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Nonlinear Gravitational-Wave Memory From Merging Binary Black Holes

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Abstract

Gravitational waves are oscillations in spacetime that propagate throughout the universe at the speed of light. They are a prediction of Einstein’s theory of General Relativity. Detectable sources of gravitational waves are typically collisions of black holes or other compact objects (neutron stars, white dwarfs). While most gravitational-wave signals are expected to be oscillatory in nature, some will exhibit a phenomenon called gravitational-wave memory. This refers to a non-oscillatory component of the gravitational wave signal that can leave a permanent distortion (or “memory”) in a gravitational-wave detector. The nonlinear memory effect is a type of memory signal that arises when gravitational waves themselves produce gravitational waves. Merging binary black holes create the strongest nonlinear memory. These memory signals are difficult to model using conventional numerical relativity simulations. To address this issue we use a semi-analytic procedure to construct the memory signal from the non-memory (oscillatory) pieces of the gravitational-wave field. We construct these memory signals using the output of several numerical simulations of non-spinning, quasi-circular black hole binaries with varying mass ratios. Our results could be used to improve the detectability and interpretation of the memory effect by ground or space-based gravitational-wave detectors.
NONLINEAR GRAVITATIONAL-WAVE MEMORY
FROM MERGING BINARY BLACK HOLES

A THESIS

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I Introduction

1.1 What are Gravitational Waves?

Since its inception in 1916 by Albert Einstein, the idea of gravitational waves (GW) has piqued the interests of many physicists and astronomers. Gravitational waves are oscillations in spacetime that propagate throughout the universe at the speed of light. They are akin to the more well-known electromagnetic (EM) waves, which are oscillations of electric and magnetic fields. While the motions of electric charges or magnets produce electromagnetic waves, gravitational waves are produced by the motions of masses. Everyday events (such as hand-waving) can produce gravitational waves, but they are far too weak to be detectable. There exist some astronomical sources (usually involving compact objects) powerful enough to produce detectable gravitational waves.

Compact objects are created when stars no longer have enough fusionable materials to produce the outward pressure needed to counterbalance their own self-gravitational attraction. When this happens, the stars undergo gravitational collapse; shrinking in radius until quantum-mechanical forms of pressure cease further collapse. The resulting object can be either a white dwarf (comparable to the size of Earth) or a neutron star (about 20 km wide) with a mass similar to that of the Sun. When much more massive stars undergo gravitational collapse, the resulting object is a black hole – an entity whose original material is compressed into an infinitely small

\[^{1}\]Parts of this section are adapted from Marc Favata's research website [http://www.astro.cornell.edu/~favata/research.html] (April 15, 2015)
volume. This creates a gravitational field so strong that not even light can escape its “surface”. The black hole itself is not made of any physical material. Instead, it is a region of spacetime that is separate from the rest of the Universe by a one-way membrane known as the event horizon, through which anything can enter, but nothing can escape.

While the collapse of a star can itself produce gravitational waves, the strongest sources are compact objects orbiting each other in a binary system. The orbital motion of the binary produces gravitational waves, causing it to shrink; this increases the orbital speed and produces even stronger gravitational waves. This is known as the inspiral phase; the first of three phases of coalescence. As the two objects slowly spiral toward each other, they eventually reach the merger phase. Orbital dynamics become unstable during this phase, and the two objects quickly merge together, forming a single object and giving off enormous amounts of gravitational waves in the process. This merger leaves a single object that is highly distorted. During the ringdown phase, the distortions are radiated away in the form of gravitational waves until the object settles down to a more quiescent state.

During each of the three phases, different calculation techniques are used to compute the gravitational waveform. For binary objects of comparable mass, post-Newtonian (PN) expansions are used to compute the waveforms during the inspiral phase, numerical relativity is used to compute the merger waveforms (as well as the late-inspiral and early-ringdown), and black hole perturbation theory is used to compute the ringdown waveforms.
Gravitational waves produced by even the strongest sources are hard to detect. In order to understand how gravitational-wave detection works, it is important to understand how gravitational waves affect matter. Much like how electromagnetic waves cause an oscillatory displacement of charged particles perpendicular to the waves' motion, gravitational waves cause an ellipsoidal stretching and squeezing of matter perpendicular to the waves' motion. Gravitational-wave detectors at the Laser Interferometer Gravitational-Wave Observatory (LIGO) aim to directly detect gravitational waves by precisely measuring the time it takes light to travel between two suspended mirrors (each oriented in a 4-km-long “L” shape) with laser light [1]. The strongest gravitational wave will move the mirrors by less than 0.1% the size of a proton; however, laser interferometry allows even the most minuscule distances to be measured.

1.2 What is Gravitational-Wave Memory?

Gravitational-wave signals typically start out with a small amplitude at early-times, grow to some maximum value, and then decay back to zero. Some sources, however, exhibit gravitational-wave memory: the signal's amplitude does not decay back to zero; rather, it asymptotes to some non-zero value at late-times [2]. In terms of detection, if we were to imagine an idealized detector made up of a ring of free-floating particles, a passing gravitational wave without memory would stretch and squeeze the particles. Once it has passed, the particles would re-assume their ring shape. A passing gravitational wave with memory, however, would cause the ring to
assume a permanent, elliptical shape long after the wave’s passage. Thus the detector has some “memory” of the gravitational-wave signal.

There are two forms of memory: linear and nonlinear [3-6]. Linear memory arises from the non-oscillatory motion of a source. Some sources of linear memory are binaries in hyperbolic orbits (i.e., where two unbound masses gravitationally scatter off one another) and mass/neutrino ejections from supernovae and gamma ray bursts. Nonlinear memory arises when gravitational waves themselves produce gravitational waves [7]. Thus every gravitational-wave source is a source with nonlinear memory [6].

1.3 Motivation

Many numerical relativity (NR) simulations have difficulty computing these memory signals, unfortunately. One of the challenges they face is that for quasi-circular orbits, the memory signals are numerically suppressed due to their relatively small size when compared to the non-memory (oscillatory) signals. To address this issue, this research uses a semi-analytic procedure to construct the memory signal from the non-memory (oscillatory) pieces of the gravitational-wave field. These memory signals are constructed using the output of several numerical simulations of non-spinning, quasi-circular black hole binaries [9,21]. While previous work has concentrated on binary black holes of equal mass [2,20], this work considers various mass ratios (i.e., 1:1, 2:1, 3:1, 4:1, 5:1, 6:1, and 8:1). These results could be used to improve the detectability and interpretation of the memory effect by ground or space-based gravitational-wave
detectors. By extension, this will give insight into highly-energetic events in the Universe, such as the merger of black holes and compact objects.

II Semi-Analytic Model

2.1 Background

When a source produces gravitational waves, they are emitted radially in all three spatial dimensions. Similar to electromagnetic waves, gravitational waves have a plus (+) and a cross (x) polarization. The polarizations of a waveform can be conveniently decomposed into

\[ h_+ - i h_x = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} h_{\ell m} \cdot -2 Y_{\ell m}(\Theta, \Phi). \tag{2.1.1} \]

The \( h_{\ell m} \) are the mode coefficients on the basis of the spin-weighted spherical harmonic functions, \(-2 Y_{\ell m}(\Theta, \Phi)\). These are defined in terms of the Wigner \( d \) functions

\[ -s Y_{\ell m}(\Theta, \Phi) = (-1)^s \sqrt{\frac{2\ell + 1}{4\pi}} s d_{\ell m}(\Theta) e^{im\Phi}, \tag{2.1.2} \]

\[ s d_{\ell m}(\Theta) = \sqrt{(\ell + m)!(\ell - m)!(\ell + s)!(\ell - s)!} \]

\[ \times \sum_{k=k_i}^{k_f} \frac{(-1)^k (\sin \frac{\Theta}{2})^{2k-m+s} (\cos \frac{\Theta}{2})^{2\ell-2k+m-s}}{k!(k - m + s)!(\ell + m - k)!(\ell - k - s)!} , \tag{2.1.3} \]

where \( s \) is the spin weight, \( k_i = \max(0, m - s) \), \( k_f = \min(\ell + m, \ell - s) \), and \((\Theta, \Phi)\) are spherical angles that indicate the direction from the source to the observer. See
the discussion in Sec. IIA of Ref. [8] and the references therein.

This mode decomposition is analogous to a Fourier series expansion where \( h_{\ell m} \) are conceptually similar to the coefficients in front of the sine and cosine terms in the sum, and \( -2Y_{\ell m}(\Theta, \Phi) \) are similar to the sine and cosine terms themselves.

The contribution from the nonlinear memory to the polarizations is given by [8]

\[
\hat{h}^{(\text{mem})} = i h^{(\text{mem})} = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \hat{h}_{\ell m}^{(\text{mem})} -2Y_{\ell m}(\Theta, \Phi),
\]

where the \( h_{\ell m}^{(\text{mem})} \) (memory) term can be computed via [8]

\[
h_{\ell m}^{(\text{mem})} = \frac{16\pi}{R} \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!}} \int_{-\infty}^{T_R} dt \int d\Omega \frac{dE_{gw}}{dtd\Omega} (\Omega) \hat{h}_{\ell m}(\Theta, \Phi),
\]

where \( R \) is the distance to the source, \( T_R \) is the retarded time, \( d\Omega \equiv \sin(\theta) \ d\theta \ d\phi \) for \( \theta \in [0, \pi] \) and \( \phi \in [0, 2\pi] \), and \( * \) denotes complex conjugation. Here \((\theta, \phi)\) are spherical polar angles with respect to a coordinate system whose z-axis lies along the orbital angular momentum direction.

In order to compute the nonlinear memory, the primary equation that needs to be evaluated is (2.1.5). In terms of the metric waveform modes, the gravitational-wave energy flux \( \frac{dE_{gw}}{dt} \) is given by [8]

\[
\frac{dE_{gw}}{dt} = \frac{1}{16\pi} \sum_{\ell'=2}^{\infty} \sum_{m'=\ell'-\ell}^{\ell'} \sum_{\ell''} \sum_{m''=-\ell'}^{\ell''} \hat{h}_{\ell' m'} \hat{h}^{*}_{\ell'' m''} -2Y_{\ell' m'}(\theta, \phi) -2Y^{*}_{\ell'' m''}(\theta, \phi).
\]

The energy flux is related to the first time derivative of the non-memory (oscillatory)
modes, which are obtained from the SXS Gravitational Waveform Database [9,21]. The limits on the \( \ell' \) and \( \ell'' \) sums can be expanded to any maximum value \( \ell_{\text{max}} \), but the SXS Catalog provides data up to \( \ell_{\text{max}} = 8 \). For instance, showing only terms out to \( \ell_{\text{max}} = 3 \), the following is obtained for the (2, 0) and (4, 0) modes

\[
\begin{align*}
\dot{h}^{(\text{mem})}_{20} &= \frac{R}{84} \sqrt{\frac{30}{\pi}} \left\{ |\dot{h}_{22}|^2 + |\dot{h}_{2-2}|^2 - |\dot{h}_{20}|^2 - \frac{1}{2} \left( |\dot{h}_{21}|^2 + |\dot{h}_{2-1}|^2 \right) \right. \\
&+ \frac{\sqrt{14}}{4} \left( \dot{h}_{21}\dot{h}_{31}^{*} + \dot{h}_{31}\dot{h}_{21}^{*} - \dot{h}_{2-1}\dot{h}_{3-1}^{*} - \dot{h}_{3-1}\dot{h}_{2-1}^{*} \right) \\
&+ \frac{\sqrt{35}}{4} \left( \dot{h}_{22}\dot{h}_{32}^{*} + \dot{h}_{32}\dot{h}_{22}^{*} - \dot{h}_{2-2}\dot{h}_{3-2}^{*} - \dot{h}_{3-2}\dot{h}_{2-2}^{*} \right) + \ldots \\
\dot{h}^{(\text{mem})}_{40} &= \frac{R}{5040} \sqrt{\frac{10}{\pi}} \left\{ |\dot{h}_{22}|^2 + |\dot{h}_{2-2}|^2 + 6|\dot{h}_{20}|^2 - 4 \left( |\dot{h}_{21}|^2 + |\dot{h}_{2-1}|^2 \right) \right. \\
&- 5\sqrt{14} \left( \dot{h}_{21}\dot{h}_{31}^{*} + \dot{h}_{31}\dot{h}_{21}^{*} - \dot{h}_{2-1}\dot{h}_{3-1}^{*} - \dot{h}_{3-1}\dot{h}_{2-1}^{*} \right) \\
&+ 2\sqrt{35} \left( \dot{h}_{22}\dot{h}_{32}^{*} + \dot{h}_{32}\dot{h}_{22}^{*} - \dot{h}_{2-2}\dot{h}_{3-2}^{*} - \dot{h}_{3-2}\dot{h}_{2-2}^{*} \right) \\
&- \frac{49}{11} \left( |\dot{h}_{31}|^2 + |\dot{h}_{3-1}|^2 + 6|\dot{h}_{30}|^2 - 7 \left( |\dot{h}_{32}|^2 + |\dot{h}_{3-2}|^2 \right) \\
&+ 3 \left( |\dot{h}_{33}|^2 + |\dot{h}_{3-3}|^2 \right) \right\} + \ldots .
\end{align*}
\]

Further discussion can be found in Section 3.2.

2.2 Numerical Differentiation of \( h_{\ell m} \)

In order to compute the energy-flux found in (2.1.6), the first time derivatives of the non-memory modes (\( h_{\ell m} \)) must be computed. The data for each \( h_{\ell m} \) mode consists of a time variable (\( t \)), a real part \( \text{Re}(h_{\ell m}) \), and an imaginary part \( \text{Im}(h_{\ell m}) \). The real
and imaginary parts of the relevant modes were numerically differentiated using the finite difference, second-order accurate scheme [10],

\[ f_i = \frac{d_{i-1}^2 f_{i+1} + (d_i^2 - d_{i-1}^2) f_i - d_i^2 f_{i-1}}{d_i d_{i-1} (d_i + d_{i-1})} + O(d^2), \]  

(2.2.1)

where \( f \) is either \( \text{Re}(h_{\ell m}) \) or \( \text{Im}(h_{\ell m}) \), \( d_{i-1} = t_i - t_{i-1} \) and \( d_i = t_{i+1} - t_i \), and \( d \) is the larger of \( |d_{i-1}| \) and \( |d_i| \). This scheme also allows for nonuniform step-size, which arises in the time parameter data from the SXS Catalog.

Once the derivatives have been computed, they are combined via

\[ \dot{h}_{\ell m} = \text{Re}(\dot{h}_{\ell m}) + i \text{Im}(\dot{h}_{\ell m}). \]  

(2.2.2)

This process is repeated for all of the non-memory \( (h_{\ell m}) \) modes. By combining the results from (2.2.2) with the time derivative of Eq. (2.1.5), the first time derivative of the memory mode \( (\dot{h}_{\ell m}^{(\text{mem})}) \) is obtained.

### 2.3 Numerical Integration of \( \dot{h}_{\ell m}^{(\text{mem})} \)

Now that \( \dot{h}_{\ell m}^{(\text{mem})} \) has been computed, the time integral

\[ h_{\ell m}^{(\text{mem})}(t) = \int_{t_0}^{t} \dot{h}_{\ell m}^{(\text{mem})}(t') \, dt' + C_{\ell m}, \]  

(2.3.1)
can be computed via a trapezoidal rule

\[
\int_{t_0}^{t} h_{\ell m}^{(\text{mem})}(t') \, dt' \approx \sum_{i=1}^{N} \frac{1}{2} \left[ h_{\ell m}^{(\text{mem})}(t_{i-1}) + h_{\ell m}^{(\text{mem})}(t_i) \right] \Delta t_i, \tag{2.3.2}
\]

where \( t_0 \) is some initial time value and \( N \) is the number of data points. The integration constants \( C_{\ell m} \) are determined via matching to analytical expressions for the memory modes during the inspiral.

### 2.4 Matching Technique

The \( C_{\ell m} \) constants are determined by matching the \( h_{\ell m}^{(\text{mem})} \) to the expressions in [8] that give the memory modes during the PN inspiral phase. These inspiral expressions are expressed as a PN expansion in \( x \equiv (M\omega)^{2/3} \), where \( \omega \) is the orbital frequency and \( M = m_1 + m_2 \). Since orbital frequency is a gauge invariant quantity (while time is not), we want the numerical and PN inspiral pieces of the memory to match at low orbital frequencies near the start of the numerical simulations (where the PN inspiral expressions will be accurate). This matching is important because the memory builds up to a non-negligible value during the entire inspiral, whereas the time integral in Eq. (2.3.1) evaluates to zero at the initial time \( t_0 \). The integration constants \( C_{\ell m} \) account for the portion of the memory that accumulates before the start of the numerical calculation.

During the inspiral phase, the non-memory modes can be broken down into a real amplitude \( A_{\ell m} \) that slowly evolves and an orbital phase \( \varphi \) that modulates the waveform;
this is given by [22]

\[ h_{\ell m} = A_{\ell m}e^{-im\varphi}. \]  

(2.4.1)

Combined with the relation \( \omega \equiv \dot{\varphi} \), the orbital angular frequency of the \( h_{22} \) mode can be obtained via

\[ \omega = -\frac{1}{2} \text{Im} \left( \frac{\dot{h}_{22}}{h_{22}} \right). \]  

(2.4.2)

The matching frequency is chosen to be the point where the time derivatives of the PN and NR expressions match (or nearly match). This ensures that the resulting function will be continuous and (moderately) smooth.

The PN form of the first time derivative of the \((2, 0)\) and \((4, 0)\) memory modes, for instance, are given to 3PN order by [8]

\[ h_{20}^{(\text{mem})} = \frac{256}{21} \sqrt{\frac{3\pi}{100R}} \eta^2 x^5 \left\{ 1 + x \left( -\frac{1219}{288} + \frac{1}{24}\eta \right) + 4\pi x^{3/2} \right. \]

\[ + x^2 \left( -\frac{793}{1782} - \frac{14023}{6336}\eta - \frac{4201}{1584}\eta^2 \right) + \pi x^{5/2} \left( -\frac{2435}{144} - \frac{23}{12}\eta \right) \]

\[ + x^3 \left[ \frac{174213949439}{1816214400} + \frac{16}{3} \left( -\frac{856}{105} (2\gamma_E + \ln 16x) + \left( -\frac{126714689}{4447872} + \frac{41}{48}\pi^2 \right) \eta \right. \right. \]

\[ \left. \left. + \frac{4168379}{123552}\eta^2 + \frac{142471}{46332}\eta^3 \right] \right\}, \]  

(2.4.3a)
\[
\dot{h}_{40}^{(\text{mem})} = \frac{64}{315} \sqrt{\frac{\pi}{10R}} \eta^2 x^5 \left\{ 1 + x \left( -\frac{10133}{704} + \frac{25775}{528} \eta \right) + 4\pi x^{3/2} \right. \\
+ x^2 \left( \frac{322533}{4576} - \frac{721593}{2288} \eta - \frac{237865}{5148} \eta^2 \right) + \pi x^{5/2} \left( -\frac{1028}{11} + \frac{11114}{33} \eta \right) \\
+ x^3 \left[ \frac{32585924257}{403603200} + \frac{16}{3} \pi^2 - \frac{856}{105} (2\gamma_E + \ln 16x) + \left( \frac{4669843}{164736} + \frac{41}{48} \pi^2 \right) \eta \\
+ \frac{16531}{52} \eta^2 - \frac{1145725}{92664} \eta^3 \right] \right\} ,
\]

where \( \eta = m_1 m_2 / M^2 \) is the reduced mass ratio, and \( \gamma_E \) is the Euler-Mascheroni constant.

Once the PN and NR curves are plotted against \( \omega \), a point where the two curves overlap, or are close together (within a \( 10^{-5} \) tolerance), at a small value of \( \omega \) is found. This point is designated as the matching point and gives us the values \( x_{\text{match}} \) and \( t_{\text{match}} \). By integrating [8]

\[
\frac{dx}{dt} = \frac{64}{5} \frac{\eta}{M} x^5 \left\{ 1 + x \left( -\frac{743}{336} - \frac{11}{4} \eta \right) + 4\pi x^{3/2} + \pi x^{5/2} \left( \frac{34103}{18144} + \frac{13661}{2016} \eta + \frac{59}{18} \eta^2 \right) \right. \\
+ x^2 \left( -\frac{4159}{672} - \frac{189}{8} \eta \right) + \frac{856}{105} (2\gamma_E + \ln 16x) \\
+ \left( \frac{56198689}{217728} + \frac{451}{48} \pi^2 \right) \eta + \frac{541}{896} \eta^2 - \frac{5605}{2592} \eta^3 \right] \\
+ \pi x^{7/2} \left( -\frac{4415}{4032} + \frac{358675}{6048} \eta + \frac{91495}{1512} \eta^2 \right) \right\}.
\]

backwards from the matching point, \( x(t) \) values are generated from an arbitrary starting time to the matching time \( t_{\text{match}} \). The generated \( x(t) \) values are then substituted
into the PN equation of $h_{\ell m}^{(\text{mem})}$ [8,11]

$$h_{\ell m}^{(\text{mem})} = 8 \sqrt{\frac{\pi}{5}} \frac{\eta M x}{R} \dot{H}_{\ell m}.$$ (2.4.5)

For the (2, 0) and (4, 0) memory modes, for instance, the polarization modes are [8,11]

$$\dot{H}_{20} = \frac{5}{14\sqrt{6}} \left\{ 1 + x \left( -\frac{4075}{4032} + \frac{67}{48} \eta \right) + x^2 \left( -\frac{151877213}{67060224} - \frac{123815}{44352} \eta + \frac{205}{352} \eta^2 \right) + \pi x^{5/2} \left( -\frac{253}{336} + \frac{253}{84} \eta \right) \right.\right.$$ (2.4.6a)

$$\left. + x^3 \left[ -\frac{4397711103307}{532580106240} + \left( \frac{700464542023}{13948526592} - \frac{205}{96} \pi^2 \right) \eta \right.\right.$$ (2.4.6b)

$$+ \frac{69527951}{166053888} \eta^2 + \frac{1321981}{5930496} \eta^3 \right\} \right.$$ (2.4.6b)

$$\dot{H}_{40} = \frac{1}{504\sqrt{2}} \left\{ 1 + x \left( -\frac{180101}{29568} + \frac{27227}{1056} \eta \right) + x^2 \left( \frac{2201411267}{158505984} - \frac{34829479}{432432} \eta + \frac{844951}{27456} \eta^2 \right) + \pi x^{5/2} \left( -\frac{13565}{1232} + \frac{13565}{308} \eta \right) \right.$$ (2.4.6b)

$$+ x^3 \left[ \frac{15240463356751}{781117489152} + \left( \frac{1029744557245}{27897053184} - \frac{205}{96} \pi^2 \right) \eta \right.\right.$$ (2.4.6b)

$$- \frac{4174614175}{36900864} \eta^2 + \frac{221405645}{11860992} \eta^3 \right\} \right.$$

Combining the generated $x(t)$ values with (2.4.5) and (2.4.6), explicit values for the memory modes vs. time during the inspiral phase are obtained $\left( h_{\ell m}^{(\text{mem}), \text{PN}}(t) \right)$. With this, the integration constants $C_{\ell m}$ are computed via

$$h_{\ell m}^{(\text{mem}), \text{PN}}(x_{\text{match}}) = h_{\ell m}^{(\text{mem}), \text{NR}}(x_{\text{match}}) + C_{\ell m}$$ (2.4.7)
where $h_{\ell m}^{(\text{mem})},_{\text{NR}}(x_{\text{match}})$ refers to the first term on the right-hand-side of Eq. (2.3.1) evaluated at $t_{\text{match}} = t(x_{\text{match}})$.

Finally, the total $h_{\ell m}^{(\text{mem})}$ mode in (2.3.1) is constructed by

$$h_{\ell m}^{(\text{mem})}(t) = \begin{cases} h_{\ell m}^{(\text{mem})},_{\text{PN}}(t) & \text{for } t < t_{\text{match}} \\ h_{\ell m}^{(\text{mem})},_{\text{NR}}(t) - h_{\ell m}^{(\text{mem})},_{\text{NR}}(t_{\text{match}}) + h_{\ell m}^{(\text{mem})},_{\text{PN}}(t_{\text{match}}) & \text{for } t \geq t_{\text{match}} \end{cases}$$

(2.4.8)

With the memory modes obtained, the polarizations of the waveform signal in (2.1.4) are computed.

### III Results and Analysis

Figure 1 shows plots of some of the $h_{\ell m}$ modes from the SXS Gravitational Waveform Database [9,21] at equal mass; displaying the oscillatory nature of the real and imaginary parts of various $h_{\ell m}$ modes in the physical regime.

![Plots of oscillatory real and imaginary parts of $h_{22}$ (top left), $h_{31}$ (top right), and $h_{22}$ (bottom) from the SXS Catalog for $\eta = 0.2500$.](image)

**Figure 1:** Plots of oscillatory real and imaginary parts of $h_{22}$ (top left), $h_{31}$ (top right), and $h_{22}$ (bottom) from the SXS Catalog for $\eta = 0.2500$. 

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The $h_{\ell m}$ modes displayed in Figure 2 show the wide range (0.1 to $10^{-12}$) of amplitudes dealt with in this study. The $h_{22}$ mode is by far the dominant one, while the higher modes decrease in amplitude. In the $\eta = 0.0988$ case, Figure 3 shows that the higher modes (e.g., $h_{31}$) have a higher amplitude than they do in the equal mass case ($\eta = 0.2500$). These higher modes become more important in the larger mass ratio cases.
3.1 Dependence of $h^{(\text{mem})}_{\ell_0}$ on $\ell$

It is important to understand the effect that $\ell$ has on the memory modes. Figure 4 shows a log plot of $|h^{(\text{mem})}_{\ell_0}|$ with $\ell \in [2, 8]$ for the equal mass case ($\eta = 0.2500$). The

![Figure 4: Plot of $|h^{(\text{mem})}_{\ell_0}|$ for $\ell \in [2, 10]$ with $\ell_{\text{max}} = 8$ and $\eta = 0.2500$.](image)

even memory modes are the most influential modes; more specifically, the $h^{(\text{mem})}_{20}$ and $h^{(\text{mem})}_{40}$ memory modes. The odd $\ell$ memory modes in Figure 4 are consistent with numerical noise. This is consistent with analytic calculations showing that odd $\ell$ modes vanish identically during the inspiral [8]. The other even modes ($h^{(\text{mem})}_{60}$ and $h^{(\text{mem})}_{80}$), contribute very minimally to the nonlinear memory; their amplitudes have orders of $10^{-5}$ and below. Therefore we will focus on the $h^{(\text{mem})}_{20}$ and $h^{(\text{mem})}_{40}$ memory modes in the remainder of the analysis.
3.2 Dependence of $h_{20}(\text{mem})$ on $\ell_{\text{max}}$

Some analysis was done on the limits of $\ell'$ and $\ell''$ in (2.1.6) to check if higher harmonics would be more influential in the waveform. This was tested by plotting $h_{20}(\text{mem})$ against time with $\ell_{\text{max}} \in [2,8]$. Figure 5 shows $h_{20}(\text{mem})$ for the equal mass ratio ($\eta = 0.2500$) case and Figure 6 shows $h_{20}(\text{mem})$ for the 8:1 mass ratio ($\eta = 0.0988$) case.

![Figure 5: Plot of $h_{20}(\text{mem})$ for $\ell_{\text{max}} \in [2,8]$ and $\eta = 0.2500$. If the $\ell_{\text{max}} = 2$ terms were the only terms used, a 10% error would be made, as opposed to using $\ell_{\text{max}} = 8$.](image1)

![Figure 6: Plot of $h_{20}(\text{mem})$ for $\ell_{\text{max}} \in [2,8]$ and $\eta = 0.0988$. If the $\ell_{\text{max}} = 2$ terms were the only terms used, a 7% error would be made, as opposed to using $\ell_{\text{max}} = 8$.](image2)
The curve for $\ell_{\text{max}} = 2$ is clearly different from the rest of the $\ell_{\text{max}}$ curves. From Eqs. (2.1.7a) and (2.1.7b) we see that if we take $\ell_{\text{max}} = 2$, the expressions for the $h_{20}^{(\text{mem})}$ and $h_{40}^{(\text{mem})}$ modes only depend on the $h_{2\pm 2}$ modes. If we set $\ell_{\text{max}} > 2$, other modes contribute to the memory. Figures 5 and 6 indicate that if we only keep the $\ell_{\text{max}} = 2$ piece of the $h_{20}^{(\text{mem})}$ mode [i.e., the first line in Eq. (2.1.7a)], this leads to an error of order 7% to 10%. Figures 7 and 8 show the $\ell_{\text{max}}$ dependence for the $h_{40}^{(\text{mem})}$ mode for $\eta = 0.2500$ and $\eta = 0.0988$, respectively. For $\eta = 0.2500$, summing up to $\ell_{\text{max}} = 2$ will yield a 50% error; while a very large error will result for $\eta = 0.0988$ due to the change in direction that arises. For the $h_{40}^{(\text{mem})}$ mode, the choice in $\ell_{\text{max}}$ is very important and is discussed further in Section 3.3. The SXS Catalog [9,21] provides the $h_{\ell m}$ modes up to $\ell = 8$, therefore further calculations will use $\ell_{\text{max}} = 8$.

Figure 7: Plot of $h_{40}^{(\text{mem})}$ for $\ell_{\text{max}} \in [2,8]$ and $\eta = 0.2500$. If the $\ell_{\text{max}} = 2$ terms were the only terms used, a 50% error would be made, as opposed to using $\ell_{\text{max}} = 8$. 
terms used, a significant error would be made, as opposed to using \( \ell_{\text{max}} = 8 \).

### 3.3 Dependence of \( h^{(\text{mem})}_{\ell 0} \) on Mass Ratio \( \eta \)

We next investigate how the \( h^{(\text{mem})}_{20} \) and \( h^{(\text{mem})}_{40} \) modes vary with the symmetric mass ratio \( \eta \). The different mass ratios used were: 1:1, 2:1, 3:1, 4:1, 5:1, 6:1, and 8:1. Figures 9 and 10 show \( h^{(\text{mem})}_{20} \) and \( h^{(\text{mem})}_{40} \) versus time for the various mass ratios, respectively. These curves have been shifted such that the merger (defined as the peak amplitude of the \( h_{22} \) mode) occurs at the same time value for each mass ratio.

![Figure 8: Plot of \( h^{(\text{mem})}_{40} \) for \( \ell_{\text{max}} \in [2,8] \) and \( \eta = 0.0988 \). If the \( \ell_{\text{max}} = 2 \) terms were the only terms used, a significant error would be made, as opposed to using \( \ell_{\text{max}} = 8 \).](image)

**Figure 8:** Plot of \( h^{(\text{mem})}_{40} \) for \( \ell_{\text{max}} \in [2,8] \) and \( \eta = 0.0988 \). If the \( \ell_{\text{max}} = 2 \) terms were the only terms used, a significant error would be made, as opposed to using \( \ell_{\text{max}} = 8 \).

![Figure 9: Plot of \( h^{(\text{mem})}_{20} \) for different reduced mass ratios (\( \eta \)).](image)

**Figure 9:** Plot of \( h^{(\text{mem})}_{20} \) for different reduced mass ratios (\( \eta \)).
These plots show that the amplitude of the memory modes decrease as $\eta$ decreases. An interesting feature of the $h_{40}^{(\text{mem})}$ mode is that for small mass ratios, the amplitudes go downward. Upon investigation, certain negative terms in the sum from (2.1.7b) for the $(4, 0)$ mode become larger at smaller mass ratios, which causes this downward effect.

### 3.4 Dependence of $h_{40}^{(\text{mem})}$ Source Angle $\Theta$

In order to construct the nonlinear memory waveform signals, the spherical angles $(\Theta, \Phi)$ in (2.1.1) must be chosen. For $m = 0$ modes, $\Phi$ does not enter the expression at all – see Eq. (2.1.2). These angles indicate the direction of the observer in the source frame. To illustrate the angular dependence of the memory piece of the polarization,
we evaluate the (2,0) and (4,0) modes at their late-time values and vary $\Theta$, or

$$h_+^{(\text{mem})}(t \to +\infty) = h_{20}^{(\text{mem})}(t \to +\infty) - \frac{1}{2} Y_2(\Theta) + h_{40}^{(\text{mem})}(t \to +\infty) - \frac{1}{2} Y_4(\Theta). \quad (3.5.1)$$

Figure 11 shows the results for all mass ratios. The value that will give us the maximal gravitational wave amplitude is $\Theta = 90^\circ$. This corresponds to an observer that sees the binary edge-on.

3.5 Nonlinear Memory Waveforms

Now that we have the memory modes $h_{\ell m}^{(\text{mem})}$, Eq. (2.1.1) allows us to construct the plus polarization of the nonlinear memory waveform for any mass ratio $\eta$ (the cross polarization for the nonlinear memory is zero). Figures 12 and 13 show the oscillatory waveforms with and without memory for the equal mass ($\eta = 0.2500$) and 4:1 mass ratio ($\eta = 0.1600$) cases, respectively.
Figure 12: Plot of $h_+$ with (blue curve) and without (red curve) memory for $\eta = 0.2500$.

Figure 13: Plot of $h_+$ with (blue curve) and without (red curve) memory for $\eta = 0.1600$.

It is easy to see that the waveforms without memory return to zero at late times, but the waveforms with memory saturate to some non-zero value.
IV Conclusions

Gravitational waves with memory serve as potential sources for pulsar-timing arrays [13-19] as well as ground-based (LIGO) and space-based interferometers [8]. The memory effect is a fascinating phenomenon because it arises from gravitational waves producing gravitational waves and it affects the waveform in a uniquely visual way. The nonlinear memory is an interesting example of how nonlinearities of general relativity arise in waveform signals. While previous work has concentrated primarily on binaries of equal mass [8,20], this research has computed the memory through the merger and ringdown for various mass ratios. As mentioned briefly in the introduction, numerical relativity simulations have trouble computing the nonlinear memory. This study serves as an example of how post-Newtonian methods can be combined with data from numerical relativity simulations to compute something that is difficult to compute directly with purely analytic or numeric methods alone [7]. LIGO has the potential to detect the memory effect, but is not as sensitive to it as low-frequency detectors. Searches for the memory effect have been underway through the use of pulsar-timing arrays [13-19]. Studies on the memory effect can also be performed with future observations from the Evolved Laser Interferometer Space Antenna (eLISA) [12] (a space-based interferometer). Our study has improved models of the time-evolution of the memory and its mass ratio dependence. This will help efforts to search for the memory or estimate its detectability. If the nonlinear memory is eventually detected, it will help to confirm the validity of general relativity in the strong field regime.
V Bibliography


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