Research Log transcriptions to provide supporting evidence. The provoking thoughts document the development of my smooth and chunky reasoning and teaching practices. The dialogues from the Critical Friend Portfolio, which I refer to as critical friend memos, reemphasized the progression of my reconceptualization of smooth and chunky reasoning.

Smooth and chunky thinking. This section begins by discussing my belief that I did not think smoothly or covariationally. I refer to my initial thinking as my preliminary benchmark on which to illustrate the progression of my smooth and chunky reasoning. In the following dialogue, I describe my doubts about my ability to think smoothly since I had previously perceived myself to be a chunky thinker. My critical friend and myself discussed my fear that I would not have anything exciting in my research about smooth and chunky thinking.

> What if I can't think smoothly? She [critical friend] said that I could not plan my findings. If nothing happens, then nothing happens; that is what I will write about. (Critical friend memo, 2/26/18)

It became apparent that by day two there was evidence of my ability to think smoothly, chunkily, and variationally (correspondence approach). At this point, I felt a sense of relief that I no longer had to worry that smooth thinking might not develop. The following day, I began to look for specific evidence of covariational and variational reasoning.

> Today, I started to see evidence that smooth and chunky thinking happens in my mind. When I focus on the ordered pairs and growth factors, I am thinking chunkily, but when I am thinking of the function in its totality I am thinking smoothly. (Critical friend memo, 2/27/18)

> During the lesson, a student's reference to a "common factor" prompted me to recognize

the opportunity to explain the concept of smooth and chunky thinking. The following vignette of a brief but vivid account occurred in the first week of my data collection. I summarize a pivotal change in my thinking.

> When figuring out the equation $y = ab^x$, I used the table to identify the a (the y-intercept) and the b (the common ratio). [I used chunky variational/coordination of values thinking.] I wanted students to figure out how to graph an exponential function. Students chose ordered pairs for example (0, 1) and (1, 3) and figured out the variables a and b [variational/ coordination of values]. I said in class, "I don't know of another way, but one of my students in another class interpreted the ordered pairs using the idea of a common factor." This is actually a mistake. In fact, he was referring to growth factor. By using the ordered pairs, I applied chunky thinking, but using the growth factors [upon reflection] I applied smooth thinking. This was huge! Consider, that if I applied factors, these growth factors go on forever and are not within an interval. Applying factors, I am thinking in a continuous fashion [smooth covariational]. (Research log, 2/27/18)

This vignette demonstrates the use of a blend of smooth, chunky, variational (correspondence approach and coordination of values), and covariational thinking within one problem. Initially, I used variational (correspondence approach) thinking, using ordered pairs as input and output. Eventually, I thought of a graph shifting without using quantities (quantitative reasoning) or a graph as a continuous exponential process. I considered growth factors not as the same intervals, but in terms of *b-ness*. For example, $y = 4^x$ rises rapidly in visual chunks (not intervals) of exponential *4-ness*. This was an example of my smooth covariational thinking.

This dialogue offered examples of the difference between my chunky thinking and smooth thinking. I envisioned smooth thinking as the totality of an idea. The reference to ordered pairs in terms of the input and output suggested variational (correspondence approach) thinking. In my mind, I linked the variational component to chunky thinking. When I envisioned the function shifting as an entity, I imagined the shape moving smoothly from one location to another. This idea of linking smooth thinking with shape and movement began to emerge at the beginning of my literature review. Therefore, it was not surprising that this idea of motion resurfaced early in the data collection.

My mind continually connected the use of motion and technology with smooth thinking, which became a common thread throughout the study. Interestingly, I related using paper and pencil to chunky thinking throughout the study, as stated in the following narrative.

> When I began this study, I thought that using motion with technology was the only way that I thought covariationally. As long as I am using pen and pencil, I am thinking both chunkily and variationally [coordination of values]. (Provoking thoughts, 3/6/18)

It was not until I realized that I used transformational shifts (e.g., parabolas and absolute value functions) more often than the input and output table of values that I thought more smoothly than chunkily, as described in the following narrative:

> I realized that I think more smoothly than chunkily. So far, most of the chunky thinking has occurred in the step functions and the introduction using a table of values. But, to me the transformations are smooth thinking. (Critical friend memo, 3/8/18)

Two weeks later, it became evident that it was the flow of the four (smooth, chunky, variational/coordination of values and covariational), moving in and out, that gave me a greater insight into the interaction between smooth, chunky, variational, and covariational thinking. In the following vignette, the students participated in a class discussion using these four types of thinking. At the end of the lesson, the students chorally described a transformation as shifting the graph up two units, which demonstrates smooth covariational reasoning. I interpreted the students' reference to moving up two units as smooth covariational reasoning because they refer to the graph as a flow of the square root function both going on forever and moving up two units. Chunky thinking would be the input and output by creating the graph of the function that was limited to a specific chunk of the function. Therefore, I interpreted this idea as a conceptual understanding of transformation, since students were not thinking from a variational (correspondence approach) interpretation.

T: A power function is a function in the form of $f(x) = k \cdot x^n$. . . Don't forget that $f(x) = \sqrt{x}$ is equivalent to $f(x) = x^{\frac{1}{2}}$.

S1: Oh, yeah

T: This is why we are reviewing. Ok, now, when I think of the square root, I think of my parent function [I drew a free-hand graph, smooth covariational] . . . Let's check to see how off I am. The graph just goes on forever in the square root shape. It does not level out. It [x and y] starts getting bigger, bigger, bigger, and bigger [smooth covariational]. It goes to positive infinity [smooth]. Ok, you ready. Let's see how close my graph is [variational/coordination of values]. If I plot $f(x) = \sqrt{0}$ what does that equal?

S1: [pause] 0

T: [I trace the point on the origin]. $f(x) = \sqrt{1}$ What does that equal?

S2: 1

T: The ordered pair would be at (1, 1). What does $f(x)$ equal; what is the next perfect square?

S3: $f(x) = \sqrt{4}$

T: What does the square root of 4 equal?

S3: 2

T: . . . So, any number for x in between [chunky] is on the line. I can put in a 1.25. I can put in a 4.87279; it does not matter [variational/coordination of values]. I am going to have some point y that corresponds to some point x on this line . . . The next thing we want to talk about is the domain [chunk]. What is my domain? Is it my x or my y?

S4: x

T: What is my range?

S5: My y

T: So, what are my x values $x \geq 0$ [chunky] . . . The first example I want to discuss is the one from the notes. $g(x) = \sqrt{x} + 2$ We will call the one we just did $f(x) = \sqrt{x}$. Now, what we are going to do is $g(x) = \sqrt{x} + 2$. What do I have to do to this graph?

S Choral: Lift it up two

T: [I perceived the students' response as smooth covariational reasoning and a procedural knowledge]. Good. (Research log, 3/12/18)

The students participated in the classroom discussion using smooth, chunky, variational (coordination of values) and covariational language. At the end of the lesson, when they were asked to describe the shift of the graph, they simultaneously replied to shift the graph up two units. This suggested to me that they represented the graph using smooth covariational reasoning. The students took the square root graph of a function and shifted the function up two units. They did not consider any specific input or output. They saw the function as the graph shifting up as a whole entity. I perceived that the students thought of the x and y moving simultaneously. Additionally, the students' response suggested to me the students demonstrated procedural knowledge. The students took their knowledge of the transformation of a parabola and applied it to a square root function, a concept that I had not taught previously. When they were asked questions, the students moved in and out of smooth, chunky, variational (coordination of values), and covariational thinking. The different questions offered students opportunities to represent the variables as an ordered pair (coordination of values), to consider them simultaneously (covariational), to think of an interval of the graph (chunky), or as a process for a graphic perspective (smooth).

As the weeks passed, I slowly began to reconceptualize my smooth and chunky reasoning in terms of transformations. I no longer believed that I thought smoothly or chunkily. My thinking about smooth, chunky, variational (coordination of values), and/or covariational thinking shifted from being compartmentalized to moving in and out of each. When graphing multiple parts of a more sophisticated graph, my thought process advanced because I considered the graph in its totality (smooth covariational) and identified specific components or characteristics of the graph (chunky variational).

> I had not thought about it, but I have students graph higher degree polynomials and find the zeros, highest degree, odd or even, and if the "*a*" is positive or negative. The students are discussing graphs smoothly because we are talking about end behaviors. But as we graph using the x-intercepts, I am looking at the function variationally [coordination of values]. Although we did not find the local maximums and minimums, the students are still visualizing if there is going to be a local maximum or minimum [chunky]. When I ask the students about what is going on with two x-intercepts when they are the same [or multiplicity of two], they need to consider the idea of a bounce [covariational reasoning]. We begin with the smooth (process-driven) reasoning and move to the chunky reasoning when we look at specific characteristics of the graph.

(Provoking thoughts, 3/16/18)

As I reflected back and considered the substantive categorical theme, I realized that as I was pinpointing examples of my smooth and chunky thinking that were in real-time, I was not planning smooth and chunky thinking. I tended to consider my smooth and chunky thinking most through reflection. However, I was focusing on procedural knowledge and conceptual understanding when graphing transformational shifts. That is, by moving in and out of smooth and chunky thinking, I perceive this ability as encouraging procedural knowledge because students must be able to understand how to determine each of the features (x-intercept or having a maximum or minimum) and conceptual understanding because they consider the totality of the function given the different features.

The following vignette demonstrates how, one month later, I asked the students to envision the length of the sides of a rectangle changing without using specific input and output values to encourage covariational reasoning.

T: Mr. Smyth uses the function $l(x) = 2x^2$ to determine the length, in feet, of his landscaping projects. He uses the function $w(x) = 5x - 2$ to determine the width, in feet, of each project. What function could Mr. Smyth use to determine the price, $p(x)$, of a rectangular project if he charges $20 per square foot? It is a rectangle, so let's draw a rectangle. What is one side?

S1: $2x^2$

T: What is the other one?

S2: $5x - 2$

T: I have to be honest with you, I have to get past the idea that $2x^2$ does not add a third dimension [I am thinking variationally/coordination of values]. If we want to find the area of this space, what do we have to do?

S3: Multiply

T: Then, what do I get?

S3: $2x^2(5x - 2)$

T: What is this?

S4: $10x^3$

T: And this is?

S5: $-4x^2$

T: What answer did you get?

S6: You have to multiply by 20 because it is $20 a square foot.

T: What do I get?

S7: $200x^3 - 80x^2$

T: That is my answer. Does that make sense? [Nods]. (Research log, 4/24/18)

Although I was asking prompting questions that seemed to encourage procedural knowledge, by not providing the students with a value of x, the problem encouraged the students to think quantitatively about the length of sides, a move that I perceive as encouraging conceptual understanding. Furthermore, my remark that "I have to get past the idea that $2x^2$ does not add a third dimension" challenged my covariational thinking, because I interpreted any variable squared as being two-dimensional.

A substantive categorical theme that occurred in the vignettes that are encoded as smooth, chunky, variational (coordination of values), or covariational, I intuitively matched smooth and covariational and chunky with variational as I taught in real time. It was not until I transcribed my lessons that I found examples of the reconceptualization of my smooth and chunky covariational reasoning. For example, when the students suggested that exponential growth was a growth factor, I connected the new idea of growth factor to smooth thinking. Finally, I realized that I did not limit my reasoning to smooth or chunky. I was moving in and out of the smooth, chunky, variational, or covariational thinking.

Two and a half months after the onset of the study, I perceived another breakthrough in the development of my smooth and chunky reasoning. My reconceptualization of smooth covariational thinking emerged when I moved or transformed two pieces of a rational function. The parts moved together in unison. Initially, I conceptualized the two pieces separately. Consider that when one is drawing the pieces of a rational function with paper and pencil between the asymptote, you draw them one at a time. Therefore, as I demonstrate in the following vignette, I saw the two parts or pieces chunkily. However, upon reflection, when I thought of the two pieces as shifting, they moved as one.

> When I began pasting my pictures into the Research Log, I realized I thought of the concept smooth covariationally. I was asking the students to take the parent function – two parts – and shift them to the left, two places, and up three. I find it interesting that I could not think about this in real time. This was my thinking process. Oh, there is a piece, but how do I take the piece and describe it smoothly when there are two parts. I never thought of it as moving them together or simultaneously. It was the images side by side in Figure 15 that allowed me to make this connection of smooth covariational thinking. Now, I only see it smoothly and not chunkily. Is it possible that once you envision a smooth process, you cannot undo the thought such as you can't undo awareness? WOW! (Provoking thought, 5/14/18)

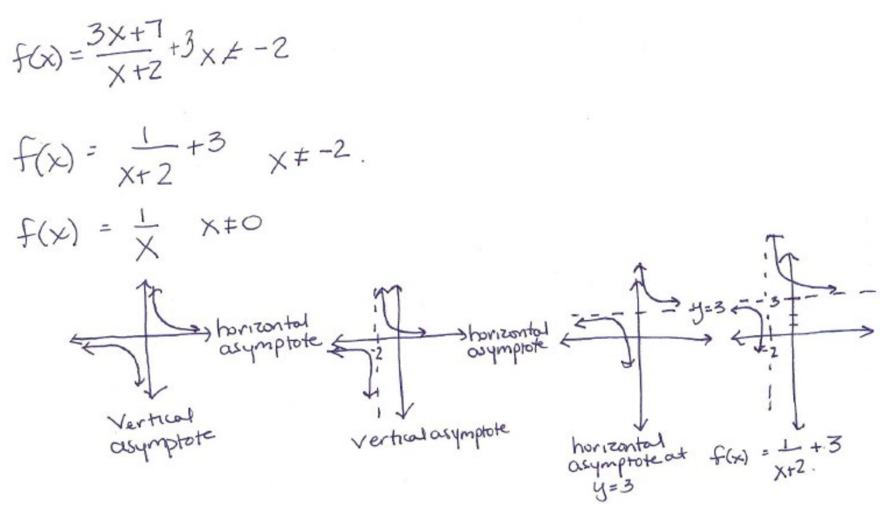


Figure 15. Development of a rational equation transformation

The breakthrough in my covariational reasoning took place because I perceived my ability to think smoothly and chunkily developed. Initially, I found any example of smooth thinking as developing. I found any example of smooth thinking as significant, but as the study progressed I found I needed to challenge my covariational thinking. When applied to rational equations, I opened my mind to alternative examples to transformation. I began to consider if there were examples of smooth and chunky thinking in advanced mathematical thinking.

Advanced mathematical thinking. In this study, I provided students with opportunities to develop advanced mathematical thinking and as expected, there were more exemplars of elementary mathematics than advanced mathematics because advanced mathematical or formal thinking takes place in advanced-courses of mathematics. To clarify, “elementary mathematics *describes* objects, whereas advanced mathematics *defines* objects. In both cases, language is used to formulate the properties of objects, but in elementary mathematics, the description is constructed from the experience of the object . . . in advanced mathematics, the properties of the object are constructed from [concept] definitions” (Tall, 1995, p. 7).

An individual transitioning from elementary mathematical thinking to the advanced mathematical thinking depends less on physical constructs, such as visuo-spatial imagery and *concept images,* and more on *concept definitions*. Elementary mathematics does not refer to mathematics learned in elementary school. For example, I am referring to visualizing end behaviors with language as increasing and decreasing. Although not stated in the following vignette, I perceived the use of language as elementary mathematics because increasing is a word that students relate to temperature increasing or walking up a hill that increases.

> T: Given an exponential growth function [I drew on the board.]. Is the graph increasing or decreasing?
>
> S1: Increasing
>
> T: Describe the end behaviors of an exponential growth function. As x increases?
>
> S Choral: y increases [covariational]
>
> T: As x decreases?
>
> S2: y approaches zero [covariational]
>
> T: . . . As x increases y increases and as x decreases y goes to zero. It approaches zero. But more importantly, I think for your notes you want to say it approaches zero, the asymptote. Does that make sense? It is approaching the asymptote. If I shifted it [the graph] down it would still approach the asymptote. Wherever that is does that make sense? . . . When they [Algebra Nation, 2017] applied this in our homework, instead of x increases, they said x goes to positive infinity and y goes to positive infinity [I still considered this elementary mathematics thinking]. What about this?
>
> S3: As x goes to negative infinity, y goes to zero.
>
> T: Does that make sense? (Research log, 2/27/18)

To encourage advanced mathematical thinking, I asked the students to use symbols to represent the end behaviors a concept that students did not connect to their life, as in the following vignette.

T: Let's talk about the end behaviors. As I go in this direction [right], I am going in the positive infinity direction. As $x \to +\infty$, what is going on with my y?

S1: It also goes to $+\infty$.

T: So, as I go this direction [left], I'm reading it from this direction, as I going toward $x \to -\infty$ what is going on with my y?

S2: y approaches zero.

T: Excellent. Are we good with the direction of what it means to approach zero? Ok . . . Is everyone ok with understanding the y approaching zero? Yes?

S3: How do you write it?

T: Oh, you write as $x \to +\infty, y \to +\infty$ [advanced mathematical thinking]. Does that make sense? Because this is a big idea, you will need to understand that in calculus. Where is it going, toward? That is the big idea. No, that [the asymptote] is an imaginary line you can't cross . . . I wanted to clarify what it means for y to approach zero. I do not want you to think that there are only three choices for y to approach zero, goes to $+\infty$, or goes to $-\infty$. [I asked the students what they thought would happen with $f(x) = 2^{x+3} - 3$.] Anyone have a guess?

S1: Shift it down 3.

T: Very good. So, my asymptote that was at 0 is now shifting down to -3. Here is my question as $x \to -\infty$. $y \to \cdots$? Nope [I could not hear response during transcription.], go ahead in the back. You realized your mistake it approaches, what? Go Head.

S2: It shifted down to -3.

T: Go ahead, yeah?

S2: It approaches -3. (Research log, 3/2/18)

I perceived these vignettes as examples of asking Algebra I students to use advanced mathematical thinking. The students explained end behaviors using symbols, referenced the asymptotes, and then described the shift of the asymptote. The students applied transformational shifts of a quadratic function to exponential function that we had not covered. When teaching end behaviors to students, I began by connecting a concept that the students related—increasing and decreasing—and then moved to a symbolic representation. However, I am not suggesting that I need elementary mathematical thinking to support advanced mathematical thinking. I am suggesting that to encourage advanced mathematical thinking in Algebra I students, they need to make connections to prior knowledge.

In another exemplar encoded as advanced mathematical thinking, I observed that I merged multiple steps during the solving the process when teaching a topic, as in the following vignette. I perceived that if a student could hold multiple steps in their mind that they were using conceptual understanding to solve the problem. When I refer to what comes next, I am not asking the student to think procedurally, but to communicate organized thinking.

T: This problem is a bit long. It is going to give you an extraneous solution. $x + 1 = \sqrt{7x + 15}$. So, what would you do first?

S1: I am going to square both sides.

T: To make this side easier, but what happened to this side?

S1: It got harder.

T: $(x + 1)^2 = (\sqrt{7x + 15})^2$ Now is when your factoring skills come into play because if you are really good, you could just look at it and know what you get? $(x + 1)(x + 1) =$

$7x + 15$ Then, $x^2 + 2x + 1 = 7x + 15$

This is what your Algebra II teacher is probably going to do. They will go from this step to this step and skip the middle step . . . Next step, what am I going to do? Now look, I want to solve for x. If I have a trinomial, what do you think you are going to have to do?

S1: Factor

T: Actually, you just need to bring all of the terms to one side.

$$\begin{array}{rcl} x^2 + 2x + 1 & = & 7x + 15 \\ -7x - 15 & & -7x - 15 \end{array}$$

If you want to do it in one step, then do it in one step. If you want to do it in two steps, then do it in two steps. $x^2 - 5x - 14 = 0$ Now, I am going to think of my two factors. Two numbers that multiply to equal -14 and add to equal -5.

S2: 7 and 2

T: What would the signs be?

S3: -7 and $+2$

T: $(x - 7)\ (x + 2) = 0$

$x = 7 \quad x = -2$

OK, now you have to check you work. How do you check you work?

S4: Use foil

T: Nope

S5: Plug it in.

S6: Where do you plug it in?

S7: Into the original equation

T: $7 + 1 = \sqrt{7(7) + 15}$

$8 = \sqrt{64}$ true

$-2 + 1 = \sqrt{7(-2) + 15}$

$-1 = \sqrt{1}$ Not true $x = -2$ is an extraneous solution. (Research log, 5/16/18)

At first, I thought this vignette represents procedural knowledge because of the phrase, “what is the next step?” In actuality, I expected students to use conceptual understanding as the students used their prior knowledge to identify the extraneous solution. My intent was to see if the students could explain the cause of the extraneous solution, as I discuss in the following vignette.

T: Where did the extraneous part come from? What step would have generated an extraneous solution?

S8: Factoring

S9: The squaring

T: The squaring generates it. Cubing or an odd exponent does not generate it. When you square something, say -8, I am always going to get a positive when raising something to an even power and that can sometimes create an extraneous solution. I did all the steps right. But when I plug the solution back into the original equation, it did not work [It was not a true statement.]. Always be wary if you are squaring a radical; when you square the radical, it changes the dynamics. You always want to be aware of those extraneous solutions. Did you actually follow along with what we did?

S Choral: Yes. (Research log, 5/16/18)

As the students solved the equation, I encouraged students to explain the development of their thinking in real time. There did not seem to be smooth or chunky covariational reasoning in my thought process. I believe this is because the content was not as familiar. In the vignette encoded as *advanced mathematical thinking*, the substantive categorical themes that surfaced were my use of language and tone to project a scholarly expectation to my students. I used a less lighthearted tone to communicate a more advanced level of mathematical thinking. There was a sense of seriousness that did not occur in the day-to-day Algebra I lectures in my classroom. I saw myself as merging procedural knowledge with conceptual understanding when solving the problem, because that is going to be expected of them in more advanced-level courses.

Process thinking. Another breakthrough in my reconceptualization of my smooth and chunky reasoning occurred at the end of the study. In the following narrative, I asked myself if it were possible for students to graph a completed inverse equation, then it would be smooth thinking. I further questioned whether my thinking of the "undoing" process of solving equations was smooth thinking.

> I want students to practice graphing inverses. Today, I seemed to focus on variational [correspondence approach and coordination of values] reasoning. But I want to figure out a way to reason in a smooth covariational way. Is it possible that if I have the students graph a completed equation, it would seem to be smooth thinking? Until now, I have depended on a transformational shift to think smoothly. Not a continuous process. When I am thinking about "undoing" the process [of inverses], is that smooth thinking? I am going from here to there and back again. I am thinking of a process. I am thinking of completed change. Therefore, it would not be smooth. It would be chunky [completed change]. But if I am considering the inverse process going on forever, that is smooth thinking. I apply the logic of process to Rover in Figure 16 to graph a line. I consider if Rover would roll up the axis to the y-intercept and then turn at a specific degree corresponding to the slope. This allows the students to acquire an understanding of slope in motion rather than slope as rise over run. (Provoking thoughts, 5/22/18)

Figure 16. Rover a car that attaches to a programmed graphing calculator

However, it was the dialogue with my critical friend that reinforced the realization of process in my mind. It was during our dialogue that I began to realize that my interpretation of process had changed. Initially, I thought of "process" as going from a starting point to there (a fictitious point). The fictitious point was never reached. Castillo-Garsow (2014) explained that smooth thinking continues on without bound and if you thought about it as a chunk of time — whether small or large — it was still a chunk. As I continually analyzed the evolution of my practice, I witnessed and allowed for a broadening of my framing of smooth and chunky thinking far beyond Castillo-Garsow's original conceptions. When we solve an equation, I am able to block out the world as I solve the problem. I do not think of the problem as steps, but a continuous flow of thought. Your thought process becomes seamless.

At the outset of this study, I understood *process* as procedural knowledge. It was not until the completion of the data collection that my understanding of process changed to thinking of it as a continuous fluid motion because it is automatic. As shown in the following vignettes, my conception of process thinking focuses on *who* initiates procedural knowledge. In the following lesson, I asked the students to find the solutions to the equation $\frac{12}{3x^2+12x} = 1 - \frac{1}{x+4}$. My words suggested that my thinking about how to use the solving process was the most efficient method. However, this conclusion does not suggest that I expect the students to replicate my thinking process.

> T: Now, let's do a more complex example . . . The very first thing you should do [suggesting my way] is if something is factorable, then find the factors. What is my factor here?
>
> S1: $3x(x + 4)$
>
> T: Right. Notice anything interesting?
>
> S1: There is another $x + 4$.
>
> T: What you want to do is multiply each of these terms by the same expression, and you want to be able to cancel out each of the terms in the denominators. What would be the two factors in the denominator? What would be the two factors I want to cancel out?
>
> S1: $x + 4$
>
> T: And a?
>
> S2: a $3x$ and a 1, but 1 is just a 1, and it does not change anything when I cancel it out.
>
> T: I am going to multiply every single term by $3x(x + 4)$ . . . Personally, I like to re-write each one so I can actually cross things out. I like to re-write it this way. [I am suggesting that this is my thinking process]. $3x(x + 4) * \frac{12}{3x^2+12x} = 1 * 3x(x + 4) - \frac{1}{x+4} * 3x(x + 4)$ Does everybody see what I am doing? Does that make sense? Now, I am going to cancel out. (Research log, 5/14/18)

The words "I like to do it this way" suggest that students should use the order of my thinking process and then possibly understand the function of my thinking process. However, this vignette did not specifically imply the students must solve the problems using the same process. Ideally, I would expect students to be able to multiply the equation through by the least common denominator $3x(x+4)$. The students might write $3x(x+4)\left[\frac{12}{3x^2+12x} = 1 - \frac{1}{x+4}\right]$; However, my experience with Algebra I students is that they find multiplying through by the least common denominator challenging because they cannot physically cancel out terms. I suggested to students that my way allowed them to cross terms out, but my past experience with solving rational equations was that some students made careless mistakes when they attempted to multiply through by the least common denominator. In the following lesson, the phrases I use to question changed from "The first thing you should do" to "What do we do first?" This type of re-phrasing suggests to students that I expected them to formulate their own process.

In the following vignette, the students explained the solving process that offered them the opportunity to elevate the conceived level of difficulty to advanced mathematical thinking.

T: Here we go. $\frac{1}{7}(x+9)^{\frac{3}{2}} = 49$ What do we do first?

S1: I first want to get rid of the $\frac{1}{7}$. Multiply both sides by 7 or 7 over one.

T: $(7)\frac{1}{7}(x+9)^{\frac{3}{2}} = 49(7)$ You can handle this [in] two different ways. What does it mean if I have something raised to the $\frac{3}{2}$?

S2: You square root it.

T: And?

S3: Raise to the power of 3 inside.

T: I did it this way. I raised everything to 3. $(x+9)^{\frac{3}{2}} = 343. (\sqrt{x+9})^3 = 343$ [Advanced Mathematical Thinking] Now, is that what you were trying to tell me? I want to get rid of something raised to the third power. I am going to?

S4: Cube root the cube.

T: Everybody see. $\sqrt[3]{(\sqrt{x+9})^3} = \sqrt[3]{343}$ So, I end up with the cube root of a cube and they cancel each other out, leaving me with a square root. $\sqrt{x+9} = 7$ [Because the bell was going to ring, I finished the problem.] To get rid of the square root, I am going to square both sides. I am going to get $(\sqrt{x+9})^2 = (7)^2$. I am going to get $x + 9 = 49$. Subtract 9, which equals $x = 40$ (Research log, 5/16/18)

Although I provided leading phrases, the previous vignette demonstrated my expectation that students would verbally explain the solving process of a problem they have not seen before, such as when I asked, what do we do first? I asked pointed questions to guide their response; ultimately, however, the students themselves were responsible for the solving process demonstrating procedural knowledge.

In the exemplars coded as *process thinking*, two main substantive themes emerged in the vignettes. One theme that emerged was my use of phrases that implied that my thinking progress was the best, whereas another theme that emerged was my use of phrases that implied the students should formulate their own processes. By suggesting that my way was the best, I did not give the students the opportunity to develop their thinking about the solving process. When I changed my phrases and, as a result, the expectations, the students took the initiative to formulate a solving process of their own. Then, they verbally explained their thinking about that solving process that I perceived as demonstrating conceptual understanding because they were using mathematical thinking and reasoning. To me, I perceived the student who thought, “What do I do next?” as demonstrating procedural knowledge and chunky thinking because they were using discrete, step-by-step procedures.

In the following narrative, I made the connection between inverse functions, process, and covariational thinking.

> I discussed my thoughts about the thinking process as covariational. My critical friend asked me when the thought came. I explained that the thought came to me as I was considering the inverse function. I know inverse is the undoing process. But I never really thought of the thinking process of solving. Yes, I thought of the order of operations [demonstrating procedural knowledge], but they were steps and not a continuous flow of thought . . . As we continued, I again went back to Rover and the process. What would Rover want me to do? [I am pushing myself to develop a conceptual understanding of the process.] My critical friend asked me, "What made me think of the process for Rover? It was a good question considering I taught inverses last year. Initially, I used Rover and did not think of process. I said that I needed "a spark." I connected an idea I was teaching and I took the two ideas and connected them. I was excited about this finding. I was reconceptualizing my thinking about process. I said, "I think about equations as a process." (Critical friend memo, 5/25/18)

In the dialogues from the critical friend memos and provoking thoughts, the reconceptualization of my smooth and chunky thinking evolved. I began by doubting whether I thought smoothly and then shifted to the realization that I thought smoothly more than chunkily. Next, I went from a compartmentalized interpretation of smooth and chunky thinking to realizing that I move in and out between them, followed by the realization that smooth thinking actually occurred with transformational shifts of graphs and solving algebraic equations. This idea of solving algebraic equations using process thinking led to the idea of process and Rover. When I was re-enacting the movement of Rover's wheels in my mind, I asked myself, "How it would move to graph a line? If I wanted to draw a line, $y = 2x + 5$, what would the process be? As stated in one exemplar, I envisioned Rover rolling to the y-intercept and then I asked myself, how do I move in a way that reflects the slope? Rover cannot move rise over run. [I am referring to the vertical and horizontal components of slope.] Then, it hit me. The slope was a degree." (Provoking thoughts, 5/22/18). This was my thought process in explaining how to graph a line as smooth thinking.

Results from Narratives

In the prior section, I used the evidence from the research logs to support the provoking thoughts and dialogues from the critical friend portfolio that referred to the organizational categories of smooth and chunky reasoning and discussed substantive categorical themes that emerged. In this section, I use the evidence from the research logs to support the narratives that depict the transformation of the organizational categories of smooth and chunky thinking by reflecting on emerging substantive categorical themes. Additionally, this section explicitly identifies additional organizational categories that surface during the coding process and the substantive categorical themes that emerge.

Making Connections. The narrative process began by discussing with my critical friend the importance of making connections in reconceptualizing how I thought about how I taught algebra. The following is an excerpt from the first narrative on the subject of the organizational category of connections.

> *Me to my critical friend:* I noticed yesterday that there were many times I was trying to help my students make connections. These connections may be new knowledge compared to prior knowledge, but the connections may also be new knowledge compared to newly created analogies. [I try to connect mathematics to the real-world using analogies.] We discussed that these connections or anchors are part of my teaching style. I really did not see that connections had a place in my self-study analysis. I thought that I

make connections to help my students understand concepts. I explained that two of my past students stopped by to visit. As we spoke, I suggested they should memorize the unit circle to help them make connections with the degrees, graphs, and ordered pairs as their teacher explains lessons. I explained I memorized trigonometry without making any connections. I seemed to just memorize everything. As I was coding tonight, it hit me why I am trying to make connections. It is not only to connect ideas for my students, but so I could connect ideas as I am teaching to help me understand the concepts. (Narrative, 5/30/18)

This narrative identified one emergent theme of why I perceived connections to be important to understanding the content taught. The second substantive emergent theme is that making connections is not only essential to teaching for understanding, it is also more efficient – and efficiency is what teachers appreciate, because the pacing of the algebra I curriculum is dense.

Anchors. One way that I help students make connections is with anchors (organizational category). *Anchors* refer to linking mathematical concepts to prior knowledge. During the coding process, two substantive themes emerged that correspond to realizations that I used two types of anchoring in my lessons: (1) I anchored ideas to students' knowledge of non-mathematical concrete objects; and (2) I anchored ideas to the students' existing knowledge of algebraic concepts that had been recently taught. In the following lesson, my objective for the students was to anchor the abstract, mathematical idea of *conic sections* to their physical representation. In class, the students shifted the sand in an hourglass to visualize a conic section.

T: Is the hourglass giving you a good visual?

S1: Yes

> T: Here we go. The definition of a *conic section* is a figure formed by the intersection of a plane and a right circular cone. A conic section can be in the form of a circle, an ellipse, a parabola, or hyperbola. Can everybody visualize the sections now? I showed you the image and handed around the hourglass. I think sometimes it is not easy to see conic sections when looking at a picture. Are there any questions about the shape? (Research log, 5/24/18)

The hourglass provided the students with a visual anchor for the abstract concept of conics. Although not explicitly stated in this vignette, each of the students spent a different amount of time interacting with the hourglass as they attempted to create each of the conic sections of the circle, ellipse, hyperbola, and parabola by tipping the hourglass and shifting the sand.

Another type of anchor that emerged during the coding process was the substantive category of anchoring new concepts to other concepts that I had recently taught. Ausubel (2000) believed that the "cognitive structure of relevant anchoring ideas, their stability, clarity, and discriminability from related internalized ideas [thinkable concepts] . . . [and] are the most prominent factors that influence meaningful learnability, the degree of learning, and retention of new potentially meaningful instructional material" (p. xi). Those whose cognitive development is based on stable prior knowledge became meaningful learners. These new concepts were not necessarily stable (re-creatable) ideas (e.g., transformations of graphs) because they were newly taught. As a result, anchoring to abstract concepts occurred more often than anchoring to physical objects. I seem to link the teaching of new algebraic concepts to abstract concepts more often than physical objects because I do not link algebraic concepts to physical objects in my mind. In other words, I have few examples of connecting algebraic concepts to physical objects in my mathematical toolbox.

In the following vignette I anchored a newly taught concept to the prior knowledge that any number divided by itself equals 1 (except 0). Although this anchor was not actually physical, it provided students with what I perceived as a stable anchor. As the lesson progressed, the students realized that the same factor existed in the numerator as in the denominator, which canceled them out. I perceived that by anchoring to prior knowledge that I was developing conceptual understanding.

T: First, what is $\frac{2}{2}$?

S Choral: 1

T: What is $\frac{3}{3}$?

S1: 1

T: Good, What is $\frac{x+4}{x+3}$?

S1: $\frac{4}{3}$

S2: $\frac{3}{4}$

T: What else?

S3: 1

S4: $\frac{4}{3}$

T: The people who said 1. Did you subtract? $4-3=1$

S4: No

T: Where do you get 1?

S5: Oh, because for the other ones you said 1?

T: . . . Let me ask you this question. What if I asked what $\frac{(x+4)}{(x+4)}$ equals?

S6: 1

T: Because?

S7: It is the same in the numerator and the denominator.

T: Good, OK. You can't break these [I pointed to $x + 4$] up. Think of these as a chunk of knowledge [abstract anchor] if you will. This chunk and this chunk [are] the same, [identical]. Therefore, anything over itself equals 1 . . . What do you think $\frac{(x+4)(x+3)}{(x+3)}$ equals? [An effective process for the compression of knowledge is chunking similar concepts. Thurston (1990) remarks that mathematics is *compressible,* which means that you can replace many ideas and think of them as one complete idea.]

S8: $x + 4$

T: Why do you think this equals $x + 4$?

S9: Because the $x + 3's$ cancel out, then $(x + 4) * 1 = x + 4$

T: Does that make sense? (Research log, 5/18/18)

In both vignettes coded as anchors, I realized that these anchors exhibited different features of anchors. Abstract anchors, as I used the code, refers not only to mental images — concepts held in the mind (a whole can be understood as the ratio of a number to itself), it also refers to anchors when they are concretized into visual representations (e.g., a transformation of a graph written on the board). While this framing of abstract is not the typical one (e.g., Bruner, 1966), it is the one I have settled on as most useful in my thinking about the value of multiple mathematical representations in learning. One vignette represented the students with a visual anchor that they could see and touch, whereas the other was an anchor to existing thoughts that are retained in our mind. Another example appeared in the earlier vignette of a student verbally stating that "$f(x) = \sqrt{x}$ is equivalent to $f(x) = x^{\frac{1}{2}}$. S1: Oh, yeah" (Research Log, 3/12/18).

The student's response suggested that they had previously anchored the equivalence of the square root and raised it to the half power in their mind.

Advance organizers. Another organizational category that I identified was making connections using advance organizers. *Advance organizers* refer to those organizers given prior to teaching a concept to assist students in learning the material. The advance organizer illustrated in Figure 17 offered students four methods of dividing polynomials: the division of two polynomials in fraction form, long division without a remainder, the remainder theorem, and synthetic division. By giving the students the organizer in advance, I was preparing the students to make connections between the four different methods of dividing polynomials. I began the lesson by explaining the objective of an advance organizer.

> We are going to start with what is called an advance organizer . . . I wanted you to see the connection between all of the different ways we can divide polynomials. The advance organizer is a nice way of putting all the information on one piece of paper. I am not going to get into the specifics. I just want to explain that we are going to divide polynomials by breaking down the numerator and denominator into factors. We are also going to use long division. (Research log, 5/10/18)

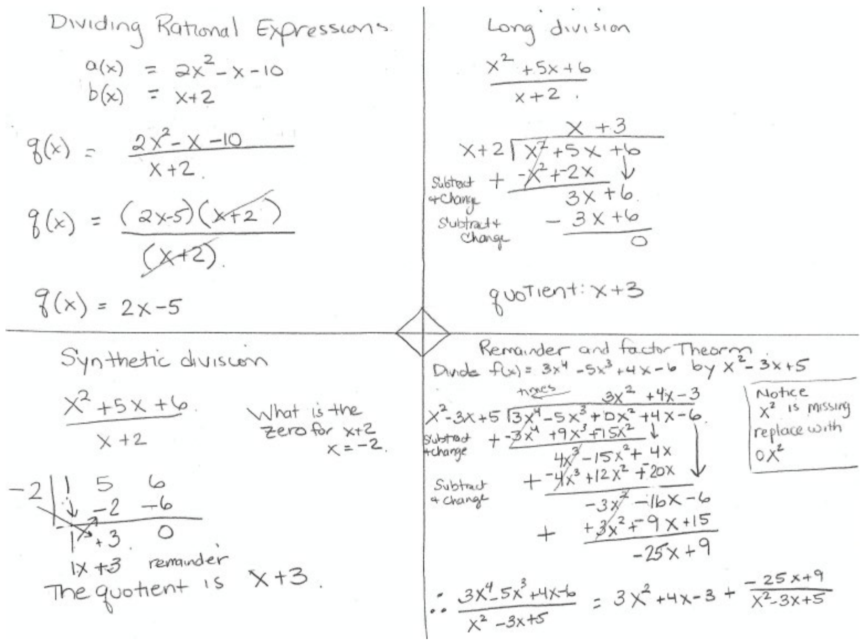


Figure 17. Division of polynomials advance organizer. (Research log, 5/10/18)

In another lesson, I distributed the advance organizer on the left of Figure 18 to the students. When I began using the *rational function advance organizer,* I realized that the complexity of the advance organizer overwhelmed both the students and me. Therefore, I decided to use the graphic organizer in Figure 18 on the right to facilitate our thinking. Both the students and I found the simplicity of the graphic organizer to be more helpful and less overwhelming than compiling all of the most important information. In the following vignette, we find the x-intercepts.

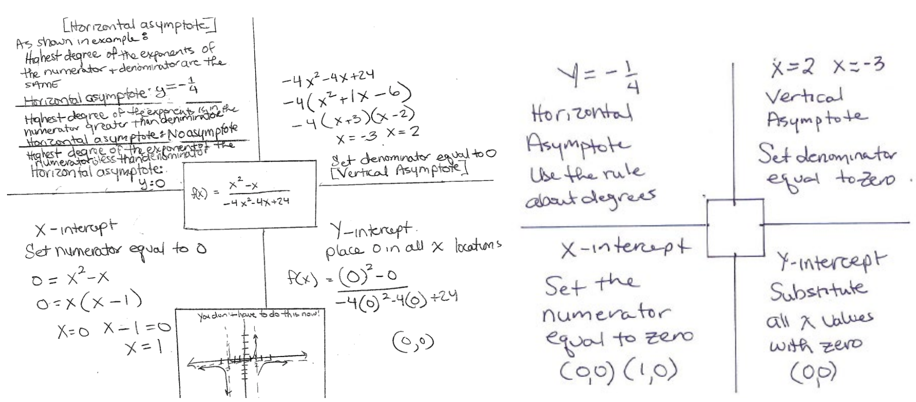


Figure 18. Advance organizer compared to a graphic organizer

$$f(x) = \frac{x^2 - x}{-4x^2 - 4x + 24}$$

T: Now, we are going to set the numerator equal to 0. We have $0 = x^2 - x$. What do we do first?

S1: Factor out the x.

T: This leaves me with $0 = x(x - 1)$. So, $x = 0$ and $x - 1 = 0$. Then, $x = 1$. When you write the x-intercepts, write them as ordered pairs in the graphic organizer. You have $(0,0)$ and $(1,0)$. Those are your zeros . . . It is ok to have $x = 0$ and $x = 1$ because we are solving for x and we know the y is 0. [I point to $x = 0$]. Are we good?

S2: Yes

T: . . . Do you see the value of the graphic organizer now? How many of you think this is helpful? [Most raise their hands.] If I could remember all the steps from the advance organizer, I would use it. I struggle as I am going [I make an over and over motion with my hand] over every single one. I have to keep referring to the advance organizer. Something is really nice about the graphic organizer. It's simple.

S3: Simple

T: We say at the same time. (Research log, 5/21/18)

In general, I realized that the goal of an advance organizer or a graphic organizer is to provide consolidated information and to facilitate the making of connections across representations. The moment that the student and I both simultaneously said that the graphic organizer was "simple" reinforced my view of the way we were actually using the graphic organizer. The response suggested that I was using the advance organizer to encourage procedural knowledge because I was referring to how easy it was to follow along. By consolidating procedures related to rational functions into the advance organizer, I may have inadvertently taken a tool to develop conceptual understanding into a tool to develop procedural knowledge.

In the following narrative, my critical friend asked me the importance of making connections.

> *Critical friend to me:* Do you think the connections you're making are changing the way you're teaching the particular concept? How are the connections you're making influencing your teaching practice? (Narrative, 6/2/18)

Answering these questions required me to think of my use of the word *connections*. More importantly, I needed to consider the purpose of making connections in my classroom. Additionally, I referred to the organizational categories of new connections and past connections.

> *Me to critical friend:* I referred to the organizational categories of new connections and past connections. Past connections refer to connections based on my past students' met-befores. [I find an inaccurate met-before presents a challenge for myself. I may be able to reconceptualize how I can teach a concept differently (using accurate met-befores) merely through discussions with my colleagues, but the students' mistaken met-befores present a greater challenge to me. I can construct models of my students' thinking, but these are second-order models (Steffe & Thompson, 2000) constructed by someone other than the model holder); thus, I am not certain. I need to determine if the student's challenge is a result of an inaccurate met-before.] As I am teaching, I use past met-befores as a toolbox of common algebra interventions for misconceptions. I am using these interventions as teaching points during a lesson … The other connections that I refer are new connections. These are based on class discussions. These I would call the "aha" moments. I never thought of looking at new connections that way before. I see

new connections as future teaching points or interventions. I see these new connections as interventions and the result is a transformation of my practice. (Narrative, 6/5/18)

The substantive themes that emerged from the narrative above reveals that the organizational category of past connections developed my students' algebraic thinking, correct misconceptions, and build a stronger mathematical foundation.

Personal history from my past. I use the phrase "personal history from my past" to refer to my teaching practices when I reflected upon my past experiences teaching algebra. The substantive categorical themes that emerged revealed that my past experiences with mathematics influenced my teaching strategies and my openness to share the strategies that I developed as a child to overcome my struggles with mathematics. In the following vignette, I state the importance of labeling ordered pairs when using the point slope form.

T: The problem asks us to find a slope and a y-intercept. [The problem provided two ordered pairs on a line and did not number the x or y-axis.]

S1: We have both [ordered pairs]; we can just write it. $y = -\frac{1}{3}x + 2$.

T: I did not even think about using the slope intercept form. That was a good idea. I was thinking of the point slope form . . . Let's practice the point slope form just to review. Point slope form says I need a point and slope. The formula would be $y - y_1 = m\ (x - x_1)$. It was funny that you thought of the slope intercept form when I thought of the point slope form. . . Remember that I said to label your points. My points would be

$$\underset{x_1,\, y_1}{(-3, 3)}\ \underset{x_2,\, y_2}{(0, 2)}$$

I like to put down the labels because when I was your age I used to get them mixed up. (Research log, 4/24/18)

In this vignette, we discussed that as a child I had mixed up the ordered pairs and the fact that the strategy of labeling corrected my error. I did not expect students to replicate the strategy unless they found it to be useful. However, the language "remember that I said to label your points" suggested replication. In the following vignette, I referenced a struggle to substitute a negative number into an equation that had a negative as a coefficient:

$$f(x) = \begin{cases} -2x + 1, & x \leq -3 \\ 3x - 1, & -3 < x \leq 5 \\ x + 5, & x > 5 \end{cases}$$

-7	2
-3	7
1	13
5	14
8	15

T: I'm going to change the first equation because it is a common error that happens to students . . . The equation we are going to use is $f(x) =- x + 1$. And I want you to let $x = -5$. Ok, write what it equals, and I am going to walk around the classroom . . . Exactly what I expected happened. Many of you had the answer right, but some of you didn't. I am going to model the common error. Most of the common errors are the errors I made when I was your age. I have a negative and I want to substitute it into my x [I make a box] plus one. I take my x, the negative 5 and place it in the box. This becomes what? Positive 5 plus 1 and that equals 6. Does everyone see why I went over this question?

S1: So, what did you do again?

T: [I repeated what I said].

S2: Can we say $-1 \cdot -5 = 5 + 1 = 6$.

T: Absolutely. Perfect. That is what I hoped you would know, then, I would settle for it is the opposite of -5. (Research log, 3/8/18)

Although it is not explicitly stated in the vignette, I thought that if I substituted $x = -5$ into $-x + 1$ that this question would be easier, because the people who made up the problem had already put the negative sign there for me. Based on this reasoning, I coded the above exemplar with the organizational category as personal history error that I had made in mathematics in my past. A substantive categorical theme that emerged was that I conceived of my errors as negative experiences in mathematics. It is possible that I only remembered the negative experiences because it took additional time to develop strategies to resolve the struggles; therefore, I considered them worthy of remembering. Upon further reflection, my past personal history played a major role in creating new strategies to rectify my mathematical confusions I had as a child. I continue to use these strategies today both in my personal thinking and my teaching practice.

Met-befores. Another type of connection that I referenced was personal history based on met-befores. The organizational category of personal history based on *met-befores* refers to the anticipated clarification necessary for my present students based on past students' struggles with algebraic concepts. For example, if while teaching, I referenced the specific common mistakes that my past students made in solving a problem, I coded it as a *met-before*. Upon reflection, I believe that I focused on specific met-befores because they represented a personal mathematical confusion from past learning experiences that only surfaced when a student asked a question. These met-befores usually revealed themselves when I struggled to answer the question using sound mathematics. Sometimes, I came up with an unusual way to keep the concept straight in my mind and it came back to me when a student asked a question. For example, when I traveled through Pennsylvania, I noticed rolled hay. This was different from the bales we used on our farm. I applied rolled hay to inequalities. When there was an inequality that was less than or equal to, I forgot if it was a closed or opened hole. I would pretend that I rolled the line under the sign into a tight circle and stuffed it in the hole. So, I remembered it was a closed hole. Other times, as in the following vignette, I would think to myself, “Oh, I never thought of that before.” The students struggled with the placement of the negative sign to represent a negative fraction. Students did not know where to place the negative sign with a slope of $m = -\frac{1}{3}$. They perceived that placing the negative on the outside, in the numerator, or in the denominator changed the value of the fraction.

T: What would I write?

S1: It is 1 over -3.

T: You would write negative 3 because it is in the negative direction. Is everybody good with that? So, $m = \frac{1}{-3}$. Can I write $m = \frac{-1}{3}$?

S2: No

T: How come? In this case, I would have to go down one [I am starting at the y-intercept]. Can I start at [(-3, 3)] this point?

S3: No

T: I can’t?

S3: Yes, I can go down one and the positive direction three.

T: Ok, so it is the same thing. They are equivalent. Another fraction that is equivalent is $m = -\frac{1}{3}$. In past years and even at the beginning of the year, you said the 1 was negative and here you told me that the 3 was negative. Remember, it is not the numerator or the denominator that is negative. It is what?

S4: It is the fraction that is negative. (Research log, 4/23/18)

Each year, the students questioned where to place the negative sign because they believe that $-\frac{1}{3}, \frac{-1}{3}$, and $\frac{1}{-3}$ as not being equivalent. I have asked where they would place each fraction on a number line. Some students will say that $\frac{-1}{3}$ is on the left of 0, $\frac{1}{-3}$ is on the right of 0, and not until you ask where would $-\frac{1}{3}$ be on the number line, do they begin to wonder. Some will say that $-\frac{1}{3}$ and $\frac{-1}{3}$ are equivalent. Although it is not stated in the exemplar, I have noticed that students who are confused often say "1 over negative 3" for $-\frac{1}{3}$. The reference to a negative number in the denominator is an indication of this confusion. Unfortunately, this confusion goes unnoticed for many students because without sharing their thoughts during discussion, I believe that students continue to believe that fractions with negative sign in the numerator, denominator, or in front of the fraction are not equivalent.

In a different problem, a previous student's met-before from the prior year prompted me to ask this class to explain the meaning of the 1 from 1.05 in the following exponential function. This exemplar demonstrated the importance of understanding the idea of the total amount, or as stated by the student, "all of it". The students needed me to rephrase the question.

> T: Consider the exponential function $f(x) = 500 \cdot 1.05^x$, which models the amount of money in Tyler's savings account, where x represents the number of years since Tyler invested the money. What does 1.05 represent [met-before]? [The students struggled.] Let's try it another way. 1.05 = 105%. What does the 100% represent? What does the 5% represent?
>
> S1: 5% represents the rate of growth.
>
> S2: The 100% represents all of it. That would be the $500.
>
> T: Does that make sense? Another way you can look at it is $1.05 - 1$ [which is the 1 total amount] = .05 = 5%. Does anyone have a question on any of the steps of the

problem? Are we good? (Research log, 3/2/18)

The students were able to come up with the answer to the question, but when I began to ask more specific questions, I realized that they did not have a conceptual understanding of what the 1 represented. In the following vignette, I used a met-before the students had at the beginning of week with synthetic division and applied it to a new long division problem. The synthetic division met-before had a missing term in the dividend. The following vignette was missing a term in the divisor. My intent was for the students to apply their prior knowledge of replacing the missing term with a 0 coefficient. As shown, the students connected the prior knowledge to a new situation.

T: Now, I am going to talk about a possible misconception. [I am dividing the $\frac{m(x)}{n(x)}$.]

$m(x) = 6x^5 - 5x^4 + x^3 - 2x^2 - 4x + 3$

$n(x) = x^2 - 1$

I am not doing this problem in its entirety. I only want to point out two little parts.

$x^2 - 1 \overline{\big)\, 6x^5 - 5x^4 + x^3 - 2x^2 - 4x + 3}$

I am going to do a tiny little window of this problem — not the whole problem.

$x^2 - 1 \overline{\big)\, 6x^5 - 5x^4}$

Is this your first step?

S Choral: Yes

T: I will just write the next term [$+x^3$] just so it is there. I want to point something out. See how there are two terms here? Where do you think you place your first term when dividing? Right above the second term here [$-5x^4$]. What times x^2 gives me $6x^5$?

S Choral: $6x^3$

T: Ok, $6x^3$ times x^2 is?

S1: $6x^5$

T: $6x^3$ times -1 is?

S1: $-6x^3$

S2: Oh. I see what it is!

S3: Oh, Oh, you would have to put the 0 there.

T: So, if you write the placeholder, you are going to catch that mistake. I have to align it over here [I use motion to show me moving the $-6x^3$ under the x^3. How many of you here would forget to align the $-6x^3$ under the x^3? Raise your hand! How many of you think you would remember?

S4: I probably wouldn't remember.

$$\begin{array}{r|l} & \quad 6x^3 \\ \hline x^2-1 & 6x^5 - 5x^{\boxed{4}} + x^3 \\ & \underline{6x^5 - 6x^{\boxed{3}}} \end{array}$$

T: So, here is what I am going to suggest: add ox to the divisor.

$$x^2 + 0x - 1 \,\overline{\big)\, 6x^5 - 5x^4 + x^3 - 2x^2 - 4x + 3}$$ (Research log, 5/14/18)

In the exemplars coded as *personal history met-befores*, each example seemed to be mathematically specific. The met-befores were mathematical concepts with little interpretation and limited explanation. A student could perform the problem procedurally with little conceptual understanding.

There was an emerging theme that surfaced when I compared the exemplars from personal history from my past and based on met-befores. The met-befores were more relatable to the students, unlike the exemplars of my past experiences that seemed more abstract and less relevant to them. It may be that these met-befores were specific to algebra I mathematics and based on procedural knowledge, whereas my personal history past was applicable to more abstract ideas and based on conceptual understanding.

Conceptual embodied and operational symbolism. The following is a continuation of the above narrative (6/15/18) on connections—with respect to relationships.

> *Me to critical friend:* I realized that I make four kinds of connections. I have connections with the mathematics using past and new exemplars. Now, I realized a third type of connection between mathematical understanding with conceptual embodiment using visualization and motion and operational symbolism that are built on actions or conceptual embodiment [I describe conceptual embodiment and operational symbolism in the paragraphs that follow the vignette.] The idea of connections is exciting. I realized I am developing my mathematical reasoning through the back and forth representations of conceptual embodiment and operational symbolism thought.
>
> The fourth connection I am noticing is the back and forth between chunky, smooth, variational [correspondence approach and coordination of values], and covariational thinking. When I began this study, I read Castillo-Garsow et al. (2013) who questioned whether starting with smooth covariational reasoning built a stronger chunky covariational reasoning. I realized in my own thinking that this is not the case for me. It is the moving back and forth between variational and covariational and smooth and chunky thinking that developed my mathematical reasoning. Smooth, chunky, variational, and covariational thinking is not the compartmentalized perspective I held at the beginning of this study. Yes, I can think any of the four ways, but it is the flexibility of moving amongst variational, covariational, smooth, and chunky thinking that develops

> or strengthens my awareness of my thinking. At one moment I might be thinking covariationally and then bounce into variational thinking. (Narrative, 6/1/18)

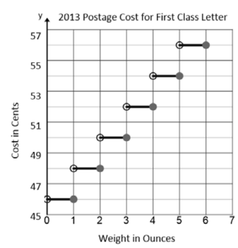
This narrative illustrates the fact that I had become aware that I could not compartmentalize smooth, chunky, variational (correspondence approach and coordination of values), and covariational thinking. Initially, I interpreted the smooth and chunky relationship to be that I think one-way and then the other. However, I found in this narrative a concept that Tall (2008a) refers to as “blending” and Castillo-Garsow et al. (2013) terms as a “pause.” This blending suggested that the lines between smooth and chunky thinking were blurred and if I stayed in the area of blurriness, then my thinking was integrated and at its strongest. This idea of a blurred line also held true for *conceptual embodied* (embodied) and *proceptual symbolic* (symbolic) cognitive development, which I explain next.

Embodied refers to Tall’s first world of mathematical cognitive development from the physical senses — visuals, motion, and movement. In the embodied world, Tall (2005) explained that we reflect on “our senses to observe, describe, define, and deduce properties developing from thought” (p. 24). Symbolic refers to Tall’s second world of cognitive development. In the symbolic world, Tall (2013) explained that mathematical procedures grow out of physical actions. Actions (such as pointing and counting) switch back and forth from *process* to do mathematics to *concepts* to think about mathematics. The pinnacle of mathematical cognitive development is axiomatic formalism or advanced mathematical thinking. Formalism is built on properties deduced by mathematical proof.

The following vignette represents the organizational category of embodied world with movement. Furthermore, I connected motion to smooth covariational thinking. It was not until I had completed the coding process that I reconceptualized my smooth covariational thinking in relation to the cognitive development of embodiment.

> "What will happen if my *h* is 3? [Pause, while students work] You've got it right. So, my vertex is (3, 2)." I had the students practice moving the absolute value function using their hands in the different directions [to represent the shifting of the absolute value function]. (Research log, 3/6/18)

In the next vignette encoded as embodied, I used hand motions to represent the step function. I thought that using the words in addition to visual hand motions of a step function in unison would create a concrete experience that would support students' abstractions of more formal ideas – a transition from embodied to symbolic.



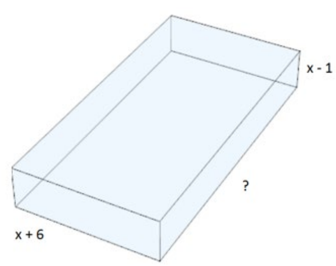
> T: [The students answered questions about the number of intervals, whether the graph represents a function, and whether or not it is linear or nonlinear.] Intervals are [using my thumb and forefinger to represent an interval] one interval, two intervals, and three intervals [embodied].
>
> S Choral: Oh [The students said in unison.]
>
> T: Are the pieces of this piecewise function linear or nonlinear? Do you know what is meant by linear?
>
> S2: No
>
> T: It means that it is straight [I am making a horizontal line in the air using my hands back and forth]; it could be this way [zero slope]; it could be that way [positive slope]; it could be this way [negative slope]. And a step function is flat. If it is nonlinear, it means

it curves. [Showing a curve with my hand] It could be quadratic, exponential [using my hand with a dramatic swoop for exponential] . . . (Research log, 3/8/18)

When I initially coded examples as *symbolic*, I conceived the examples to be more "advanced" mathematics because of the symbolism. As I re-read my transcriptions, I realized that the organizational category of symbolic world could be integrated with the cognitive development of embodied world referred to by Tall (2008a) as *embodied symbolic*. Therefore, *embodied symbolic* is the blurred line between embodied and symbolic worlds. It became clear that there was also an element of interpretation with conceptual understanding when solving the problems. I conceived the organizational category of the symbolic world with motion or visualization as being less challenging – and therefore more accessible – because the students were engaged with motion or visual representations throughout the discussion process. I still expected discussion with conceptual understanding to ensue, as in the following example.

T: Given $V = x^3 + 13x^2 + 34x - 48$ that represents the volume.

Find the factor of the remaining side.



Would you say $x = -6$ here. Wouldn't you say $x = 1$ here. Good. Let's use those as the outside numbers. First, I am going to use 1. What are my coefficients?

S1: 1, 13, 34, and -48

T: Perfect. Bring down the 1. What do I get here?

S Choral: 1

T: What do I get?

S1: 14

T: What do I get?

S Choral: 14

S1 and S2: 48

S1: Then, you get 48.

T: And, then 0. Ok. Now look. Let’s put it this way. Think about what they are saying or telling you. The length of this side is $x + 6$ and this is $x - 1$ deep. I have to get a 0 here because I am using the 1 they gave me. This 1 should also give me 0. Here is where it is fun. These are my new numerators. In other words, this is saying $x^2 + 14x + 48$ is my next factor. I am going to find the factors of this. I am going to write a negative 6 on the outside.

$$\begin{array}{r|rrrr} 1 & 1 & 13 & 34 & -48 \\ & & 1 & 14 & 48 \\ \hline & 1 & 14 & 48 & 0\ \mathit{remainder} \end{array}$$

$$\begin{array}{r|rrr} -6 & 1 & 14 & 48 \\ & & -6 & -48 \\ \hline & 1 & 8 & 0\ \mathit{remainder} \end{array}$$

I am going to use 1, 14, and 48 as my new coefficients. Bring down the 1. What do I write here?

S1: -6

T: What do I write here?

S1: 8.

T: What do I write here?

S2: -48

T: This gives me a remainder of 0. The other factor is $x + 8$. There you go. Did I lose anybody at any spot? (Research log, 5/16/18)

Although not stated in the vignette, in two of the classes, the students looked at the visual of the box and automatically knew the length of the side was $x + 8$ because they said $(6)(8)(-1) = -48$. As the lesson was about, synthetic division, I did not encourage multiplying the constants to find the last side. I perceived this strategy of finding the third side as guess and check. The process of synthetic division provides students an opportunity for procedural knowledge. However, my intention for providing the box was for students to connect the volume of the box and the use of synthetic division to build the conceptual understanding that when given a volume question in the future the students might break the volume down into factors. This breaking down can be performed by synthetic division.

In the next exemplar, I developed a process of using a box and parenthesis as a holding cell for the composite function because some of the students struggled with the variables $f, g,$ and x. By replacing the x with a box, the students did not end up with an extra x.

$f(x) = 4x - 1$ $\qquad\qquad$ $g(x) = \frac{1}{4}x + \frac{1}{4}$

What we are going to do is put one inside of the other. We are going to find:

$f(g(x))$ $\qquad$ and $\qquad$ $g(f(x))$

$f(g(x)) = 4\boxed{} - 1$ $\qquad\qquad$ $g(f(x)) = \frac{1}{4}\boxed{} + \frac{1}{4}$

$= 4(\qquad) - 1$ $\qquad\qquad$ $= \frac{1}{4}(\qquad) + \frac{1}{4}$

What do we put inside of the parenthesis? $g(x)$ and the other $f(x)$

$= 4(\frac{1}{4}x + \frac{1}{4}) - 1$ $\qquad\qquad$ $= \frac{1}{4}(4x - 1) + \frac{1}{4}$

When I distribute, I get?

$=1x + 1 - 1$ $\qquad$ $=x - \frac{1}{4} + \frac{1}{4}$

$= x$ $\qquad$ $= x$ (Research log, 5/22)

When the students were using the boxes, I perceived some were using a step-by-step process. However, I also perceived that they were in between the worlds of embodied and symbolic referred to as embodied symbolic. To me, the in-between world of embodied symbolic was more accessible than the symbolic world because I perceive my students drew from the embodied world of physical senses —visual, motion, or movement—when they used the box or parenthesis to connect the procedures in the symbolic world. Therefore, the embodied symbolic world, metaphorically speaking, is not necessarily standing on the bridge between the two worlds. It is the use of one world, in this case, the embodied world to support the other, the symbolic world. In the vignette, it was possible for some students to follow the procedural process despite lacking a conceptual understanding. Others demonstrated a conceptual understanding because most recognized that the symbols represented composite functions and the dependent variable of one is the independent variable of the other.

In the following vignette, the students reside in the symbolic world. The discussion required a more developed interpretation of symbolic representations. The students read a symbolic equation on the board and interpreted the remainder theorem based on their understanding of the symbolic representations used during discussions on long division. The students initially stated that they did not know what each symbol represented, but when they were asked one at a time about the symbols, they became more confident as became evident from the choral response.

T: Long division is written in the form of $\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$. What do you think $q(x)$ represents?

S1: Quotient

T: What do you think $r(x)$ represents?

S Choral: Remainder

T: What do you think $d(x)$ represents?

S1: Dividend

S Choral: Divisor

T: I heard it. Divisor

S3: Is the other $d(x)$ the dividend?

T: No, this is also the divisor. We will just call this the . . . what should I call this?

S4: The original

T: The original function, OK . . . Is everybody good with the remainder theorem? Does everyone understand I can have it written in multiple ways? Now, if there is no remainder, then this part will be truncated. (Research log, 5/16/18)

As shown in the transcription of the discussion between the students and teacher, “We will just call this the…what should I call this? S4: The original, the original function, Ok”. The students gave the symbol its own name. I interpreted this to mean that this example demonstrated the symbolic world because they were able to take an abstract concept and provide it with a descriptive name.

Each of the vignettes demonstrated the substantive categorical themes of cognitively moving in and out of the two different worlds of conceptual embodied and proceptual symbolic. When there was a blurred line of the two, Tall (2008b) referred to the phenomenon as *embodied symbolic*. I conceived the integration of embodied symbolic with interpretation of motion or visualization with symbolic interpretation as being conceptually more accessible than proceptual symbolic when the students needed to interpret an abstract concept verbally or when it was written without motion or a visual representation. Ultimately, I found that the vignettes fell on a spectrum between the two worlds and rarely lent themselves to a neat designation as one or the other.

I perceived the use of movement as more accessible when it was visual (conceptual embodied) as compared to a symbolic representation (proceptual symbolic), however as time passed, I realized that the world I resided did not dictate whether or not I used motion. I could use motion in advanced levels of mathematics. I am beginning to consider that motion is a perceptual representation (whether visual or tactile), just as physical manipulatives are.

Critical friend to me: When referring to movement, how would you define it? (6/2/18)

Me to critical friend: When I began this study, I considered using movement as a form of visualization when thinking about algebra as a less developed (child-like) understanding as compared to procedural or conceptual understanding. However, I realized that end-behaviors challenged my thinking about movement. Motion did not imply a less developed reasoning process. I consider end behaviors as advanced mathematical thinking because the end behavior topics are in algebra II and beyond. I used motion as a method to anchor or connect end behaviors to the coordinate plane. I used motion to give students a visual of an abstract idea to the symbolic interpretation of an algebraic equation. Therefore, I have developed my interpretation of sensorimotor as simply using your senses to assist in the reasoning or understanding of a concept. (Narrative, 6/2/18)

Tall based his paradigm of the “three worlds of mathematics” cognitive development as nested. When Tall referred to “conceptual embodied” cognitive development, the use of senses was not limited to that specific world. For example, a person in the axiomatic formalism world can still use visuals and draw pictures to understand the mathematics they perform. By understanding cognitive development, I am beginning to question my original perception that my thinking can be compartmentalized. At the onset of this study, I did not believe that I thought smoothly. Then, I believed I could think chunkily or smoothly. I then believed I move in and out between them. However, Tall uses the language of blurred lines to indicate that I do not exist in one world or another, but exist within the worlds. Is there a possibility of a blurred line between smooth or chunky thinking? I can only speak for my thinking, but when I must identify my thoughts at any moment in time. I am able to pinpoint smooth or chunky thinking.

> *Critical friend to me:* Are certain topics more appropriate to understand than others when the process is variational/covariational or chunky/smooth? (Narrative, 6/2/18)
>
> *Me to critical friend:* Initially, I seem to move among variational/covariational and smooth/chunky more so in graphing transformations. I fall back on variational (correspondence approach) more when I feel less comfortable with the topic as in the multiple "step" problems such as completing the square or solving inverse or rational equations. These would be the problems I think of as belonging to the proceptual symbolic world. These topics tend to provide a broader interpretation [not necessarily a discrete, step-by-step process], allowing me to switch back and forth between variational/covariational and smooth/chunky. I can see the problem from many angles or perspectives. (Narrative, 6/5/18)

The narrative process caused me to delve deeper into my beliefs about variational/covariational and smooth/chunky relationships. The narrative process required me to stop and ask where the beliefs about mathematics that I hold so strongly originated.

Living educational theory. Living Educational Theory is the organizational category that refers to beliefs that do not align with the implementations of my practice. For example, when teaching mathematics, I did not consider demonstrating solving an equation or graphing a function with a discrete, step-by-step process as encouraging conceptual understanding. My thoughts about student understanding a concept required me to teach for conceptual understanding. In my mind, I thought I needed to explain in words the topic for students to apply to a new model of a problem. Although order of operations may not be the best example of this dilemma, I did not realize that I fell back on the convention of using order of operation to encourage procedural knowledge until the concern really came to mind when I was dealing with teaching literal equations. To my dismay, as in the following vignette, I seemed to fall back on procedural knowledge and the message I sent to students when using order of operation was that the process was most important.

T: Samuel has been given the formula for an arithmetic sequence. Samuel needs to solve the formula $s = \frac{n}{2}(a + L)$ for L. We are going to solve for L. If I want to solve for L, [I wrote P E MD and AS on the board] what does this tell me if I am going from the bottom up? What do I do last? I do the parenthesis last. First, I want to do what is outside of the parenthesis. So, how do you get rid of what is outside of the parenthesis? Multiply by its ____?

S1: Reciprocal (Research log, 4/23/18)

The vignette demonstrated my use of procedural steps by following the acronym for the order of operations to solve for L. More importantly, I did not explain why I was preforming the steps. Nevertheless, I prompted students to provide guidance. This exemplar demonstrated that I taught procedurally in contrast to my educational philosophy of teaching for conceptual understanding. During the coding process, I also realized that the substantive categorical theme that emerged was that I did not always define steps consistently. Sometimes, they included a step-by-step process such as order of operations and at other times I thought in terms of an organized process. In the following vignette, I explained the mathematical logic behind the phrase of "keep change flip." We were discussing the following function, $f(x) = -4 \cdot 3^{x-1}$. I asked the students about $x = 0$ and this lead to 3^{-1}. The following discussion ensued about the definition of reciprocal.

T: Do you know if I have $\frac{1}{3}$ and I want to know what the reciprocal is, what are you going to tell me — to take $\frac{1}{\frac{1}{3}}$, how do I clear out the denominator $\frac{1}{3}$? Wow. I heard "flip it." [Now, each year I explain to my students that the reason behind what they are doing is mathematically correct. However, I never expected them to learn to replicate the method.] Now, we are back to the same idea of needing the reciprocal to clear the $\frac{1}{3}$ in the denominator. What do I want my denominator to equal?

S1: 1

T: Good. What do I multiply this guy $\frac{1}{3}$ to equal 1 $\frac{1}{\frac{1}{3}\cdot\frac{3}{1}=1}$. What is the reciprocal of $\frac{1}{3}$, what do I multiply it by to get 1?

S2: $\frac{1}{3} \cdot \frac{3}{1} = 1$

T: I also have to multiply the numerator by $\frac{3}{1}$. In a way, your explanation was kind of simplistic, but it still depended on knowing what reciprocal means. (Research log, 3/2/18)

Although I took the time to explain why we change division to multiplication and change the second number to its reciprocal, it seems that the students did not have a conceptual understanding – or at least they did not bring it to bear on the problem. They seemed to fall back on their procedural knowledge of "keep, change, flip." I had to ask myself whether procedural knowledge is enough in some cases or is conceptual understanding more important? These are the moments that I live in contradiction to my living educational theory.

In another vignette of the overall solving process happened when the students solved a new equation with little pause and only their toolbox of knowledge.

T: Use the Factor Theorem to verify that $x + 4$ is a factor of $f(x) = 5x^4 + 16x^3 - 15x^2 + 8x + 16$; in other words, $x = -4$ is a root or 0. What would I expect the remainder to be?

S1: 0

T: Yes, I expect the remainder to be 0. What is the number on the outside?

S Choral: -4

T: What are my coefficients?

S Choral: $5, 16, -15, 8,$ and 16

T: Perfect. Did I lose anybody there? These are your coefficients. What do I do first?

S Choral: Pull down the 5.

T: What do I write?

S Choral: -20

T: What do I write?

S1: -4

T: What do I write?

S1: 16

T: What do I write?

S1: 1

T: What do I write?

S1: -4

T: What do I write?

S1: 4

T: What do I write?

S1: -16

T: Therefore, this is ____?

S Choral: 0

$$\begin{array}{r|rrrrr} -4 & 5 & 16 & -15 & 8 & 16 \\ & & -20 & 16 & -4 & -16 \\ \hline & 5 & -4 & 1 & 4 & 0\ \mathit{remainder} \end{array}$$

The answer is yes; the remainder is zero, so the Factor Theorem says that $x + 4$ is a factor of $5x^4 + 16x^3 - 15x^2 + 8x + 16.$ (Research log, 5/16/18)

In this vignette, I had a sense that the students demonstrated both procedural knowledge and a conceptual understanding of the process. I perceived students possessed procedural knowledge because I did not provide any prompting questions. I also perceived that the students who answered possessed conceptual understanding because they understood at the onset that they needed to find the zero of the factor without any prompting. They knew the remainder needed to be zero. But, most important to me was the speed and confidence at which many students called out the process. In my classroom, students who do not understand when I am asking them to perform a task will not speak. There is silence and they look down at their papers. Those students who are confident with the task will call out the answers or process. They do not need encouraging or prompting. In this case, they seemed to answer in unison.

In the vignettes coded as Living Educational Theory, two substantive categorical themes emerged. One theme referred to the inconsistencies in my beliefs. I expected my students to solve equations conceptually such as in the previous vignette using the Factor Theorem when the students answered the questions with confidence and did not require prompting rather than procedurally such as in the literal equation vignette when the students required constant prompting and fell back on the convention of using order of operations. This is because I thought that the students had a better understanding of the process when they did not use steps.

I expected my students to understand balancing equations: what they do to one side they do to the other and to understand the difference between combining like terms and balancing equations. I expected a deeper conceptual understanding from them. I thought they should be able to demonstrate conceptual understanding by applying the knowledge that I taught them to a new and unique problem as stated in the standards such as in the previous vignette on using the Factor Theorem.

The second theme revealed in the vignettes coded as Living Educational Theory was that my definition of “steps” did not always refer to a step-by-step process; sometimes, “steps” referred to an organized and logical thinking process. This refers to students applying their overall understanding of the solving process. When I compared the two themes, I realized that students could solve using a discrete, step-by-step process and still understand the process as a whole as demonstrated in the previous vignette on using the Factor Theorem.

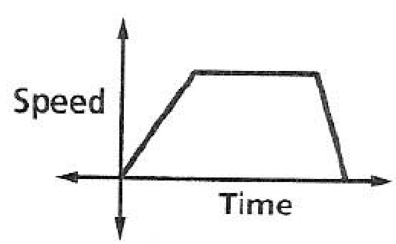
The discussions in the narratives challenged me to realize that mathematics does not use neat, orderly, and linear thinking; rather, it is messy, chaotic, and non-linear thinking. When I think of mathematics as linear, it is because I had taught myself to consider solving mathematical problems as one step and then the next (sequential). The steps can vary, but the path the problem takes is linear. This is counterintuitive to the way my mind usually works, which is chaotic, with information bombarding me from all directions. I realize that there is a place for thinking about mathematics from multiple, diverse perspectives. I do not have to think in an organized process. I can let my mind wrap around a concept.

Developing new connections. In the following narrative, my critical friend challenged me to describe exactly how the transformation of my teaching practice occurred.

> *Critical friend to me:* Did you answer your research question on reconceptualizing your thinking about your practice? (6/2/18)
>
> *Me to my critical friend:* I began this process looking for evidence of the reconceptualization of my thinking through the lens of smooth and chunky thinking. Taking the lens of smooth and chunky covariational reasoning process was the main benchmark I used to understand the transformation of my practice, but I also analyzed the reflections of my practice. I transformed my thinking about the lessons as I thought smoothly, but it was the coding and analysis process that provided me with the lens to examine my teaching story (e.g., the transcription, my words, the reasoning behind my words, and the challenge to my thinking by the critical friend and the expectations of my committee members). I heard the voices of my committee challenging my thinking by asking me," What do you mean by that *code*?" For example, what do you mean by *steps*? Are you referring to procedural knowledge? I heard my critical friend reminding me to write down the thought before I forgot. As we debriefed during the analysis, the questions provoked me to think about the evidence of my thinking. (Narrative, 6/5/18)

Throughout the sections, I discussed connections. I made connections to my prior knowledge and I also built new connections. Past connections refer to teaching practices based on making connections to my prior knowledge and connecting a new concept to a concept that I had taught earlier in the year. Toward the end of the study, I began to question my logic as to whether or not students found it helpful to connect new knowledge to earlier taught concepts. I assumed because I remembered the concepts I taught previously that the students could also remember the concepts taught. But, students need to do more than remember the concept that I refer. They need to have a conceptual understanding of the concepts. I possess a conceptual understanding of the concepts that I anchored to, but what if the students had not. Ausubel (1967) explains that if the knowledge anchorage were not stable then the knowledge would be obliterated or forgotten. Therefore, I have to ask myself the question of whether anchoring to newly taught algebraic concepts to algebra I students are a stable anchor which to build meaningful knowledge upon?

In the following vignette, the students reviewed a function that had been covered at the beginning of the year and were asked to 1) determine the slopes of the pieces of the function, and 2) to identify the type of function as piecewise linear. The students demonstrated confidence in answering this question. I believe that would not have been possible if I had asked the same question at the beginning of the year.



T: You don’t think it is linear, and you don’t think it is exponential. Why are you shaking your head?

S1: It is a constant rate.

T: Beautiful!

S1: It is building at a constant rate of change. Then, it stays the same, and then, it decreases at a constant rate of change, a linear constant rate of change.

T: Right, that is why it is not exponential. Anyone want to get specific on what type of linear function that it would be? Yes?

S2: A piecewise

T: I would call it a linear piecewise.

S3: One goes [accelerates], one stays constant and one goes [decelerates].

T: This is a piece, this is a piece, and this is a piece. Makes sense, nicely done.

(Research log, 3/16/18)

The students demonstrated their conceptual understanding and followed up with a response in their own words – constructing an answer by making connections to their prior knowledge. In the following vignette, the students connected the absolute value graph to the characteristic they applied to a parabola, a concept that I conceived as being anchored in their minds. The parabola is a concept that we have been reviewing, since the first marking term until present (the forth marking term).

T: We are going to graph the absolute value. Does anyone want to tell me the characteristics that the absolute value and the parabola share?

S1: Yes, they both have a vertex.

T: Anything else?

S2: They both have an axis of symmetry.

T: These are a couple of really good ones. Here is the thing. Absolute value is really considered a piece wise function. It has two pieces. There is this line and this line. They happen to be symmetric. When we had the parabola, I said we can go up one and over

> one if the *a* is 1, but after that, you can't do it. The absolute value is not the same because the pieces of the absolute value function are both lines you can go up one, over one, up one, over one, if your *a* was 1. I am referring to the slope of the pieces. (Research log, 3/6/18)

In the vignettes coded as past connections, the substantive theme that emerged was that the connections I made were to their prior knowledge, such as slope or rate of change, or connecting a new abstract idea (e.g., graphing a parabola) to the past connections that I taught this year and that I believed to be a stable anchor to build upon. It was not until the analysis stage that I questioned whether connecting new concepts to past connections taught this year might not be a stable anchor for improving the students' understanding. By anchoring concepts to past connections and, in turn, re-visiting and inevitably re-teaching, I seemed to be strengthening the past connection and building new connections.

The code *new connections* refer to how I created the new connections to promote conceptual understanding. Of the organizational category of new connections that I created, some I connected to objects, such as a roller coaster to an asymptote, which is demonstrated in the following vignette.

> A student struggled with the idea of asymptotes and brought up the idea of a roller coaster for exponential growth. I explained that asymptotes are like the rails that guide the graph to go where we want it to go. The asymptote is an imaginary line [boundary] that cannot be passed. I showed her the example of $f(x) = 2^x - 3$, and the asymptote is at -3. (Research log, 3/1/18)

My goal was to connect a concept they understood to a mathematical concept.

In the following vignette, a student made a connection to the shape of a parabola that I had not considered prior to the lesson. I interpreted the problem variationally (coordination of values) because I used an input and output table and then drew the parabola. The student saw the problem in terms of symmetry. A student explained their justification, while drawing in the air. They drew the vertex below the symmetric points, facing the parabola upward.

x	y
0	5
1	0
2	-3
3	-4
4	-3

T: We know it is a quadratic if it is a parabola. Right. But, looking at this table of values, does anybody see a parabola screaming at them? No?

S1: Nods. Yes.

T: Do you kind of? Well that is good. [I waited for their explanation]

S1: You look at the -3, -4, and -3 [They are pointing in the air, seeing this area here.]

T: That is good. [I was not really following]

S1: The -3 and -3 over here [symmetry] and the -4 dropped down below it.

T: That is the first time today that I actually see that. Nicely done. (Research Log, 3/15/18)

By listening to this student's justification of their answer, they convinced me of their conceptual understanding. By asking students to explain their thinking, Harel and Sowder (2005) explained that students revealed important ways of understanding. In the majority of vignettes, I noticed that students answered in a sentence or less. The student connected a visual representation of a parabola and the representation of symmetry with the y-values on the table of values. They also convinced me that their conceptual reasoning was mathematically sound. The student on this vignette helped me develop a new connection that I, in turn, shared with students in my other classes.

In the following vignette, when factoring, I connected the AC and area model. The majority of students were successful at the initial steps of the AC method. They were also successful with the area model. I decided to combine the two methods.

T: I call this the AC area model because I combine the two factoring methods. The question is $f(x) = -16s^2 + 64s - 60$. You can set the function equal to 0 now or you don't have to. $-16s^2 + 64s - 60 = 0$. I don't want to multiply -16 times 60. Is there a common factor?

S1: 2 goes into all of them.

T: What else?

S2: 4

T: Just so you know, if the *a* is negative, you should factor out the negative, so -4. I end up with $-4\ (4s^2 - 16s + 15) = 0$. What is A times C? I am going to leave the -4 on the outside for now. [I realize when transcribing that I never went back and discussed the -4, I am going to have to deal with that in the next class.] What is A times C?

S3: 60

T: What does my sum have to be?

S4: -16

T: What are the numbers?

S5: $-10, -6$

T: Did I lose anybody? Now, make the box. What goes in the first box?

2s	$4s^2$	$10s$
3	$6s$	15
	2s	5

The first term goes in the first box, and the last term goes in the last box. Then, the 10s and 6s go in the other boxes, no signs. Now, I am going to factor this way [horizontally]. What do these have in common?

S6: $2s$ and 3

T: Now factor [vertically]. What do these have in common?

S7: $2s$ and 5

T: Write down $(2s \quad 3)(2s \quad 5)$. What is 3 X 2s? It is 6s, and we said that 6s would be negative and 2s times 5 is 10s and that is negative. $(2s\ -3)\ (2s\ -5)$. Am I done?

S8: No, they want the zeros.

T: $(2s\ -3) = 0\ (2s\ -5) = 0$. Add 3. I get $s = \frac{3}{2}$ and I add 5. I get $s = \frac{5}{2}$. What do you think? Is this method going to help you guys? (Research log, 4/24/18)

Of the new connections, I found the following issue challenging, which was my attempt to connect a transformation of an odd function into an algebraic proof of $-f(-x) = f(x)$, illustrated in Figure 19. A discussion with my critical friend helped to reframe my thinking. The new connections raised my thinking to a new level of understanding.

T: Assume, $f(x) = x^3$. I like to look at it this way.$-f(-x) = f(x)$. Are you having trouble with this guy [pointing to the odd function]? Then, it hit me. All of a sudden, I move quickly. You can see by my actions, it clicks. This is a cube root, right. What is 2 cubed, 8. So, if I put a 2 in here, I get an 8. Then, at -2, I have 8 and then I take the reflection of it that is negative 8. Now, these are not abstract numbers, is it making sense? This function here is $f(x) = x^3$. If I put a 2 in for my *x,* I get 8. If I put in -2, I get -8, but the opposite of negative 8 is positive 8. Now, they are equal. I will do it one more time. [I think I am convincing myself, more than my students.]. If I put a 2 in here, [in the x cubed], it equals 8. If I put a -2, in here I get -8, then I take the opposite of it, which is positive 8. Then, $8 = 8$. It is something you need to practice, but this is a good example. I found this problem challenging. It took me all day before it finally clicked. This is an example of advanced mathematical thinking. (Research log, 3/5/18)

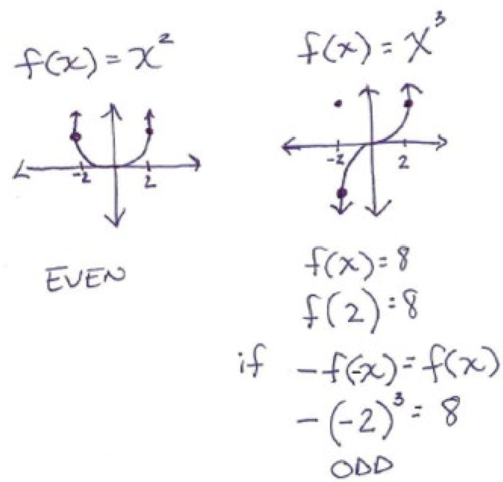


Figure 19. Board work on odd and even functions

When I began to solve this problem, I thought that the way to transform an odd function was to reflect over the y-axis and then reflect again over the x-axis. Therefore, I had a positive 8 and a negative 8. The problem arose in my thought process when I solved it algebraically and found 8 = 8. I kept questioning how one made algebraic sense while one made transformational sense. However, they did not make sense when I combined them. My discussion with my critical friend helped me understand that equivalence when solving equations algebraically was not the same as transformational shifts.

The new connections provided evidence of the transformation of my practice and the new knowledge that I brought to future lessons.

> I noticed the experiences of the past that influenced my present knowledge, during transcriptions. But I never considered the definitions behind the words to me. What does *process* mean to me? What does *smooth* and *chunky* mean to me? Throughout this paper, my committee asked me to define experts' words such as what does Castillo-Garsow (2012) mean by smooth and chunky thinking or process, Ausubel (1967; 2000) mean by meaningful learning, Tall (1991) mean by advanced mathematical learning, or Thompson (2012) mean by covariational reasoning … I interpreted experts' words through their lens. I never considered allowing myself to define their words through my lens. (Narrative, 6/5/18)

The narratives, that reflect new connections, reveal that in an attempt to reconceptualize my smooth and chunky thinking that I was able to transform the ways I thought about my teaching practices. I discussed the importance of building new connections for my students and me in attempting to understand mathematics. I realized that I interpreted smooth and chunky covariational reasoning differently than the experts. Ultimately, I realized that we learn mathematics in a way that makes the concepts meaningful to ourselves.

Chapter VI

Implications and Conclusions

This chapter begins by summarizing this self-study and ends with the conclusions drawn from the data analysis. Next, I review the methodology and unique characteristics of a self-study. I present a comprehensive conclusion based on a network of organizational and substantive categories that I illustrated using a concept map that highlights the relationship between the acquisitions of meaningful learning with building new algebraic connections. Through narratives, Critical Friend Portfolio dialogues, and transcriptions from Research Logs as supporting evidence, I explore the progression of the transformation of my teaching practice. I refer to Thompson and Carlson's (2017) levels of covariation reasoning and Tall's (2008b) Three Worlds of Mathematics cognitive development to identify specific examples of the reconceptualization of my smooth and chunky covariational thinking. Based on these findings, I found that because I learned *how* I thought about mathematics, I reconceptualized my smooth and chunky covariational reasoning.

In the final section, I suggest implications for action by encouraging the use of multiple methods to transform teaching practices. I connect the transformational elements that arose from the critical incidents and describe the shifts in my practice.

Summary of the Study

The intention of this study was to develop my smooth and chunky covariational thinking to ultimately transform my teaching practices. One concern was the use of student dialogue from which researchers could draw data and attempt to understand students' perceptions of smooth and chunky covariational reasoning. Therefore, I proposed that I reflect on my covariational reasoning, perhaps leading to a more comprehensive analysis. I hoped that the knowledge from such reflection might then provide the groundwork for identifying a stronger relationship between chunky and smooth reasoning to facilitate more meaningful learning. Therefore, the purpose of this study was to examine the development of my smooth and chunky covariational reasoning to transform and improve my practice through self-study. The following was the question I posed to guide this research:

How does analyzing the development of my acquisition and implementation of the lens of smooth and chunky covariational reasoning through reconceptualizing algebraic relationships promote the transformation of my teaching practice?

At the onset of this study, I maintained that the self-study methodology provided me a venue to narrate and explain my thinking in a more sophisticated manner. To provoke a transformation in my teaching practice, I examined the development of my smooth (process-driven) and chunky (interval-driven) reasoning, while continuing to perform my usual classroom responsibilities. I made no other modifications prior to this self-study other than to initiate the development of my smooth and chunky covariational reasoning.

Review of Methodology

A self-study methodology design has its own particular characteristics. The design is self-initiated and focused, aims to improve practice, possesses exemplar-based validation, employs multiple qualitative methods, and interacts with others and texts (LaBoskey, 2004). This study was self-initiated and focused on the self. Specifically, I aimed to transform my teaching practice and thinking processes. Based on the objective of improving my practice, I facilitated my students' meaningful learning as they built new algebraic connections. The narratives and Critical Friend Portfolio dialogues with supporting evidence from the transcriptions of the Research Logs provided the data for analysis. The multiple qualitative methods used in this self-study provided me with opportunities for diverse analytic perspectives, including personal history, living educational theory, narrative, and dialogue.

The final characteristic key to self-study is the interaction with others. In my case, a critical friend played that role whose purpose was to ask thought-provoking questions (Costa & Kallick, 1993), to provide an alternate lens to view data (Samaras, 2011), and to validate assumptions and interpretations (Samaras, 2011). Schuck and Russell (2005) suggested a "frank and thorough discussion before the start of the project… [that address] expectations and concerns by both parties" (p. 112). Prior to the study, I discussed with my critical friend that I was hesitant to analyze the data to form categories and collapse it too quickly. Our dialogue and narratives concentrated on the present lessons and what *is* happening rather than what those happenings might mean.

Pinnegar, Hamilton, and Fitzgerald (2010) argued that "fundamental to self-study of practice research is the fact that researchers using this methodology are centrally focused on what is happening in their practice and they are seeking to understand themselves and their practice better in relationship to each other" (p. 203). This study maintained that *what is happening* in the moment is of interest, but equally important was the creation of an overall depiction of the transformation of my teaching practice. The conclusions drawn from the concept map provided a comprehensive interpretation of the data.

Major Findings

In this section, I present my findings as organized by organizational and substantive categories. I depict these findings in a concept map (see Figure 20). Then, I provide critical incidents related to these categories, and these will be depicted in excerpts of that concept map.

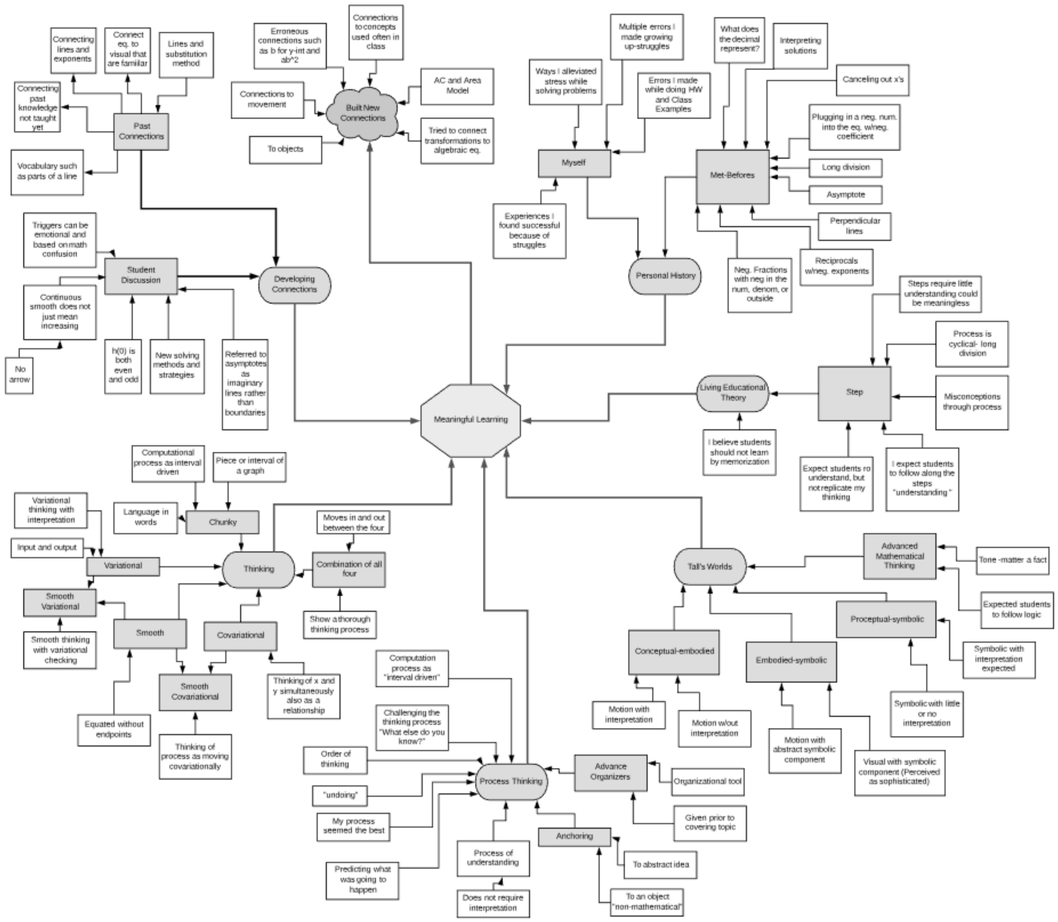


Figure 20. The concept map of the organizational and substantive categories

To organize the transcriptions, I created a concept map based on the reflections of the organizational and substantive categories from the research logs, narratives, and dialogues. Because of the depth and richness of the data, the use of a concept map allowed for the highlighting of a network of themes that emerged between the organizational and the substantive categories. The organizational and substantive categories (see Appendix D for completed example) included thinking (smooth, chunky, covariational, and variation), Tall (conceptual-embodied, proceptual-symbolic, and advanced mathematical thinking), personal history (my past and met-befores), thinking process (anchoring and advance organizers), living educational theory (steps), and connections (new connections, past connections, and student discussion).

During the coding process, I realized that I took a moment to reflect as I determined the category for each element of transcription. I recognized that the *critical incidents* created a moment of reflection and provided me with a more comprehensive analysis. Tripp (1993) described critical incidents as “essentially cognitive responses of ‘surprise’ and ‘perplexity’” (p. xiii). Throughout the analysis process, I realized that each critical incident created a “chipping away” of existing teaching practices to make room for a new awareness about my teaching practice. My findings emerged as a result of critical incidents. For example, the intent of the curriculum was to begin teaching Algebra II topics after the EOC test. The critical incident that surfaced was questioning whether attempting to facilitate topics that I perceived to be advanced mathematical thinking throughout the year seemed reasonable.

Another critical incident created a moment of pause or reflection when I considered that although it initially seemed to contrast my constructivist views on learning, basing new connections on prior knowledge – if that knowledge is actually rote, unstable knowledge – might not be helpful to improving student understanding. For example, when describing the transformation of an exponential function, I referenced the transformation of a quadratic function. To a student with a conceptual understanding of quadratic transformations, the student applied prior knowledge to new concepts taught. Consider the student in the same class who did not develop a conceptual understanding of quadratic transformations; maybe they performed the transformation task by rote. By attempting to facilitate the building of new connections on this student’s unstable prior knowledge, new understanding may not result. In other words, in aiming to make such a superficial connection, I was not providing the student an opportunity to learn meaningfully.

In addition to the relationship between the concept map and the critical incidents, I discussed my analysis of the concept map with the critical friend as a form of inter-rater reliability check. *Inter-rater reliability* refers to another individual's interpretation of the finding in the same way.

The critical friend and I discussed the justification for the analysis of my coding and my interpretation of the concept map. I gathered through the provoking questions that my critical friend expected me to delve more deeply into the underpinnings of my thinking about my findings and explain the reasoning for my coding. In Figure 21, I provide an excerpt from the Inter-rater Reliability Discussion used as an authentic representation of my understanding about my teaching practice (see Appendix E for a completed example).

Inter-rater Reliability Discussion
Critical Friend: How do you see the concept map? *Me:* As an organizational tool. *Critical Friend:* Geared to help. *Me:* Motion and also connections. *Critical Friend:* Why are connections important? *Me:* If you understand the big picture, you can solve word problems. The problems are meaningful. I never understood the big picture. *Critical Friend:* What affects the big picture? *Me:* Having an advance organizer allows me to maximize teaching. Too see the totality (smooth) not pieces (chunky). My visualization evolved.

Figure 21. Excerpt from the Inter-rater Reliability Discussion

Thinking: Smooth, chunky, variational, and covariational. In Chapter 2, I provided the following statement by Thompson and Carlson (2017): "Covariation happens in the minds — students, teachers, and researchers. When a researcher reports that a situation involves variation or covariation, he is saying how he [the researcher] conceives the situation" (p. 461). This means that through a questioning process, the researcher asks a student to explain their reasoning, but only the student has access to the knowledge in their mind. I wanted to understand the connection between smooth, chunky, variational, and covariational reasoning in order to transform my teaching practices. The concept map excerpt in Figure 22 illustrates how this connection emerged in my teaching. In what follows, I explain what is significant about how the connections represented in the map came to shape my teaching.

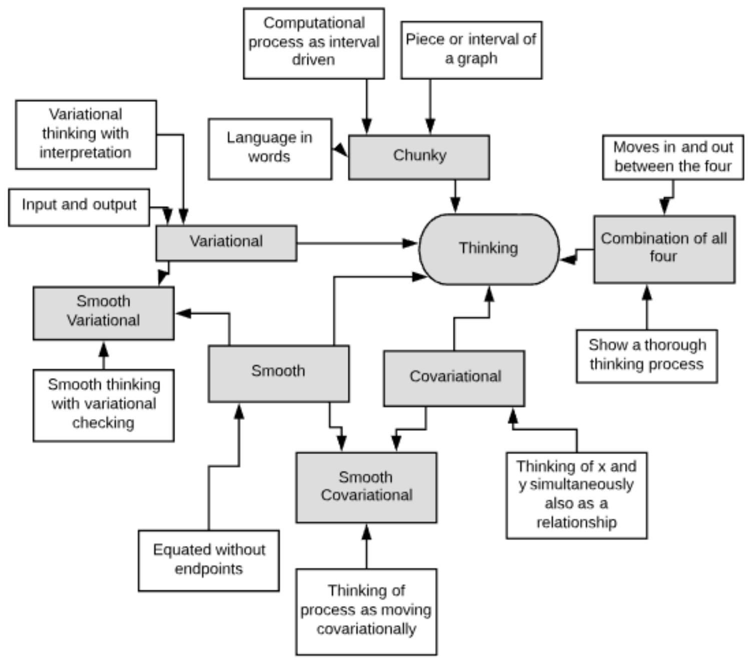


Figure 22. The network from the concept map of smooth, chunky, variational, and covariational

Smooth covariational reasoning is more sophisticated than variational (correspondence approach and coordination of values) reasoning. That is, one must simultaneously conceptualize multiple variables at one time rather than conceptualizing the value of the variable as having a specific value. We think of a situation, as I perceived variationally (thinking about variable, correspondence approach or coordination of values) or covariationally (thinking about the variables, simultaneously) in terms of smooth or chunky reasoning. A situation denoted as smooth (process-driven) or chunky (interval-driven) suggests envisioning the qualitative reasoning covariationally. Thompson and Carlson (2017) described the levels of chunky and smooth covariational quantitative reasoning as hierarchically ordered in terms of their sophistication (see Figure 5). They go on to explain that the individual's initial thinking determined their covariational level and suspected that the levels are developmental. This next section describes my own development of smooth covariational reasoning.

I began this study by identifying a preliminary benchmark on which to illustrate the progression of my smooth and chunky reasoning. I described my doubts about my ability to think smoothly, since I had previously perceived myself to be a chunky variational (correspondence approach and coordination of values) thinker.

Me to critical friend: What if I can't think smoothly? (Critical friend memo, 2/26/18)

I considered myself a chunky variational [coordination of values] thinker (interval-driven) because I considered a graph as a chunk (relative to its domain and range), and I thought in terms of input and output to create graphs. I perceived my use of chunky variational thinking to be more easily accessible compared to chunky and smooth (process-driven) covariational thinking.

It became apparent that by day 2 of the data collection that there was evidence of my ability to think smoothly, chunkily, variationally (correspondence approach and coordination of values), and covariationally. At this point, I felt a sense of relief that I no longer had to worry that I thought smoothly.

> When I focus on the ordered pairs [variational/coordination of values thinking] and growth factors, I am thinking chunkily. But when I am thinking of the function in its totality, I am thinking smoothly. (Critical friend memo, 2/27/18)

I realized that when graphing an exponential growth function, there was a variational component because of the ordered pairs, but I recognized that when I looked at the function as going to positive infinity, I was thinking smoothly. I was not thinking about input and output. I thought of the path of the function in the present time, similar to a finding by Castillo-Garsow (2012), who explained that in one teaching episode, Derek stated, "It's growing constantly . . .

And keeps going." The student imagined that the account is growing in the present. The reference to thinking in the present implies thinking smooth covariationally because the student is thinking of the growth process increasing (smoothly) and the variables change simultaneously (covariationally).

Although the covariational levels may begin as developmental, as my study progressed, I realized that I moved in and out of the levels, moment by moment. Furthermore, the form of covariational reasoning with which I operated depended on the questions students asked. They reflected how I solved the problem at that moment and were also just as informed by classroom discussion that ensued.

> At one moment I might be thinking covariationally and then bounce into variational [coordination of values] thinking. (Narrative, 6/1/18)

Castillo-Garsow (2014) explained, "*Chunky reasoning* is forming an image of completed change" (p. 158). Consider chunky reasoning as a completed thought with a beginning and an end. Castillo-Garsow (2014) continued, "*Smooth reasoning* is forming an image of a dynamic change in progress" (p. 158). Based on these distinctly different definitions, I interpreted the smooth and chunky relationship as being mutually exclusive. That said, I was not always cognizant of my thinking in real-time.

Critical Incident: I moved in and out of smooth and chunky thinking.

Upon reflection, the critical incident that surfaced was that as the study progressed, I seemed to move in and out of smooth and chunky covariational thinking easily, as if blending the two. This suggests to me that the lines were blurred, something that Castillo-Garsow et al. (2013) acknowledged as the hybrid nature of smooth and chunky thinking, the back and forth or moving in and out. I began this study believing I thought chunky variationally (correspondence approach), but stronger smooth covariational thinking ensued that represented a contradiction to starting with chunky thinking.

> Castillo-Garsow et al. (2013) who questioned whether starting with smooth covariational reasoning built a stronger chunky covariational reasoning. I realized in my own thinking that this is not the case for me. It was the moving back and forth between variational (correspondence approach and coordination of values) and covariational and smooth and chunky thinking that developed my mathematical reasoning. (Narrative, 6/1/18)

As I continued to consciously apply smooth covariational thinking to transformations, smooth covariational thinking became more accessible to me. I preferred smooth covariational thinking, because I did not have to think about using ordered pairs. It was less work. Consider the following dialogue:

> T: A power function is a function in the form of $f(x) = k \cdot x^n$ . . . Don't forget that $f(x) = \sqrt{x}$ is equivalent to $f(x) = x^{\frac{1}{2}}$.
>
> S1: Oh, yeah . . .
>
> T: It [y] starts getting bigger, bigger, bigger, and bigger [smooth covariational]. It goes to positive infinity [smooth]. (Research log, 3/12/18)

Similar to the language used in a teaching episode in Castillo-Garsow (2012) to demonstrate smooth continuous reasoning, the student described how they imagined the rate of change: "It would start slow, keep getting faster and faster." My use of the smooth covariational language "getting bigger, bigger, bigger, and bigger" should provide my students with an opportunity to develop a conceptual understanding of square root functions.

Critical Incident: Was I too focused on smooth covariational thinking and not enough on chunky thinking?

Upon reflection, I am not convinced that I am giving an Algebra I student enough practice creating an exact square root graph to develop a conceptual understanding of the square root function. Sfard (1991) made a related assertion: “A person must be skillful in performing algorithms in order to attain a good idea of the “objects” involved in these algorithms” (p. 32). That is, a student must have computational practice (that I perceive as chunky thinking) and understand the concepts that underlie the computations. I am not saying that encouraging a conceptual understanding about square root functions is not necessary. Rather, I question if my students just view the situation as the rote transformation of a few quick sketches of arbitrary shapes. Sfard (1991) continued, “One ability cannot fully develop without the other” (p. 32), with those abilities referring to performing the algorithm and understanding the process of the algorithm.

Ausubel (1967) argued that learning occurs when “symbols are non-arbitrarily relatable to existing content in the learners’ cognitive structure” (p. 21). For instance, suppose I drew a square root graph using an input and output table. If I were to consider that graph to be a “chunky variational (coordination of values) graph”, because it was formed from a discrete collection of points, would the student develop stronger smooth covariational thinking because they developed a cognitive scheme for a square root function? I question the relationship between the act of constructing the graph and then shifting the function using smooth and chunky covariational thinking. Was there a cost to my students that I spent more time transforming quick sketches of functions to encourage smooth covariational reasoning rather than allowing them to practice graphing accurate functions that I perceived as chunky variational (correspondence approach and coordination of values), as in the following vignette?

T: The first example I want to do is $g(x) = \sqrt{x} + 2$, the one in the notes. We will call

> the one we just did $f(x) = \sqrt{x}$. Now, what we are going to do is $g(x) = \sqrt{x} + 2$. What do I have to do to this graph?
>
> S Choral: Lift it up two
>
> T: [I perceived the student response to be smooth covariational]. Good. (Research log, 3/12/18)

The students described a transformation as shifting the graph up two, which demonstrated smooth covariational reasoning. Different types of questions provided students with numerous opportunities to represent the variables as an ordered pair (variationally), to consider variables simultaneously (covariational), to think of an interval and domain of the graph (chunky), and to shift the graph (smooth continuous covariational).

One month later, the students solved an algebraic word problem in which the length of the sides changed, without providing input and output values, to encourage the students to think of the length of sides covariationally. This approach was similar to that used in Moore and Carlson (2012) in the sense that the student's response suggested that he "imagined a fixed length and width of a box that did not depend on or vary with the length of the side of the square cutout" (p. 52). My challenge was that I thought of a variable squared as a two-dimensional image rather than as the length of a side.

> I have to be honest with you, I have to get past the idea that $2x^2$ does not add a third dimension [My first thought was variational (coordination of values), but I was also juggling my thinking of an exemplar from Carlson (1998) of a bottle filling.]. (Research log, 4/24/18)

As I taught, I grappled with my internal variational thoughts and simultaneously considered Thompson and Carlson (2017), who suggested that the conceptualization of a quantitative structure supported the covariation of the quantity's values. This study concurred with the position of Castillo-Garsow et al. (2013), who recognized the moving in and out of variational and covariational and/or smooth and chunky thinking. This study also concurred with the position of Thompson and Carlson (2017), who clarified that students may go back and forth in response to a situation.

Critical incident: The situation of the context affects the use of smooth or chunky thinking.

The moving in and out of the levels of covariational reasoning may not be solely developmental or situational. The conclusion drawn from this study is that the complexity of our mind permits a network of factors to influence the hybrid nature of variational or covariational reasoning. In my case, the moving in and out of variational and covariational and/or smooth and chunky thinking occurred because of self-reflective thought during phenomena such as the transcriptions of the lessons, student questions during class discussions, the context of problems such as transformations, and my understanding of problems (such as my interpretation of a variable raised to the power of two as representing an area that I perceive as two-dimensional space).

Embodied, symbolic, and advanced mathematical thinking. Tall's (2008b) cognitive mathematical development comprised of the three worlds: conceptual embodied, proceptual symbolic, and axiomatic formalism (as shown in Figure 23). This study considered the conceptual embodied (embodied) and proceptual symbolic (symbolic) worlds used in elementary or school mathematics (Tall, 2008b) and the world of axiomatic formalism (formal) used in advanced courses of mathematics (Tall, 2008b).

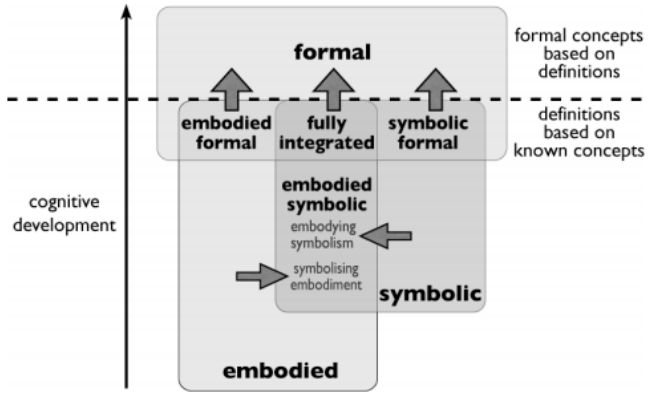


Figure 23. Cognitive development via the three worlds of mathematics (Tall, 2008b , p. 8)

The concept map excerpt in Figure 24 illustrates the connection between the worlds. Although Algebra I students may find formal concepts based on definitions complex, I provided students with opportunities to practice formal or advanced mathematical thinking.

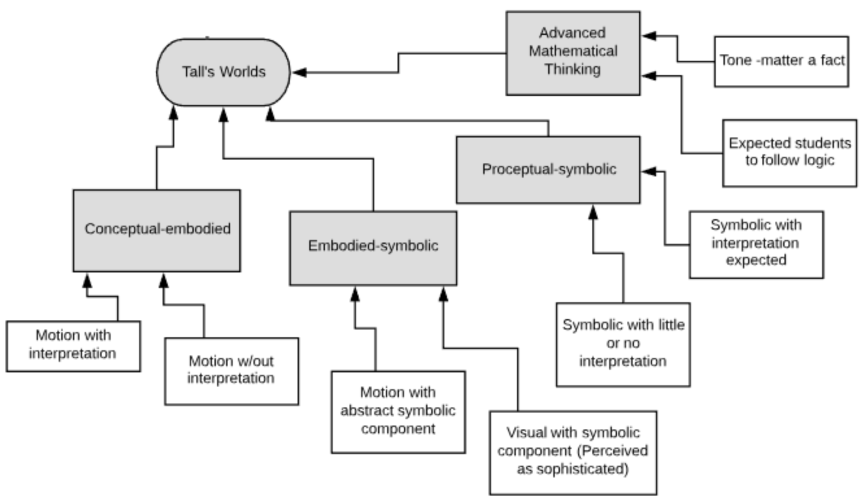


Figure 24. The network from the concept map of Tall's Three Worlds of Mathematics

The embodied world that Tall (2008b) depicted was the "perception of and reflection on properties of objects, initially seen and sensed in the real world, but then imaging in the mind" (p. 7). I interpreted *embodied* to refer to motion and visual cues and that cognitive development is initially grounded in and then abstracted from the physical world. Tall (2008b) explained that the symbolic world grows out of the embodied world through action, symbolized as thinkable concepts that function. I interpreted *symbolic* as representing the manipulation of algebraic symbols. During the coding process, I considered the difference between the two worlds of embodied and the symbolic and reflected on the delineation of those worlds.

Critical Incident: Students can reside in one world, but draw on another.

I realized that the transformations that used body movements (e.g., the gesture of a hand quickly motioning upward to represent exponential growth) and a visual representation (e.g., the graph of a function) on the board are exemplars of the embodied world. The use of algebraic symbols represents the symbolic world. The level of difficulty of the questions did not determine whether or not the student resides in the symbolic world. Consequently, a student may reside in the symbolic world, but find it necessary to draw a representation that I perceived as reflective of the embodied world.

Critical Incident: The world one resides in does not imply procedural knowledge or conceptual understanding.

A student who resides in the symbolic world did not necessarily imply conceptual understanding; that is to say, a student who relied on procedural knowledge to solve a long division polynomial problem (symbolically) may actually have little conceptual understanding of the essence of the problem. However, a student who solved a problem symbolically and then was able to explain the meanings underlying that process demonstrated conceptual understanding.

Axiomatic formalism is the third world that Tall (2008b) discussed as a more advanced form of mathematics seen in advanced mathematics courses. In an attempt to challenge my students' thinking, I choose topics that I perceived as advance for Algebra I students. The topics that I perceived to be examples of providing students with opportunities for advanced levels of mathematical thinking for Algebra I students included describing end behaviors, graphing higher-degree polynomial functions, and explaining what it was that yielded an extraneous solution when solving some equation. In the following dialogue between the students and me, I asked them to identify each part of a term that I had perceived as being an element of advanced mathematical thinking for Algebra I students. I had not used these symbols prior to asking this question. The students wanted to know the difference between $\frac{f(x)}{d(x)}$ and $\frac{r(x)}{d(x)}$.

T: Long division is written in the form of $\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$ . . .

S3: Is the other $d(x)$ the dividend?

T: No, this is also the divisor. We will just call this the . . . what should I call $\left[\frac{f(x)}{d\,(x)}\right]$?

S4: The original

T: The original function, OK . . . Is everybody good with the remainder theorem? Does everyone understand that I could have it written in multiple ways? Now, if there is no remainder, then this part will be truncated. (Research log, 5/16/18)

As shown in the transcription of the discussion between the students and me, the students gave the symbol its own name. I interpreted students creating a name for a symbolic representation as advanced level of mathematical thinking, upon reflection, this example illustrated the symbolic world because they took an abstract concept and provide it with a descriptive name. I began to question the difference between the symbolic world and providing opportunities for advanced levels of mathematical thinking.

Critical Incident: What does it mean to me to provide opportunities for advanced mathematical learning?

Although the intent of the curriculum was to begin teaching Algebra II topics after the

EOC test, I questioned whether I perceived teaching the Algebra II content (e.g., the remainder theorem) at the end of the year in an Algebra I class as advanced levels of mathematics as reasonable, whereas teaching the same concepts throughout the year when the students initially learned the material would be regarded as unreasonable. Because topics that required advanced levels of mathematical thinking occurred at the end of the year, there were some students who learned the topics meaningfully, because they had the requisite Algebra I background, that is, the stable meanings of more prior concepts, to do so.

Based on classroom discussions, I noticed that the students who developed the necessary computational skills demonstrated a convincing argument when answering challenging questions that I had posed and perceived as in the realm of advanced mathematical thinking. I came to the position that posing questions that encompass both procedural knowledge and conceptual understanding is one way of providing opportunities for advanced mathematical thinking. Ultimately, I realized that advanced mathematical thinking requires a student to be able to perform the algorithms and understand the process of the algorithms. They had to have a conceptual understanding to provide a convincing argument or justification of a concept that challenged them to understand situations that were deeper than the curriculum explicitly required.

Personal history: Met-befores and my past. Personal history reflected the effect that my past exerted on my current teaching practices. Past personal history includes my childhood experiences. The concept of met-befores refers to "a current mental facility based on specific prior experiences of the individual" (Tall, 2008a, p. 6). The personal history concept map excerpt in Figure 25, illustrates the connection between met-befores and my past, which I refer to as "Myself."

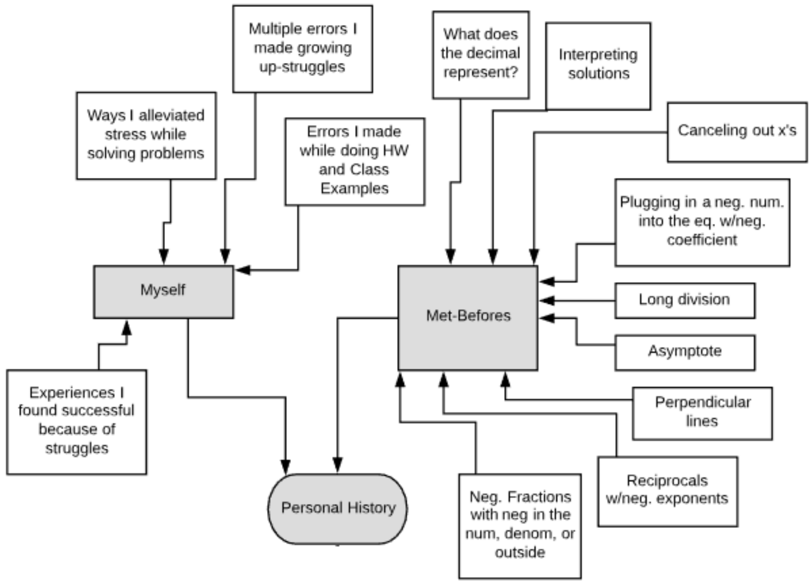


Figure 25. The network from the concept map of personal history

Tall (2008b) explained that sometimes met-befores affect new learning positively, while at other times, they caused internal confusion that obstructs learning. When I aimed to guide the lesson in accordance with my own personal experiences, the lesson was less effective than when it was informed by the students' own prior knowledge. I may be well positioned to anticipate errors and confusions based on my own learning experiences, but it is more important to elicit and respond to students' own prior knowledge. Tall (2008a) elaborated: "Experts may have forgotten how they thought when they were young and are likely to need to reflect on how different students' met-befores affect their way of learning" (p. 6) or my way of learning. The findings indicate that I focused on met-befores that, in fact, hindered learning.

I could not help wondering why I referenced specific met-befores when I taught. What made certain met-befores memorable? In this study, many of the met-befores that I referenced as I taught were based on the students' questions during class that jogged my memory about my past experiences with mathematics. Therefore, the met-befores and my personal history were aligned on some occasions and in conflict at others.

The following Personal History example was both from my past and a met-before. I remember as a child that I struggled with the expression '$-x$.' When a problem on the test required me to substitute a value into the expression, $-x$, I became anxious. I came up with this mantra: I drew parentheses and asked myself, "What is the x? What value goes in the parentheses?" This strategy always worked. In the following vignette, I referenced a struggle to substitute a negative number into an expression containing a variable with a coefficient of –1:

> T: I'm going to change the first equation because it is a common error that happens to students . . . The equation we are going to use is $f(x) = -x + 1$. And I want you to let $x = -5$. Ok, write what it equals, and I am going to walk around the classroom . . . Exactly what I expected happened [met-before]. Many of you had the answer right, but some of you didn't. I am going to model the common error. Most of the common errors are the errors I made when I was your age. (Research log, 3/8/18)

In this lesson, I recognized the type of equation that I struggled with as a child.

Critical Incident: What was the connection between my past struggles and met-befores?

I noticed during this study that I sometimes refer to "common errors" instead of saying, "This is a common error I made as a child." I believe this is because sometimes I did not realize until I reflected later that they actually were *my* common errors. Although it was not explicitly stated in the vignette, I thought that if I substituted $x = -5$ into $-x + 1$, this question would be easier, because whoever made up the problem had already inserted the negative sign there for me. Based on this reasoning, I coded the above example with the organizational category as personal history from my past and a met-before.

The greatest influence from my past was making weaknesses in mathematics into strengths because of an internal drive for perfection. Samaras, Hicks, and Berger (2004) offer that "personal history — the formative, contextualized experiences of our lives that influence how we think about and practice our teaching — provides a powerful mechanism for teachers wanting to discern how their lived lives impact their ability to teach or learn" (p. 905). My past mathematical experiences had created a lack of mathematical confidence, because I believed the other students found the mathematical concepts to be more accessible. They did not have to come up with silly mnemonics like I did. Samaras, Hicks, and Berger (2004) continued, "Who we are as people affects who we are as teachers and consequently our students' learning" (p. 906).

Critical Incident: Connecting past struggles with meaningful learning.

Although, I am not certain if understanding why I perceive struggles as negative is relevant to my teaching, this is a self-study and requires me to ask myself the difficult questions. Therefore, a substantive categorical theme that emerged during the final coding process was that I conceived my errors to be negative experiences in mathematics because I perceived them to be a personal weakness. The following is a dialogue from an inter-rater reliability discussion.

> *Critical Friend:* Why do you see your struggles in the past as negative? You overcame your struggles. You persevered.
>
> *Myself:* I remember very little growing up, but they [struggles] are the only ones I remember. I know *negative* is not the right term. I just don't seem to see any positive memories. Some [struggles] I didn't even realize until I was teaching, and I would say, oh yeah, I struggled with this. (IRR Discussion, 6/16/18)

I find it interesting that I perceived developing these strategies as a weakness and not strength. I seemed to remember the negative mathematical experiences, possibly because it took additional time to develop strategies to resolve the struggles. As a result, I considered them to be worth remembering. Upon further reflection, my past personal history played a major role in creating new strategies to rectify the mathematical confusions I had as a child. I believe that even as a child I realized that each student has an individualized interpretation of meaningful learning in mathematics. As a teacher, I may share my childhood strategies to enhance my students' learning, but I also realize that my strategies might not be meaningful to others and may be perceived as just one more thing for students to learn.

Connections: Past connections and classroom discussion. When referring to developing connections, I considered the importance of past connections and classroom discussion. The concept map excerpt in Figure 26 illustrates the connection. Past connections refer to the connections made between one math concept and another. Classroom discussion refers to sharing mathematical thinking.

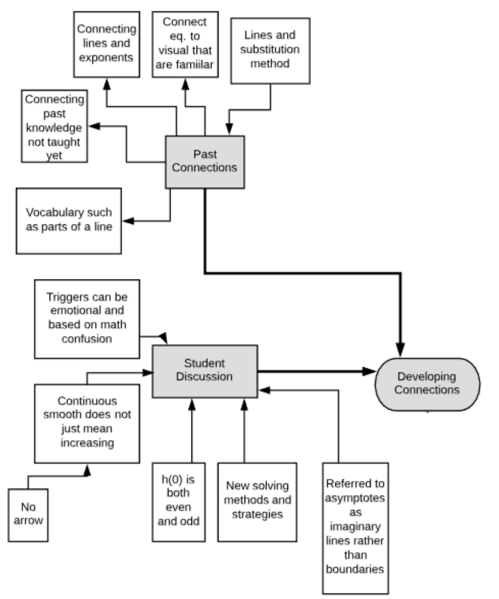


Figure 26. The network from the concept map of developing connections

Past connections anchored the relatively new concept of translating the graph of a parabola, particularly one that is the graph of a square root function. Earlier we had discussed the covariational reasoning associated with this transformation. However, I also perceived this example to be making connections to past concepts taught.

T: The first example I want to discuss is the one from the notes: $g(x) = \sqrt{x} + 2$. We will call the one we just did $f(x) = \sqrt{x}$. Now, what we are going to do is $g(x) = \sqrt{x} + 2$. What do I have to do to this graph?

S Choral: Lift it up two (Research log, 3/12/18)

The students' response suggested their procedural knowledge, which they transferred from quadratics to the context of square root functions.

Critical Incident: The importance of stable prior knowledge.

Upon reflection, I examined my decision-making regarding the process of building connections to recently taught information that might not help to improve student understanding. My intention in connecting the new concepts to concepts taught earlier in the current year was to connect the *big* ideas. For example, by considering the common characteristics of graphs, students applied their knowledge of transformations to an "unfamiliar graph." However, it was not until I completed my analysis of the concept map that I realized that the prior knowledge that I sought to build upon might be unstable. I assumed because I taught the information in the past, albeit in the relatively recent past, that the information was stable. I had not considered whether the students had a conceptual understanding. For instance, consider two students in the same class; both may apply the new information. My concern is not with the student who developed a conceptual understanding of transformations. My concern is for the student who performed the task of transformation by rote, therein enacting a directly taught procedure. By building new connection on this student's unstable prior knowledge, I am not providing the student an opportunity to learn the new information meaningfully. In turn, I am tolerating and perpetuating the student's rote learning.

Ponte and Quaresma (2016) found that the key elements in classroom communication include "inviting students to explain their solutions, to make connections, to make conjectures, and to justify assertions" (p. 64). When the teachers encourage students to explain relationships among varying quantities (see Ellis, 2011), they build stronger foundations in quantitative reasoning. Furthermore, Ellis (2011) suggested that the teacher's role is to shape the discussion by posing appropriate questions and encouraging students to generalize relationships that are grounded in meaning that they have. In the following vignette, a student described their understanding of a parabola from its representation in a table of values that was different from the way I thought about the problem. They justified their assertion of symmetry in the parabola as they gestured a drawing in the air. They drew the vertex below the symmetric points, facing the parabola upward.

x	y
0	5
1	0
2	-3
3	-4
4	-3

Figure 27. Table of values for a quadratic function

T: We know it is a quadratic if it is a parabola. Right? But, looking at this table of values, does anybody see a parabola screaming at them? No?

S1: Nods. Yes.

T: Do you kind of…? Well, that is good. [I waited for their explanation]

S1: You look at the -3, -4, and -3 [They are pointing in the air to the y-values.]

T: That is good. [I was actually not following.]

S1: The -3 and -3 over here [symmetry] and the -4 dropped down below it.

T: That is the first time today that I actually see that. Nicely done. (Research log, 3/15/18)

In listening to this student's justification for their answer, they convinced me that they had a conceptual understanding of quadratic symmetry, and their ability to explain their reasoning to me demonstrated meaningful learning of quadratic symmetry. Harel and Sowder (2005) suggested that students reveal important ways of understanding as they explain their thinking. As the student explained their thought process, in this vignette I built on my past connections and learned meaningfully. Both of the above questions revealed that the students and I were seeking our own particular, meaningful learning opportunity.

Critical incident: My interpretation of meaningful learning differed from that of my students.

Initially, I thought that I needed to provide more opportunities for meaningful learning, but during the analysis I recognized that my interpretation of my own meaningful learning differed from that of my students. I observed students' meaningful learning as they demonstrated conceptual understanding (e.g., by applying new concepts to past concepts already taught) and procedural knowledge (e.g., by transferring procedures from a quadratic function to an square root function), but I also needed to do this kind of mathematical work in relation to my own teaching. Therefore, when this student discussed quadratic symmetry, I began to consider where I could apply that same way of thinking to other instances. For example, given a piece-wise absolute value function, I could discuss the role that symmetry plays by reflecting its two pieces over the line of symmetry.

Living education theory and steps. At the onset of this study, I limited the scope of living education theory to my belief that teachers needed to explain the "big" idea or concept for students to understand mathematics. The concept map for living educational theory excerpt in Figure 28, illustrates the evolution of the connection between my living educational theory and steps.

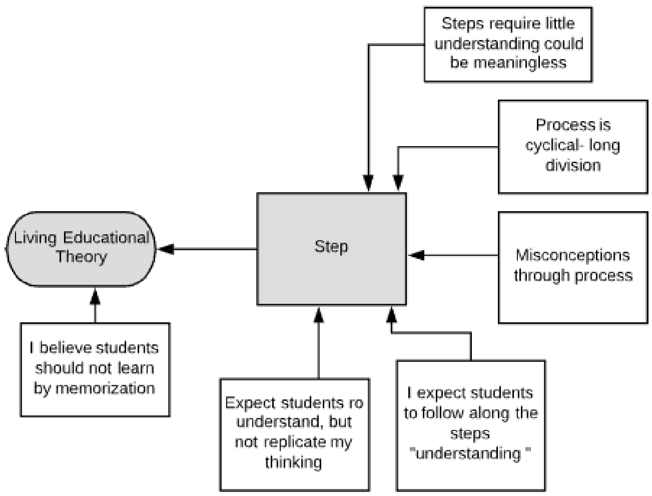


Figure 28. The network from the concept map of living education theory

I considered teaching with steps such as making a list on the board to encourage procedural thinking ("skills to carry out procedures flexibly" (NRC, 2001, p. 116)). I considered conceptual understanding ("comprehension of mathematical concepts" (NRC, 2001, p. 116)) to be paramount. As the study progressed, I realized that my beliefs about teaching with steps shifted. Sometimes, students needed to know what steps to take when solving a problem. After re-evaluating my beliefs, I came to understand the significance of Hamilton and Pinnegar's (1998) ideas about living educational theory. "It is living because, as people engage in understanding it, they learn more and their theory changes as they understand more. Furthermore, because they are living what they learn, new knowledge emerges" (p. 242).

My actions in the classroom exemplified a "living" change about my beliefs that I did not teach with steps. However, as became obvious to me, my beliefs did not match my actions, a phenomenon that Alderton (2008) warned us about. I realized that some students wanted to be told the procedures: first you add or subtract and then you multiply and divide. I taught this process through the application of inverse operations, which is a process that *all* of my students *needed* to know.

Critical incident: My teaching beliefs and actions did not align.

It was the global interpretation illustrated by the concept map that highlighted a broader understanding of “step.” When, I stepped back and reflected on the steps I used during class lessons, I realized that I used “steps” differently. In some cases, I used the word “steps” to suggest a prescribed sequential order, as a to do list. I perceived that students who seemed to be asking for this type of “steps” lacked a conceptual understanding of the solving process. That is, they lacked the foundation necessary to solve multi-step equations, such as those that require the application of the distributive and other properties. As a result, they followed my thinking process rather than developing their own: “You can think of it this way,” (Research log, 2/28/18). In other cases, I used the word “steps” to request a sequential order proposed by the students of a non-routine solving process. I perceived students who developed their own steps for solving a non-routine problem as possessing conceptual understanding. I realized that I responded with admiration when I perceived that students were developing their own problem-solving strategies, “I didn’t even think of that was a good idea. Guess what *I* was thinking?” (Research log, 4/23/18)

I also realized that I used language that suggested that my thinking process was best because I, unnecessarily, stated the way I think. Upon reflection, I realized that it was not my intention for students to recreate my thinking, such as when I was dealing with teaching literal equations.

T: Samuel has been given the formula for an arithmetic sequence. Samuel needs to solve the formula $s = \frac{n}{2}(a + L)$ for L. We are going to solve for L. If I want to solve for L, [I wrote P E MD and AS on the board] what does this tell me if I am going from the bottom up? What do I do last? I do the parentheses last. First, I want to do what is outside of the parenthesis. So, how do you get rid of what is outside of the parentheses? Multiply by its ____?

S1: Reciprocal

T: Now, You could multiply by 2 and then divide by n, which way do you want to multiply by the reciprocal or multiply by 2 and then divide? I am going to do it in two steps, then. If I multiply both sides by 2, I get $2s = n(a + L)$. Any question on that step? Yes?

S2: Why don't you get na and nL?

T: Because I did not distribute it. I am only dealing with the n/2. It felt weird right? Ok, this is what you would have to do. What if I distributed it? I would get this. $2s = \left(\frac{n}{2}a + \frac{nL}{2}\right)2$. Then cancel out the 2's. Then, it's the same thing.

(Research log, 4/23/18)

The vignette demonstrated my use of procedural steps by following the acronym for the order of operations to solve for L. More importantly, I did not explain why I was preforming the steps. Nevertheless, I prompted students to provide guidance. This exemplar demonstrates that I taught procedurally in contrast to teaching for conceptual understanding. I know that they needed to understand the process to perform the procedure, but they lacked the understanding behind the procedure. These were the moments I experienced that contradicted my living educational theory.

Critical incident: Do I always encourage a conceptual understanding?

I realized that I missed an opportunity for the students to develop their own process when I went back to $2s = n(a + L)$. Although continuing with $\left(2s = \frac{n}{2}a + \frac{nL}{2}\right)2$ would have been challenging to the students, they could have developed the reasoning behind the process. I worried that I would confuse some of the students. I had to ask myself whether procedural knowledge is enough in some cases or is conceptual understanding more important? I want students to understand why the processes they perform are mathematically sound, but there are times when I have to make compromises, so the procedure becomes adequate. Is it enough to say, this is the most efficient method?

I am aware that there are times when proficiency seems to be only about procedures, such as order of operations, perhaps it is because that order of operations has more to do with convention than logic. I encouraged procedural thinking when I used the phrase, “First, I want to do what is on the outside of the parentheses” (Research log, 4/23/18) by applying the order of operations. To promote conceptual understanding, I could have compared the difference between multiplying by the reciprocal and using the distributive property.

With respect to living educational theory, the evolution of my beliefs changed in real-time through my teaching practices. My practice was transformed through the critical incident reflective process as I grappled with my use of the term “steps” in addition to the insight that steps reflect both procedural knowledge and conceptual understanding. Ultimately, I recognize that I need to balance procedural knowledge and conceptual reasoning.

Process thinking. The concept map excerpt in Figure 29 illustrates connections between process thinking and tools used to develop process thinking. I interpret process thinking in two ways. The first way I used the concept of *thinking about process* was using tools of advance and graphic organizers to develop process thinking.

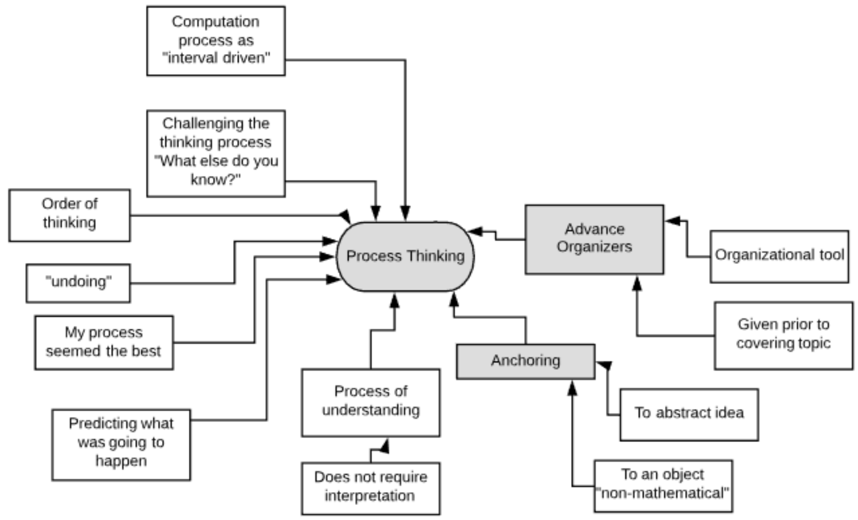


Figure 29. The network from the concept map of process thinking

One tool used to develop students' process thinking is to provide them with advance organizers – "organizers produced in advance of the learning material itself and… presented at a higher-level of abstraction" (Ausubel, 1967, p. 26). Initially, I provided the advance organizer because I thought it would be useful for promoting conceptual understanding. Later, I realized it was actually used more often to support procedural knowledge.

Generally speaking, advance organizers assisted students in scaffolding new information to prior knowledge. In the following vignette, I gave the students an advance organizer on the left of Figure 30, which I envisioned using to connect various characteristics of rational functions. I soon realized, during the lesson, that there was an overwhelming amount of information on the advance organizer. Immediately, I decided to use the graphic organizer on the right of Figure 30 to facilitate their thinking instead.

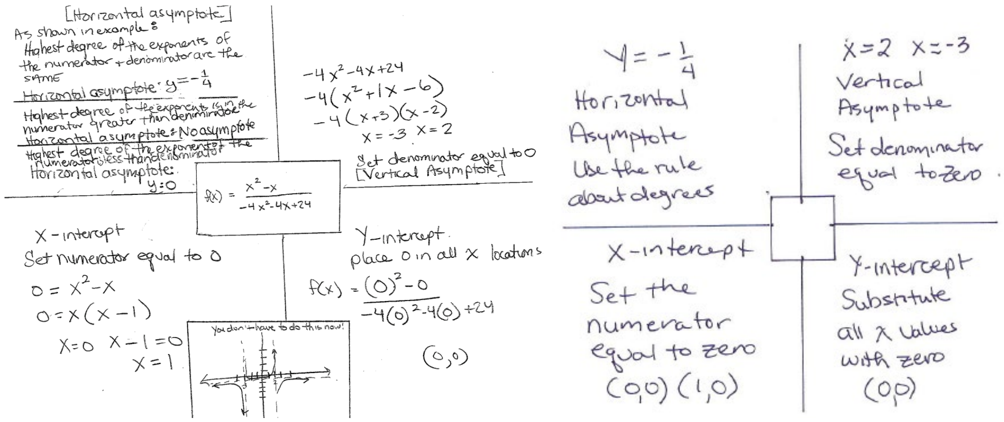


Figure 30. Advance organizer compared to the graphic organizer

$$f(x) = \frac{x^2 - x}{-4x^2 - 4x + 24}$$

T: . . . Do you see the value of the graphic organizer now? How many of you think this is helpful? [Most raise their hands.] If I could remember all the steps from the advance organizer, I would use it. I struggle as I am going [I make an over and over motion with my hand] over every single one. I have to keep referring to the advance organizer. Something is really nice about the graphic organizer. It's simple.

S3: Simple

T: We say at the same time. (Research log, 5/21/18)

Both the students and I found that the graphic organizer to be more helpful and less overwhelming than compiling all of the most important information on one paper. I realized that the goal of an advance organizer or a graphic organizer is to provide information to the students as well as to facilitate the making of connections across representations. The moment that the students and I simultaneously stated that the graphic organizer was "simple" reinforced my view of the way that we were processing information using the graphic organizer.

Critical Incident: Advance and graphic organizers gather, connect, and apply information.

I thought advance organizers were tools to gather and connect information, but for the students to make connections they needed to be able to apply all the information. However, by consolidating the information that related to rational functions in the advance organizer to assist students in applying the information, I might have inadvertently changed a tool for developing conceptual understanding into a tool for developing procedural knowledge.

To gather, connect, and apply the information, the students needed to anchor new knowledge to organize cognitive structures. This allowed students to learn the new information in chunks. By anchoring an abstract idea to stable (Ausubel, 1967) prior knowledge, the students made connections and provided opportunities for meaningful learning. That is, an organizer created for the purpose of anchoring students' prior knowledge to new information allowed students to develop their meaningful learning.

The second way in which I used the concept of thinking about process was my interpretation of process thinking compared to interval thinking as in smooth and chunky covariational thinking. I am thinking in terms of a break in thought (chunky) as interval-driven. However, my interpretation of process evolved throughout the study. Initially, I thought of process as going on forever to positive or negative infinity as in smooth thinking. By the end of the study I thought of process as the "flow of thought."

The idea of solving algebraic equations using process thinking led to the idea of process and Rover (an electronic car you program with a graphing calculator). I asked myself, "What would Rover want me to do? [I am pushing myself to develop a conceptual understanding of the process." (Critical friend memo, 5/25/18)] When I was re-enacting the movement of Rover's wheels in my mind, I asked myself how it would move to graph a line. If I wanted to draw a line, what would the process be? As I stated in one exemplar, "I envisioned Rover rolling to the y-intercept and then I asked myself, how do I move in a way that reflects the slope? Rover cannot move rise over run. [I am referring to the vertical and horizontal components of slope.] Then, it hit me. The slope was a degree." (Provoking thoughts, 5/22/18). This was my thought process in explaining how to graph a line as smooth thinking.

Through provoking thoughts and dialogues with my critical friend, my smooth and chunky thinking evolved and was reconceived. At one point, I doubted whether I thought smoothly. At another, I had shifted to the realization that I thought smoothly more often than chunkily.

Critical incident: Does smooth and chunky thinking occur when solving algebraic equations?

Extending the interpretation of process beyond Thompson and Carlson (2017) and Castillo-Garsow et al. (2013), I envisioned solving an equation as a solving process and considered chunky thinking as the steps it took to solve that equation. What would represent smooth thinking, then? Castillo-Garsow et al. (2013) referred to chunky thinking as a completed thought with a beginning and end, and smooth thinking as a continuous process that does not end. Reflecting back on old coursework, I thought of the use of differential equations to solve one problem across multiple pages as exemplifying a process (smooth reasoning). There was a point at which I forgot about the beginning and end and only lived in the moment as time passed. This represents smooth reasoning to me. However, if I stopped to contemplate the next step, it would exemplify chunky reasoning. Taken together, when solving an algebraic equation, the solving process was smooth reasoning (solving without pause) and each step (asking what comes next?) was chunky reasoning.

The following question exemplified the usefulness of my critical friend in challenging my thinking:

> *Critical friend:* Are you suggesting that chunky and smooth thinking are interwoven? Are your students able to seamlessly go from one process of thinking to the other because you've modeled it? (Narrative, 6/2/18)
>
> *Me:* Yes, I believe they are interwoven, but my concern is that my depth of knowledge affects whether or not I am able to "seamlessly" move back and forth between the two. (Narrative, 6/5/18)

This response referred to the depth of knowledge with respect to the problems I am solving at any given time. When I first began teaching, I practiced problems at home in advance of teaching a lesson and examined each step closely (chunky thinking). However, as the years passed, my confidence increased and I solved questions automatically and without pause (smoothly). Anytime a student would ask a "What if?" question regarding some problem, I would inevitably solve that problem quickly and efficiently, never second-guessing my solution. I now realize that even though I explained each step (chunkily) to the students, I *could* have solved the same question without pause (smoothly).

My analysis of the data revealed to me that my thinking reflected variational or covariational situations with smooth or chunky thinking. I transformed my understanding of process as a continuous flow of thought either in the context of graphing functions or in solving algebraic equations. My teaching practice was transformed as my understanding of process evolved in that, for example, I no longer thought of shifting a graph as reconstructing that graph using new points. Instead, my image of shifting the image was representative of smooth covariational thinking.

Building new connections. The concept map excerpt in Figure 31 illustrates the new connections I believe I have made. I developed these connections to reconceptualize the way I think about mathematical concepts. They offer evidence of the transformation of my practice as well as the new knowledge that I brought to future lessons.

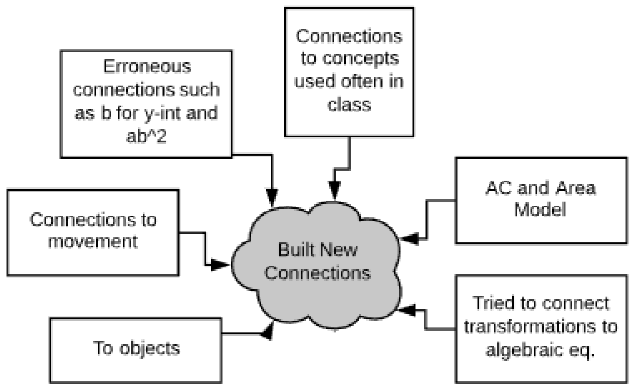


Figure 31. The network from the concept map of building new connections

In the following narrative, I realize that I am building new connections as I transform my thinking about experts' words.

> I noticed the experiences of the past that influenced my present knowledge as I transcribed. But I never considered what the definitions behind the words meant to me. What does *process* mean to me? What does *smooth* and *chunky* mean to me? Throughout this study, my committee asked me to define experts' words, such as what does Castillo-Garsow (2012) mean by smooth and chunky thinking or process, what does Ausubel (2000) mean by meaningful learning, what does Tall (1991) mean by advanced mathematical learning, and what does Thompson (2012) mean by covariational reasoning … I interpreted experts' words through *their* lens. I never considered allowing myself to define their words through *my* lens. (Narrative, 6/5/18)

The narratives that reflect new connections reveal that in an attempt to reconceptualize my smooth and chunky thinking. I was able to transform the ways I thought about my teaching practice.

Critical incident: I built new connection in an effort to better understand mathematics.

I discussed with my critical friend the importance of building new connections for my students and me as we sought to better understand the mathematics. Eventually, I realized that my conception of smooth and chunky covariational thinking was different from the experts. Initially, I interpreted chunky thinking as an interval over time and smooth thinking as a process that goes on forever, as in the case of end behaviors. In my mind, smooth and chunky thinking belonged in a geometry class, because I did not see the connection between process and algebraic equations. When my thinking about process changed, I considered smooth thinking as a continuous flow of thought. I realized that by considering process in relation to algebraic equations, I was thinking of the solving process conceptually and not attending to procedural knowledge. Ultimately, I acknowledged and accepted that I understand smooth covariational thinking in a way that makes the concepts meaningful to me.

The reflective process offered me unique insights into the development of my thinking about my practice and the teaching of mathematics. Because features of this self-study were recursive, the hermeneutics interpretive process allowed reflection to focus inward and outward (teacher and learner) as well as backward and forward (past, present, and future). I looked at my practice not only in the present, but what could I take from the present and apply to future lessons. This means that as I was teaching I could build new connections. This finding reveals the necessity of anchoring new knowledge to stable prior knowledge to promote meaningful learning, and as a result, providing students with legitimate opportunities to build new connections.

Critical Incident: Why did I think that building new connections was important?

I challenged myself to examine the effects of building new algebraic connections. These newly built connections were my attempt to help students grasp complex concepts. They were also anchored to stable, meaningfully learned prior knowledge. During a discussion with my critical friend, we discussed how I perceived building new connections: “I am equating connections with meaningful learning. Which is odd, but it is the way I think” (IRR Discussion, 6/16/18). The purpose of building new connections is to enrich learners’ understandings of the concepts being taught and thereby learn meaningfully. Ausubel (1967) stated, “By influencing the cognitive structure variables we should be able to control the accuracy, clarity, longevity in memory, and transferability of the concepts to be assimilated” (p. 22). In other words, by making new connections, my students and I are taking active control of our own learning. Our discussion continued.

> *Critical friend:* Why is meaningful learning important?
>
> *Myself:* Because, I never understood [as a child]. I guessed on a lot of answers. I did not really understand . . . I am pushing myself to develop new connections, because I equate making connections with understanding. If I do not make connections, I am memorizing. (IRR Discussion, 6/16/18)

Being a product of the 1970s, to me, the most important part of mathematics was getting the correct answer. An understanding of why the mathematics worked was never discussed. In *Principles to Actions: Ensuring Mathematical Success for All*, Leinwand (2014) viewed mathematical tasks as those that make high cognitive demands of students in order for them to meaningfully connect concepts and understanding. This is in contrast to procedures that rely on memorized facts, which only require low cognitive demand of students. That is, in his opinion, high-demand tasks promote the building of new connections, which constitutes meaningful learning.

Synthesis of Critical Incidents

When I consider the more critical moments in the transformation of my teaching practice, I find that they took place during the analysis phase. As I coded the transcriptions, I continually stopped to revisit and reflect upon the transcription. I realized that during the moments of reflection, critical incidents surfaced and I asked myself the question of why I thought about the coded words the way I did. In this section, I have shared the thinking about my practice that was provoked by these critical incidents. I described the transformation of my smooth and chunky, variational, and covariational thinking. Next, I discussed the development of my thinking about Tall's Three Worlds of Mathematics. Then, I explained meaningful learning and the importance of making connections to me. Finally, I reflected on the progression of my living educational theory.

Smooth, chunky, variational, and covariational thinking. I began this study believing that I thought chunky variationally (correspondence approach). The days prior to the beginning of this study, I worried that my thinking might not develop. As I have reported earlier, I came to realize that I thought both smoothly and chunkily. I moved in and out of smooth and chunky covariational thinking easily, as if blending the two. This blending aligned with Castillo-Garsow et al. (2013), who acknowledged the hybrid nature of smooth and chunky thinking, the back and forth or moving in and out. As the study continued, I recognized that I preferred smooth covariational thinking because I found the dynamic shifting of the parent functions in transformations more of an authentic representation of the transformation of a graph than creating a table of values and graphing each order pair.

Then I began to wonder if I was not providing my students enough computational practice such as completing a table of values or plotting points on a graph, because of my preference for smooth covariational thinking. I questioned if my students just viewed transformations of quick sketches as moving the shapes around the coordinate plane. Might there be an unintended consequence to my students because I privilege smooth covariational reasoning over practicing computation skills?

I found that there was a situational component to the moving in and out of the levels of covariational reasoning. My covariational reasoning interacted with my students' questions during class discussions, the problem contexts (e.g., transformations), and my interpretations of abstract representations (e.g., a variable raised to the power of two represents an area). More broadly, what I experienced through this self-study was that a network of factors influenced and were influenced by the emerging hybrid nature of my smooth and chunky reasoning.

Extending the interpretation of process further, I envisioned the solving of equations as solving *processes* and chunky thinking as the steps it took to solve that equation. What would represent smooth thinking, then? Castillo-Garsow et al. (2013) referred to chunky thinking as a completed thought with a beginning and end, and smooth thinking as a continuous process that does not end. When solving an equation smoothly, there was a point at which I forgot about the beginning and end and only lived in the moment as time was passing. This represents smooth reasoning to me. However, if I stopped to contemplate the next step, it would exemplify chunky reasoning. Taken together, when solving an algebraic equation, the solving process was smooth reasoning (solving without pause) and each step (asking what comes next?) was chunky reasoning.

Since the early days of this study, I asked myself if I could think of solving algebraic problems either smoothly or chunkily. There was a point in the study that my interpretation of *process* changed. My interpretation of process thinking compared to interval thinking as in smooth and chunky covariational thinking. I am thinking in terms of a break in thought (chunky) as interval-driven. However, my interpretation of process evolved throughout the study. Through provoking thoughts and dialogues with my critical friend, my smooth and chunky thinking evolved and was reconceived. My analysis of the data revealed to me that I transformed my understanding of process as a continuous flow of thought. I thought smoothly and chunkily either in the context of graphing functions or in solving algebraic equations.

The Three Worlds of Mathematics. When I began this study, I thought that our thinking would reside in only one world at a time. The transformations of different functions I described using body movements and visual representations are exemplars of the embodied world, and algebraic symbols in problem solving are elements of the symbolic world. It was not until the coding process that I realized a student could reside in one world, but draw on a representation from another world. When I attempted to connect a transformation of an odd function (embodied) to an algebraic proof (symbolic) of $-f(-x) = f(x)$, I moved back and forth between the symbolic world and the embodied world as I tried to draw on a visual interpretation of transformations to understand the symbolic proof.

Residing in one world does not imply that the student understands information conceptually or with procedural knowledge. This means that my perception that a student with advanced mathematical thinking automatically had conceptual understanding of a concept such as end behavior was problematic. When I began to question the difference between the symbolic and the formal world, I realized that providing opportunities for advanced levels of mathematical thinking meant providing them with what I perceived to be opportunities to justify an argument based on their developed procedural knowledge and conceptual understanding of topics covered beyond Algebra I.

Meaningful learning. Known for his research on meaningful learning, Ausubel (1954) stated that meaningful learning is personal because the student is "translating and rephrasing new ideas into his own terms and relating them to his own experience, personal history, and system of values" (p. 494). A student applies new knowledge to what they perceive as being important. Ultimately, Ausubel (1954) explained that whereas the teacher can only "present ideas as meaningfully as possible… the actual job of articulating new ideas into a personal frame of reference can only be performed by the learner" (p. 494).

At the beginning of the study, I thought I needed to provide students opportunities for meaningful learning. The construction of the concept map allowed me to see that a rich network of influences on my teaching practice converged at meaningful learning.

Connections. Throughout the study, I made connections to my personal history. Many times, I shared with my students that as a child I struggled with a concept. Then I would share how I thought of the concept. There were other times that I did not realize I struggled with a concept until I transcribed the lesson. It was not until I created the final analysis of the concept map that I recognized that in connecting the present lesson to my past struggles, I perceived these struggles as weaknesses. I was trying to build a connection between my students and me by relating their struggles to my past struggles. My personal history played a major role in creating new strategies to rectify the mathematical confusions I had as a child. Upon reflection, I believe that even as a child I was trying to make meaning when learning mathematics to overcome my struggles.

My intention of building connections was to instill in my students the same drive that I possessed as a child to make meaning when learning mathematics. I thought that by helping the students make or build connections I was sharing how I make meaning. To build connections, I took the ideas taught in class and anchored them to a big idea, as if to demonstrate how we make connections. However, it was not until I completed my analysis of the concept map that I realized that the prior knowledge that I built upon could be unstable. I assumed because I had a stable anchor to the prior knowledge; so did my students. I had not considered whether the students had the same conceptual understanding that I did. By building new connections to a student's unstable prior knowledge, I am perpetuating the student's rote learning. It was my perception that if the student understood the concepts, then they would not memorize the concepts as I did as a child.

Living educational theory. When I began this study, I asked myself about my beliefs about teaching. The two beliefs I found most important were that 1) it is important to think of teaching mathematics as the development of conceptual understandings of big ideas, and 2) I do not teach mathematics as a process of carrying out sequential steps. The evolution of my beliefs changed in real-time through my teaching practices. As I grappled with my use of the term "steps" in the coding process, I experienced a contradiction to my living educational theory and gained the insight that "steps" can reflect both procedural knowledge and conceptual understanding. Ultimately, I recognized that procedural knowledge did not imply a lack of understanding.

Concluding Summary

The research question that framed this self-study is as follows: *How does analyzing the development of my acquisition and implementation of the lens of smooth and chunky covariational reasoning through reconceptualizing algebraic relationships promoted the transformation of my existing teaching practices?* I aimed to answer this question by examining the development of my smooth and chunky thinking as I taught and reconceptualized transformations using smooth covariational thinking. As my smooth covariational reasoning developed, I further refined my thinking until I answered the question that emerged: How do I actually reason smoothly about an algebraic process? The answer to the question about reasoning smoothly about an algebraic process emerged as I compared the motion of a calculator-based "Rover" with the process of solving an algebraic equation. Eventually, I came to understand *process* as continuous flow of thought in our minds, which we can see, for example, in particular interpretations of a graphical transformation of the solving of an algebraic equation. Understanding a student's thinking about solving an algebraic equation allows the teacher to further advance that thinking using either steps or a continuous process, thereby encouraging meaningful learning.

The overall conclusion drawn from the network of the concept map was that the process of reconceptualizing algebraic relationships through the lens of smooth and chunky covariational thinking transformed my teaching practice and, as a result, I built new connections that encouraged meaningful learning. The essential elements for meaningful learning revealed in this study were the following:

- Monitoring the types of phrases used to prompt students to answer questions;
- Advanced mathematical thinking promotes cognitive development;
- Reviewing past students' met-befores offers them relatable exemplars;

- Advance or graphic organizers are tools for gathering, connecting, and applying new information;
- Because unstable anchors are unreliable nodes in a network of understanding, making connections to the past or building new ones offers students stable anchors;
- Engaging in frequent classroom discussions provides a forum in which to negotiate connections.

The essential elements for meaningful learning facilitated building new algebraic connections. As I mentioned, by building new connections, I was able to transform my teaching practice. I realized that when building new connections to stable prior knowledge, I retained the information because I had made these connections. Ausubel (1967) suggested that students who learn by rote build unstable anchors and, because rote learning is not meaningful, they are likely to forget new information. This rote process does not suggest that the mathematical knowledge lacks value because it is meaningless, it only identifies that the conceptual knowledge is missing. This is not to say that there is not a need for procedural knowledge in mathematics. The challenge for teachers is to balance procedural knowledge and conceptual knowledge to provide a stable anchor to build upon. Therefore, the implications for action suggest that building new algebraic connections creates experiences for learning meaningfully.

Implications

I began this study with the intention of reconceptualizing my covariational reasoning to transform my teaching practice. More specifically, if I could understand how I conceived of my covariational reasoning, educators and researchers who wanted to understand covariational thinking could benefit from having a more sophisticated model of smooth and chunky thinking to build upon. However, I did not limit the scope of this study to my own covariational reasoning.

It is obvious that smooth and chunky covariational reasoning may not be a universal area of interest in mathematical reasoning. An educator or researcher might not be in the field of mathematics. Therefore, this study describes a well-organized self-study methodology that can be used to transform teaching practices in almost any field. To encourage multiple methods for validity, this self-study included the use of narratives, dialogues, personal history, living educational theory, and the developmental portfolio method.

The question educators might continually ask themselves is, "How can I transform my teaching practice?" I discovered that as an educator I brought my past and present experiences into the classroom with the hopes of transforming my teaching practice. However, in my case, without the dialogue with the critical friend who challenged my thinking, the reflection on my teaching practices would have been a shallow experience. The critical friend asked questions that I had not asked myself. Furthermore, it was after the discussion with the critical friend, who asked how I had determined the substantive categories, when I realized that my reflections provoked periodic moments of disequilibrium. During these moments, the critical incidents surfaced that prompted me to consider the transformation of my thinking about my practice.

Critical Incidents and the Shifts They Provoked

In this next section, I connect the transformational elements that arose from the critical incidents and describe the shift in my thinking that I perceive to be the major findings. I organize this section by the nodes from the concept map.

Smooth, chunky, variational, and covariational thinking. *Critical Incident: I moved in and out of smooth and chunky thinking.* Throughout this study, I continually developed and refined my smooth, chunky, variational, and covariational thinking. Originally, I considered my thinking to be chunky (interval-driven) variational. Quickly, however, I discovered my thinking to also be smooth (process-driven). For instance, when I created a graph of a parabola, I began to tend toward thinking smoothly as I referenced the parent function and then performed the transformational shift. As the study progressed, however, I found that I shifted my thinking, moving in and out, moment by moment, from chunky to smooth thinking or variational to covariational reasoning.

Critical incident: The situation of the context affects the use of smooth or chunky thinking. For example, at one moment, I thought chunkily because the student had asked a question about a piece of a function. The next moment, I thought smoothly because students asked about the end behaviors of polynomial functions. I found that I did not differentiate between the levels of covariational reasoning, since I seemed to move back and forth based on my students' questions and/or the context of the problem. This study convinced me the power of variational and covariational thinking and that our mind could accommodate their hybrid nature.

Critical Incident: Was I too focused on smooth covariational thinking and not enough on chunky thinking? Toward the end of the study, I began to question whether students lacked the skills for graphing quadratics using a table of values, because I had taught graphing transformations using smooth thinking. I seemed to go from one extreme (chunky thinking) to the other (smooth thinking). I realized that I needed to utilize a balance of smooth and chunky thinking when I was teaching the transformation of functions. The students needed to be able to graph an equation using a table of values, but they also needed to be able to shift a parent function continuously in the coordinate plane.

Process thinking. *Critical Incident: Advance and graphic organizers gather, connect, and apply information.* Initially, I thought that introducing an advance organizer into my lessons assisted students in processing the big ideas. The advance organizer was the tool I used to gather and connect these ideas for my students. For example, a student might know that there are different methods for dividing polynomials, because I gathered the information on the advance organizer, and that there are different ways to enact the computational skills of division, because I facilitated connections across those skills. Nevertheless, this does not mean that given one of these tasks in the context of an application problem, the student knows how to apply their knowledge to solve it. I found that for students, connecting information is not the same as applying the information to a specific situation.

Critical incident: Does smooth and chunky thinking occur when solving algebraic equations? Throughout this study, I went back to my original understanding of the word *process*. That is, I envisioned a dynamic function moving toward infinity without bounds. When I envisioned chunky thinking, I thought of a *chunk* or *interval*. Throughout the study, I challenged myself to imagine solving algebraic equations smoothly. During the last few days of the study, I reconceptualized my definition of the word *process* to be a continuous flow of thought. With this shift, I considered solving an equation both smoothly and chunkily.

Embodied, symbolic, and advanced mathematical thinking. *Critical Incident: Students can reside in one world, but draw on another.* I began this study believing that a student resides in one world at a time. During the coding analysis, I realized that even within one lesson, I taught symbolically and sketched pictures or graphs on the board. That is, as I taught in the symbolic world, I still drew on the embodied world. I now realize that I was actually building connections *between* the worlds. I found that although a student might reside in one world, she might draw on another world as she furthers her understanding.

Critical Incident: What does it mean for me to provide opportunities for advanced mathematical learning? When I considered advanced mathematical thinking at the onset of this study, I thought of proofs or topics from Algebra II. I envisioned advanced mathematical thinking in the formal world. However, when I asked myself what I really meant by advanced mathematical thinking, I identified three elements in this study: 1) The students performed the algorithms and understood the algorithm process; 2) They had a conceptual understanding of the concept taught, and 3) They provided a convincing argument to justify their procedural and conceptual understanding.

Personal history: Met-befores and my past. *Critical Incident: What was the connection between my past struggles and met-befores?* It was obvious that my personal history played a major role in creating new strategies to rectify the mathematical confusion I had as a child. Many times in class I referenced these struggles and my strategies. When I asked if these examples helped, some students raised their hands to affirm. However, when I asked if clarification of a student's met-befores helped, many more students raised their hands. Therefore, I found that strategies used in student met-befores were not only helpful to other students; they were, in fact, more meaningful to them than my own.

Connections: Past connections and classroom discussion. *Critical Incident: The importance of stable prior knowledge.* Although at the beginning of this study I took for granted that connecting new knowledge to prior knowledge allowed students to scaffold information, I did not recognize the importance of the stability of the prior knowledge. However, during the coding analysis, I found that I connected new information taught to the topics that I had recently taught, such as the new connection of shifts of exponential functions to the prior knowledge of shifts of quadratic functions. Yet, I wasn't at all mindful of the stability of prior knowledge. It wasn't a lens I had before. Now, I recognize the need for stable prior knowledge to build upon in mathematics classrooms. That said, the questions I continue to struggle with are: *How do I ensure that prior knowledge is actually stable?* And, *What do I do if it isn't?*

Building new connections. *Critical Incident: Why did I think that building new connections was important?* Personal history and connections seemed to overlap for me. I perceived that by building connections I made meaning of my learning. As a child, I relied on memorized facts or procedures to "learn" mathematics. Although I do believe I had some features of a conceptual understanding of the concepts associated with those procedures, I'm sure that network of understanding wasn't sufficiently rich to be useful to me in applying that knowledge to authentic problems. As a result of this study, I realized that I continue to build new algebraic connections.

Living education theory and steps. *Critical incident: My teaching beliefs and actions did not align.* I began this study with the belief that I did *not* teach using steps, an approach I have referred to as a step-by-step (procedural knowledge) process. I believed that students should know the *big ideas* (conceptual understanding). As the study progressed, I incorrectly inferred that students who learned using procedures lacked a conceptual understanding as a result. I now recognize the difference between procedural knowledge and conceptual understanding. Procedural knowledge requires a conceptual understanding of the process of the algorithms and performing the algorithm, whereas conceptual understanding requires an understanding of big ideas.

Theoretical Implications for Teachers and Mathematics Teachers/Researchers

Freire (2000) proposed that individuals who embrace transformational thinking see the world not as a static reality, but as a reality in process, so as to reflect on their actions and in turn change future actions. As my thinking about mathematics and its teaching was transformed, my beliefs about my practice shifted as well. The theoretical conclusions that I drew from this study represent important implications for teachers, as well as mathematics teacher/researchers.

Implications for teachers. When undertaking the process of transforming thinking, situations in the classroom or in the writing of a narrative influence the folding in and out of interactive dialogue between the teacher, the critical friend and their students. That is, teachers alter their thinking and understanding based on *where* and *with whom* they are communicating. Authentic engagement between the critical friend and the teacher is critical to and allow for this folding back and forth. That is, as the critical friend asks the teacher provoking and challenging questions, the critical friend also internalizes the dialogue, and reciprocal learning occurs as responses are crafted and discussed. At the same time that learning takes place, a mutual trust is developing between the critical friend and the teacher that derives from a shared sense of vulnerability as they both share their teaching practices.

When teachers deepen their understanding of how they think about concepts, they are better able to understand how their students potentially think about such concepts. This is not as straightforward as it might sound. In addition to planning for teaching by thinking about how new ideas will build on students' current thinking, it is also useful for teachers to interrogate their own thinking about concepts so that they can better imagine what new connections they need to facilitate. Moreover, teachers should realize the major role that their personal history plays in building new connections. If they do not take the time to explore the qualities of their own thinking, they may find themselves unknowingly projecting meaningless learning into their teaching practice.

Teachers should reflect before, during, and after teaching a lesson. Prior to teaching a lesson, the teacher may reflect back on past years of teaching the same concept. Reflection during the lesson promotes opportunities for the teacher to formatively assess their students' thinking and respond in real-time. Reflecting after the lesson provides the teacher with an opportunity to identify any errors in need of improvement. As teachers reflect on their practice, they start to notice what is going on around them. These noticings are source material for further reflection. Further reflection is further practice for noticing, which then gets easier over time.

Implications for mathematics teacher/researchers. As teacher/researchers aim to assess and develop students' thinking as smooth covariational, it is useful for the teacher/researchers to imagine the *process* as continuous and not restrict the notion of continuous to mean "without bounds."

Teacher/researchers should realize that the self-study methodology allows for theorizing on one's own terms. This theorizing requires some boldness in that it allows the teacher to appropriate experts' and researchers' language. For example, a teacher's use of terms like smooth and chunky covariational reasoning, or embodied or symbolic worlds, may stray from the researchers' or experts' intentions, as long as such theorizing is based on both self and teaching practice.

Contrary to the ways that procedures and concepts are portrayed in the literature, teachers can hold views of procedural knowledge that entail an understanding the algorithm as well as the process behind solving the algorithm. That is, one can reject a false dichotomy between procedural knowledge and conceptual understanding. For instance, it is indeed possible for students to demonstrate their understanding as they use a taught algorithm to solve equations.

When new mathematical ideas are grounded in conceptual understanding and then followed up with the development of procedural knowledge, students are able to make sense of the concept — both procedurally and conceptually. By balancing the teaching of a concept both procedurally and conceptually, teachers can teach for understanding at the same time that they minimize the time it takes to teach a lesson. Students are afforded the opportunity to learn meaningfully when the procedures that can be taught with conceptual connections are taught prior to the procedures. .

Recommendations for Further Research

My ongoing analysis of the data compiled in this self-study reveal that I transformed my teaching practice as I reconceptualized my covariational reasoning. Although my transformation of practice is tailored to my experience, I propose that the use of multiple methods (e.g., narratives, dialogues, personal history, living educational theory, and the developmental portfolio method) in this self-study is worth following to determine the extent to which it might be relatable to other self-studies. That said, I make no claims that my living educational practice and my personal history are generalizable to other studies. In relation to the proposed implications of this study, this could be seen as a limitation.

A recommendation for a future study is to invite the critical friend into the classroom. This study found that the critical friend asked questions I had not thought to ask myself. For example, I could discuss the mathematics with the critical friend in real-time. The critical friend could also add a first-hand perspective of the happenings in the classroom, allowing for more specific questions, so as to delve further into underpinnings of my thinking. The purpose of the critical friend is to provide another view of a shared classroom lesson. As expected, we will each experience the classroom differently.

By discussing the critical friend's interpretation of their experience, the researcher will gain better insight into how another thinks about the mathematics and in turn, prompt new critical incidents.

Concluding Remarks

As I reflect back on this lengthy self-study, I realize that my initial intention was to describe the development of my smooth covariational reasoning, which I thought conflicted with the ways I have been thinking. Examining my actual daily teaching practice provoked me to question my beliefs about my practice, such as the importance of teaching to promote conceptual understanding. As I analyzed the classroom videos, I saw in real-time the reward of students grasping a concept and the consequences when my teaching missed the objective of the lesson. At times, it was a difficult process to watch. The critical friend discussed the day-to-day lessons and, by posing provoking questions, challenged me to verbalize my internal thoughts and dilemmas. More than once I expressed my thoughts that since this self-study took place in an algebra classroom that I would not feel a sense of accomplishment if I could not apply smooth or chunky thinking to algebraic equations. I had given up. Then, at the end of the study I used Rover (a calculator-based robot). I was ecstatic, because this provided me with an opportunity to test out and ultimately reconceptualize me covariational thinking. This is not the same as breaking a habit, that at some point you go back. When you reconceptualize your thinking, the thoughts cannot be undone.

Another of the exciting moments was the creation of the concept map. I planned to create a concept map using paper and pencil. However, when I created the map on my computer, a complex web that represented the overall study was illuminated. That map exemplified the fact that my teaching and way of thinking of self—past and present—in fact, encouraged meaningful learning and facilitated my students' building of new and stable algebraic connections. To see all the networks meet gave me a sense of pride that by envisioning the importance of building connections I created a learning experience that was meaningful.

This self-study taught me that I needed to understand the way I thought about mathematics. Throughout this study, I thought of procedural knowledge as just an algorithm that a student performs. By the end of the study, I took on a view of procedural knowledge as not simply a sequence of steps, but as a sequence of steps that could be meaningful and understood. Now, I have a basis for how my students might understand mathematics. I find I am more likely to have students justify and explain the steps of any procedure they choose to use. I recognize the importance of waiting long enough to answer questions until I am sure I am answering the questions the students' want answered so that I may develop more coherent and complete understandings of their thinking. As the student explains their thought process, we – both the students and I – build on our past connections so that we can learn meaningfully.

References

Alderton, J. (2008). Exploring self-study to improve my practice as a mathematics teacher educator. *Studying Teacher Education*, *4*(2), 95–104. doi:10.1080/17425960802433587

Algebra Nation. (2017). *Online Algebra Nation Workbook*. Retrieved from https://web.algebranation.com/#/

Amrein-Beardsley, A. (2014). *Rethinking value-added models in education: Critical perspectives on tests and assessment-based accountability*. New York, NY: Routledge.

Ausubel, D. P. (1954). *Theory and problems of adolescent development.* Oxford, England: Grune & Stratton.

Ausubel, D. P. (1967). *Learning theory and classroom practices.* Toronto, Canada: Ontario Institute for Studies in Education Bulletin.

Ausubel, D. P. (2000). *The acquisition and retention of knowledge: A cognitive view*. Dordrecht, Netherlands: Kluwer Academic Publishers.

Ausubel, D. P., & Robinson, F. G. (1969). *School learning: An introduction to educational psychology*. New York, NY: Holt, Rinehart, and Winston.

Ball, D. L. (2011) Forward. In M. Sherin, V. Jacobs, & R. Philips (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. xx-xxiv). New York, NY: Routledge.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389–407.

Baxendall, B. W. (2003). Consistent, coherent, creative: The 3 C's of graphic organizers. *Teaching Exceptional Children*, *35*(3), 46–53. doi:10.11177/004005990303500307

Belenky, M., Clinchy, B., Goldberger, N., & Tarule, J. (1997). *Women's ways of* *knowing: The development of self, voice, and mind.* New York, NY: Basic Books, Inc.

Bernard, H. R., & Ryan, G. W. (2010). *Qualitative data: Systematic approaches*. Thousands Oaks, CA: Sage Publications.

Berry, A., & Taylor, M. (2017). Personal experience methods in practitioner research. In D. Wyse, N. Selwyn, E. Smith & L. E. Suter (Eds.). *The Bera/Saga handbook of educational researcher*, (pp. 589–608). London, UK: Sage Publication. Retrieved from https://trove.nla.gov.au/version/241168857

Bogdan, R., & Biklen, S. K. (2007). *Qualitative research for education*. Boston, MA: Allyn & Bacon.

Bruner, J. S. (1966). *Toward a theory of instruction*. MA: Harvard University Press.

Bullough Jr, R. V., & Pinnegar, S. (2001). Guidelines for quality in autobiographical forms of self-study research. *Educational Researcher*, *30*(3), 13–21. doi:10.3102/0013189X030003013

Carpenter, T. P., & Lehrer, R. (1999). Teaching and learning mathematics with understanding. In E. Fennema & T. R. Romberg (Eds.), *Mathematics classrooms that promote understanding*, (pp. 19–32). Mahway, NJ: Erlbaum.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth, NH: Heinemann. Retrieved from https://eric.ed.gov/?id=ED474452

Carlson, M. P. (1998). A cross-sectional investigation of the development of the function concept. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), *Research in collegiate mathematics education*, *III. Issues in mathematics education, (*pp. 114–162). Washington, DC: Mathematical Association of America.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, 33(5), 352–378. doi:10.2307/4149958

Castillo-Garsow, C. (2010). *Teaching the Verhulst model: A teaching experiment in covariational reasoning and exponential growth* (Unpublished doctoral dissertation). Arizona State University, Tempe, Arizona.

Castillo-Garsow, C. (2012). Continuous quantitative reasoning. In R. Mayes, R. Bonillia, L. L. Hatfield & S. Belbase (Eds.), *Quantitative Reasoning and Mathematical Modeling: A Driver for STEM Integrated Education and Teaching Context. WISDOMe Monographs, Volume 2*, (pp. 55–73). Laramie: University of Wyoming. Retrieved from http://www.uwyo.edu/wisdome/_files/documents/castillo_garsow.pdf

Castillo-Garsow, C. (2014). Beside the iterable unit: Reply to Steffe et al. In L. P. Steffe, K. C. Moore, & L. L. Hatfield (Eds.), *Epistemic algebraic students: Emerging Models of Students' Algebraic Knowing Papers from an Invitational Conference, WISDOMe Monographs, Volume 4*, (pp. 157–186). Laramie: University of Wyoming. Retrieved from http://www.yeolcoatl.net/research/2014cwcg_iterableunit-wordonly-assembled_version3.6.pdf

Castillo-Garsow, C., Johnson, H. L., & Moore, K. C. (2013). Chunky and smooth images of change. *For the Learning of Mathematics*, *33*(3), 31–37. Retrieved from www.jstor.org/stable/43894859

Cochran-Smith, M., & Boston College Evidence Team. (2009). "Re-culturing" teacher education: Inquiry, evidence, and action. *Journal of Teacher Education*, *60*(5), 458–468.

doi:10.1177/0022487109347206

Cochran-Smith, M., & Lytle, S. L. (1993). Research on Teaching and Teacher Research: The issues that divide. In M. Cochran-Smith & S. L. Lytle (Eds.), *Inside/outside: Teacher research and knowledge,* (pp. 5–22). New York, NY: Teachers College Press. Doi:10.2307/1176596

Cohen, L. M., & Manion, L. L., & Morrison, K. (2011). *Research methods in education* (7th ed.). London, UK: Falmer Press.

Common Core State Standards Initiative. (2010). *Common Core State Standards for mathematics*. Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Confrey, J., & Smith, E. (1995, January). Splitting, covariation, and their role in the development of exponential functions. *Journal for research in mathematics education*, 26(1), 66–86. doi:10.2307/749228

Connelly, F. M., & Clandinin, D. J. (1990). Stories of experience and narrative inquiry. *Educational Researcher*, *19*(5), 2–14. doi:10.3102/0013189X019005002

Costa, A. L., & Kallick, B. (1993). Through the lens of a critical friend. *Educational Leadership*, *51(2)*, 49–52. Retrieved from http://www.educandojuntos.cl/wp-content/uploads/2017/08/through_the_lens_of_a_critical_friend.pdf

Cornu, B. (1983). *Apprentissage de la notion de limite: Conceptions et obstacles [Apprenticeship of the limit notion: Conceptions and obstacles]* (Published doctoral thesis) University of Grenoble, Grenoble, France.

da Ponte, J. P., & Quaresma, M. (2016). Teachers' professional practice conducting mathematical discussions. *Educational Studies in Mathematics*, *93*(1), 51–66.

doi:10.1007/s10649-016-9681-z

de Beer, H., Gravemeijer, K., & van Eijck, M. (2015). Discrete and continuous reasoning about change in primary school classrooms. *ZDM*, *47*(6), 981–996. doi:10.1007/s11858-015-0684-5

Denzin, N. K., & Lincoln, Y. S. (1994). *Handbook of qualitative research*. Thousands Oaks, CA: Sage Publications.

Dewey, J. (1933). *How we think: A restatement of the relation of reflective thinking to the educative process*. Boston, MA: Heath.

Ellis, A. B. (2007). Connections between generalizing and justifying: Students' reasoning with linear relationships. *Journal for Research in Mathematics Education*, 38(3), 194–229. doi: 10.2307/30034866

Ellis, A. B. (2011). Algebra in the middle school: Developing functional relationships through quantitative reasoning. In J. Cai & E. J. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives*, (pp. 215–238). New York, NY: Springer Press. doi:10.1007/978-3-642-17735-4_13

Freire, P. (2000). *Pedagogy of the oppressed*. New York, NY: Bloomsbury Publishing.

Goodell, J. E. (2006). Using critical incident reflections: A self-study as a mathematics teacher educator. *Journal of Mathematics Teacher Education*, *9*(3), 221–248. doi:10.1007/s10857-006-9001-0

Goodell, J. E. (2011). Reforming mathematics teacher education through self-study. In S. Schuck & P. Pereira (Eds.), *What counts in teaching mathematics? Adding value to self and content,* (pp. 111–125). Dordrecht, Netherlands: Springer Press. doi: 10. 1007/978-94-007-0461-9_8

Gould, J., & Gould, E. J. (1999). *Four square: Writing method for Grades 3-5.* Carthage, IL: Teaching and Learning Company.

Grandau, L. (2005). Learning from self-study: Gaining knowledge about how fourth graders move from relational description to algebraic generalization. *Harvard Educational Review*, *75*(2), 202–221. doi:10.17763/haer.75.2l10727jj72082n10

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A" proceptual" view of simple arithmetic. *Journal for Research in Mathematics Education*, 25(2), 116–140. doi:10.2307/749505

Guilfoyle, K., Hamilton, M. L., Pinnegar, S., & Placier, P. (2004). The epistemological dimensions and dynamics of professional dialogue in self-study. In J. J. Loughran, M. L. Hamilton, V. K. LaBoskey & T. Russell (Eds.), *International handbook of self-study of teaching and teacher education practices,* (pp. 1109–1167). Dordrecht, Netherlands: Springer Press. doi:10.1007/978-1-4020-6545-3_28

Guðjónsdóttir, H., & Kristinsdóttir, J. V. (2011). Team-teaching about mathematics for all: Collaborative self-study. In S. Schuck & P. Pereira (Eds.), *What counts in teaching mathematics? Adding value to self and content,* (pp. 29–43). Dordrecht, Netherlands: Springer Press. doi:10.1007/978-94-007-0461-9_3

Hamilton. M. L., & Pinnegar, S. (1998). The value and the promise of self-study. In M. L. Hamilton, S. Pinnegar, T. Russell, J. Loughran, & V. LaBoskey (Eds.), *Reconceptualizing teaching practice: Self-study in teacher education*, (pp. 235–246). London, UK: Falmer Press.

Harel, G., & Sowder, L. (2005). Advanced mathematical-thinking at any age: Its nature and its development. *Mathematical Thinking and Learning*, *7*(1), 27–50.

doi:10.1207/s15327833mtl0701_3

Herr, K., & Anderson, G. L. (2005). The continuum of positionality in action research. (2005). In K. Herr & G. L. Anderson (Eds.).,*The action research dissertation: A guide for students and faculty* (pp. 29–48). Thousand Oaks, CA: SAGE Publications Ltd. doi: 10.4135/9781452226644

Hjalmarson, M. A. (2017). Learning to teach mathematics specialists in a synchronous online course: A self-study. *Journal of Mathematics Teacher Education*, *20*(3), 281–301. doi:10.1007/s10857-015-9323-x

Hunt, D. E. (1987). *Beginning with ourselves: In practice, theory, and human affairs*. Cambridge, MA: Brookline Books.

Johnson, H. L. (2010). Making sense of rate of change: Secondary students' reasoning about changing quantities. Unpublished doctoral dissertation. University Park, PA: The Pennsylvania State University.

Johnson, H. L. (2015). Together yet separate: Students' associating amounts of change in quantities involved in rate of change. *Educational Studies in Mathematics*, *89*(1), 89–110. doi:10.1007/s10649-014-9590-y

Klein, E. J., & Taylor, M. (2017). Teacher educators struggling to make complex practice explicit: Distancing teaching through video. *Studying Teacher Education*, *13*(3), 312–330. doi:10.1080.17425964.2017.1366307

Koch, T. (1999). An interpretive research process: Revisiting phenomenological and hermeneutical approaches. *Nurse Research*, 6(3), 20–34. doi:10.7748/nr1999.04.6.3.20.c6085

LaBoskey, V. K. (2004). The methodology of self-study and its theoretical underpinnings. In J. J.

Loughran, M. L. Hamilton, V. K. LaBoskey, & T. Russell (Eds.), *International handbook of self-study of teaching and teacher education practices,* (pp. 817–869). Dordrecht, Netherlands: Springer Press. doi:10.1007/978-1-4020-6545-3_21

Laverty, S. M. (2003). Hermeneutic phenomenology and phenomenology: A comparison of historical and methodological considerations. *International Journal of Qualitative Methods*, *2*(3), 21–35. doi:10.1177/160940690300200303

Leinwand, S. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: National Council of Teachers of Mathematics.

Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic inquiry*. Beverly Hills, CA: Sage Publication.

Lortie, D. (1975). *Schoolteacher: A sociological analysis.* Chicago, IL: University of Chicago Press.

Loughran, J. J. (2004). A history and context of self-study of teaching and teacher education practices. In J. J. Loughran, M. L. Hamilton, V. K. LaBoskey, & T. Russell (Eds.), *International handbook of self-study of teaching and teacher education practices,* (pp. 7–39). Dordrecht, Netherlands: Springer Press. doi:10.1007/978-1-4020-6545-3_1

Lovin, L. H., Sanchez, W. B., Leatham, K. R., Chauvot, J. B., Kastberg, S. E., & Norton, A. H. (2012). Examining beliefs and practices of self and others: Pivotal points for change and growth for mathematics teacher educators. *Studying Teacher Education*, *8*(1), 51–68. doi:10.1080/17425964.2012.657018

Marin, K. A. (2014). Becoming a teacher educator: A self-study of the use of inquiry in a mathematics methods course. *Studying Teacher Education*, *10*(1), 20–35. doi:10.1080/17425964.2013.873976

Mason, J. (1998). Enabling teachers to be real teachers: Necessary levels of awareness and structure of attention. *Journal of Mathematics Teacher Education*, *1*(3), 243–267. doi:10.1023/A:1009973717476

Mason, J. (2002). *Researching your own practice: The discipline of noticing*. London: Routledge Falmer.

Mason, J. (2011). Noticing: Roots and branches. In M. Sherin, V. Jacobs, & R. Philips (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 35-–60). New York, NY: Routledge.

Mason, J., Burton, L. & Stacey, K. (1982). *Thinking mathematically*. London, UK: Addison-Wesley.

Maxwell, J. A. (2009). Designing a qualitative study. In L. Bickman & D. J. Rog (Eds.), *The SAGE handbook of applied social research methods*, (pp. 214–253). Thousand Oaks, CA: Sage Publications. Retrieved from http://noles.skrivesenteret.no/wp-content/uploads/2016/10/Maxwell-Designing-a-qualitative-study.pdf

Maxwell, J. A., & Miller, B. A. (2008). Categorizing and connecting strategies in qualitative data analysis. In P. Leavy & S. Hesse-Biber (Eds.), *Handbook of emergent methods*, (pp. 461–477). New York, NY: Guilford Press.

McGowan, M., & Tall, D. (1999). Concept maps and schematic diagrams as devices for documenting the growth of mathematical knowledge. In O. Zaslavsky (Ed.), *Proceedings of the 23rd International Group for the Psychology of Mathematics Education, (281–288)* Haifa, Israel. 3. Retrieved from https://files.eric.ed.gov/fulltext/ED466963.pdf

Meehan, M., O’Shea , Ann, & Breen, S. (2017). “What you see and don’t see, shapes what you do and don’t do”: Noticing the first year mathematics lectures. *Thematic Working Group 14 Paper presented at the 10th Congress of European Research in Mathematics*

Education, Dublin, Ireland. Available from https://keynote.conference-services.net/resources/444/5118/pdf/CERME10_0332.pdf

Merriam, S. B. (2009). *Qualitative research: A guide to design and interpretation.* San Francisco, CA: Jossey-Bass.

Mishler, E. (1990). Validation in inquiry-guided research: The role of exemplars in narrative studies. *Harvard Educational Review*, *60*(4), 415–443. doi:10.17763/haer.60.4.n4405243p6635752

Mittapalli, K., & Samaras, A. P. (2008). Madhubani Art: A journey of an education researcher seeking self-development answers through art and self-study. *The Qualitative Report*, *13*(2), 244–261.

Moore, K. C., & Carlson, M. P. (2012). Students' images of problem contexts when solving applied problems. *The Journal of Mathematical Behavior*, *31*(1), 48–59. doi:10.1016/j.jmathb.2011.09.001

Munby, H., & Russell, T. (1994). The authority of experience in learning to teach: Messages from a physics methods class. *Journal of Teacher Education*, *45*(2), 86–95. doi:10.1177/0022487194045002002

National Council of Teachers of Mathematics (Ed.). (2000). *Principles and standards for school mathematics* (Vol. 1). Reston, VA: National Council of Teachers of Mathematics.

National Research Council. (2001). Adding it up: Helping children learn mathematics. In J Kilpatrick, J. Swafford, & B. Findell (Eds.), *Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education*, (pp. 115–155). Washington, DC: National Academy Press.

Novak, J. D. (2011). A theory of education: Meaningful learning underlies the constructive integration of thinking, feeling, and acting leading to empowerment for commitment and responsibility. *Meaningful Learning Review*, *1*(2), 1–14. Retrieved from http://www.if.ufrgs.br/asr/artigos/Artigo_ID7/v1_n2_a2011.pdf

Owolabi, J., & Adaramati, T. F. (2015). Effects of graphic organizer on students' achievement in algebraic word problems. *Journal of Education and Practice*, *6*(5), 39–44. Retrieved from https://files.eric.ed.gov/fulltext/EJ1083649.pdf

Piaget, J. (1953). *The origin of intelligence in the child*. London, UK: Routledge and Kegan Paul.

Piaget, J. & Inhelder, B. (1956). *Child's conception of space.* New York, NY: Norton.

Pinnegar, S. Hamilton, M. L., & Fitzgerald, L. (2010, August). Guidance in being and becoming self-study of practice researchers. In *Proceedings of 8th International Conference on S-STEP. Herstmonceux, UK*. Retrieved from https://s3.amazonaws.com/academia.edu.documents/30925450/8thInternationalSSTEPconference2010Proceedings.pdf?AWSAccessKeyId=AKIAIWOWYYGZ2Y53UL3A&Expires=1526167719&Signature=2aD%2FF%2BkompIbqTJYWfvR%2BNtDK%2Bk%3D&response-content- disposition=inline%3B%20filename%3DEngaging_with_ethical_praxis_A_study_of.pdf#page=203

Ponte, J. P., & Quaresma, M. (2016). Teachers' professional practice conducting mathematical discussions. *Educational Studies in Mathematics*, 93(1), 51–66. doi:10.1007/s10649-016-9681-z

Russell, T. (2004). Tracing the development of self-study in teacher education research and practice. In M. L. Hamilton, S. Pinnegar, T. Russell, J. Loughran, & V. LaBoskey (Eds.), *International handbook of self-study of teaching and teacher education practices,* (pp. 1191–1210). Dordrecht, Netherlands: Springer Press. doi:10.1007/978-1-4020-6545-3_30

Samaras, A. P. (2011). *Self-study teacher research: Improving your practice through collaborative inquiry*. Thousand Oaks, CA: Sage Publications.

Samaras, A. P., & Freese, A. R. (2006). *Self-study of teaching practices.* New York, NY: Peter Lang Publishing.

Samaras, A. P., Hicks, M. A., & Berger, J. G. (2004). Self-study through personal history. In J. J. Loughran, M. L. Hamilton, V. K. LaBoskey, & T. Russell (Eds.), *International handbook of self-study of teaching and teacher education practices,* (pp. 905–942). Dordrecht, Netherlands: Springer Press. doi:10.1007/978-1-4020-6545-3_23

Schwandt, T. A. (2007). *The Sage dictionary of qualitative inquiry* (3rd ed.). Thousands Oaks, CA: Sage Publications.

Schoenfeld, A. H. (2011). Noticing matters. A lot. Now what. In M. Sherin, A. Jacobs, & R. Phillipp (Eds.). *Mathematics teacher noticing: Seeing through teachers' eyes*, (pp. 223–\238). London, UK: Routledge.

Schön, D. A. (1983). *The reflective practitioner how professionals think in action.* New York, NY: Basic Books.

Schuck, S. (1999, April). *Driving a mathematics education reform with unwilling passengers*. Paper presented at the annual meeting at the American Educational Research Association, Montreal, Quebec, Canada. Retrieved from https://files.eric.ed.gov/fulltext/ED431734.pdf

Schuck, S., & Russell, T. (2005). Self-study, critical friendship, and the complexities of teacher education. *Studying Teacher Education*, *1*(2), 107–121.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational studies in mathematics*, *22*(1), 1–

36. doi: 10.1007/bf00302715

Sherin, M., Jacobs, V., & Philipp, R. (2011). Situating the study of teacher noticing. In M. Sherin, V. Jacobs, & R. Philips (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 3–13). New York, NY: Routledge.

Smith, J., & Thompson, P. W. (2007). Quantitative reasoning and the development of algebraic reasoning. In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the Early Grades*, (pp. 95–132). Mahwah, NJ: Erlbaum.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education,* (pp. 267–306). Hillsdale, NJ: Erlbaum.

Tall, D. (1989). Concept images, generic organizers, computers, and curriculum change. *For the Learning of Mathematics*, *9*(3), 37–42. Retrieved from www.jstor.org/stable/40248161.

Tall, D. (1991). The psychology of advanced mathematical thinking. In D. Tall (Ed.), *Advanced Mathematical Thinking*, (pp. 3–21). Dordrecht, Netherlands: Kluwer.

Tall, D. (1995). Cognitive growth in elementary and advanced mathematical thinking, plenary address. In L. Meira & D. Carraher (Eds.), *Proceedings of the 19th Psychology of Mathematics Education, (*61–75). Recife, Brazil 1. Retrieved from http://digilander.libero.it/leo723/materiali/algebra/dot1995b-pme-plenary.pdf

Tall, D. (2005). The transition from embodied thought experiment and symbolic manipulation to formal proof. In M. Bulmer, H. MacGillivray & C. Varsavsky (Eds.), Proceedings of Kingfisher Delta: *Vol. 5. Fifth Southern Hemisphere Symposium on Undergraduate Mathematics and Statistics Teaching and Learning (pp. 23-35). Australia: Fraser Island. Retrieved from* https://www.researchgate.net/profile/David_Tall/publication/228981514_The_transition_from_embodied_thought_experiment_and_symbolic_manipulation_to_formal_proof/links/00b4952b363783da23000000/The-transition-from-embodied-thought-experiment-and-symbolic-manipulation-to-formal-proof.pdf

Tall, D. (2006a, December). Encouraging mathematical thinking that has both power and simplicity, plenary address at the APEC – Tsukuba International Conference, "*Collaborative Studies on Innovations for Teaching and Learning Mathematics in Different Cultures (II)–Lesson Study Focusing on Mathematical Thinking*-" CRICED: *University of Tsukuba, Tokyo.* doi:10.1.1.510.6483

Tall, D. (2006b). A theory of mathematical growth through embodiment, symbolism, and proof. In *Annales de Didactique et de Sciences Cognitives,* IREM de Strasbourg, 11, 195–215. Retrieved from https://pdfs.semanticscholar.org/8b66/78300961be114ba653c5a5da3ab595eaede0.pdf

Tall, D. (2007, August). Setting lesson study within a long-term framework of learning. *Presented at APEC Conference on Lesson Study*, Khon Kaen: Thailand, 1–17. doi:10.1142/9789812835420_0003

Tall, D. (2008a). The transition to formal thinking in mathematics. *Mathematics Education Research Journal*, *20*(2), 5–24.

Tall, D. (2008b). Using Japanese lesson study in teaching mathematics. *Scottish Mathematical Council Journal*, *38(1)*, 45–52. https://homepages.warwick.ac.uk/staff/David.Tall/pdfs/dot2008d-lesson-study.pdf

Tall, D. (2013). *How humans learn to think mathematically: Exploring the three worlds of mathematics*. NY: Cambridge University Press.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*(2), 151–169. doi:10.1007/BF00305619

Thompson, P. W. (2012). Advances in research on quantitative reasoning. In R. Mayes, R. Bonillia, L. L. Hatfield, & S. Belbase (Eds.), *Quantitative Reasoning: Current State of Understanding. WISDOMe Monographs, Volume 3*, (pp. 143–148). Laramie: University of Wyoming. Retrieved from http://www.uwyo.edu/wisdome/_files/documents/thompson.pdf

Thompson, P. W. (2013). In the absence of meaning. In K. Leatham (Ed.) In *Vital directions for mathematics education research* (pp. 57–93). New York, NY: Springer Publications. doi:10.1007/978-1-4614-6977-3_4

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium of research in mathematics education,* (pp. 421–456). Reston, VA: National Council of Teachers of Mathematics.

Thurston, W. P. (1990). Mathematical education. *Notices of the American Mathematical Society*. 37(7), 844–850. Retrieved from https://arxiv.org/pdf/math/0503081.pdf

Tripp, D. (1993). *Critical incidents in teaching*. London, UK: Routledge.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced mathematical thinking*, (pp. 65–81). Dordrecht, Netherlands: Springer Press.

Whitehead, J. (1989). Creating a living educational theory from questions of the kind, how do I improve my practice? *Cambridge Journal of Education*, *19*(1), 41–52.

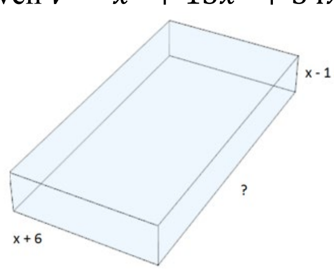
doi:10.1080/0305764890190106

Zollman, A. (2009). Mathematical graphic organizers. *Teaching Children Mathematics*, 16(4), 223–229. Retrieved from http://www.jstor.org/stable/41199444

Appendix A

Sample of Coded Research Log

Code	Date: 5/16/18
	Decision Points: What are my coefficients? S Choral: 6, -5, 4, -17. Anybody lost on any of that? S2: Nope. So these are my coefficients. These are the numbers [I am pointing to the coefficients] there are no letters or variables here. S3: Student asks why I used 3. We are finding the root or zero.
LET4 STP4	They are also telling you it is 3 here. [Pointing to $x = 3$]. This is the original question, what it is saying is this [fraction version]. I am going to pull down my 6. What do I get here? S1:18 I am multiplying 3 time 6 is 18. What do I get here? Positive 13. 3 times 13 is 39. I add this what do I get? S4: 43. 3 times 43 equals? S5: 129. For a total of? 112 is my remainder. Everybody good with that? S7: Yes. $6 \quad 13 \quad 43 \quad 112\ remainder$ My Answer is $f(3) = 112$. Everybody good with that? Questions? Yes?
CE4	S8: Is the last box the remainder. Yes, the last box is always the remainder. That is why I like to put it in a box to remind me to be focusing on the remainder. If it was a zero there would be no remainder. And we are going to talk about that right now. Is everybody ok with what the remainder theorem is? Also, the Remainder Theorem is also known as the little Bezout's Theorem. I came across this name while I was looking for a nice problem for you and thought I would share it. Use the Factor Theorem to verify that $x + 4$ is a factor of $f(x) = 5x^4 + 16x^3 - 15x^2 + 8x + 16$ in other words $x = -4$ is a root or zero. Remember these are for your notes purposes. If I want to verify that $x + 4$ is a factor and $x = -4$ is a root. What would I expect the remainder to be? S1: 0 Yes, I expect the remainder to be 0. What is the number on the outside? S Choral: -4, -4. It is ok. What are my coefficients? S Choral: 5, 16, -15, 8, 16. Perfect. Did I lose anybody there? These are your coefficients.
LET4 STP4	What do I do first? S Choral: pull down the 5 What do I write? -20 What do I write? S1: -4 What do I write? S1: 16 What do I write? S1: 1 What do I write? S1: -4 What do I write? 4 What do I write? S1-16 Therefore, this is? S Coral: 0 The answer is yes, the remainder is zero, so the Factor Theorem says that $x + 4$ is a factor of $5x^4 + 16x^3 - 15x^2 + 8x + 16$. Long division is written in the form of $\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$.
OS4	Ok guys I am going to ask you the big question and show you how smart you are. Ready? What do you think q(x) represents? S1: quotient What do you think r(x) represents? S Choral: remainder What do you think d(x) represents? S1: dividend S Choral: divisor I heard it. Divisor S3: Is the

PH4 MB4 OS4	other d(x) the dividend? No, this is also the divisor. We will just call this the…what should I call this? S1: the original, The original function Ok S1: I don't get it. This is what they give you, the f(x) they give you. In one class I refer to it as the dividend. I will point this one out $\frac{6x^3-5x^2+4x-17}{x-3}$ this [numerator] is your function divided by you divisor- this [x-3] is going to be your denominator. Did that help anybody? It confused you more. S2: I really did not need that. Here is the thing, depending on the teacher you have in algebra II, it can really make a difference what they use. When I first started teaching algebra II, I was into putting all of this stuff down, all the definitions. There are teachers that want you to remember all that stuff, now I just want you to be successful in math. I did not even teach you this until I was sure that most of you understood how to do long division and synthetic division. Now, I am just saying it as a refresher. Depending on your teacher depends on how they will go with that.
	$\frac{\text{Original Function}}{\text{divisor}} \quad \frac{f(x)}{d(x)} = \underbrace{q(x)}_{\text{quotient}} + \frac{r(x)}{d(x)} \quad \frac{\text{remainder}}{\text{divisor}}$
CE4	Is everybody good with the Remainder Theorem? Does everyone understand I can have it written in multiple ways? Factors are important because I am going to show you how you can use it. And the division just fits in there. Now, if there is no remainder then this part will be truncated. It won't be there. Everybody good with that? Great. I am going to do a geometry component. Woo Hoo. I want you to see where geometry comes into play. You think you are never going to need this in geometry. How many people think that way? You all think you might. That is good to know. Here is a geometry component. Then, we will move onto the next topic.
	Given $V = x^3 + 13x^2 + 34x - 48$ that is the volume.  $x = -6$ Find the factor of the remaining side.
LET4 STP4	Let's see if that actually works. Everybody ready? First thing I am going to use is synthetic division to find this. I want to teach you something new and the value of synthetic division. Would you say $x = -6$ here. Wouldn't you say $x = 1$ here. Good let's use those as the

NCN4 ACH4 OS4	outside numbers. First I am going to use 1. What are my coefficients? S1: 1, 13, 34, -48. Perfect. Bring down the one. What do I get here? 1. What do I get? S1: 14. What do I get? S Choral: 14. S1 and S2: 48 S1: then you get 48. And then 0. Ok. Now look. Let's put it this way. Think about what they are saying they are telling you the length of this side is $x + 6$ and this is $x - 1$ deep. I have to get a zero here because I am using the one they gave me. This one should also give me zero. Here is where it is fun. I now, think of it as a tower if you will. These are my new numerators. In other words this is saying $x^2 + 14x + 48$ is my next factor. I am going to find the factors of this. Ok, let's come over here. I am going to put a negative six on the outside. $\begin{array}{r\|rrrr} 1 & 1 & 13 & 34 & -48 \\ & & 1 & 14 & 48 \\ \hline & 1 & 14 & 48 & 0\ remainder \end{array}$ $\begin{array}{r\|rrr} -6 & 1 & 14 & 48 \\ & & -6 & -48 \\ \hline & 1 & 8 & 0\ \ remainder \end{array}$ I am going to use these as my new coefficients. 1, 14, 48. Bring down the one. What do I put here? S1: -6. What do I put here? S1: -12, 8 aha. What do I put here? -48 This gives me a remainder of 0. The other factor is $x + 8$. There you go. Did I lose anybody at any spot? Anybody?
LET4 STP4 CE4	Advanced mathematical opportunities: Now, what we are going to do is the next lesson. Here were go. Solve Radical Equations. Here is the thing. If I am taking the problem from the worksheet, then I will tell you the problem I am taking. In this case, this is problem number 7. The questions is $-2\sqrt{24x} + 13 = -11$. What we want to do is solve for x. But, here is what is important after we solve for x, we always have to double check to make sure we do not have any extraneous solutions. Here is another perspective of extraneous solutions. Just for you who want to remember order of operations P E MD AS Backwards we always start with adding or subtracting when solving. Let's do this. First question. What do I do first? S1: minus 13. This is not meant to be difficult. $-2\sqrt{24x} = -24$. Now what. S1: subtract 2. [I crinkle my nose.] S1: I mean add 2 S Choral: divide by -2. Divide by negative 2. Think of it this way. We just did adding or subtracting. Now, is there multiplication or division? Working inward to the x. [I make a open hug movement to show inward]. Now, I get $\sqrt{24x} = 12$

PT7 PH4 MB4	How do I get rid of a square root? S2: You square it. Square it good. So, I am going to square it S3: plus and minus it. We don't have to plus or minus it. $(\sqrt{24x})^2 = (12)^2$ We only plus or minus it if we square root it. Side work: the square root of 24x is equal to $(24x)^{\frac{1(2)}{2(1)}=1}$ That is why it undoes it. We don't just say it undoes it, we have to say the math reason. So this side equals 24x and this side equals 144. Next, what do I do? S Choral: divide by 24, which is 6 [x=6]. Now, we are going to check.
LET4 STP4	$-2\sqrt{24(6)} + 13 = -11$ $-2\sqrt{144} + 13 = -11$ $-2 * 12 + 13 = -11$ $-24 + 13 = -11$ Yes, it checks. It works no extraneous solutions. What do you think it would be, if it were the cube root of something?
PT4	This one you do not have $\sqrt[3]{2x+7} = 3$. What do you think gets rid of the cube root of something? S1: you put it to the power of 3. Let me re-write it.$(\sqrt[3]{2x+7})^3 = (3)^3$ Side work: what this is saying is
LET4 STP4	$(2x+7)^{\frac{1(3)}{3(1)}=1}$ That is why the cube root raise to the third power is raised to 1. I end up with $2x + 7 = 27$ 3 times 3 is 9 times 3 is 27. Now, what do I do? S1: minus 7 that gives me $2x = 20$ then divide by 2 and
VAR4	$x = 10$. Now, I have to check to see if $\sqrt[3]{2(10)+7} = 3$. Is $\sqrt[3]{27} = 3$. Yes, so it is correct. This is a tiny bit long, but this one it going to give you an extraneous solution.
LET4 STP4	$x + 1 = \sqrt{7x+15}$ So what would you do first? I am going to square both sides. Because this side just got easier, but what happened to this side? S1: It got harder. $(x+1)^2 = (\sqrt{7x+15})^2$ Now, is when your fabulous factoring skills come into play because if you are really good you could just look at it and know what you get? If you are not so good you have to do this middle step here. You are going to end up with $(x+1)(x+1) = 7x + 15$
LET4 STP4	$x^2 + 2x + 1 = 7x + 15$ This is what your algebra II teacher is probably going to do. They will go from this step to this step and skip the middle step. Here is what I did if I was not sure how to go from this step to this step I left a blank space in my notes. Does every one see? Then, when I went home, I would look to see how to fill in that space. You are going to have teachers that are more than happy to explain every step to you and then you will have teachers with an attitude with filling in all of those steps. You are going to know if you feel comfortable or not. Next, step what am I going to do? Now, look I want to solve for x if I have a

	trinomial what do you think you are going to have to do? S1: Factor. You just need to bring all of this to this side. $x^2 + 2x + 1 \quad = \quad 7x + 15$ $\quad -7x - 15 \quad\quad -7x - 15$
PHM4	If you want to do it in one step, then do it in one step. If you want to do it in two steps, then do it in two steps. I tend to do it in one step. $x^2 - 5x - 14 = 0$ Now, I am going to think of my two factors. Two numbers that multiply to equal -14 and add to equal -5. 7 and a 2. What would the signs be? S1: -7 and a +2. $(x - 7)\ (x + 2) = 0$ $x = 7 \ \ x = -2$
VAR4 OS7	Ok, now you have to check you work. How Do you check you work? S2: Use foil. Nope. S3: Plug it in. S4: Where do you plug it in? Into the original equation. $7 + 1 = \sqrt{7(7) + 15}$ $8 = \sqrt{64}$ true $-2 + 1 = \sqrt{7(-2) + 15}$ $-1 = \sqrt{1}$ Not true $x = -2$ is an extraneous solution
AMT4	Where did the extraneous part come from? What generated the extraneous-ness? What step would have generated an extraneous solution? S1: factoring S2: The squaring. The squaring generates it. Cubing or an odd exponent does not generate it. When you square something, say -8. I am always going to get a positive. Or the forth power. Somehow when I raise something to an even exponents that is what creates an extraneous solution. I did all the steps right, but when I plug it back in it did not work. Always be weary if you are squaring a radical when you square the radical it changes the dynamics. You always want to be aware of those extraneous solutions. Did you actually follow along with what we did? S Choral: yes. Here we go.
LET4 STP4	$\frac{1}{7}(x + 9)^{\frac{3}{2}} = 49$ What do we do first? I first want to get rid of the $\frac{1}{7}$. Multiply both sides by 7 or 7 over one. $(7)\frac{1}{7}(x + 9)^{\frac{3}{2}} = 49(7)$ You can handle this two different ways. What does it mean if I have something raised to the $\frac{3}{2}$? S1: you square root it. And? Raise to the power of 3 inside. I did it this way. I raised everything to 3. $(x + 9)^{\frac{3}{2}} = 343$
PT4	

PT4	$(\sqrt{x+9})^3 = 343$ Now, is that what you were going to tell me. I want to get rid of something raised to the third power. I am going to cube root the cube. Everybody see. $\sqrt[3]{(\sqrt{x+9})^3} = \sqrt[3]{343}$ So, I end up with this cube root of a cube cancels out leaving me with a square root. $\sqrt{x+9} = 7$ Guys we literally have two minutes left of class. To get rid of this [square root] I am going to square both sides. I am going to get $(\sqrt{x+9})^2 = (7)^2$ I am going to get $x + 9 = 49$ Subtract 9 which equals $x = 40$

Appendix B

Critical Friend Narrative

Critical Friend Narrative Discussion during Analysis
Research question: How does analyzing the development of my acquisition and implementation of the lens of smooth and chunky covariational reasoning through the reconceptualizing algebraic relationships promote the transformation of my existing teaching practices? Thread 1: 5/30/18 *Me:* I noticed, yesterday, that there are many times I am trying to help my students make connections. These connections may be new knowledge to old knowledge, but the connections may also be new knowledge to analogies. We discussed, these connections or anchors are part of my teaching style. I really did not see that connections had a place in my self-study analysis or my research question. I thought that I make connections for my students. Today, two past students stopped by to visit. In passing, I asked if they learned the unit circle this year, a topic I mentioned to my students because of slope and Rover. As we spoke, I suggested they should memorize the unit circle to help them make connections with the degrees, graphs, and ordered pairs as their teacher explains lessons. I explained that when I took trigonometry, I had to self-teach because I did not take it in high school. I explained I memorized trigonometry without making any connections. I seemed to just memorize everything. As I was coding tonight, it hit me why I am trying to make connections. It is not only to connect ideas for my students, but so that I connect ideas I am teaching to help me understand. I am pushing myself to make connections or develop new connections because I equate making connections with understanding. If I do not make connections, I am only memorizing. Ausubel would say that the knowledge is meaningless. Therefore, I also equate connections to meaningful learning. *6-2-18: Critical Friend- That's a great discovery. Do you think the connections you're making are changing the way you're teaching the particular concept? Or, are your probing questions changing, your evaluation of student understanding? How are the connections you're making influencing your existing teaching practice?* *6/5/18: Me:* As I was writing my definitions of my codes, I referred to my coding as new connection and past connections. Past connections referred to connections based on past students met-befores. As I am teaching I use these met-befores as common misconceptions. I am using these met-befores, not as evaluations, but as interventions or teaching points as I am teaching a lesson. I also use the met-befores as examples to cover in class. I tend to think of these questions geared for struggling students and am always surprised when even the students I perceive as non-struggling students have difficulty.

"The students had to draw a model of $f(x) = 625 - 16x^2$" It was a falling model. I remembered that students found falling "overtime was confusing to students and that it was a bow shape" not a straight line falling. Therefore, this year, we discussed that when the stick falls straight down "there wasn't any time to drop". The other connections that I referred to were new connections. These were based on class discussions. These I would call the "aha" moments. I never thought of looking at it that way before. I see these as future teaching points or interventions. I see these new interventions are the true transformations of my practice. An example of a new connection was during a discussion $y = ab^x$. "Consider, that if I am using growth factors these factors go on forever and are therefore not within the interval and also if they are factors they are a continuous [process] idea. A student said, but I did not hear it until my transcription." Upon reflection, I realized that I never equated the common ratio with factors. When I think of ratio I think of multiplying a number, but when I think of factors I think of a measureable chunk. I am not really sure why this is the case.

Thread 2: 6/1/18

Me: During the coding process tonight, I realized that I have four kinds of connecting. I have connections of the cognitive mathematics using past examples and mathematics using analogies. Now, I realized a third type of connection between cognitive mathematical understanding using both conceptual-embodied (sensorimotor that includes visualization and motion) and proceptual-symbolic (that builds on actions or conceptual embodiment). This idea of connection is exciting. During the coding process, I realized I am developing my cognitive mathematical reasoning through the back and forth representations of conceptual-embodied and proceptual-symbolic thought.
The fourth connection I am noticing in the coding process is the back and forth between chunky and smooth and variational and covariational. When I began this study, Castillo-Garsow questioned whether starting with smooth covariational reasoning builds a stronger chunky covariational reasoning. I realized in my own thinking that this is not the case for me. It is the moving back and forth between variational and covariational and smooth and chunky thinking that develops my cognitive reasoning. This is not linear as I thought at the beginning of this study. Yes, I can think any of the four ways, but it is the flexibility of moving amongst variational, covariational, smooth, and chunky thinking that develops or strengthens ones' awareness of the thinking. At one moment I might be thinking covariationally and then bounce into variational thinking.

6-2-18: Critical friend-Are certain topics more appropriate/easier to understand than others when the process is variational/covariational or chunky/smooth? How does being able to switch from one process to the other help you with your understanding and or your teaching practice?

6/5/18: Me: Initially, I seem to move amongst variational/covariational and smooth/chunky more so in graphing of transformations. This question required me to reflect deeper and you have a point. It seems that I fall back on variational more so when I feel less comfortable with the topic such as the multiple "step" problems such as completing the square or solving inverse or rational equations. These would be the problems as perceived from the world of proceptual-symbolic. As for smooth and chunky, I will have to get back to you if there is a specific focus, as I really can't answer the question. I do believe that the topics I tend to switch back and forth I seem to have a broader view of the questions. I can see it from many angles or perspectives.

Thread 3: 6/1/18

Me: I believe it was not until I understood my interpretation of the definition of process that I was not ready to truly understand the idea of smooth thinking. I thought smooth thinking was, as shown in my transcription, continuation of thought. Not continuation in terms of mathematics continuous or discontinuous. I meant a continuous flow of thought demonstrated by my graphs going to infinity. But, it was not until I programmed Rover that I realized the continuous flow of thought begins and does not have steps in thinking (e.g., order of operations). From a past mathematical example, it is exemplified by discrete mathematics when we discussed networking and not picking up a pencil. It is a continuous flow of thought. But, just because I am not picking up my pencil when drawing a network, does not mean that I am unaware of the complexity of my task and whether I am getting closer to my ending point. I am aware that I am reaching another point. Smooth and chunky thinking is the same. I may be thinking smoothly from the beginning of the process and will continue on, but I am still aware of points as I move along the process. The process I consider the covariational thinking. I know where I am going coordinately speaking, but I am not necessarily thinking about the exact location or ordered pair.

When writing the programs for Rover I knew my intent of direction, but was not as concerned with the exact location. As I watched my students write and tweak their programs, I realized they were drawing shapes and attempting to close off the shape, but where they placed Rover was not as important as the path they created. The thought process was smooth even though they were creating chunks or segments for their turns.

6-2-18: Critical friend-Are you suggesting that chunky and smooth thinking are interwoven? Are your students able to seamlessly go from one process of thinking to the other because you've modeled it? Do they have a deeper understanding? Or, is using Rover allowing them to experience mathematics through motion? What changes in your practice, if any, encouraged your students to explore with motion?

6/5/18: *Me:* Yes, I do believe they are interwoven, but my concern is that my depth of knowledge effects whether or not I am able to "seamlessly" move back and forth between the two. Again, I need time to process my thoughts.
6/6/18: *Me:* You asked if I modeled smooth and chunky thinking seamlessly going from on to the other. Yes, when I was using Rover, I was moving back and forth. Although, I do not think the students were aware of the changing thought process. I believe they were attempting to ask themselves, what action do I need to take to get the desired reaction. I am not taking away from the students thinking process. As a matter a fact, I was a bit angry when students presented their efforts to another class and made light of

their efforts. I am not sure of the long-term affect that this activity had on my students, but for myself it changed my interpretation of process. In my mind, when I thought of process, I still had a visual understanding. I could visualize the line going on forever. I could also think of process as doing and undoing. However, it was not until I was driving in my car that I deepened my understanding of chunky thinking.

I was considering my code of "STP." I was trying to image the reader interpreting my explanation of steps. I envisioned the reader equating steps with solving an equation as procedural knowledge. This is not my intention. What word would I use to describe a step-by-step process that is not referring to knowledge, but the actual process? I am comparing a fluid process of thought and a broken process of thought. That is when it occurred to me, I am talking about solving an equation using intervals. I am not referring to a measureable interval. I am talking about an interval of thought.
"We are going to have $-4x^2 - 4x + 24$. Tell me what happens if the denominator equals zero. S3: It is zero Which means? You can't do that right? That is why we want to know about those. How do I simplify this? <u>What do I do first</u>? [I am not suggesting procedural knowledge. I am asking what do they [the students] want to do? [I am empowering them to make the choice. They could factor out a negative four or they could choose to use the AC and area model to find the factors. They can make the choice. I am giving the students options.] Go ahead. S4: Factor out a -4. I get $-4\,(x^2 + x - 6)$. Now, I want to find the factors. I want to know what value of x will make the denominator equal to zero because it can't be equal to zero. We can't do that because it is undefined. What about this guy [looking for the factors]? What is the first factor? [I remind students to find the factors they can think of] two numbers that multiply to equal -6 and add to equal 1. S3: 2 and 3 Two and three. Which one is positive and which one is negative? Positive 3 and negative 2? $-4(x + 3)(x - 2)$ What are my zeros? $x + 3 = 0$ and $x - 2 = 0$ Yes. $x = -3$ and $x = 2$." When I am referring to steps, I am actually breaking down a problem in multiple chunks of process. I began this teaching practice when I went to the high school to give multiple students an opportunity to answer questions. Additionally, this teaching practice challenges students to follow another student's thinking process. To answer your question, I do seem to be modeling chunky and smooth thinking in my classroom.

6-2-18: Critical friend-Does being able to switch from one process to the other help you with your understanding and or your teaching practice? [Myself: I copied this question In this thread because the chunky and smooth component fits nicely.]

Me: I find that by empowering the students to direct the solving process. The students challenge my thinking as I teach. There are many times I might us a different process. In the case, that I perceive my method more efficient. I will use my method as a follow up teaching moment. Sometimes, as in an example that I planned on using the point slope form and students used the slope intercept form. I stated that I had not thought of that method and gave them verbal praise. Other times, they make errors and I am able to correct their thinking process. For example, when a student using $y = ab^x$ perceived the b as the y-intercept. Now, I have to figure out the "switch" between the two. I have chunky interval thought process and I have smooth thought process. How am I connecting the two thoughts in my mind? I sometimes say, that when I am solving it seems like a mindless task when you get really good at, for example, solving or factoring. But, how did that occur?

6-2-18: Critical friend-When referring to sensorimotor, how would you define it? Are you referring to Piaget's or Tall's interpretations? What is your interpretation of sensorimotor?

Me: When I began this study, I had a rather simplistic version of sensorimotor. I saw the baby picture in Tall's illustration of the Three Worlds of cognitive mathematical thinking and took it literally. When I refer to sensorimotor as simplistic, I am referring to a less developed level of thinking, almost child-like. Today, our discussion of end-behaviors challenged my thinking that movement does not imply a less developed reasoning process. Consider, I coded end behaviors as advanced mathematical thinking because I consider the end behavior topic as algebra II and beyond. I used motion as a method to anchor or connect end behaviors to the coordinate plane. I used motion to give students a visual of an abstract idea to the symbolic interpretation of an algebraic equation. Therefore, I have developed my interpretation of sensorimotor as simply using your senses to assist in the reasoning or understanding of a concept.

6-2-18: Critical friend-Did you answer the question on reconceptualizing your thinking? How did the transformation happen?

Me: I began this process looking for evidence of reconceptualizing my thinking through the lens of smooth and chunky thinking. However, I needed to broaden the lens. To reconceptualize my thinking, it took more than a thinking process as a lens to view my practice. It took the methodology of self-study. Yes, I reflected on my practice, I transformed my lessons and even my thinking behind the lessons, but it was the coding and analysis process that provided me with the lens to examine my story (e.g., the transcription, my words, the reasoning behind my words, and the challenge to my thinking by the critical friend and the expectations of my committee members). I may not have discussed my analysis at a specific moment with my committee, but I heard them in my mind challenging my thinking by asking me what do you mean by that code? For example, what do you mean by steps? Are you referring to procedural knowledge? I heard my critical friend reminding me to write that thought before you forget. As we discussed my analysis through the narratives, the questions provoked me to think about the evidence and my thinking. To often, I ask my students to explain their thinking. My goal is to fill in those missing gaps in knowledge. As educators, we do not ask ourselves the same question. I transcribed pages of lessons, but never looked past my superficial base knowledge. I noticed the experiences of the past that influenced my present knowledge, during transcriptions. But, what I never really thought of was the definitions behind the words. What does process mean to me? What does sensorimotor mean to me? What does smooth and chunky mean to me? Throughout this paper, my committee asked me to define expert's words. I interpreted their words through the experts' lens. I missed the memo that asked me to define their words through lens.

Appendix C

Critical Friend Portfolio

How does analyzing the development of my acquisition and implementation of the lens of smooth and chunky covariational reasoning
2/26: Critical friend memo: We also discussed my fear that I would not having anything exciting in my research about smooth and chunky thinking. What if I can't think smoothly? She always says, that I can't plan my findings. If nothing happens, nothing happen, that is what I will write about. ***** 2/27: Critical friend memo: We discussed how she was correct that the smooth and chunky thinking would come naturally. Today I started to see evidence in my realizing that smooth and chunky thinking does happen in my mind. When I focus on the ordered pairs, I am thinking chunkily and when I am thinking of the growth factors I am also thinking chunkily, but when I am thinking of the function in its totality I am thinking smoothly. ***** 3/6: Provoking thoughts: Why do I go back to old questioning styles? Does using a communicator function as asking specific questions? When I began the research, I thought that motion is the only way that I can think covariationally. More specifically, using movement through technology and as long as I am using pen and pencil that I am thinking both chunkily and variationally. ***** 3/8: Critical friend memo: I also realized that I think more smoothly than chunkily. So far, most of the chunky thinking has come from the step functions and also the introduction using a table of values. But, the transformations are smooth thinking. **** 3/10: Reflection Log prior to EOC Camp: When I began this research, I would argue that I only thought chunkily and not smoothly. I also thought variationally and not covariationally. I thought that I needed movement to think covariationally. This is correct to a certain extent. But, when I meant movement, I really meant technology. This did not turn out to be the case. I now realize that making such a blanket statement is not realistic. I am finding that when I graph a function that I innately use covariational and smooth thinking. I think of the parent function and then shift it around the paper. One could argue that I am transforming as a topic of instruction, but it is more than the lesson content. Yesterday, I was given both a table of values to complete and a blank coordinate plane. Although I did fill in the table because I wanted to make sure every student could complete a table of values and reference the coordinating order pairs. I found this tedious. I really just wanted to take the parent function and shift it. As a matter of fact, the only reason I was thinking about numbers at all is to transform them the function in this case an absolute value function to the location the equation dictated. The only reason one could say that variational reasoning may have applied is because the parent function originated at the origin. However, this is not my thinking. If the function began at say

(2,4). I would have used that point at the starting point. Thus, I can say I was not thinking variationally.
Another important point of my thinking is that I am not thinking of chunks unless the problem dictated it. I am thinking smoothly, as a continuous process maybe this is because we are discussing end behaviors everyday. I am not certain.
As for variational thinking, I realize that I use variational thinking more when I need to use a specific value. For example, when using the critical value initially I like to see when the pieces share the critical value. But, when provided without a graph this did not seem as accurate at taking the crucial value and plugging it into the equations provided. Again, I seem to use variational reasoning more with an equation that expects a specific value. Today, I am working with the struggling students and look forward to seeming if I tend to use variational reasoning more, both as in instruction and also in word problems.
******
3/12: Provoking thoughts: I really had not thought about the idea of a transformation as an upward or down ward shift as having the same x value. I only thought of it as a shift of the entire graph. Does this mean that as I am getting stronger at smooth thinking I am now developing my chunky thinking?
*****
3/16: Provoking thoughts: I had not thought about it, but when I am having students graph higher degree polynomials and they find the zeros, highest degree, odd or even, and if the a is positive or negative. They are taking a smooth idea of a processed thinking because we are taking about end behaviors. But, as we graph using the x-intercepts, I am looking at the function chunkily. Although, we did not find the local maximums and minimums, the students are still determining if there is going to be a local maximum or minimum. When I ask the students about what is going on with two x-intercepts that are the same, they need to consider the idea of a bounce [multiplicity of two]. We begin with the smooth process driven reasoning and move to the chunky reasoning when we look at specifics of the graph.
*****
3/18: Provoking thoughts: Revisited my use of body language to represent functions. Talking about the analysis and how I may want to interpret the analysis. People who represent the idea from within. Motion of me moving as the function is taking a process thought and representing through body language. Multiple representation expressing ideas in my hand and making students see them in there own way. Some listening, swish language, talk out my thinking… and I call it my dance.
*****
4/23: Critical friend memo: We are closing in on the Algebra I EOC test. I have two weeks of review left. One topic that students struggle is literal equations. I am not sure how this will fit into my study, but I am going to record. Maybe it will fit into past and present. I am going to review literal equations and graphing inequalities (a student choice topic) on a coordinate planes today.
When I taught these topics before, especially literal equations, I taught it procedurally using the idea of undoing order of operations. Today, I look forward to students who feel more comfortable with solving. As for graphing inequalities and systems of inequalities this may fit more into thinking using covariational reasoning, but I am not convinced. I expect I will think of it variationally. However, I look forward to interpreting my

execution of my lesson. Typically, I identify the y-intercept and the slope and graph the line and shade. Then, I find the region they overlap. I am going to try to infuse covariational reasoning by shifting the line. I will just have to see when I can add a covariational twist. As for advanced mathematical thinking, I think that I can kick the lesson up a notch if I add in a constraint situation. Yes, I can use three equations. Maybe I can interpret the equation using words. Such as no negative time. I am glad I decided to journal first because I would not have thought of the constraint idea for today's lesson.
****
5/14: Provoking thoughts: When I transcribed my prior lesson, I realized I made an error that I had not caught during the lesson. I referred to the vertical and horizontal asymptotes as the x-axis and y-axis for the parent function. At the time, I reversed them. Although uncomfortable, the correction was necessary. I thought after I had moved onto long division. I wish I had explained the transformation smoothly or chunkily. But, I did not see any connection. When I began pasting my pictures into a word document preparing for the Research Log, I realized I had taught the concept smoothly. I was asking the students to take the parent function – two parts – and shift them to the left two places and up three. I find it interesting that I could not think on the spot. This was my thinking. Oh, there is a piece but how do I make the piece as describing it smoothly when there are two parts. I never thought of it as I was moving them together or simultaneously. It was the images side by side that allowed me to make this connection of smooth thinking. Now, I only see it smoothly and not chunkily. Is it possible that once you see a smooth process? You cannot undo the thought such as you can't undo awareness. WOW!
At the beginning, I said I only saw chunky variational.
*****
5/22: Provoking thoughts: I never thought of a student struggling with one to one correspondence. I also want student to practice when graphing the inverses. Today, I seemed to focus on variational reasoning. But, I want to figure out a way to think in terms of smooth covariational. Is it possible that if I have the students graph a completed equation it would seem to be smooth thinking? Until now, I have depended on a transformational shift to think smoothly. Not a continuous process. When I say undo the process. Is that smooth thinking? Because I am going from here to there and back again. I am thinking as a process. I am thinking of completed change. Therefore, it would not be smooth. It would be chunky. But, if I am considering the inverse going on forever. That is smooth thinking. If I apply the logic of process to Rover. When, graphing a line, it would be moving up the axis to the y-intercept and then turn at a specific degree for the slope. This allows the students the true understanding of slope.
******
5/23: Provoking thoughts: I never really thought about why a parabola did not have an inverse function. I am glad that I used this example instead of a line that I typically use. The most exciting points today was during the $x = y$ moment. I also realize that I was thinking smoothly and covariationally except when I was looking at points for exactness. I want to develop the idea of process using Rover. Today, I decided to introduce the unit circle to the students. I want them to consider how they could represent the movement of a line using movement to the y-intercept and then the degrees based on the unit circle.
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5/25: Critical friend memo: After I read my thoughts about the thinking process as covariational. My critical friend asked me where the thought came. I explained that the thought came to me as I was considering the inverse function. I know inverse is the undoing process. But, I never really thought of the thinking process of solving. [yes, I thought of order of operation, but that was still steps not a continuous flow of thought.] She suggested that this paper was challenging my thinking through reflection or paying attention to my learning. I said I have always thought of teaching in a form of reflection, but that this form of reflection seemed to take on a different depth. As we continued, I again went back to Rover and the process. What would Rover want me to do? My critical friend asked me what made me think of the process for Rover. I said it was when I taught inverses. It was a good question considering I taught inverses last year and on May 9th the initial time I used Rover I did not think of process. I said that I needed "a spark." I connected an idea I was teaching and I took this idea out of context. I was excited about this finding. I was reconceptualizing my thinking. I said, "I think about equations as a process" my thought was that I can generalize this to everyone, but then I added " someone else might think about it chunky." This was a disappointing realization that I can reconceptualize my thinking about algebraic equations and processes. I can even explain my thinking to my students. But, it is the "other" who determines the way they will internalize my thinking. ***** 5/25: Provoking thoughts: I want to write a program with Rover and use a marker. The students should then measure the angles to see if the degrees are as expected. I am going to alter my program to test the motion of both 90 degrees and also the slope of the line. First change calculators into degrees using mode then change radian into degree. I am going to use https://www.inchcalculator.com/rise-run-degrees-calculatorto show student they can use $tan^{-1}\left(\frac{3}{4}\right) = 36.87°$ Therefore if I want to graph a line of $y = \frac{3}{4}x + 6$ I should know to find the degrees I use the above and to convert 6 inches into meters I would multiply the number of inches by .0254 to find the part of a meter. Although, I do not think I need to concert the y intercept into inches, I just need to remind students if they are going to graph a line process the y-intercept should be relatively small.
Reconceptualizing algebraic relationships
3/5: Critical friend memo: I love that my critical friend is an upper level mathematics person because I can't ask colleagues questions. I was struggling with the idea that odd functions and applying it to an equation of a situations. She explained I was mixing up transformation, which is movement and the idea that when I solve algebraic equations it is about equivalence. We discussed that it was easier when I had these types of questions, when I taught in high school because here I cannot really ask anyone except my critical friend. ***** 3/6: Critical friend memo: We discussed how using technology I am able to introduce concepts to my students before I actually teach it. This allowed them to self explore. Since, we discussed I only used technology in two classes, she asked if I thought the students who did not have this technology lesson would struggle more with the ideas. I will probably have a different type of lesson in terms of thoroughness.

***** 3/7: Critical friend memo: We discussed that she was correct that when I taught the students who used technology the prior day it was a review, but the other classes I literally taught them every concept. We also discussed that the ideas that I am teaching do not seem to be the same topics we covered in Algebra I when we went to school.
Promote the transformation of my existing teaching practices
3/2: Critical friend memo: Today, we discussed my use of graphs to help students understand exponential growth and decay. I said how I thought by teaching it in this manner before using the formula that the students seem to understand the information better, than how I taught it in past years with the formula. We discussed how they really struggle with the idea of what 1 means in the formula ***** 5/18: Provoking thoughts: Prior to the lesson, I decided to use the advance organizer to organize the students thinking when it came to the degrees and asymptotes. I think the best course of action on Monday is to review factoring before introducing the graphs. I look forward to using the idea of smooth and chunky thinking when graphing. **** 5/18: Critical friend memo: I asked about the connection between the vertical asymptote and the denominator set equal to zero. She said that I should think of factoring and the importance of the zero and the same holds true for synthetic division. I discussed that I was not sure what to teach. She gets that there are assembles and field trips. I was asking about dividing polynomials. We in the past discussed that I tend to anchor new material to old knowledge. She suggested building the cancelling out of numbers then move on to canceling out factors but to be sure to use parenthesis so students do not think they can cancel the x to x and number to number. My first response is that I do that, but upon recollection, I only use parenthesis if there are two factors. **** 5/21: Provoking thoughts: I had not thought about thinking of testing points as variational thinking. I was so focused on smooth and chunky thinking. I also realized today that a more simple organizer is more helpful to the students. More is just more confusing.
Living Educational Theory
2/22: Critical friend memo: We discussed that no matter how hard I try to get students to understand the shift right and left, they seem to fall back on memorization as the opposite direction. Even though I keep trying different perspectives. Are there always going to be a few who fall back on the opposite of? It worked for us when we when to school, it is a different generation of learning. ***** 4/23: Provoking thoughts: Not really sure if I really need to video my review for the EOC test, but my thoughts are that maybe I can compare the method I taught the topics compared to the review. This transcription might play a part with living theory. When I began to video, I soon realized that during the review I have a tendency to talk about my learning experience. I decided to make another box that I refer to as transcription. It may not fall into any of the above topics, but I do believe that this transcription will add to my narrative story. Tomorrow, I have a two-hour block. I intend to focus on systems of

inequalities and quadratics. I will see where the review takes me. To be transparent, I take a picture after each question and then send it to me so each class I teach the same review. The reason I make the review uniform is to organize my scattered thinking and also to make sure that I can say to students that they all had the same review.
Personal History Method
4/25: Provoking thoughts: Yesterday was a difficult day. I had expectations of knowledge and the students did not meet that goal. When I was young, I challenged my thinking and studied to be prepared. Some of my students just seem to float along. I do not understand how students can expect knowledge to be handed to them. Of, course many of my student did know what to do. My greatest concern is that many could not graph a line. How can this be? Is it that they just did not remember? Tomorrow I give a quiz. The next day I will ask them to draw a line and see if it was only that they were rusty.

Appendix D

Interpretative Summary for Concept Map

Code	Descriptive Summary
PT	I realized my evidence of process thinking was based on a loose definition of process. I identify evidence that implies my thinking or solving process is best. The evidence shows the order of the thinking process and understanding of the thinking process. There is evidence that I ask students what they know as if challenging their thinking, which contradicts others that I ask what is going to happen and then say they do not need to interpret. In two of pieces of greater value, I ask the undoing process that seems to be procedural. The second takes the "undoing' process and the realization that I am thinking of the process as an entire or "complete" idea. Yet, the actual process I perceive broken into intervals or a chunks of the thinking process.
PCN	The evidence identifies a similarity between making connections to prior knowledge and comparing a concept to a concept that I already taught in the present year. However, I would ask whether or not students would find it is effective to connect new knowledge to prior knowledge taught. Although my intention, is to use past connects that I found successful in past. Isn't it possible, even more than likely that students may struggle with the new concepts that I am connecting to prior knowledge?
NCN	It seems that new connects strictly reflected how I developed new connections. Develop in the sense they are newly created. I only created three new connections to objects. There were two connections based on erroneous efforts by students. The majority of new connections were based on connections to abstract knowledge, transformations used in class, and concepts often referred to in class. I also connected the AC and area model. My exciting connection was that when transforming or solving an odd function that I cannot compare the representations. The new connections seemed to move my thinking to a new level of understanding. As compared to past connections that did not seem to elevate my understanding of the concepts.
ACH	Anchoring to an idea is different from making a connection. Anchoring I see as attaching to an existing idea. Connecting I seem to be using as more as a comparison. Interestingly, I tend to anchor to a non-mathematical object or a algebraic symbolic concept. Although, I really do not think that anchoring takes place if the original idea is not a stable idea. In one example, I anchored to any fraction over itself equals one. This to me is stable, but stating "it is falling into everything you are doing" is not a stable idea.
ADV	I used the advance organizer to introduce multiple topics that are connected. This also organized students thinking. Additionally, I used a concept map to organize my thinking. I used the concept map as a synopsis of the most important information, necessary.
RC LFD	Reconceptualized thinking as a result of learning from classroom discussion. The first common piece of evidence came from students answering questions in a differently than I expected. Three pieces of common mathematical

	knowledge that I learned was one h(0) is both even and odd. I really should not call an asymptote an imaginary line that you can't touch. Students may confuse imaginary numbers and imaginary lines. I should refer to the asymptote as boundaries. I was surprised with the confusion over continuous functions. Students perceived a function without arrow as not being continuous. They also perceived continuous as only increasing. The most valuable piece of evidence was when I realized that a trigger could bring forth a feeling or expectation. It can also bring forth a mathematical concept of confusion.
PHM	It seems my past experiences affect both the strategies I use in class to solve problems and also the proactive methods I use in preparing my students for the state test. I point out errors I made as a young student and errors while completing the homework and class problems. I openly discuss strategies I found successful to alleviate stress during a stressful problem and steps used when I struggled with types of problems. I found it interesting that each of my past experiences were based on negative situations that I turned around to show successful solutions.
PH BM	It seems that the met-befores that I have experienced with past students has affected my teaching. The met-befores I addressed are very specific mathematic concepts. These include perpendicular lines, negative exponents, where to place the negative sign to represent a negative fraction, plugging in a negative number into an equation that has a negative coefficient, long division of polynomials, asymptotes, canceling out x's when simplifying a rational equation, interpreting the answers to a word problem, and what does a the one of 1.0275 represent. Although these examples seem extensive, upon reflection, these examples seem minimal compared to the bulk of the data set. Unlike the reflection of my past experiences for the coding of PHM, the met-befores seem to be more relatable to the students. My students seem to share more of the met-befores with my students than my past experiences as a student. It may be that the met-befores are specific to mathematics, whereas the PHM seems to be more abstract.
LET STP	Living Education Theory asks if I my beliefs align with my execution of in this case solving equations. I would have to say in general they definitely do not know if they align. I ask students to understand and not memorize the process. I cannot infer what my students are thinking. I can say that my definition of steps is not necessarily procedural. When I am saying what is the next step, I am thinking in terms of the chunk of the process. However, I have separated the steps into five categories. The first I see the step as cyclical, literally going over and over when solving long division. There is the step that asks students to interpret a misconception when the exponents do not align. The next category is when I explain the steps. Yet, I do not expect that my students will replicate the steps in the near future. For example, when I explained to students the mathematical logic of why keep change flip, although I do not like hearing the phrase in my class, holds true. The next category I ask students the steps with very little interjection or explanation. I ask, what is next? Finally, the last category, I ask the students the steps, but they explain along the way. The last two categories, one could argue that the last demonstrates greater understanding, but I could also argue that they could also be completely solid in

	their understanding and that is why they are able to solve an equation with little pause.
CE OS	I am writing the summary together as they are two of the cognitive worlds. Initially, I thought there would be little to discuss. Coded them as motion and visual as CE and any symbolic representations as OS. When I started to separate them I realized there was an element of interpretation. The first category was motion without interpretation, motion without requiring thinking. They could be perceived as motion that could be mimicked without understanding. The next classification was motion that required interpretation, I expected a discussion to ensue. The next two categories was motion shared CE and OS on the same slip. Interesting, initially I separated them into one I perceived as a more advanced symbolism. When I re-read I realized the two categories were split further. The one I perceived as less symbolically challenging, the CE I used was motion. The symbolism I perceived as more challenging, the CE was visual and without motion. The last two categories were coded as OS, one required interpretation or discussion and the other was symbolic with little or know interpretation or discussion.
SM CH VAR COV	I envision smooth thinking as thinking of the continuation of an entire process without bound. When I think of chunky thinking, it seems to be in the form of intervals or pieces either in language or graphs. However, it is the combination of SM and VAR or SM and COV that seems more comfortable for my thinking. Such as, thinking of a graph without using quantities (quantitative reasoning) and the shifting of the graph or thinking of a continuous process and what happens at certain points. However, it becomes evident that it is the flow of the four, moving in and out, that gives me the greatest understanding of my thinking. It may be that my mind is no longer thinking linearly. Another possibility is that my mind is a flow of thought and I need to represent an idea from many perspectives to explain it in its entirety.
VAR	I isolated the VAR from the SM or CH section. These examples were either input or output or required an interpretational component. The interpretational component simply means that when the reasoning took more than plugging in a variable and getting a response. For example, determining the value when given an interval or piecewise function or determining an extraneous solution.
AMT	In this section, there were minimal comments. My one observation was that my tone I used was more straightforward when discussing the topic. Although, I do not believe I can make a blanket statement, as there were many AMT codes throughout the other sections. These exemplars did not have another home.

Appendix E

Inter-Rater Reliability Discussion

Final Concept Map Discussion
Excerpt from the final concept map discussion to demonstrate inter-rater reliability of coding. The discussion lead to the importance of meaningful learning. Thread: Struggle *Me:* I commented that when I look at my past history. I see the struggles. I see the struggles as negative? *Critical Friend:* Why do I see the struggles in the past as negative? *Me:* Why do I only remember the struggles? Critical Friend: You overcame your struggles. You persevered. You had to overcome your struggles. *Me:* I remember very little growing up, but they [struggles] are the only ones I remember. I know negative is not the right term. I just don't seem to see any positive memories. Some I did not even realize until I was teaching and I would say oh yeah I struggled with this. Then, I would come up with different strategies. I think that is the positive aspect when you overcame your struggles. *Critical Friend:* I do not see your struggles as negative. *Me:* I did not write them as negative, but in my mind I perceive them as negative. There has to be a reason I am teaching certain met-befores. I find it interesting then I find it challenging, figuring out a way to teach it. Which is saying, I am equating met-befores to? *Critical Friend:* You are equating your met-befores with your students' met-befores. *Me:* If I am coming up with interventions I have to come up with some kind of a transformation and I am equating it to meaningful. I am finding I am equating a lot of words. For example I am equating connections with meaningful learning. Which is odd, but it is the way I think. No, its not. *Critical Friend:* Isn't the whole purpose is that you reflect on yourself. That is what they mean to you. It would seem that when you make these connections you are finding your pieces and that for you is meaningful learning. *Me:* It would be, but I find it interesting that as you look at the connection page. Look at my connections. There is something different about it. I am just not seeing it. Originally, I was seeing a difference. Past I see, I feel like this is very specific. *Critical Friend:* These are chunks of things you are connecting. You know how one of your things you said you went from one to another. *Me:* Yes? *Critical Friend:* This [new connections] you are talking about ideas. It is following. It is abstract. You are making abstract connections. Where over here you are I am seeing you are connecting specifics. You are making it more concrete. You are trying to visualize it. You are changing it to a visual representation. *Me:* I feel a chunky-ness to it. Over here [new connections] I am seeing smoothness to it. I feel I am covering it in its totality. I thought it was interesting that I brought the student discussion into this part.

Critical Friend: But, isn't it possible that the student discussion is I do not want to say forcing you,
Me: But it challenges my thinking.
Critical Friend: Exactly. It is challenging you to think differently.
Me: I realized by discussion, I was triggering my math confusion. When I think of past, I was always thinking of what was confusing to me. So, it is kind of negative.
Critical Friend: I still do not see it as a negative. I get that you see it as a negative because you struggled, but I see that perspective. For me, I do not see it as a negative. I see it as you are making connections and you are trying to make it easier for your students.
Me: You forget who I was.
Critical Friend: Who were you?
Me: When I was a child, I needed to be the perfect child and student.
Critical Friend: That is why you see it as negative. You are not allowed to struggle that is imperfection. Things should come easy. We have those expectations for our students.
Me: I connected struggle with imperfection. I seem to equate Negative= Struggles.

New Thread: Compartmentalization
Critical Friend: In the past, you stated that you were able to compartmentalize. Is it possible that is the reason you only saw chunky or smooth reasoning?
Me: It might be, but now I seem to think in terms of a blend with smooth and chunky. The blend also occurs with conceptual-embodied and proceptual-symbolic. By the end of this study, I realized that conceptual-embodied would also occur in axiomatic formalism. I seem to have a greater understanding of the three worlds of mathematics.

Thread: Struggle
Me: I think it is about the way I think. I see my thinking as a barometer to all thinking. It was my belief as I grew up that "I was never smart enough" compared to those I perceived as the smart students. Therefore, I perceived struggles as negativity. My past effects the way I teach because my goal is to alleviate my students struggles by my sharing mind. But, I think it is also about filling in students gaps, so they can avoid future struggles. My goal is for meaningful learning. I can accomplish meaningful learning by transforming my practice. I equate connections=understanding=meaningful learning and I can accomplish this goal by transforming my teaching practice.
Critical Friend: What contributes to this transformation?
Me: The implementation (teaching practices) and Living Educational Theory (implementation of teaching practice) setting expectations. Why do I equate steps as not equal to meaningful learning?
Critical Friend: Because it goes back to personal history.
Me: I did not believe I understood. I just memorized and studied all the examples.
Critical Friend: But, not why.
Me: I saw steps as routines, such as order of operation.
Critical Friend: Can the core be meaningful learning?
Me: Yes, if that is what they have in common.
Critical Friend: How do you see the concept map?
Me: As an organizational tool.

Critical Friend: Geared to help.
Me: Motion and also connections.
Critical Friend: Why are connections important?
Me: If you understand the big picture you can solve word problems. The problems are meaningful. I never understood the big picture.
Critical Friend: What affects the big picture?
Me: Having an advance organizer allows me to maximize teaching. Too see the totality (smooth) not pieces (chunky). My visualization evolved.
Critical Friend: Why is meaningful learning important?
Me: Because I never understood. I guessed on a lot of answers. I did not really understand. Meaningful learning effects practical [conceptual] understanding of a continuous process not steps.