An Investigation into Lagrangian Coherent Structure Detection for Low Reynolds Number Flows

Kyle Fitzsimmons
AN INVESTIGATION INTO LAGRANGIAN COHERENT STRUCTURE DETECTION FOR LOW REYNOLDS NUMBER FLOWS

by

Kyle Fitzsimmons

A Master's Thesis Submitted to the Faculty of
Montclair State University
In Partial Fulfillment of the Requirements
For the Degree of
Master of Science with a Concentration in Pure and Applied Mathematics
August 2016

School College of Science and Mathematics Thesis Committee:

Department Mathematical Sciences

Dr. Eric Forgoston
Thesis Sponsor

Dr. Ashwin Vaidya
Committee Member

Dr. Bogdan Nita
Committee Member
ABSTRACT

Title of Thesis: AN INVESTIGATION INTO LAGRANGIAN COHERENT STRUCTURE DETECTION FOR LOW REYNOLDS NUMBER FLOWS

Kyle Fitzsimmons, Master of Science, 2016

Thesis directed by: Dr. Eric Forgoston
Department of Mathematical Sciences

In order to better understand and make use of complex fluid systems, regions of similar behavior can be categorized, thus reducing the problem to one of identifying the important structures within the fluid system, and examining their interaction. An important category of fluid structures are those known as Lagrangian Coherent Structures. An experimental setup has been constructed in order to study methods for improving the detection of Lagrangian Coherent Structures, and experimental data has been analyzed in order to verify the experimental setup. Several modifications to the basic gyre flow have been analyzed in order to provide direction, and a point of reference to built upon in the future.
AN INVESTIGATION
INTO LAGRANGIAN COHERENT STRUCTURE DETECTION
FOR LOW REYNOLDS NUMBER FLOWS

A THESIS

Submitted in partial fulfillment of the requirements
For the degree of Master of Science in Pure and Applied Mathematics

by

KYLE FITZSIMMONS

Montclair State University

Montclair, NJ

August 2016
Acknowledgments

The author was supported by the National Science Foundation (Award Numbers DMS-1418956 and CMMI-1462884). The content is solely the responsibility of the author and does not necessarily represent the official views of the National Science Foundation.
# Contents

1 **Introductory Theory** 5  
1.1 Motivation and Applications 5  
1.2 Lagrangian Coherent Structures 5  
1.3 The Finite Time Lyapunov Exponent 6  
1.3.1 False Positives and Shear 8  
1.3.2 Forward and Backward FTLE 9  
1.4 Particle Image Velocimetry 9  

2 **Methodology** 10  
2.1 Software 10  
2.2 Hardware 12  
2.2.1 Dye Experiments 13  
2.3 Data Collection 15  

3 **Analysis** 15  
3.1 Areas of Investigation 15  
3.2 Validity of a System 17  
3.2.1 Establishing a Baseline 17  
3.2.2 Backward FTLE 21  
3.3 Variations in Computation 21  
3.3.1 Comparison of Integration Techniques 21  
3.3.2 Grid Size 25  
3.4 Variations in Flow 26  
3.4.1 Basic Filling 26  
3.4.2 Filling with Motor Stop 29  
3.4.3 Wire Blocking 30  

4 **Future Work** 30  

5 **Bibliography** 32
List of Figures

1. An attracting material line (in blue) and a repelling material line (in red) come together to form a Hyperbolic LCS. The image depicts how a Hyperbolic LCS acts on a region of fluid as time progresses [1].

2. The image depicts trajectory separation along a repelling material line (in blue). Fluid particles located nearby at time $t_0$ separate as time progresses. The FTLE is a quantification of this trajectory separation, with regions that have a large amount of trajectory separation having a high FTLE value [2].

3. Seed particles are tracked in groups by PIV software. The algorithm that creates the velocity field looks at groupings of particles since there is a tendency to maintain relative positions between individual particles. Each grid point is assigned a velocity based on this advection of local groupings of particles. [3]

4. Example velocity field for a steady state double gyre flow, and the resulting FTLE field [4].

5. Computationally generated FTLE field for a double gyre flow with cyclical perturbations. Shearing effects generating high FTLE regions are noticeable spiraling towards the center of the gyres.

6. Experimental setup overview showing camera, laser, and tank positioning.

7. Experimental tank with motor groups marked.

8. Experimental tank illuminated by laser sheet.

9. Partially dyed glycerine advected by motors. Dye was initially placed at a few points on the middle band of 8 discs (One at the center of the rightmost grouping of 4, one on the top edge of the second disc in the second row, and one at the center of the first disc on the third row). Dye tends to follow along the suspected attracting and repelling material lines. This is especially evident in the lower left region of the tank.

10. Streamlines and velocity field for steady state flow with alternating disc direction generated by DaVis software (used to calculate FTLE field). Velocity vectors are to scale, relative to one another.

11. FTLE field for a nearly steady state alternating disc multi-gyre flow. Particle position is calculated out 2 seconds into the future (approximately 0.5 complete rotations of the discs).

12. This FTLE field is Figure 11) with marked triangle shaped regions (white), and '+' shaped disc centers (black).

13. FTLE field for a nearly steady state alternating disc multi-gyre flow. Particle position is calculated out 4.5 seconds into the future, this covers just over one full rotation of the discs. Note the absence or decreased prominence of the elements marked in Figure 12.

14. The FTLE field displayed here is identical to Figure 13, with marked regions indicating a possible explanation for the triangle regions observed in Figure 12.

15. This FTLE field is generated by negating the velocity fields that generated Figure 13, and then running the same FTLE algorithm. This shortcut to compute backwards FTLE only works for steady-state flows.
16 FTLE field for a nearly steady state alternating disc multi-gyre flow. Particle position is calculated out 4.5 seconds into the future using Matlab’s ‘ode45’ solver. FTLE ridges seen in earlier figures are notably absent. .................................................. 22

17 This figure is computed from the same data set as Figure 16), however, it features an altered interpolation function. ........................................................... 23

18 This FTLE field is nearly identical to Figure 17, however, the final particle positions in this version are computing using a homebuilt RK4 solver. The integration algorithms uses in both examples were built independently from each other. ........................................................... 23

19 In this figure, a cap is applied to the FTLE field in Figure 18, in an attempt to ensure that some noisy peaks were not dominating the scaling. ...................... 24

20 FTLE field for a nearly steady state alternating disc multi-gyre flow. Particle position is calculated out 4.5 seconds into the future on a 54 x 34 PIV grid. .......... 25

21 FTLE field for a alternating disc multi-gyre flow. Particle position is calculated out 4.5 seconds into the future while fluid is being poured into the bottom-right corner of the tank .................................................. 26

22 FTLE field for a alternating disc multi-gyre flow. Particle position is calculated out 2 seconds into the future while fluid is being poured into the bottom-right corner of the tank .................................................. 27

23 This FTLE field is generated using the same data set and integration run time as Figure 22, however, the computation is started approximately 1 second further into the future. .................................................. 28

24 Identical FTLE field to Figure 23 with high FTLE ridges marked. .............. 28

25 FTLE field for a alternating disc multi-gyre flow. Particle position is calculated out 2 seconds into the future while fluid is poured into the bottom-right corner of the tank, top 8 discs stopped midway through the integration ............................. 29

26 FTLE field for a nearly steady state alternating disc multi-gyre flow. Particle position is calculated out 2 seconds into the future. A tangled bundle of thin wire which allows for some liquid to pass through is fixed at the fluid surface in the position marked by the black line. .................................................. 30
1 Introductory Theory

1.1 Motivation and Applications

Today, applications of unmanned vehicles are growing. It is frequently cheaper, safer, and more reliable to use an unmanned vehicle instead of a human in order to complete some desired task. One category of vehicle to consider is those that operate on the surface of the ocean [5], [6], and [7]. In order to use such vehicles effectively, solutions to control problems on the surface of fluid bodies must be found [8].

Some applications may require constant sensory information at a specific point on the ocean’s surface. In this case, it becomes important for the autonomous vehicle to hold as a specified position. Some problems arise if this is attempted. Changes in the ocean currents need to be corrected for in real time. Working directly against currents may not be time or energy efficient. In some cases, it may be beneficial to make use of the underlying structure of the surface flows to find a lower energy, or shorter time method of returning to a desired position. In some cases, it may be possible to predict the future behavior of the fluid system, and preemptively correct for this [9].

Similarly, given an autonomous vehicle that is required to move to a different, specific, location, a straight line is unlikely to be either the shortest-time, or most cost effective path. Again, it may be desirable to make use of underlying patterns to find a path optimized for one of the above criteria. The underlying structures can be pinpointed through knowledge of markers known as Lagrangian Coherent Structures (LCS). However, the detection of LCS may require complete velocity field information, which might not be easily obtainable in real applications.

Currently, theory to determine the locations of LCS from local velocity measurements has been developed [10]. One problem at the moment is that the mass of the autonomous vehicles may cause inertial effects to be relevant when considering their motion. In particular, vehicles with mass may aggregate along inertial LCS rather than a non-inertial LCS. In order to facilitate study of this area, an experimental setup that can simulate ocean-like flows is required. Further studies of Lagrangian Coherent Structures must be undertaken to move the research towards a comprehensive theory that can solve the aforementioned control problems.

1.2 Lagrangian Coherent Structures

Fluid systems, and in particular, geophysical fluid systems are extremely complicated. Despite this, there are regions within flows which can be loosely grouped as behaving in similar ways. This is advantageous for those who wish to study a particular fluid system, as it reduces the problem of understanding the flow overall from one of understanding the behavior at each point on a mesh to one of understanding the location and behavior of a few larger meta-structures.

These structures, Lagrangian Coherent Structures can exist in any number of dimensions [11] and [12], but we will concern ourselves with those that exist in two dimensions, using planar flow to approximate behavior on the surface of the ocean. When looking at a
Figure 1: An attracting material line (in blue) and a repelling material line (in red) come together to form a Hyperbolic LCS. The image depicts how a Hyperbolic LCS acts on a region of fluid as time progresses [1].

flow, there are some standard patterns that may be present. Examples include whirlpools, wakes, and still areas. These can be thought of as a consequence of the underlying Lagrangian Coherent Structures (LCS). More specifically, LCS can serve as groupings of material lines with similar trajectories. These material lines can tend towards, or away from, and area, which gives rise to the different types of LCS.

LCS come in a few different varieties: hyperbolic, parabolic, and elliptical. Hyperbolic LCS can be broken down into two components, namely an attracting material line, and a repelling one. An attracting material line is a trajectory that nearby streamlines tend towards, while a repelling material line is the opposite. Elliptical and Parabolic LCS fall outside the scope of this work [13].

An attracting LCS, together with a repelling LCS forms an hyperbolic LCS, as seen in Figure 1. The blue line represents an attracting LCS, while the red represents a repelling one.

Given knowledge of the LCS within a flow, but absent the knowledge of a complete velocity field [10], one could make a very good guess as to the behavior of the fluid within at any point in the flow. In this respect, they are a tool that allows for easier understanding of complicated fluid problems. Additionally, it can be very difficult to obtain complete velocity field information for real flows, but obtaining the partial information required to determine the positions of the LCS in the flow may be easier. Overall, LCS are useful as a language by which complex fluid problems can be simplified. For a more comprehensive treatment, refer to [14] and [15].

1.3 The Finite Time Lyapunov Exponent

In many dynamical systems the asymptotic behavior of trajectories as time goes to infinity is an area of interest, particularly as an indicator of the stability of the system. The Lyapunov Exponent is one way of quantifying the asymptotic behavior. Specifically, it quantifies the separation of two trajectories that start off infinitesimally close to each other.
Figure 2: The image depicts trajectory separation along a repelling material line (in blue). Fluid particles located nearby at time $t_0$ separate as time progresses. The FTLE is a quantification of this trajectory separation, with regions that have a large amount of trajectory separation having a high FTLE value [2].

We can also consider separations that do not occur on an infinite time scale. These are quantified using a subcategory of Lyapunov Exponents known as Finite Time Lyapunov Exponents (FTLE). As described by the name, FTLE measure trajectory separation over finite time periods. Due to this, the time scale over which they are calculated becomes important. It is possible for trajectories that are initially nearby to diverge, and then later re-converge. Here, we will address the usage of Lagrangian FTLE, but see [16] for a development of an Eulerian approach to FTLEs, and a comparison between the two methods.

The FTLE in regions of high trajectory separation is higher than the FTLE in regions where trajectories remain close. Figure 2 shows a region where trajectories diverge, resulting in high FTLE. Regions of high FTLE typically correspond to the Lagrangian Coherent Structures [17] and [18].

One can compute the FTLE field given knowledge of a flow's velocity field. Consider the flow map $\phi_{t_0}^T: \bar{x} \mapsto \phi_{t_0}^T(\bar{x})$ which represents the advection of a fluid particle, $\bar{x}$. Consider another fluid particle in the neighborhood of $\bar{x}$, denoted $\bar{y} := \bar{x} + \delta \bar{x}$ Note that the position of fluid particles $\bar{x}$ and $\bar{y}$ vary as a function of time, therefore, the infinitesimal $\delta \bar{x}$ does as well. To determine the distance at an arbitrary point in the future, consider:

$$\delta \bar{x}(T) = \phi_{t_0}^T(\bar{y}) - \phi_{t_0}^T(\bar{x})$$

(1)

Taylor Expansion gives

$$\delta \bar{x}(T) = \frac{d\phi_{t_0}^T(\bar{x})}{d\bar{x}} \delta \bar{x}(t_0) + R$$

(2)

Where R represents higher order remainder terms that may be discarded. The magnitude of the resulting distance can be expressed as:
Note that the quantity \(\frac{d\phi^T_{t_0}(x)}{dx}\) is a matrix of partial derivatives. When multiplied by the transpose (denoted with *), we obtain the Cauchy-Green deformation tensor.

\[ \Delta = \frac{d\phi^T_{t_0}(\vec{x})}{dx} \ast \frac{d\phi^T_{t_0}(\vec{x})}{dx} \]  

Using properties of the inner product,

\[ \|\delta \vec{x}(T)\| = \sqrt{\left\langle \frac{d\phi^T_{t_0}(\vec{x})}{dx} \ast \frac{d\phi^T_{t_0}(\vec{x})}{dx}, \delta \vec{x}(t_0) \right\rangle} \]  

This is due to the inner product necessitating that one of the vectors be transposed to enable the vector multiplication.

If we are interested in the maximum difference between \(\vec{x}\) and \(\vec{y}\), this will occur in the direction of the maximum eigenvalue of \(\Delta\). This is a scalar that can be extracted out of the inner product. The FTLE is then expressed as:

\[ \sigma^T_{t_0}(\vec{x}) = \frac{1}{|T - t_0|} \log(\sqrt{\lambda_{max}}) \]

This value is a scaling factor between the initial and final differences in position of \(\vec{x}\) and \(\vec{y}\).

A more detailed treatment of FTLE can be found in [19], [20], [17], [18], [11], and [21].

### 1.3.1 False Positives and Shear

The divergent behavior described above is the primary way of detecting an FTLE ridge. However, there are cases under which false positives can result. Consider a vortex, where the speed of the fluid increases as the radius increases. After enough time has passed, fluid particles that started out near one another will end up offset by some angle as a consequence of the differences in speed. Computing the FTLE on such a region can indicate the presence of a ridge, or more likely, an entire region of high FTLE. However, this shearing effect is not indicative of the presence of a Lagrangian Coherent Structure.
Figure 3: Seed particles are tracked in groups by PIV software. The algorithm that creates the velocity field looks at groupings of particles since there is a tendency to maintain relative positions between individual particles. Each grid point is assigned a velocity based on this advection of local groupings of particles. [3]

1.3.2 Forward and Backward FTLE

Intuitively, a high FTLE region can be thought of as nearby fluid particles that end up separated after a long period of time has passed. The reverse of this procedure is also worth considering. In the standard case, only the repelling material line is significant, however, in the reverse-time case, the attracting material line behaves like a repelling one, which can be detected [22]. Fluid that starts out spread apart at far distance may come together, and end up in a closely knit region. This type of behavior can be calculated by running a numerical integration backwards in time, at which point, the computation becomes identical to the standard version. Another convenient way of calculating the backward-time FTLE (only for the case of steady-state flow), is by reversing the direction of the velocity field.

1.4 Particle Image Velocimetry

A common technique used to study fluids is Particle Image Velocimetry (PIV). PIV is used to extract a velocity field from a real flow. PIV works by adding negligible mass tracer particles to a fluid of interest. A high speed camera is set up to record the flow, and a laser sheet is created orthogonal to the direction the camera is pointing in. The laser sheet illuminates the tracer particles, whose motion is captured by the camera, and translated into a velocity field using software. It is possible for other light sources to be used, however, a laser sheet is the widespread standard.

One of the concerns with the use of our PIV system, is that it only collects data in a plane where the illumination is present. If 3D effects are present within the fluid, is is possible that they will distort the result, because tracer particles will be entering and
leaving the illuminated layer.

Given a recording of illuminated fluid particles, the software works by iterating over the image, and taking small regions (of specifiable size and shape). Within each region, particles grouped in a certain way have some tendency to remain in that pattern. This allows individual identity of particles to be tracked by the software across time by tracking small groups of particles. A raw pixel scale velocity can then be calculated, and converted to more standard units automatically based on the calibration of the system. Figure 3 shows how a PIV system may track a certain group of particles.

Conducting only one iteration when generating the velocity field is generally inadvisable, and may result in errors. Boundaries between regions may cause particle groups to not be tracked, or may cause particles to appear to belong to a certain group when they do not. A second iteration, using overlapping area, can typically solve these problems.

2 Methodology

In order to study LCS of different flows, we make use of both an experimental hardware setup, as well as computer simulations.

2.1 Software

Consider a flow defined by the stream function \( \phi(x, y) = \sin(\pi x)\sin(\pi y) \) on the rectangle \([0, 2] \times [0, 1]\). The differential equations giving velocity at every point in the region are:

\[
\begin{align*}
\dot{x} &= -\frac{\delta \phi}{\delta x} \\
\dot{y} &= \frac{\delta \phi}{\delta y}
\end{align*}
\]
The result is a velocity field for double gyre flow which will serve as a building block and reference point for later flows that will feature more gyres (see Figure 4). Figure 4 also shows the FTLE field computed from the steady state velocity field. Note the characteristic high FTLE ridge located between the gyres, which corresponds to a repelling material line.

Figure 5 shows a time dependent FTLE field, to generate this, a slightly modified stream function is used [4].

\[
\phi(x, y) = \sin(\pi f(x, t))\sin(\pi y) \\
f(x, y) = \varepsilon\sin(\omega t)x^2 + (1 - 2\varepsilon\sin(\omega t)x
\]

Where \( \varepsilon \) and \( \omega \) are parameters that dictate the magnitude and frequency of perturbation respectively.

![Figure 5: Computationally generated FTLE field for a double gyre flow with cyclical perturbations. Shearing effects generating high FTLE regions are noticeable spiraling towards the center of the gyres.](image)

As a precursor to handling experimental data, computer simulations of a two-gyre flow were performed. These serve to establish a baseline for expected behavior, and allow for contrasts to be drawn with experimental results.

The FTLE computation software is a combination of three main ideas.

- A module to establish initial conditions, and to create or load the velocity field
- An integration algorithm to determine final positions of fluid particles that begin at each grid point
- The FTLE computation, which uses the final positions grid, and carries out the computation outlined previously in the FTLE section

The simulation differs slightly from the code that will be later used to handle experimental data in that there is no file loading required. When velocity is required by the integration algorithm, it is obtained by evaluation of the differential equations which define the idealized flow.
The traditional method for FTLE computation involves incrementally updating particle positions and velocities to track particle motion into the future and see if nearby particles end up diverging. An example of a non-steady state double gyre flow is shown in Figure 5. The regions of high FTLE correspond to LCS.

A different approach to compute the locations of LCS is developed in [23]. Because the FTLE field may contain false positives for LCS, Farazmand and Haller outline a set of criteria to apply in order to check for the presence of LCS within a flow. These criteria are checked by integrating along the streamlines of the flow, and considering the eigenvectors and eigenvalues of the strain tensor at various points in the flow. A comprehensive software package has also been developed by Haller's nonlinear dynamics group, to facilitate study of these types of problems [24].

2.2 Hardware

In order to study the properties of real flows, an experimental tank has been constructed. A wooden base serves as support for the metal and plastic tank. A second metal guidepiece serves to hold the gear shafts straight, and functions as an attachment point to the wooden base. Adhesive facilitates the attachment, and can easily be removed for maintenance, then reapplied afterwards. Without it, the connections between the motor gear shafts, and the vertical shafts driving the discs would be unreliable.

A consideration in motor selection was that the shafts linked to the discs fit snugly through a machined piece of metal that served as the base of the new tank. This fit needed to be tight to prevent leaks. However frictional effects meant that correspondingly high torque was required from the motors.

The assembly features sixteen rotating discs, placed in a four by four grid, in an open-
top tank walled off by clear plastic to allow for laser penetration. Each corner, consisting of four discs is controlled by a high torque DC motor. Figure 7 shows the motor groupings, each is marked by a black box. The motors can be controlled independently. Gearing is chosen such that the direction of rotation of the discs alternates within each grouping. This results in a system that is capable of producing a variety of flows of interest, including different gyre arrangements and jet flows, among others.

Accompanying the physical tank, is a high-speed camera mounted overhead (as seen in Figure 6)). The camera is connected to a controller, which is in turn connected to a computer running DaVis PIV software. The PIV software extracts velocity field data for later processing. The fluid of choice for these experiments is glycerine. Glycerine has a viscosity that is roughly three orders of magnitude greater than that of water.

To image flows in the tank, the fluid filling the tank is pre-seeded with tracer particles. It is important to thoroughly mix the particles into the glycerine solution, but then follow this by giving the fluid time to settle so that any air bubbles present will dissipate. Air bubbles, or clumps of seed particles can interfere strongly with the normal detection of tracer particles during PIV, resulting in imperfections in the resulting velocity field. These particles are illuminated by a laser sheet, as seen in Figure 8, and their motion is captured by the high speed camera. This method gives data by which a velocity field to conduct processing on is generated.

2.2.1 Dye Experiments

For a qualitative look at some of the behaviors of fluid within the tank, before moving to more rigorous PIV imaging, the tank was run with partially dyed glycerine (see Figure 9).
Figure 8: Experimental tank illuminated by laser sheet.

Figure 9: Partially dyed glycerine advected by motors. Dye was initially placed at a few points on the middle band of 8 discs (one at the center of the rightmost grouping of 4, one on the top edge of the second disc in the second row, and one at the center of the first disc on the third row). Dye tends to follow along the suspected attracting and repelling material lines. This is especially evident in the lower left region of the tank.
The following are some observations pertaining to the dye test.

- Regions of dyed fluid located between discs spread out, with some following the curve of each disc. This is the type of behavior that the FTLE computation seeks to detect.
- Reversing the direction of flow mostly undoes the spreading effect. This is due to the fact that the flow has a low Reynolds number of approximately 1 or 2.
- Some dyed sections of the fluid tend to slowly spiral towards the center of a rotating disc. The probable cause is that velocity is lost by shearing frictional effects. In later sections, we will keep this in mind.

2.3 Data Collection

The experimental setup can be used to collect a wide variety of datasets. In this section, we will briefly cover those that will be used for analysis in later sections. The basic flow case is the one that consists of steady state alternating direction discs. This serves to produce several areas that are similar to a more basic two-gyre flow, which allows for comparison within the produced velocity field in order to make sure the collected data is usable.

In a prior section, potential application of this area of study are discussed. Here, we wish to conduct some experimental trials in order to examine the two most relevant conditions. Specifically, the introduction of a particle into the flow, and the composition of flows. To this end, trials were conducted by slowly adding fluid at points in the baseline flow to simulate a point source, or a larger, 'river' or 'waterfall' type of source. In addition, heavier particles were introduced into the flow, in order to later examine the FTLE readings near such particles. Finally, larger, static objects were introduced to the surface of the flow, again, to examine the resulting FTLE field in the vicinity.

The other aspect to feature prominently into data collection is the software and hardware settings for the experimental setup. Certain settings, such as the laser amperage, imaging frequency, or camera focus and zoom, are adjusted iteratively, until the extracted velocity field is well resolved. Others, such as the duration captured, and the motor speed, can be varied if needed, but as those topics are not covered in much depth here, we note that these are held constant for all further analyses that will be conducted. In particular, the motor speed is such that a disc makes a full rotation is approximately 4 seconds.

3 Analysis

3.1 Areas of Investigation

Initial experimental goals involve testing the validity of the experimental setup, and a comparison of software analysis techniques. More specifically, the following questions will be addressed.

- Does the velocity field produced by the experimental setup generate an FTLE field that is consistent with expectations? The following sections will test the validity of the experimental setup.
Figure 10: Streamlines and velocity field for steady state flow with alternating disc direction generated by DaVis software (used to calculate FTLE field). Velocity vectors are to scale, relative to one another.
• How does varying analysis methods, while leaving input parameters constant, impact the resulting FTLE field. More technically, we will compare various numerical integration techniques, and examine a backward FTLE field.

• Given a reasonable understanding of the basic flow from the previous analysis topics, how do modifications to it (the introduction of particles, or a fluid source) impact it.

3.2 Validity of a System

3.2.1 Establishing a Baseline

We begin by establishing some criteria that together are indicative of a successful experimental setup. First, we need to convert the output from the PIV system to a usable format. In this case, as the analysis will be conducted using the MATLAB software package, we will need to convert to the .mat file format. At this point, the script to compute FTLE can be run, giving image-based output.

We know from prior works, and we have verified via software simulation that an FTLE ridge indicating the presence of a Lagrangian Coherent Structure should occur between two discs that are rotating in opposite directions. We also know that it is possible that shearing effects of the fluid moving around the exterior of the discs may result in a high FTLE area that is not indicative of the presence of an LCS. Therefore, we would not be surprised to see high FTLE regions in rings around the discs.

Finally, we expect that the velocity field from all the discs will be present and, roughly equal in strength. Errors in this area would likely be indicative of a lack of seed particles for the PIV system to capture, or poor lighting rendering those tracer particles invisible to the camera.

We take as a base case, a steady state flow produced by an alternating grid of discs (see Figure 11).

Immediately, we notice several positives.

• We see 16 roughly circular regions with low FTLE, corresponding to the centers of the discs.

• We see horizontal or vertical FTLE ridges between adjacent discs.

• There is no interference generated by anything around the tank, in other words, the background isn’t disruptive.

• Laser penetration and particle seeding are likely good, the patterns do not break down in any area of the tank, which might have been indicative of poor data quality.

• The values that make up this scalar field that we claim is an FTLE field appear to be within standard ranges.

However, there are also some potential problems.

• FTLE ridges near the center of the tank appear to consist of slightly higher FTLE values, and are more clearly defined.
• The centers of some discs appear to have more of an '+' shape than a circular one (see the regions circled in black in Figure 12).

• Roughly triangular regions of lower FTLE value exist at the centers of 4 disc groupings. These exist in pairs, separated by a high FTLE ridge (see the regions circled in white in Figure 12).

In order to address the discrepancy in strength between the center, and exterior FTLE ridges, briefly refer back to Figure 10. Note that this does not correspond to the velocity field which has generated our current FTLE field. Between the second and third discs in either the second or third row from the top, or between the third and fourth discs in the bottom row, we can see that the divergent behavior present in other areas (for example between the first and second, or second and third discs on the second row from the top, or that we saw in the dye experiments), is not present.

We conjecture that material conservation properties are playing a role to move fluid along paths that it might not take in the absence of such a property. If more fluid is being drawn out of an area than is being replaced, fluid pressure forces will work to correct for this, potentially overcoming the effect of the velocity field generated by the rotating discs.

To understand the triangle regions, we can consider an example of an FTLE field where we consider the evolution over a longer period of time (see Figure 13). In this example, the integration time has been increased from 2 seconds to 4.5 seconds.
Figure 12: This FTLE field is Figure 11) with marked triangle shaped regions (white), and '+ ' shaped disc centers (black).

Figure 13: FTLE field for a nearly steady state alternating disc multi-gyre flow. Particle position is calculated out 4.5 seconds into the future, this covers just over one full rotation of the discs. Note the absence or decreased prominence of the elements marked in Figure 12.
Figure 14: The FTLE field displayed here is identical to Figure 13, with marked regions indicating a possible explanation for the triangle regions observed in Figure 12.

The effect of this increase is that we are considering the position of the particles at a point further out in time to be the final position. In this example, the triangle regions are gone in some areas, and barely present in others.

This suggests that the following behavior is present at the center of four disc groupings:

Slow moving fluid is pushed into the area from two of the four regions in between the discs, while the remaining two areas slowly draw fluid out. Fluid that ends up nearer to one of the outward paths is very likely to use that path, while fluid in the middle may take either path. This creates the FTLE ridge between the triangle structures. If the final fluid positions are not taken to be far enough into the future, the fluid that is drawn in from one side has not yet fully passed between the pair of discs, meaning it still remains close together, and therefore has a lower FTLE. If we consider a later final position, the fluid has been spread out when it leaves the region between the two discs.

Note the boxed regions in figure 14. The white box is the starting location. The black box marks where the fluid is after a shorter integration time, and the Pink box marks where it may be after a longer integration time.

Finally, consider the tendency for the centers of the discs to end up being ‘+’ shaped. They remain present in the longer integration time field. Using the information we just discovered regarding fluid speed, we conjecture that shearing effects are more prominent near the region between four discs, compared to the faster moving fluid between the individual pairs of discs. Fast moving fluid adjacent to slower, or stationary fluid will suffer shear,
Figure 15: This FTLE field is generated by negating the velocity fields that generated Figure 13, and then running the same FTLE algorithm. This shortcut to compute backwards FTLE only works for steady-state flows.

and frictional forces, resulting in fluid spreading out, and giving a false positive FTLE, as the differences in final position are not due to the fluid being located on the boundary of adjacent Lagrangian Coherent Structures.

### 3.2.2 Backward FTLE

Figure 15 shows a simplistic computation of a backward FTLE conducted by negating the velocity at all points, over the entire time interval.

The clear identifier that this is a backward FTLE is that orientation of the lower-FTLE triangles in the center (see Figure 12 for comparison).

Despite this being a backwards FTLE calculation, the resulting ridge arrangement is nearly identical. Given that the fluid in a near enough approximation of steady state flow, for fluid region about to spread apart in the forward time direction, there is a fluid region that has already been spread apart. During the calculation of the backward FTLE field, these come together, indicating the high FTLE areas.

### 3.3 Variations in Computation

#### 3.3.1 Comparison of Integration Techniques

In order to determine the final position of a fluid particle it is necessary to apply a numerical integration algorithm. In the preceding sections, a first order Euler’s Method is
Figure 16: FTLE field for a nearly steady state alternating disc multi-gyre flow. Particle position is calculated out 4.5 seconds into the future using Matlab’s 'ode45' solver. FTLE ridges seen in earlier figures are notably absent.

used, but perhaps the usage of a more sophisticated integration algorithm will improve the results.

The two algorithms that will be addressed in this section are a 4th-Order Runge Kutta method, and Matlab’s built in 'ODE45' function, which uses a variable step size Runge Kutta method. We begin by examining the FTLE field generated by the built in solver (see Figure 16).

This result has a number of glaring flaws. There are no FTLE ridges located between the centers of the discs. Furthermore, the range of values that the FTLE takes are much lower than those generated previously Note that this trial was run with a 4.5 second integration time, which is near the upper end of what is supported by the data that had been collected. Shorter integration times yielded similar results. Attempts to modify the interpolant, or adjust the options flags for the solver, were met with little success, resulting in figures similar to Figure 17.

In order to diagnose the issue, we will implement and test a 4th-Order Runge Kutta solver (RK4). Given a sequence of experimentally obtained data at a set of times with time step size $dt$, we use the RK4 algorithm and interpolation to advect a set of particles according to the experimental velocity field data.

Repeated applications trace out a trajectory forward into time, giving the final particle positions that are necessary to compute the FTLE.
Figure 17: This figure is computed from the same data set as Figure 16, however, it features an altered interpolation function.

Figure 18: This FTLE field is nearly identical to Figure 17, however, the final particle positions in this version are computing using a homebuilt RK4 solver. The integration algorithms uses in both examples were built independently from each other.
Figure 19: In this figure, a cap is applied to the FTLE field in Figure 18, in an attempt to ensure that some noisy peaks were not dominating the scaling.

The result of the RK4 integration are seen in Figure 18. Defined FTLE ridges are missing, but currently, some of the noise from the non-tank area is dominating the computations. Perhaps removing those, and rescaling to a different max FTLE value will yield better results. We should also note that this image is nearly identical to the result from figure 17, even having the same FTLE peaks in the noisy area outside of the tank.

Figure 19 still shows a FTLE field that is lacking ridges between the discs. It can be considered a marginal improvement, due to the improved clarity of the shearing effects near the edge of each disc. The lack of ridges has occurred using both the inbuilt ode45 solver, and an implementation of RK4, and these two different methods have given a nearly identical FTLE field. This suggests that it is possible, but unlikely that the error is due to a code bug, since that bug would need to have been reproduced in two different implementations. At this point, we conjecture that using a more sophisticated integration misses the detection of certain high FTLE areas, when used with the collected datasets, and consider why this might be the case.

In order to diagnose the issue, we consider what improvements a more sophisticated integration scheme offers, in comparison with Euler's Method. At the most basic level, the more sophisticated schemes seek to better minimize error. However, the FTLE itself is a measure of deviation. It is possible that on the time-scales which are being worked with, fluid particles integrated using Euler's Method rapidly show deviation from a trajectory, while other methods do not. In the case that we consider particles that being in close proximity to one another, the trajectories are likely to be close to each other.
This theory has some drawbacks though. How can the low FTLE areas in the FTLE field computed with Euler’s Method be explained. If the ‘correct’ trajectories are so similar, why wouldn’t deviations occur in a consistently similar way, still resulting in similar ending positions. Another possibility is that the grid size for the velocity and position data is interacting unfavorably with a step-based approach. If interpolations are too inaccurate, perhaps there is a tendency for everything traveling between certain grid sections to end up at a point nearby on the boundary. Euler’s method, which does not use the midpoint interpolation that RK4 does, would be immune to this effect.

In the following section, we will use Euler’s method as the numerical integration method of choice. The usage of the different integration schemes remains an area of potential future investigation.

3.3.2 Grid Size

Recall that while discussing PIV, we noted that the recording is overlaid onto a grid in order to track groupings of seed particles. Previous figures, such as Figure 13 are on an 80 x 50 grid, while the FTLE field in Figure 20 is computed on a 54 x 34 grid. This figure is not just a re-computation of the PIV; it is computed using data from a different experimental trial. We note the similarity with figures such as Figure 13, and consider this as further evidence for the validity of the experimental setup.

At the most basic level, altering the grid fineness gives stronger or weaker knowledge of the velocity field as a whole. A finer grid better approximates a stream function, by providing velocity field data at more points, and closer neighbors (in general) if interpolation is required. A coarser grid moves in the opposite direction. Interpolations may be
Figure 21: FTLE field for an alternating disc multi-gyre flow. Particle position is calculated out 4.5 seconds into the future while fluid is being poured into the bottom-right corner of the tank.

less precise, since fewer specific velocity data points are known.

Grid size has implications for the PIV system when it comes to determining the velocities of particle groups as well. If a grid is too fine, it renders the PIV ineffective, since particle groups may not be easily contained within a grid square, or there may not be enough space to track a group of particles over time.

3.4 Variations in Flow

Up until this point, we’ve focused on testing the validity of the experimental setup. At this point, we have some confidence in the results produced by the process, and the direction of inquiry shifts to address some other types of flow.

3.4.1 Basic Filling

We begin by considering the FTLE field displayed in Figure 21. Fluid is being poured in between the third and fourth discs in the fourth column, along the edge of the tank. There are heavy similarities between this field, and those generated earlier, such as Figure 11. We notice ridges between adjacent discs, and shearing effects near the boundaries of some discs. We expect that there will be a general flow towards the top left of the tank as the glycerine seeks a level. The area where the fluid is being added is highly disrupted, resulting in an entire region of high FTLE values. These high values do not entirely overshadow the effects of the discs in the area.
Several discs maintain a lower FTLE region in the center of the disc. This suggests that the existing Lagrangian Coherent Structures associated with each disc are somewhat resistant to fluid flowing across them.

Furthermore, it is likely that vertical effects are present within the flow, and it is very likely that tracer particles are moving into or out of the laser sheet. This is a potential cause for the moderate FTLE region in the top left of the tank, where, in theory, some fluid should be coming to a stop. We can now alter parameters of this flow, and examine what effects that may have. Figure 22 displays an FTLE field generated using a shorter integration time (two seconds).

In this case, several discs, especially those in the top left, maintain a low FTLE region in the center. It seems that 2 seconds is not enough time for the fluid to spread evenly across the tank. An FTLE field that is calculated starting at a point roughly 1 second further into the future is shown in Figure 23.

In this image, the flow has broken down more, and we begin to see greater similarity to Figure 21. This suggests that the longer integration time FTLE field looks the way it does because it is capturing flow that happens once a large amount of fluid has propagated across the tank. Starting a shorter integration later achieves a similar effect.

Additionally, while FTLE ridges between discs do exist to some degree, there appears to be a larger trend, of three high FTLE regions dividing the rest of the tank into wedges.
Figure 23: This FTLE field is generated using the same data set and integration run time as Figure 22, however, the computation is started approximately 1 second further into the future.

Figure 24: Identical FTLE field to Figure 23 with high FTLE ridges marked.
Figure 25: FTLE field for a alternating disc multi-gyre flow. Particle position is calculated out 2 seconds into the future while fluid is poured into the bottom-right corner of the tank, top 8 discs stopped midway through the integration (see Figure 24). These wedges likely serve as low resistance paths by which the new fluid spreads across the tank.

3.4.2 Filling with Motor Stop

We briefly examine the case where the motor stops in the middle of a PIV session (see Figure 25).

We observe the same division into wedges that is present in Figure 24. In this case though, the regions of rotational motion caused by the discs has largely broken down. At the point in time that the calculation of this FTLE field begins from, the top eight discs have recently stopped rotating. There is some tendency for rotational motion to continue; this is especially apparent in the top left corner of the tank.

A motor stopping can be compared to running a shorter integration time. During standard motion, fluid particles follow typical behavior, and after a stop, rapidly come to a halt. Once they do so, there is little further divergence, so the final positions are essentially what they were at the point where the motors stopped. Integration from that point forward means little, unless the motors resume motion.

Overall, stopping motors for a period of time tends to simply shorten the integration time. We can only make this claim for the glycerine that was used in experiment; another fluid may retain inertia better.
Figure 26: FTLE field for a nearly steady state alternating disc multi-gyre flow. Particle position is calculated out 2 seconds into the future. A tangled bundle of thin wire which allows for some liquid to pass through is fixed at the fluid surface in the position marked by the black line.

### 3.4.3 Wire Blocking

In Figure 26, black lines mark the approximate position of a bundle of wire fixed to the edge of the tank, and resting at the surface. The wire bundle is able to be passed through by the fluid, and is acting similarly to a screen or mesh, but with greater width. The region to the top right is partially blocked by the end of the wire, which is attached to the side of the tank.

Compare the FTLE ridge between the first and second columns of discs with the ridge located between the second and third. The ridge where the wire is present is not as clearly defined. The permeable wire bundle serves to lower the FTLE values by a little bit. The wire serves to lower velocity, and introduce some turbulence into the flow. This creases the possibility that fluid particles that would have previously diverged may end up nearby, due to the interference of the wire.

At first, this seems to contradict earlier observations regarding tank filling and unpredictability. However, in that case, the regions that were affected were already mostly low FTLE regions.

### 4 Future Work

Now that an experimental setup has been constructed, and we have some confidence in the data that can be extracted from it, the tests in the previous section serve as a starting point for continuing investigation into various modifications to a basic flow.
• The filling experiments serve as a starting point into investigating the presence of FTLE ridges in overlaid flows, or using their presence to detect if one flow in a region dominates the others.

• Numerical integration techniques need to be studied more thoroughly. Specifically, the question of the absence of FTLE ridges generated by higher order integration methods needs to be addressed.

• Particle motion, and irregular boundaries require study, in order to better understand how FTLE ridges, and Lagrangian Coherent Structures behave near boundaries. Real flows may exist in the presence of such factors.

• Finer data grids, generated through the PIV software, may tie in with the integration technique questions. Current data on a finer grid attempts to produce in-memory objects that are hundreds of GB in size. Significant software modifications could be made, which could allow for the use persisted data. By accessing only a small amount of it at a time, it should be possible to circumvent this issue.

• Now that there is some more substantial groundwork completed, especially pertaining to the experimental setup, it may be time to refocus efforts onto the inertial effects problem.
5 Bibliography

References


