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Interlace Polynomial of a Special Eulerian Graph

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Interlace Polynomial of a Special Eulerian Graph

by

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Abstract

In a recent paper, Arratia, Bollobás and Sorkin discussed a graph polynomial defined recursively, which they call the interlace polynomial. There have been previous results on the interlace polynomials for special graphs, such as paths, cycles, and trees. Applications have been found in biology and other areas. In this research, I focus on the interlace polynomial of a special type of Eulerian graph, built from one cycle of size n and n cycle three graphs. I developed explicit formulas by implementing the toggling process to the graph. I further investigate the coefficients and special values of the interlace polynomial. Some of them can describe properties of the considered graph. Aigner and Holst also defined a new interlace polynomial, called the Q-interlace polynomial, recursively, which can tell other properties of the original graph. One immediate application of the Q-interlace polynomial is that a special value of it is the number of general induced subgraphs with an odd number of general perfect matchings. Thus by evaluating the Q-interlace polynomial at a specific value, we determine the number of general induced subrgaphs with an odd number of general perfect matchings of the considered Eulerian graph.

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Contents

1 Acknowledgments

I would like to thank Dr. Aihua Li for introducing me to the study of interlace polynomials, and all the time and effort she put into helping me develop this thesis. I would like to thank my committee members Dr. Cutler and Dr. Song, for their time to make my thesis a completed work. I would also like to thank my parents Teresa and Pawel, for the love and support during my whole life, but especially during the last few years of my undergraduate and graduate schooling. I would also like to thank my sister Sylvia for being an inspiration to achieving my goals. I would also like to thank my uncle Adam, for gving me motivation during graduate school. Last but not least, I would like to thank my friends for being there when I needed them to be.

2 History

The origin of studying Eulerian graphs came from Euler's solution to the Konigsberg Bridge Problem in 1735; he proved that it was not possible to cross each of the seven bridges exactly one time and return to the orginal starting point [13]. Through solving this problem, Euler established the foundation of graph theory. He defined the *Eulerian graph* to be a graph containing an Eulerian circuit, which is a circuit that includes each edge of a graph exactly once, starting and ending at the same vertex. His negative result on the *k Bridge Problem* was the first theorem in graph thoery [13].

In recent years, an abundance of graph polynomials have been studied. The two most prominent graph polynomials that have been studied are the *Tutte Polynomials* and the interlace polynomial, which resembled the *Martin Polynomial* [12]. The Tutte polynomial, a two-variable graph polynomial, has the important universal property that essentially any multiplicative graph invariant with a deletion/contraction reduction must be an evaluation of it [8]. These deletion/contraction operations are natural reductions for many network models arising from a wide range of problems at the hearts of computer science, engineering, optimization, physics, and biology [8]. *Martin Polynomials* can be considered as a specific type of a *Tutte Polynomial*, by making the two variables equal, which is how we can connect interlace polynomials to *Tutte Polynomials.*

Eulerian circuits are directly connected in DNA sequencing by hybridization; counting the number of 2-in, 2-out digraphs (Eulerian, directed graphs in which each vertex has in-degree two and out-degree two) [3]. The interlace polynomial can count the number of Eulerian circuits in a 2-in, 2-out digraph [3] and models the interlaced repeated subsequences of DNA that can interfere with the unique reconstruction of the original DNA strand [8]. The interlace polynomial can also be used to successfully count the k-component circuit partitions of a graph [3]. Interlace polynomials share similar properties as Martin Polynomials and Kauffman polynomials, which encode information about the families of closed paths in Eulerian graphs [4]. Since Euler circuits have a lot to do with DNA sequencing and other graph polynomials, I choose to observe a certain type of Eulerian graph and investigate if they share common properties with other graph polynomials. In this paper I focus on the interlace polynomial for a specific Eulerian Graph, Γ_n, and investigate how the graph polynomials can be specialized and generalized, and if they can encode any information relevant to physical applications. Aigner and Holst defined two interlace polynomials [1]. Since each one holds its own specific properties, I develop the recursive and explicit formulas for both interlace polynomials for Γ_n.

3 Preliminaries

In this paper, we work with graphs, which are represented by an ordered pair of sets of vertices and edges.

Definition 1. *A graph G is an ordered pair $G = (V, E)$ such that:*

1. *V is a set, called the vertex set;*

2. *E is a set of two-element subsets of V, called the edge set, that is* $E \subseteq \{\{u, v\} : u, v \in V\}$.

The number of edges incident to a specific vertex, $v \in V$, is called the degree of v, denoted $deg(v)$. The maximum degree of a graph G is the highest degree of all the possible vertices in the graph G. From existing results, we know that for any graph, the number of vertices with odd degree is always even. Furthermore, the degree sum formula tells us the sum of the degrees of all vertices in a graph G is equivalent to 2 times the cardinality of the edge set of G.

Lemma 3.1. *(Degree Formula) Given a graph $G = (V, E)$,*

$$\sum_{v \in V} deg(v) = 2 \mid E \mid$$

.

In this paper we focus only on simple graphs. A graph G is called a simple graph if there are no multiple edges joining the same pair of vertices, as well as no loops, which would be an edge that starts and ends at the same vertex. In order to uderstand the structure of a graph, we state common elements a graph can have, which can be found in any graph theory text book.

Definition 2. *Given a graph $G = (V, E)$, where $V = \{v_0, v_1, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$:*

1. *A walk of length k is a sequence $v_0, v_1, \ldots v_k$, which contains k edges of the form $\{v_0, v_1\}, \{v_1, v_2\}, \cdots, \{v_{k-1}, v_k\}$.*
2. *A path of length k is a walk with $k+1$ distinct vertices, denoted by P_k.*
3. *A cycle of size k, is a path with k vertices, with an additional edge between v_k and v_1, making a closed path, denoted by C_k.*
4. *A sequence of distinct edges $e_1 e_2 \cdots e_k$ is called a trail if we can take a continuous walk in our graph, first walking through the edge e_1, then the edge e_2, and so on. In addition, if we start and end at the same vertex, we have a closed trail.*
5. *An Eulerian Trail is a trail that covers each edge of G exactly once.*
6. *An Eulerian Circuit is an Eulerian Trail that starts and ends at the same vertex, covering each edge exactly once.*

Note that the difference between a trail and a path is the uniqueness of edges and vertices. A trail has distinct edges while a path has distinct vertices.

Definition 3. *[7] If for any two vertices x and y in a graph G, one can find a path from x to y, then we say that G is a connected graph.*

When dealing with graphs, we can have directed or undirected graphs. A graph where each edge is assigned a direction is a *directed graph.* When the edges of a graph are not assigned a specific direction, we are dealing with an *undirected graph.*

A graph G' is called the *subgraph* of the graph G if the set of vertices and edges of the graph G' form subsets of the vertices and edges of the original graph G. In other words, $G' = (V', E')$ where $V' \subset V$ and $E' \subset E$. An *induced subgraph* of G by a subset $S \subseteq V(G)$ is the subgraph $G' = (S, E')$ where for $u, v \in S$, $uv = e \in E' \iff e \in E(G)$. We must take all and only those edges present in G between the specified vertices in S. That is $E' = \{uv \mid \ u, v \in S, \ uv \in E(G)\}$. Below we define special graphs that we will use further in the paper.

Definition 4. *Special Graphs*

1. *A Bipartite Graph is a graph whose vertex set is decomposed in two disjoint sets, called partite sets, such that no two graph vertices within the partite set are adjacent.*

2. *A Complete Graph, denoted K_n, is a simple undirected graph which every pair of distinct vertices is connected with a unique edge.*

3. *A Complete Bipartite Graph, denoted $K_{m,n}$, is a bipartite graph such that every pair of graph vertices in different partite sets are adjacent.*

4. *An Eulerian Graph, is a graph containing an Eulerian circuit, which is only possible if all vertices are of even degree.*

When finding the interlace polynomial of a graph we need to recall a few definitions. If $x \in V(G)$, $G \setminus \{x\}$ is the resulting graph after removing the vertex x and all edges of G incident to x. Further, we need to recall the pivot of a graph [13].

Consider an undirected graph G and $a, b \in V(G)$, with $ab \in E(G)$. The edge ab will divide the neighbors of a or b into three classes: (1) vertices adjacent to both a and b, (2) vertices adjacent to a alone, and (3) vertices adjacent to b alone. Hence when we toggle an edge in between any two of these three classes, xy will be an edge of the new graph if and only if xy is not an edge of G.

Definition 5. *[3] (Pivot) Let G be any undirected simple graph and ab an edge of G. The pivot of G with respect to ab, denoted G^{ab}, is the resulting graph after toggling all pairs x, y where x, y are from different classes of (1), (2), (3) described above, shown in Figure 1.*

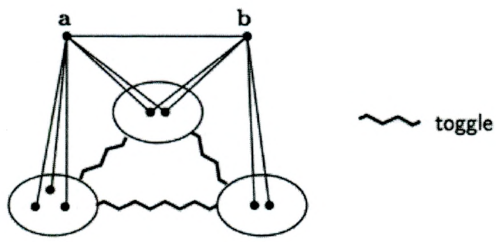


Figure 1: Neighborhoods of a, b and the Toggle Operation

The formula for finding the interlace polynomial of a graph G is given recursively by:

Definition 6. *(Interlace Polynomial) For any undirected graph G with n vertices, the interlace polynomial of G is defined recursively by:*

1. $q(G, x) = x^n$ *if* $E(G) = \varnothing$;
2. $q(G, x) = q\left(G \setminus \{a\}, x\right) + q\left(G^{ab} \setminus \{b\}, x\right)$ *where* $ab \in E(G)$;
3. $q(G, x) = q(G_1, x)q(G_2, x) \cdots q(G_n, x)$ *if* $G = G_1 \uplus G_2 \uplus \cdots \uplus G_n$.

Note that it is shown [3] that the interlace polynomial is the same no matter what edge is toggled. Below we state the existing results of interlace polyomials of certain graphs.

Lemma 3.2. *The interlace polynomials are known for the following graphs [3]:*

1. *(complete graph K_n)* $q(K_n, x) = 2^{n-1}x$;
2. *(complete bipartite graph $K_{m,n}$)*

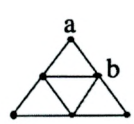
   $q(K_{m,n}, x) = (1 + x + \cdots + x^{m-1})(1 + x + \cdots + x^{n-1}) + x^m + x^n - 1$;
3. *(path P_n with n edges)* $q(P_1, x) = 2x$, $q(P_2, x) = x^2 + 2x$, *and for* $n \geq 3$,

   $q(P_n, x) = q(P_{n-1}, x) + xq(P_{n-2}, x)$;
4. *(small cycles)* $q(C_3, x) = 4x$ *and* $q(C_4, x) = 3x^2 + 2x$.

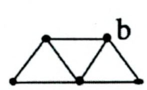
Note that the interlace polynomial of an isolated vertex is simply x. Below is an example of the pivot process for a graph G.

Example 1. *Developing the Interlace Polynomial of* Γ_3 *using the pivot process.*

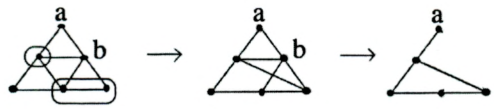
Consider the following graph, called Γ_3. *We pivot with respect to ab.*



$\Gamma_3 \setminus \{a\}$:

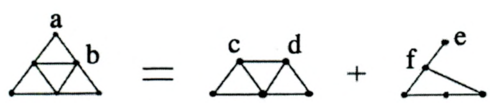


$\Gamma_3^{ab} \setminus \{b\}$: *In this step we toggle between* $N_1 = N(a, b)$ *and* $N_2 = N(b) \setminus (\{a\} \cup N(a))$, *then remove b and its adjacent edges.*

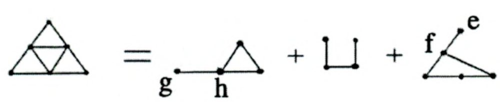


We will now use this pivot process multiple times, in order to find the interlace polynomial for Γ_3*:* $q(\Gamma_3, x) = q\left(\Gamma_3 \setminus \{a\}, x\right) + q\left(\Gamma_3^{ab} \setminus \{b\}, x\right)$.

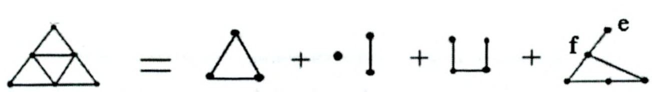
Step 1: Using the pivot process from above we obtain the two graphs $\Gamma_3 \setminus \{a\}$ *and* $\Gamma_3^{ab} \setminus \{b\}$.



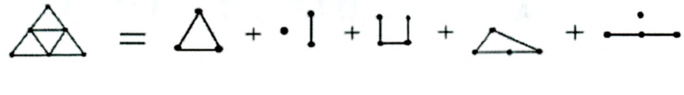
Step 2: Pivoting $\Gamma_3 \setminus \{a\}$ *with respect to cd, we obtain a graph we call* Λ_1 *and* P_3.



Step 3: Pivoting Λ_1 *with respect to gh gives us* C_3 *and* xP_1.



Step 4: Pivoting $\Gamma_3^{ab} \setminus \{b\}$ *with respect to* ef *gives us* C_4 *and* xP_2.



Now we are able to substitute the existing results for the graphs left in *Step 4* and determine the interlace polynomial for Γ_3. From our last step in the pivot process of Γ_3, we have $q(\Gamma_3, x) = q(C_3, x) + xq(P_1, x) + q(P_3, x) + q(C_4, x) + xq(P_2, x)$.

$$
\begin{aligned}
q(\Gamma_3, x) &= 4x + x(2x) + (3x^2 + 2x) + (2x + 3x^2) + x(x^2 + 2x) \\
&= x^3 + 10x^2 + 8x.
\end{aligned}
$$

This specific example is actually an example of the type of graph we work with, and is denoted by Γ_n. I define this type of graph further in this section, but before I do, I want to point out another graph from *Step 2* of Example 1. This graph consists of a triangle and an edge from *Step 2*, and is called Λ_1. By the breakdown in *Step 3* we see that $q(\Lambda_1, x) = 2x^2 + 4x = 2x(x + 2)$. This result will be used in section 4.

Lemma 3.3. *The interlace polynomial for* Λ_1 *is* $q(\Lambda_1, x) = 2x(x + 2)$.

In this thesis, I focus on a special type of an Eulerian graph, Γ_n. An Eulerian graph is a graph that contains an Eulerian circuit. Recall that an Eulerian circuit is a trail that starts and ends at the same vertex and uses each edge exactly once. A certain vertex can be repeated throughout the circuit, but an edge cannot be repeated. The graph Γ_n has all vertices with degree two or four. It consists of a cycle graph C_n, where each edge contributes to a cycle graph C_3 along the perimeter of C_n, and the third vertex is of degree two. I use pivoting and other techniques to find the interlace polynomial for this graph of any size n. Further, I discuss any recognizable patterns noticed during the pivoting process. After finding a explicit formula for the interlace polynomial of Γ_n, I focus on the meaning of the coefficients, properties at specific values of x, properties dealing with different paraties of x, the relationship with matrix theory, and other applications of the interlace polynomial.

4 Basic Properties of Γ_n

Definition 7. *For* $n \geq 3$, *define* $\Gamma_n = (V, E)$, *where* $V = \{v_1, v_2, \cdots, v_{2n}\}$ *and* $E = \{v_1v_2, v_2v_3, \cdots, v_{2n-1}v_{2n}, v_{2n}v_1, v_1v_3, v_3v_5, \cdots, v_{2n-3}v_{2n-1}, v_{2n-1}v_1\}$.

Example 2. *The graph* Γ_n *(see Figure 2): the inner part is* C_n

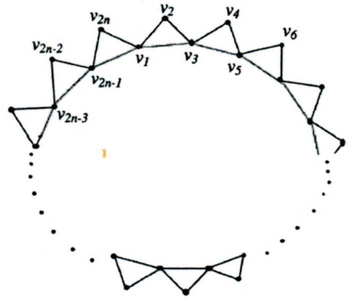


Figure 2: Γ_n with n three-cycle graphs outside of the cycle graph C_n

An example of Γ_4 is provided below.

Example 3. *The graph Γ_4 consists of the graph C_4, with a graph C_3 associated to each edge in C_4, around the outside perimeter: (see Figure 3).*

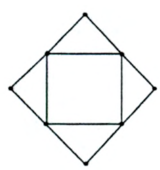


Figure 3: Γ_4

Proposition 4.1. *The graph Γ_n has $2n$ vertices and $3n$ edges, with n of each, degree 2 and degree 4 vertices.*

Taking a look at the characteristics of the graph Γ_n gives us a better perspective on how to relate the meaning of our graph at certain values of x. The following characteristics were observed and proved:

Theorem 4.2. *Properties of Γ_n.*

1. *The independent number of Γ_n is n.*

2. *Edge-connectivity and vertex-connectivity of Γ_n are both 2.*

3. *The circumference of* Γ_n *is* $2n$*, which is the cardinality of the vertex set, or* $|V(\Gamma_n)|$.

4. *Diameter*$(\Gamma_n) = \begin{cases} \frac{n+2}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$.

5 The Interlace Polynomial of Γ_n

Throughout, for any graph G, we will represent the interlace polynomial $q(G,x)$ as $G(x)$. Using the toggling approach and some basic results about interlace polynomials, we first describe the interlace polynomial of Γ_n for small values of n. The polynomials are shown below for n ranging from 3 to 7.

Lemma 5.1. *The interlace polynomials for* Γ_n*, with* $3 \le n \le 7$*, are as follows:*

1. $\Gamma_3(x) = x^3 + 10x^2 + 8x$;

2. $\Gamma_4(x) = x^4 + 8x^3 + 32x^2 + 24x$;

3. $\Gamma_5(x) = x^5 + 10x^4 + 40x^3 + 96x^2 + 64x$;

4. $\Gamma_6(x) = x^6 + 12x^5 + 60x^4 + 160x^3 + 272x^2 + 160x$;

5. $\Gamma_7(x) = x^7 + 14x^6 + 84x^5 + 280x^4 + 560x^3 + 736x^2 + 384x$.

While working on the polynomials for the specific values of n, I started to notice a couple of patterns in the graphs that resulted from the toggling process. For n greater than or equal to four, the toggling process produced very similar graphs, resulting in particular, three special types of graphs defined below.

Definition 8. *The graphs* M_n*,* Λ_n*, and* Δ_n*, with* $n \ge 1$.

1. *The graph* $M_n = (V(M_n), E(M_n))$*, where* $V(M_n) = \{v_1, v_2, \cdots, v_{2n}, v_{2n+1}\}$ *and* $E(M_n) = \{v_1v_2, v_2v_3, \cdots, v_{2n}v_{2n+1}, v_1v_3, v_3v_5, \cdots, v_{2n-1}v_{2n+1}\}$.

2. *The graph* $\Lambda_n = (V(M_n) \cup \{v_0\}, E(M_n) \cup \{v_0 v_1\})$.

3. *The graph* $\Delta_n = (V(\Lambda_n) \cup \{v_{2n+2}\}, E(\Lambda_n) \cup \{v_{2n+1} v_{2n+2}\})$.

Below we provide an example of each graph.

Example 4. *The graphs* M_5*,* Λ_5*, and* Δ_5*.*

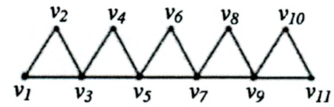


Figure 4: M_5 made of 5 adjacent C_3 graphs.

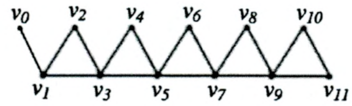


Figure 5: Λ_5 made of M_5 and an additional edge

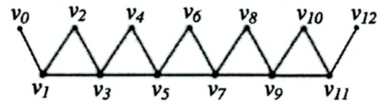


Figure 6: Δ_5 made of M_5 and two additional edges, one at each end.

Proposition 5.2. *The recursive formula for* $\Gamma_n(x)$*, with* $n \geq 4$*, is:*

$$\Gamma_n(x) = 2\Gamma_{n-1}(x) + \Lambda_{n-2}(x) + x\Delta_{n-3}(x).$$

Proof. For $n \geq 4$, applying the toggling process for Γ_n, with respect to ab, creates $\Gamma_n \setminus \{a\}$ and $\Gamma_n^{ab} \setminus \{b\}$. Applying the toggling process to these resulting graphs, with respect to cd and ef results in the graphs Λ_{n-2}, two Γ_{n-1} graphs, and Δ_{n-3} with an isloted vertex. The process is shown below.

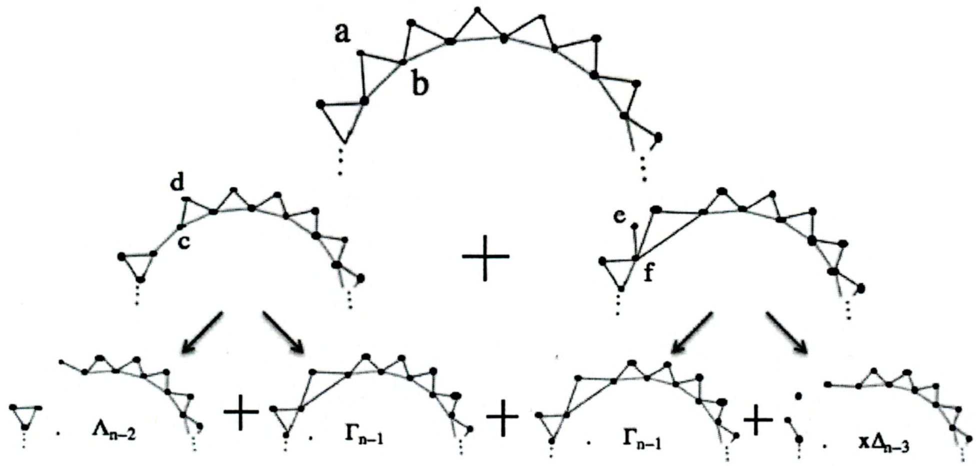


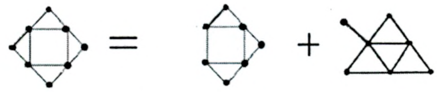
Figure 7: Toggle proccess for Γ_n.

□

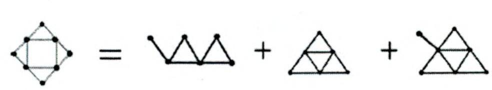
We demonstrate this process for Γ_4, which also provides a proof for the formula for $\Gamma_4(x)$ shown in Lemma 5.1.

Decomposition of Γ_4:

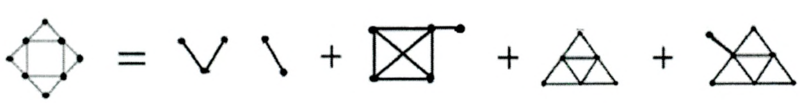
Step 1: Toggling Γ_4 into $\Gamma_4 \setminus \{a\}$ and $\Gamma_4^{ab} \setminus \{b\}$



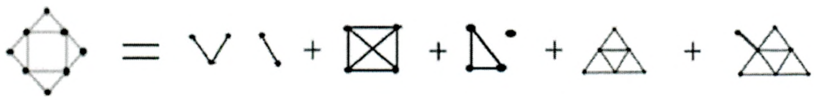
Step 2: Toggling $\Gamma_4 \setminus \{a\}$ to show Λ_2 and Γ_3.



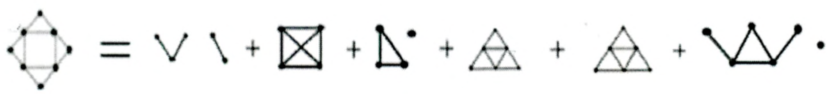
Step 3: Further toggling Λ_2 to show P_1 with P_2 and a new graph.



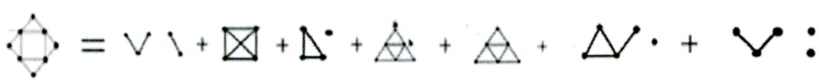
Step 4: Further, toggling the second graph in *Step 3* results in K_4 and C_3, with an isolated vertex.



Step 5: Toggling the last graph in *Step 4* we achieve Γ_3 and a Δ_1 with an extra vertex.



Step 6: Finally, we toggle Δ_1 in *Step 5* to achieve Λ_1 the graph P_2.



Now using the formulas from Lemma 3.2, the result for $\Gamma_3(x)$ in Lemma 5.1, and the result for $\Lambda_1(x)$ in Lemma 3.3, the interlace polynomials for the graphs from *Step 6* can be expressed by:

$$\begin{aligned}\Gamma_4(x) &= 2x(x^2+2x)+8x+x(4x)+2(x^3+10x^2+8x)+x(4x+2x^2)+x^2(x^2+2x)\\ &= x^4+8x^3+32x^2+24x.\end{aligned}$$

If we take a look at *Step 6* for the breakdown of Δ_1, we see that $\Delta_1(x) = x(x+2)^2$. We will use this result in Section ??.

Lemma 5.3. *The interlace polynomial for* Δ_1 *is:* $\Delta_1(x) = x(x+2)^2$.

In order to achieve an explicit formula for $\Gamma_n(x)$, we need to find an explicit formula for each of $\Lambda_n(x)$ and $\Delta_n(x)$.

5.1 Recursive Formula for $\Lambda_n(x)$

Applying the toggling process to the graph Λ_n, with respect to ab, gives us two graphs (see Figure 8): M_n and Λ_{n-1}, with an additional vertex.

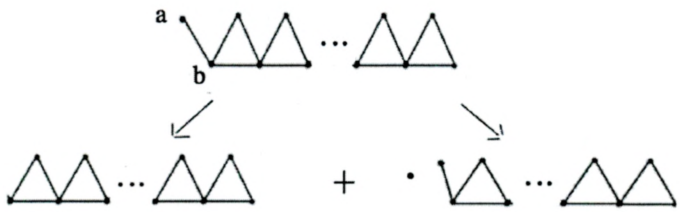


Figure 8: Toggling Λ_n with respect to ab

Since toggling Λ_n results in M_n we further our process by toggling M_n with respect to ab, giving us two Λ_{n-1} graphs (See Figure 9).

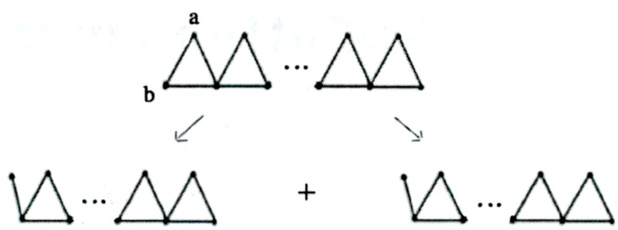


Figure 9: Toggling M_n with respect to ab.

From these two steps, we have a recursive formula for $\Lambda_n(x)$, which we use to find an explicit formula, $\Lambda_n(x) = M_n(x) + x\Lambda_{n-1}(x) = 2\Lambda_{n-1}(x) + x\Lambda_{n-1}(x) = (x+2)\Lambda_{n-1}(x)$.

Lemma 5.4. *The recursive formula for $\Lambda_n(x)$:*

$$\Lambda_n(x) = (x+2)\Lambda_{n-1}(x).$$

We can now expand on our recursive formula to achieve an explicit formula for $\Lambda_n(x)$:

$$\Lambda_n(x) = (x+2)\Lambda_{n-1}(x) = (x+2)^2\Lambda_{n-2}(x) = (x+2)^3\Lambda_{n-3}(x) = \cdots = (x+2)^{n-1}\Lambda_1(x).$$

From Lemma 3.3, we substitute in for $\Lambda_1(x) = x(x+2)^2$ to achieve an explicit formula.

$$\Lambda_n(x) = (x+2)^{n-1}(2x^2+4x) = (x+2)^{n-1}(2x)(x+2) = 2x(x+2)^n.$$

This formula will be formally mentioned in the next section.

5.2 Recursive Formula for $\Delta_n(x)$

Before I introduce the recursive formula for $\Delta_n(x)$, I would like to share a known result for a given power series we will see. The finite sum $\sum_{k=0}^{n} r^k$ can be expanded as: $\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$. Applying the toggling process to the graph Δ_n, with respect to ab, gives us the graph Λ_n, and Δ_{n-1} with an extra vertex. We demonstrate this process with Δ_5 (see Figure 10).

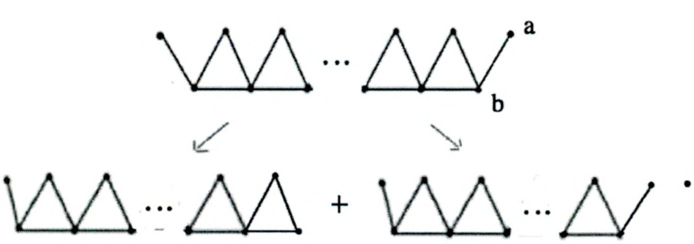


Figure 10: Toggling Δ_n with respect to *ab*.

From the toggling process, we create a recursive formula for $\Delta_n(x)$ and use it to find the explicit formula.

Lemma 5.5. *The recursive formula for $\Delta_n(x)$ is:*

$$\Delta_n(x) = \Lambda_n(x) + x\Delta_{n-1}(x) = 2x(x+2)^n + x\Delta_{n-1}(x).$$

We can now expand the recursive formula to achieve an explicit formula for $\Delta_n(x)$.

$\Delta_n(x) = 2x(x+2)^n + x[2x(x+2)^{n-1} + x(\Delta_{n-2}(x))]$

$$
\begin{aligned}
&= 2x(x+2)^n + 2x^2(x+2)^{n-1} + x^2\Delta_{n-2}(x)\\
&= 2x(x+2)^n + 2x^2(x+2)^{n-1} + x^2[2x(x+2)^{n-2} + x(\Delta_{n-3}(x))]\\
&= 2x(x+2)^n + 2x^2(x+2)^{n-1} + 2x^3(x+2)^{n-2} + x^3(\Delta_{n-3}(x))\\
&\ \vdots\\
&= x^{n-1}(\Delta_1(x)) + 2x^{n-1}(x+2)^2 + 2x^{n-2}(x+2)^3 + \cdots + 2x(x+2)^n\\
&= x^{n-1}(\Delta_1(x)) + 2x\sum_{k=2}^{n}(x+2)^k x^{n-k} = x^{n-1}(\Delta_1(x)) + 2x\sum_{k=2}^{n} x^n(\tfrac{x+2}{x})^k\\
&= x^{n-1}(\Delta_1(x)) + 2x^{n+1}\sum_{k=2}^{n}(\tfrac{x+2}{x})^k.
\end{aligned}
$$

Now we can expand our finite sum by the known result mentioned above, but we will have to modify our expansion since we are starting with $k=2$ instead of $k=0$:

$$
\begin{aligned}
2x^{n+1}\sum_{k=2}^{n}(\tfrac{x+2}{x})^k &= 2x^{n+1}\left[\frac{1-(\frac{x+2}{x})^{n+1}}{1-\frac{x+2}{x}}\right] - (2x^{n+1})(\tfrac{x+2}{x})^0 - (2x^{n+1})(\tfrac{x+2}{x})^1\\
&= 2x^{n+1}\left[\frac{1-(\frac{x+2}{x})^{n+1}}{\frac{-2}{x}}\right] - (2x^{n+1})(\tfrac{x+2}{x})^0 - (2x^{n+1})(\tfrac{x+2}{x})^1\\
&= x^{n+1}\left[1-\frac{(x+2)^{n+1}}{x^{n+1}}\right](-x) - (2x^{n+1})(\tfrac{x+2}{x})^0 - (2x^{n+1})(\tfrac{x+2}{x})^1\\
&= -x^{n+2} + x(x+2)^{n+1} - 4x^{n+1} - 4x^n;
\end{aligned}
$$

Now going back to the previous formula $\Delta_n(x) = x^{n-1}(\Delta_1(x)) + 2x^{n+1}\sum_{k=2}^{n}(\frac{x+2}{x})^k$ and apply the result for the finite sum and the result for $\Delta_1(x)$, from Lemma 5.3, to receive:

$$
\begin{aligned}
\Delta_n(x) &= x^{n-1}(x^3+4x^2+4x) - x^{n+2} + x(x+2)^{n+1} - 4x^{n+1} - 4x^n\\
&= x^{n+2} + 4x^{n+1} + 4x^n - x^{n+2} + x(x+2)^{n+1} - 4x^{n+1} - 4x^n\\
&= x(x+2)^{n+1}.
\end{aligned}
$$

Lemma 5.6. *Respectively, the explicit formula for $\Lambda_n(x)$ and $\Delta_n(x)$ are:*

1. $\Lambda_n(x) = 2x(x+2)^n, \quad n \geq 1.$

2. $\Delta_n(x) = x(x+2)^{n+1}, \quad n \geq 1.$

Proof. 1. $\Lambda_n(x) = 2x(x+2)^n$, for $n \geq 1$. By Induction:

Let $n = 1$, $\Lambda_1(x) = 2x(x+2)^1 = 2x(x+2)$.

From Lemma 3.3, we know this is true.

From Lemma 5.4, we know $\Lambda_n(x) = (x+2)(\Lambda_{n-1})$. Assume $\Lambda_{n-1}(x) = 2x(x+2)^{n-1}$.

$\Lambda_n(x) = (x+2)(\Lambda_{n-1}) = (x+2)(2x(x+2)^{n-1}) = 2x(x+2)^n$.

$\therefore \Lambda_n(x) = 2x(x+2)^n$, for all $n \geq 1$.

□

Proof. 2. $\Delta_n(x) = x(x+2)^{n+1}$, for $n \geq 1$. By Induction:

Let $n = 1$, $\Delta_1 = x(x+2)^2$.

From Lemma 5.3, we know this is true.

From Lemma 5.5, we know $\Delta_n(x) = 2x(x+2)^n + x(\Delta_{n-1})$. Assume $\Delta_{n-1}(x) = x(x+2)^n$.

$$\begin{aligned}\Delta_n(x) &= 2x(x+2)^n + x(\Delta_{n-1}) = 2x(x+2)^n + x(x(x+2)^n)\\ &= x(x+2)^{n+1}.\end{aligned}$$

$\therefore \quad \Delta_n(x) = x(x+2)^{n+1}$, for all $n \geq 1$.

□

5.3 Explicit Formula for $\Gamma_n(x)$

We can now use the recursive formula for $\Gamma_n(x)$ from Lemma 5.2 to find an explicit formuala for $\Gamma_n(x)$.

$$\begin{aligned}\Gamma_n(x) &= 2\Gamma_{n-1}(x) + \Lambda_{n-2}(x) + x\Delta_{n-3}(x) = 2\Gamma_{n-1}(x) + 2x(x+2)^{n-2} + x^2(x+2)^{n-2}\\ &= 2\Gamma_{n-1}(x) + x(x+2)^{n-1} = 2^2\Gamma_{n-2}(x) + 2x(x+2)^{n-2} + x(x+2)^{n-1}\\ &= 2^3\Gamma_{n-3}(x) + 2^2x(x+2)^{n-3} + 2x(x+2)^{n-2} + x(x+2)^{n-1}\\ &\vdots\end{aligned}$$

$$= 2^{n-4}\Gamma_4(x) + 2^{n-5}(x+2)^4 + 2^{n-6}(x+2)^5 + \cdots + 2^0 x(x+2)^{n-1}$$
$$= 2^{n-4}\Gamma_4(x) + \sum_{k=4}^{n-1} 2^{n-1-k}x(x+2)^k = 2^{n-4}(\Gamma_4(x)) + 2^{n-1}x \sum_{k=4}^{n-1} (\tfrac{x+2}{2})^k.$$

Note that $\sum_{k=4}^{n-1} (\frac{x+2}{2})^k = \sum_{k=0}^{n-1} (\frac{x+2}{2})^k - \sum_{k=0}^{3} (\frac{x+2}{2})^k$.

$$2^{n-1}x \sum_{k=0}^{n-1} (\tfrac{x+2}{2})^k = (x+2)^n - 2^n;$$
$$2^{n-1}x \sum_{k=0}^{3} (\tfrac{x+2}{2})^k = 2^{n-4}(x+2)^4 - 2^n.$$

We go back to our previous equation to apply the power series formula and the result for $\Gamma_4(x)$, from Lemma 5.1.

$$\Gamma_n(x) = 2^{n-4}(x^4 + 8x^3 + 32x^2 + 24x) + (x+2)^n - 2^n - [2^{n-4}(x+2)^4 - 2^n]$$
$$= 2^{n-4}(8x^2 - 8x - 16) + (x+2)^n = 2^{n-1}(x^2 - x - 2) + (x+2)^n.$$

Theorem 5.7. *The explicit formula for $\Gamma_n(x)$, with $n \geq 3$, is:*

$$\Gamma_n(x) = 2^{n-1}(x^2 - x - 2) + (x+2)^n.$$

Proof. By Induction:

Let $n = 4$, $\Gamma_4(x) = 2^3(x^2 - x - 2) + (x+2)^4 = x^4 + 8x^3 + 32x^2 + 24x$.

It is confirmed by Lemma 5.1.

From Proposition 5.2 and Lemma 5.6 we know $\Gamma_n(x) = 2\Gamma_{n-1}(x) + x(x+2)^{n-1}$.

Assume $\Gamma_{n-1}(x) = 2^{n-2}(x^2 - x - 2) + (x+2)^{n-1}$, then

$$\Gamma_n(x) = 2\Gamma_{n-1}(x) + x(x+2)^{n-1}$$
$$= 2(2^{n-2}(x^2 - x - 2) + (x+2)^{n-1}) + x(x+2)^{n-1}$$
$$= 2^{n-1}(x^2 - x - 2) + 2(x+2)^{n-1} + x(x+2)^{n-1}$$
$$= 2^{n-1}(x^2 - x - 2) + (x+2)^n.$$

$\therefore \quad \Gamma_n(x) = 2^{n-1}(x^2 - x - 2) + (x+2)^n$ is true for all $n \geq 3$. $\square$

6 Properties of $\Gamma_n(x)$

The interlace polynomial of a graph is a special graph invariant that can tell us different information about the graph. We are specifically interested in the coefficients and some special values of $\Gamma_n(x)$. Do the coefficients give us any meaning towards the graph Γ_n itself, or any of the subgraphs within Γ_n? What do special values of $\Gamma_n(x)$ tell us and what can that information be used for? Furthermore, what kind of relation is there with $\Gamma_n(x)$ to the adjacency matrix of Γ_n? Within this section, I analyze the meaning of the interlace polynomial for Γ_n and correlate the information to certain applications.

6.1 Coefficients of $\Gamma_n(x)$

From the explicit formulas given in Lemma 5.6 and Theorem 5.7, we are able to relate the coefficients between $\Gamma_n(x)$, $\Lambda_n(x)$, and $\Delta_n(x)$. The relation is made using generating functions to define the coefficients. Obviously the constant term of any interlace polynomial is zero. From the explicit formula of $\Gamma_n(x)$ (Theorem 5.7) we can see that the polynomial is of degree n. Using the fact that the constant term is zero and expressing $(x+2)^n$ by the binomial formula, we can rewrite $\Gamma_n(x)$ as:

$$\Gamma_n(x) = 2^{n-1}x^2 - 2^{n-1}x + \left[\sum_{k=1}^{n} \binom{n}{k} 2^{n-k}x^k\right], \quad n \geq 3. \tag{1}$$

Definition 9. *Consider the polynomials $\Gamma_n(x)$, $\Lambda_n(x)$, and $\Delta_n(x)$. We use $a_{n,k}$, $l_{n,k}$, and $d_{n,k}$ to represent the coefficients for each polynomial respectively. That is,*

$$\Gamma_n(x) = \sum_{k=1}^{n} a_{n,k}x^k, \quad \Lambda_n(x) = \sum_{k=1}^{n+1} l_{n,k}x^k, \quad \textit{and} \quad \Delta_n(x) = \sum_{k=1}^{n+2} d_{n,k}x^k.$$

Lemma 6.1. *The coefficients of $\Gamma_n(x) = \sum_{k=1}^{n} a_{n,k}x^k$, with $n \geq 3$, are given by*

$$a_{n,k} = \begin{cases} 2^{n-1}(n-1) & \textit{if } k=1 \\ 2^{n-3}(n^2-n+4) & \textit{if } k=2 \\ 2^{n-k}\binom{n}{k} & \textit{if } 2<k\le n \end{cases}.$$

Let us take a look at an example of the coefficients of $\Gamma_3(x)$. We know the degree of the polynomial is three, so we will only need to concentrate on the formula for our coefficients $a_{3,k}$ with $1 \le k \le 3$.

Example 5. *Coefficients of* $\Gamma_3(x)$*:*

$$a_{3,1} = 2^2(2) = 8, \quad a_{3,2} = 2^0(9-3+4) = 10, \quad a_{3,3} = 2^0\binom{3}{3} = 1.$$

This gives $\Gamma_3(x) = x^3 + 10x^2 + 8x$*. It is confirmed by Lemma 5.1.*

We can express the coefficients for $\Lambda_n(x)$ and $\Delta_n(x)$ in a similar manner. Recall, taking a look at Lemma 5.6, the degree of $\Lambda_n(x)$ is $n+1$ and the degree of $\Delta_n(x)$ is $n+2$.

Lemma 6.2. *Coefficients of* $\Lambda_n(x)$ *and* $\Delta_n(x)$ *are, respectively:*

$$l_{n,k} = 2^{n+2-k}\binom{n}{k-1} \quad 1 \le k \le n+1; \tag{2}$$

$$d_{n,k} = 2^{n+2-k}\binom{n+1}{k-1} \quad 1 \le k \le n+2. \tag{3}$$

Proof. (2). Using the binomial formula:

$$\Lambda_n(x) = 2x(x+2)^n = 2x\sum_{k=0}^{n}\binom{n}{k}2^{n-k}x^k = \sum_{k=0}^{n}\binom{n}{k}2^{n+1-k}x^{k+1} = \sum_{k=1}^{n}\binom{n}{k-1}2^{n+2-k}x^k.$$

□

Proof. (3). Using the binomial formula:

$$\Delta_n(x) = x(x+2)^{n+1} = \sum_{k=0}^{n} \binom{n+1}{k} 2^{n+1-k} x^{k+1} = \sum_{k=1}^{n} \binom{n+1}{k-1} 2^{n+2-k} x^k.$$

$\square$

Let us look at an example for the coefficients of the polynomials $\Lambda_2(x)$ and $\Delta_1(x)$.

Example 6. *Coefficients of* $\Lambda_2(x)$*:*

$$l_{2,k} = \begin{cases} 2^{2+2-1}\binom{2}{0} = 8 & \text{for } k = 1 \\ 2^{2+2-2}\binom{2}{1} = 8 & \text{for } k = 2 \\ 2^{2+2-3}\binom{2}{2} = 2 & \text{for } k = 3 \end{cases}.$$

This gives $\Lambda_2(x) = 2x^3 + 8x^2 + 8x$*, which can be confirmed by Lemma 5.6:* $\Lambda_2(x) = 2x(x+2)^2 = 2x^3 + 8x^2 + 8x$.

Example 7. *Coefficients of* $\Delta_1(x)$*:*

$$d_{,k} = \begin{cases} 2^{1+2-1}\binom{2}{0} = 4 & \text{for } k = 1 \\ 2^{1+2-2}\binom{2}{1} = 4 & \text{for } k = 2 \\ 2^{1+2-3}\binom{2}{2} = 1 & \text{for } k = 3 \end{cases}.$$

This gives $\Delta_1(x) = x^3 + 4x^2 + 4x$*, confirmed by Lemma 5.6:* $\Delta_1(x) = x(x+2)^2 = x^3 + 4x^2 + 4x$.

Recall from Proposition 5.2, the formula $\Gamma_4(x) = 2\Gamma_3(x) + \Lambda_2(x) + x\Delta_1(x)$. From our examples, 5, 6, and 7, we have the following expression for the polynomial $\Gamma_4(x)$, proving the result from Lemma 5.1.

$$\Gamma_4(x) = 2(x^3+10x^2+8x)+(2x^3+8x^2+8x)+x(x^3+4x^2+4x) = x^4+8x^3+32x^2+24x.$$

Obviously, since $\Gamma_n(x)$ is made from $\Lambda_n(x)$ and $\Delta_n(x)$, the direct relationship can be shown among the coefficients of each polynomial.

Corollary 6.3. *The direct relationship of coefficients between* $\Gamma_n(x)$*,* $\Lambda_n(x)$*, and* $\Delta_n(x)$ *is:*

$$l_{n,k+1} = 2a_{n,k}, \quad \textit{for } n \geq k > 2; \tag{4}$$

$$d_{n,k} - l_{n,k} = a_{n,k-2} \quad \textit{for } n \geq 4 \textit{ and } k \geq 5. \tag{5}$$

Proof. To prove equation (4), we proceed as follows:

From equation (2), we have

$$l_{n,k+1} = 2^{n+1-k}\tbinom{n}{k} = 2(2^{n-k}\tbinom{n}{k}) = 2a_{n,k}.$$

To prove equation (5), we proceed as follows:

From equations (2) and (3), we have:

$$d_{n,k} - l_{n,k} = 2^{n+2-k}\left[\tbinom{n+1}{k-1} - \tbinom{n}{k-1}\right];$$

Using Pascals Recurrence Relation, which states $\binom{n}{k-1} + \binom{n}{k-2} = \binom{n+1}{k-1}$,

$$d_{n,k} - l_{n,k} = 2^{n+2-k}\left[\tbinom{n+1}{k-1} - \tbinom{n}{k-1}\right] = 2^{n-(k-2)}\tbinom{n}{k-2} = a_{n,k-2}.$$

□

Concentrating on the coefficients for $\Gamma_n(x)$, we notice a few properties. The first observation made is that our leading coefficient, $a_{n,n}$, is always one. Another observation we can directly see is the coefficient of x^{n-1}, or $a_{n,n-1}$, is $2n$, which represents the cardinality of the vertext set, or $|V(\Gamma_n)|$. This value is also the circumference of Γ_n.

6.2 Properties of $\Gamma_n(x)$ at Different Values of x

Research has been done on the values of interlace polynomials at $x = 1$ and -1 [1]. We discuss the importance of these values for $\Gamma_n(x)$ in this section. Using the known results from previous papers, we determine some characteristics for $\Gamma_n(x)$.

Corollary 6.4. *(Known results from [1] and [6] respectively.) Let G be a graph with n vertices.*

1. *$G(1)$ = number of induced subgraphs of G with an odd number of perfect matchings (including the empty set).*

2. *$G(3)$ is divisble by $G(-1)$ and the quotient is an odd integer.*

3. *Let A be the adjacency matrix of G, $n = |V(G)|$ and let r be the rank of the matrix $I + A$ over the field $\mathbb{Z}_2$ of two elements. Then*

$$G(-1) = (-1)^r 2^{n-r}.$$

We eveluate $\Gamma_n(1)$ and $\Gamma_n(-1)$ then correlate the meaning to these results.

Corollary 6.5. *The number of induced subgraphs of Γ_n with an odd number of perfect matchings is $3^n - 2^n$.*

The number of induced subgraphs for the complete graph K_{2n} is 2^{2n}. It is obvious that the number of induced subgraphs of Γ_n is less than the number of induced subgraphs of the complete graph K_{2n}.

Theorem 6.6. *Let A_n be the adjacency matrix of Γ_n. The matrix $B_n = I + A_n$ has a full rank, or $rank(B_n) = 2n$, over $\mathbb{Z}_2$.*

Proof. Note that A_n, B_n, and I are $2n \times 2n$ matrices. Let $r = rank(B_n)$ over $\mathbb{Z}_2$ with $0 \le r \le 2n$.

From Theorem 5.7,

$$\Gamma_n(-1) = 2^{n-1}(1^2 - (-1) - 2) + (-1+2)^n = 1 \quad \forall\, n \geq 3.$$

From Corollary 6.4,

$$1 = (-1)^r (2)^{2n-r}.$$

This leaves us with only one solution, $r = 2n$, because 2 cannot divide one. Thus B_n is of full rank.

□

Let us look at the $2n \times 2n$ adjacency matrix A_n of Γ_n:

$$A_n = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 \end{bmatrix}_{2n \times 2n};$$

$$I + A_n = B_n = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & \cdots & 1 & 1 & 1 \end{bmatrix}_{2n \times 2n}.$$

Since B_n has a full rank, we know the determinant of $B_n \neq 0$ and the matrix B_n is invertible.

6.3 Parity of $\Gamma_n(x)$

From Corollary 6.4, we know $\Gamma_n(3)$ must be an odd integer since $\Gamma_n(-1)$ is 1. Now we determine parity of x. Recall $\Gamma_n(x) = 2^{n-1}(x^2 - x - 2) + (x+2)^n$. For $n \geq 3$, the first term in $\Gamma_n(x)$ is a multiple of 2, resulting in an even number. The parity of the second term depends on x, since x is adding to an even number. Any power of an even number stays even and any power of an odd number stays odd. Therefore, the parity of $\Gamma_n(x)$ is the same as that of x.

Proposition 6.7. *$\Gamma_n(x)$ is odd if x is odd and $\Gamma_n(x)$ is even if x is even.*

Lets take a look at a couple values of $x \geq 3$ for $\Gamma_n(x)$.

$$\Gamma_n(3) = 2^n(2) + 5^n;$$

$$\Gamma_n(4) = 2^n(5) + 6^n = 2^n(2) + 5^n + 2^n(3) + 6^n - 5^n = \Gamma_n(3) + 2^n(3) + 6^n - 5^n;$$

$$\Gamma_n(5) = 2^n(9) + 7^n = \Gamma_4 + 2^n(4) + 7^n - 6^n.$$

We can visibily see the pattern for the parity of $\Gamma_n(x)$ depends on x. Also, from the pattern of x values and values inside the first term, I create a formula for finding the next value of $\Gamma_n(x)$:

$$\Gamma_n(x+1) = \Gamma_n(x) + 2^n(x) + (x+3)^n - (x+2)^n.$$

It can be proved easily by applying Theorem 5.7.

7 Applications

In this section, we show applications of the interlace polynomial towards linear algebra and a related application in biology.

7.1 Linear Algebra

If someone was given the matrix $B_n = A_n + I$, where A_n is the adjacency matrix for Γ_n and I is the identity matrix, and was asked to find the determinant of the matrix, there would be multiple steps to find the solution. By Theorem 6.6, the rank of B_n is $2n$, which is a full rank. One way to find this result using Linear Algebra is by showing the $2n \times 2n$ matrix has a nonzero determinant, or $det(B_n) \neq 0$. Let us examine at the process to show the matrix B_5, for Γ_5, has a full rank.

The adjacency matrix A_5 and $I + A_5$, respectively, for Γ_5 is:

$$A_5 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}_{10\times 10}$$

$$B_5 = \begin{bmatrix} 1&1&0&0&0&0&0&0&0&1 \\ 1&1&1&1&0&0&0&0&0&1 \\ 0&1&1&1&0&0&0&0&0&0 \\ 0&1&1&1&1&1&0&0&0&0 \\ 0&0&0&1&1&1&0&0&0&0 \\ 0&0&0&1&1&1&1&1&0&0 \\ 0&0&0&0&0&1&1&1&0&0 \\ 0&0&0&0&0&1&1&1&1&1 \\ 0&0&0&0&0&0&0&1&1&1 \\ 1&1&0&0&0&0&0&1&1&1 \end{bmatrix}_{10\times 10}$$

One popular way to find the determinant is to use cofactor expansion multiple times, in order to reduce the matrix into a 3×3 matrix, or a triangular matrix, to easily compute the determinant.

Definition 10. *[11] Let $A \in M_{n\times n}(F)$. For $n \geq 2$, we define $det(A)$, or $\mid A \mid$, recursively as*

$$\mid A \mid = \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \cdot \mid A_{ij} \mid .$$

By the cofactor expansion on the first column of B_5, we obtain:

$$\mid B_5 \mid = det\left(\begin{bmatrix} 1&1&1&0&0&0&0&0&1 \\ 1&1&1&0&0&0&0&0&0 \\ 1&1&1&1&1&0&0&0&0 \\ 0&0&1&1&1&0&0&0&0 \\ 0&0&1&1&1&1&1&0&0 \\ 0&0&0&0&1&1&1&0&0 \\ 0&0&0&0&1&1&1&1&1 \\ 0&0&0&0&0&0&1&1&1 \\ 1&0&0&0&0&0&1&1&1 \end{bmatrix}\right) - det\left(\begin{bmatrix} 1&0&0&0&0&0&0&0&1 \\ 1&1&1&0&0&0&0&0&0 \\ 1&1&1&1&1&0&0&0&0 \\ 0&0&1&1&1&0&0&0&0 \\ 0&0&1&1&1&1&1&0&0 \\ 0&0&0&0&1&1&1&0&0 \\ 0&0&0&0&1&1&1&1&1 \\ 0&0&0&0&0&0&1&1&1 \\ 1&0&0&0&0&0&1&1&1 \end{bmatrix}\right)$$

$$-det\left(\begin{bmatrix}1&0&0&0&0&0&0&0&1\\1&1&1&0&0&0&0&0&1\\1&1&1&0&0&0&0&0&0\\1&1&1&1&1&0&0&0&0\\0&0&1&1&1&0&0&0&0\\0&0&1&1&1&1&1&0&0\\0&0&0&0&1&1&1&0&0\\0&0&0&0&1&1&1&1&1\\0&0&0&0&0&0&1&1&1\end{bmatrix}\right).$$

In order to have all the matrices reduced to 3×3 or trianglar matrices, this process would need to be continued multiple times. It is tedious and time consuming. This is where the interlace polynomial for this graph comes extremely useful. For $\Gamma_5(-1)$, we use the formula from 6.4 and show that the matrix has a full rank.

$$\Gamma_5(-1) = 1 = (-1)^r 2^{10-r} \Rightarrow r = 10, \text{ where } r = rank(B_5) \text{ over } \mathbb{Z}_2.$$

Hence we know the $\mid B_5 \mid \neq 0$ and B_5 has a full rank. Using Maple Software, we can show $\mid B_5 \mid = 1$, confirming the result.

7.2 Biology

As mentioned in Section 2, the study of interlace polynomials grew from trying to reconstruct DNA strings. String reconstruction is the process of reassembling a long string of symbols from a set of its subsequences together with some sequencing information [9]. For example, fragmenting and reassembling messages is a common network protocol, and reconstruction techniques might be applied when the network protocol has been disrupted, yet the original message must be reassembled from the fragments [9]. Sequencing by hybridization is a method of reconstructing a long DNA string from knowledge of its short substrings. Unique reconstruction is not always possible, and the goal is to study the number of reconstructions of a random string [4]. This is where the interlace polynomial for 4-regular Eulerian digraphs come to play. The different types of reconstructions can be looked at through the different Eulerian circuits in that graph, which the interlace polynomial can tell about. The probability of correctly sequencing the original strand is thus the reciprocal of the total number of Euler circuits in the graph [9].

A *de Bruijn graph* is a directed graph representing overlaps between sequences of symbols in string reconstruction [5]. A DNA string is represented by a 2-in 2-out de Bruijn Graph. Tracing the original DNA sequence of this graph can be represented by a Eulerian Circuit that starts at the vertex representing the beginning and end of the strand [9]. Therefore, once we have the de Bruijn Graph of the DNA sequence, we look at one of the Eulerian Circuits that begins with the orginating vertex in the strand. Then we represent it by a chord diagram, and construct the circle graph from the chord diagram. Given this circle graph, we calculate the interlace polynomial of the graph, and relate it to the Circuit Parition Polynomial to calculate the number of Eulerian Circuits in the orignial graph.

The coefficient of x in a Circuit Partition Polynomial counts the number of Eulerian circuits for the graph. The Circuit Partition Polynomial, represented by $f(G, x)$, where G represents the *de Bruijn* 2*-in* 2*-out digraph*, can be represented by the interlace polynomial of the circle graph, created from the Eulerian Circuit of the 2-in 2-out digraph. We modify the relation to deal with only one component graphs

Proposition 7.1. *[5] If G is a* 2*-in* 2*-out Eulerian digraph, C is any Eulerian circuit of G, and H is the circle grah of the chord diagram determined by C, then* $f(G, x) = xH_n(x + 1)$.

This is how the interlace polynomial is applied to DNA sequencing, but unfortunately the graph Γ_n does not represent a circle graph so it does not represent any type of DNA string. Fortunately, research has shown a modification of the interlace polynomial can show more properties on induced subgraphs and specifically induced Eulerian subgraphs. This modified interlace polynomial is defined in the next chapter, and also defined explicitly for Γ_n.

8 A New Interlace Polynomial

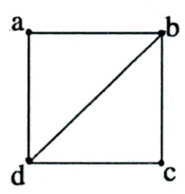
A related interlace polynomial is introduced by Aigner and Holst [1], which tells us a few more distinct properties about the graph. We represent this new interlace polynomial by $Q(G, x)$. The difference between the previous interlace polynomial and $Q(G, x)$ is an additional term in the formula. The graph $G * a$ is obtained from G by interchanging edges $\leftrightarrow$ non-edges in the neighborhood of a [1]. The modified definition for $Q(G, x)$ is defined below.

Definition 11. *[1] Let G be a simple graph, where $G = \{V, E\}$. The Q-interlace polynomial, $Q(G, x)$ is given by:*

$$Q(G, x) = Q(G \setminus \{a\}, x) + Q(G * a \setminus \{a\}, x) + Q(G^{(ab)} \setminus \{b\}, x) \text{ where } a, b \in V(G) \text{ and } ab \in E(G).$$

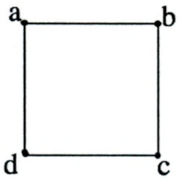
As we see, respectively the first and last term for $Q(G, x)$ follow the same process as our previous interlace polynomial. The additional term is what makes the new interlace polynomial different. An example is shown below.

Example 8. *Consider a graph G and a vertex a of G shown below.*



The neighborhood of a, or $N(a)$, is $\{b, d\}$, and $bd \in E(G)$. For $G * a$, we must interchange the edges $\leftrightarrow$ non-edges within the neighborhood of a. Therefore, the edge bd does not exist in $G * a$, shown below.

$G * a$:



In order to find $Q(\Gamma_n, x)$, we needed to work with more graphs, and the method was a lot more tedious. The formula is more complicated than the previous polynomial, but can still be used to calculate an explicit $Q(\Gamma_n, x)$. Below we discuss the breakdown of Γ_n using definition 11.

8.1 The Q-Interlace Polynomial $Q(\Gamma_n, x)$

In order to avoid confusion, in this section, we write $Q(G_n)$ for any graph G, to represent the Q-interlace polynomial $Q(G_n, x)$.

Theorem 8.1. *The Q-Interlace Polynomials for Γ_3, Γ_4, and Γ_5 are:*

1. $Q(\Gamma_3, x) = 2x^3 + 47x^2 + 84x$;
2. $Q(\Gamma_4, x) = 2x^4 + 64x^3 + 363x^2 + 468x$;
3. $Q(\Gamma_5, x) = 2x^5 + 100x^4 + 800x^3 + 2421x^2 + 2388x$.

Before introducing the recursive formula for $Q(\Gamma_n)$, I define a new graph we will see in our breakdown. If we take the graph Γ_n and eliminate any two edges, not in C_n, that form the same C_3 graph around the perimeter of C_n, we are left with Ω_{n-1}.

Definition 12. *The graph* $\Omega_n = (V(\Gamma_n) \setminus \{v_2\}, E(\Gamma_n) \setminus \{v_1v_2, v_2v_3\})$.

Figure 11: Ω_n with n C_3 graphs.

We begin by giving the initial recursive formula for $Q(\Gamma_n)$.

Lemma 8.2. *The recursive formula for $Q(\Gamma_n)$, for $n \geq 4$, is:*

$$Q(\Gamma_n) = 2Q(\Gamma_{n-1}) + Q(\Omega_{n-1}) + Q(M_{n-1}) + xQ(\Delta_{n-3}).$$

Proof. Breaking down Γ_n with respect to definition 11:

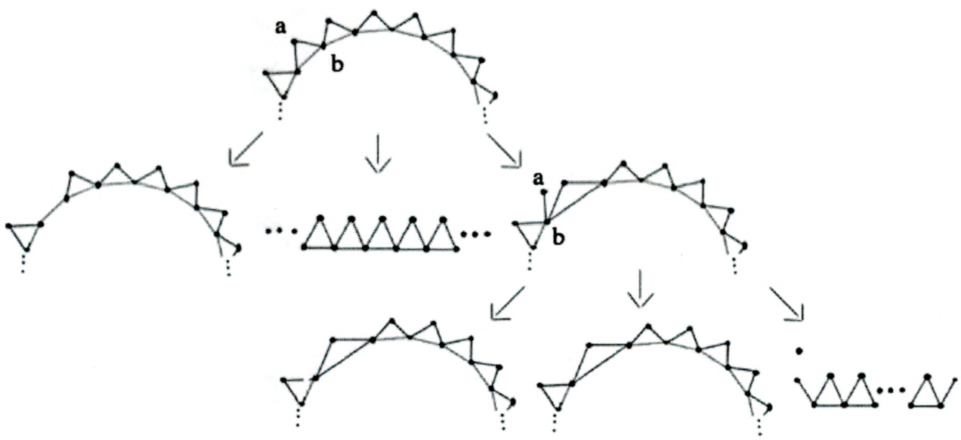


Figure 12: Respectively from left to right, the graph Ω_{n-1}, M_{n-1}, Γ_{n-1}, Γ_{n-1}, and $x\Delta_{n-3}$.

□

In the recursive formula for $Q(\Gamma_n)$, we deal with two graphs that were defined in the previous sections, M_n and Δ_n. Also, when using the toggling process for these two graphs, we achieve another familiar graph, Λ_n. During the devlopment for an explicit formula for $Q(\Gamma_n)$, I use the Q-interlace polynomials at specific values of n, for Λ_n, M_n, Δ_n, and Ω_n.

Lemma 8.3. *Q-interlace polynomials at specific n:*

1. $Q(\Lambda_0) = 3x$.

2. $Q(\Lambda_1) = 5x^2 + 12x$.

3. $Q(\Lambda_2) = 7x^3 + 44x^2 + 48x$.

4. $Q(M_0) = x$

5. $Q(M_1) = x^2 + 6x$.

6. $Q(M_2) = x^3 + 16x^2 + 24x$.

7. $Q(\Delta_1) = x^3 + 16x^2 + 24x$.

8. $Q(\Delta_2) = x^4 + 30x^3 + 112x^2 + 96x$.

9. $Q(\Omega_2) = 15x^2 + 36x$.

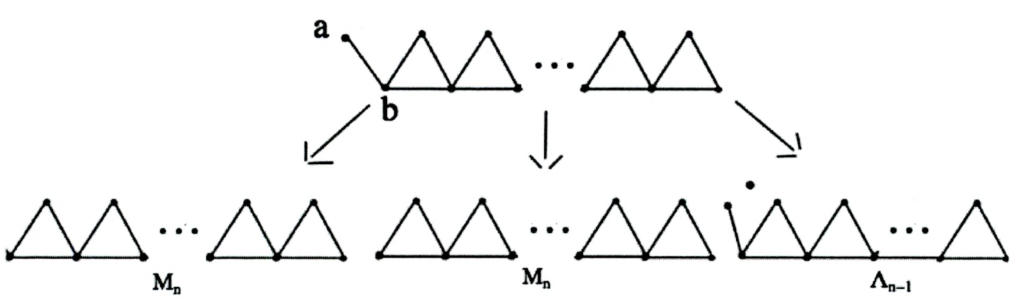
10. $Q(\Omega_3) = 14x^3 + 133x^2 + 204x$.

We start by working for an explicit formula for $Q(\Lambda_n)$.

Lemma 8.4. *The recursive formula for $Q(\Lambda_n)$, with $n \geq 1$, is:*

$$Q(\Lambda_n) = xQ(\Lambda_{n-1}) + 2Q(M_n).$$

Proof. Looking at $Q(\Lambda_n)$:



Obviously, the statement is true.

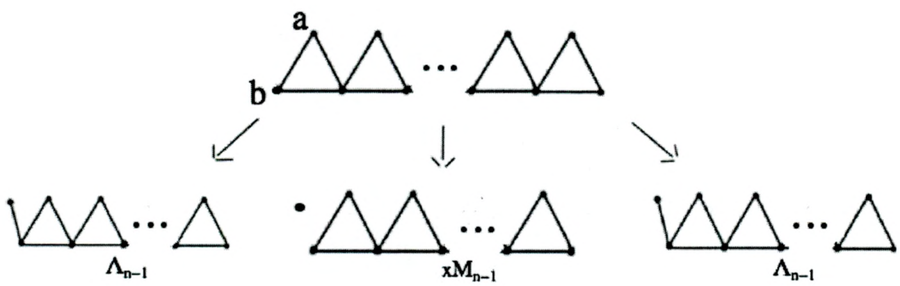
$\square$

Since we have the graph M_n inside our recursive formula, let us take a look at the recursive formula for $Q(M_n)$ and try to relate them.

Lemma 8.5. *The recursive formula for $Q(M_n)$, with $n \geq 1$, is:*

$$Q(M_n) = xQ(M_{n-1}) + 2Q(\Lambda_{n-1}). \tag{6}$$

Proof. Looking at $Q(M_n)$:



Obviously, the statement is true.

$\square$

We substitute equation 6 into $Q(\Lambda_n)$ and solve for $Q(M_{n-1})$.

$$Q(\Lambda_n) = xQ(\Lambda_{n-1}) + 4Q(\Lambda_{n-1}) + 2xQ(M_{n-1});$$
$$Q(M_{n-1}) = \frac{Q(\Lambda_n) - (x+4)Q(\Lambda_{n-1})}{2x}.$$

Now plug this into the recursive formula for $Q(\Lambda_n)$.

$$Q(\Lambda_{n)} = xQ(\Lambda_{n-1}) + 2\left(\frac{Q(\Lambda_{n+1}) - (x+4)Q(\Lambda_n)}{2x}\right);$$
$$xQ(\Lambda_n) = x^2Q(\Lambda_{n-1}) + Q(\Lambda_{n+1}) - (x+4)Q(\Lambda_n);$$

$$Q(\Lambda_{n+1}) = (2x+4)Q(\Lambda_n) - x^2 Q(\Lambda_{n-1}). \tag{7}$$

We represent this recurrence relation by its characteristic equation and solve for the roots, shown below.

$$y^2 - (2x+4)y + x^2 = 0 \quad \Rightarrow \quad y = \frac{2x+4 \pm \sqrt{(2x+4)^2 - 4x^2}}{2}.$$

The solutions to the characteristic equation are:

$$y = x + 2 \pm 2\sqrt{x+1}.$$

Definition 13. *The roots of the characteristic equation for $Q(\Lambda_n)$ are defined as:*

$$y_1(x) = x + 2 + 2\sqrt{x+1}, \quad y_2(x) = x + 2 - 2\sqrt{x+1}. \tag{8}$$

We use these roots to express Λ_n explicitly, shown below.

$$Q(\Lambda_n) = c_1(x)(y_1(x))^n + c_2(x)(y_2(x))^n.$$

We know the values for $Q(\Lambda_0)$ and $Q(\Lambda_1)$, so we use them to find our values for the two coefficient functions $c_1(x)$ and $c_2(x)$.

$$3x = c_1 + c_2;$$

$$5x^2 + 12x = c_1(x + 2 + 2\sqrt{x+1}) + c_2(x + 2 - 2\sqrt{x+1}).$$

Solving this set of linear equations, I find the values for our coefficient functions of x.

$$c_1(x) = \frac{x^2 + 3x + 3x\sqrt{x+1}}{2\sqrt{x+1}}; \tag{9}$$

$$c_2(x) = \frac{-x^2 - 3x + 3x\sqrt{x+1}}{2\sqrt{x+1}}. \qquad (10)$$

In order to simplify the explicit formula for $Q(\Lambda_n)$, I determine relations between the coefficient functions and the roots to the characteristic eqution.

Proposition 8.6. *Relationship between $y_1(x)$, $y_2(x)$, $c_1(x)$, and $c_2(x)$.*

$$\begin{aligned}
c_1(x) &= \frac{3x\sqrt{x+1} + x^2 + 3x}{2\sqrt{x+1}}, \qquad c_2(x) = \frac{3x\sqrt{x+1} - x^2 - 3x}{2\sqrt{x+1}}; \\
y_1(x) + y_2(x) &= 2x + 4, \quad y_1(x)y_2(x) = x^2; \\
y_1(x) - y_2(x) &= 4\sqrt{x+1} \quad \textit{and} \quad ((y_1(x) - x)(y_2(x) - x) = -4x; \\
c_1(x) + c_2(x) &= 3x \quad \textit{and} \quad c_1(x)y_2(x) + c_2(x)y_1(x) = x^2; \\
\Lambda_n(x) &= c_1(x)(y_1(x))^n + c_2(x)(y_1(x))^n.
\end{aligned}$$

Notice that since $c_1(x)$ and $c_2(x)$ are fractions, we must make sure the denominator is never zero. It is easy to see that when $x = -1$, the denominators are zero. $Q(\Lambda_n, -1)$ can be obtained seperately:

$$\begin{aligned}
y_1(-1) &= (-1) + 2 + 0 = 1; \\
y_2(-1) &= (-1) + 2 - 0 = 1; \\
y_1 &= 1 = y_2.
\end{aligned}$$

Since the roots are the same, the solution looks like:

$$\begin{aligned}
Q(\Lambda_n, (-1)) &= c_3(-1)(1)^n + c_4(-1)n(1)^n; \\
Q(\Lambda_n, (-1)) &= c_3(-1) + c_4(-1)n;
\end{aligned}$$

Applying values at $Q(\Lambda_1, -1)$ and $Q(\Lambda_2, -1)$:

$$Q(\Lambda_1, (-1)) = -7 = c_3 + c_4;$$

$$Q(\Lambda_2, (-1)) = -11 = c_3 + 2c_4;$$

Solving this system of linear equations, we obtain:

$$c_3(-1) = -3 \text{ and } c_4(-1) = -4.$$

Theorem 8.7. *The interlace polynomial $Q(\Lambda_n, x)$ is:*

1. $Q(\Lambda_n, x) = \sum_{m=0}^{\lfloor n/2 \rfloor} 4^m (x+1)^m (x+2)^{n-2m-1} \left[(3x^2+6x)\binom{n}{2m} + (2x^2+6x)\binom{n}{2m+1}\right]$ *for $x \neq -1$;*

2. $Q(\Lambda_n, (-1)) = -3 - 4n.$

Proof. Note that

$$\begin{aligned} Q(\Lambda_n) &= c_1(x)y_1^n(x) + c_2(x)y_2^n(x) \\ &= \frac{3x\sqrt{x+1} + x^2 + 3x}{2\sqrt{x+1}} \cdot y_1^n(x) + \frac{3x\sqrt{x+1} - x^2 - 3x}{2\sqrt{x+1}} \cdot y_2^n(x) \\ &= \frac{3x}{2}\left(y_1^n(x) + y_2^n(x)\right) + \frac{x^2+3x}{2\sqrt{x+1}}\left(y_1^n(x) - y_2^n(x)\right). \end{aligned}$$

$$y_1^n(x) = (x+2+2\sqrt{x+1})^n = \sum_{k=0}^{n} \binom{n}{k} (x+2)^{n-k} 2^k (x+1)^{k/2},$$

$$y_2^n(x) = (x+2-2\sqrt{x+1})^n = \sum_{k=0}^{n} \binom{n}{k} (x+2)^{n-k} 2^k (-1)^k (x+1)^{k/2},$$

Thus set $k = 2m$.

$$\begin{aligned} y_1^n(x) + y_2^n(x) &= \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) (1 + (-1)^k)(x+2)^{n-k} 2^k (x+1)^{k/2} \\ &= 2 \sum_{m=0}^{\lfloor n/2 \rfloor} \left(\begin{array}{c} n \\ 2m \end{array} \right) 4^m (x+1)^m (x+2)^{n-2m}. \end{aligned}$$

Similarly, let $k = 2m + 1$.

$$\begin{aligned} y_1^n(x) - y_2^n(x) &= \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) (1 - (-1)^k)(x+2)^{n-k} 2^k (x+1)^{k/2} \\ &= 4(x+1)^{(1/2)} \sum_{m=0}^{\lfloor n/2 \rfloor} \left(\begin{array}{c} n \\ 2m+1 \end{array} \right) 4^m (x+1)^m (x+2)^{n-2m-1}. \end{aligned}$$

From here you can combine the above and obtain the formula in Theorem 8.7. Note that equation (2) in Theorem 8.7 can be proven by looking at the development prior to the theorem. $\square$

We now use the explicit formula for $Q(\Lambda_n)$ to find the explicit formulas for $Q(M_n)$ and $Q(\Delta_n)$. Recall, from Lemma 8.4, we have

$$Q(M_n) = \frac{Q(\Lambda_n) - xQ(\Lambda_{n-1})}{2}. \tag{11}$$

Theorem 8.8. *The interlace polynomial $Q(M_n, x)$, for $n \geq 1$, with $y_1(x)$ and $y_2(x)$ given by equation 8 is:*

1. $Q(M_n) = \left[(x^2 + 6x) + \frac{4x^2+6x}{\sqrt{x+1}}\right] \cdot y_1^{n-1}(x) + \left[(x^2 + 6x) - \frac{4x^2+6x}{\sqrt{x+1}}\right] \cdot y_2^{n-1}(x)$ *for* $x \neq -1$*;*

2. $Q(M_n,(-1)) = -1-4n$.

Proof. 1.

Note that

$$y_1 - x = 2(1+\sqrt{x+1}) \quad \text{and} \quad y_2 - x = 2(1-\sqrt{x+1}).$$

Then

$$\begin{aligned}
Q(M_n) &= \frac{Q(\Lambda_n) - xQ(\Lambda_{n-1})}{2} \\
&= \frac{c_1(x)y_1^n(x) + c_2(x)y_2^n(x) - xc_1(x)y_1^{n-1}(x) - xc_2(x)y_2^{n-1}}{2} \\
&= \frac{c_1(x)y_1^{n-1}(x)(y_1(x)-x)}{2} + \frac{c_2(x)y_2^{n-1}(x)(y_2(x)-x)}{2} \\
&= \frac{\left(3x\sqrt{x+1}+x^2+3x\right)\left(1+\sqrt{x+1}\right)}{2\sqrt{x+1}} \cdot y_1^{n-1}(x) \\
&\quad + \frac{\left(3x\sqrt{x+1}-x^2-3x\right)\left(1-\sqrt{x+1}\right)}{2\sqrt{x+1}} \cdot y_2^{n-1}(x) \\
&= \frac{(x^2+6x)\sqrt{x+1}+4x^2+6x}{\sqrt{x+1}} \cdot y_1^{n-1}(x) \\
&\quad + \frac{(x^2+6x)\sqrt{x+1}-4x^2-6x}{\sqrt{1+x}} \cdot y_2^{n-1}(x).
\end{aligned}$$

□

Proof. 2. By mathematical induction. Assume $x=-1$.

Our inital case for $n=1$ holds true.

Since $Q(M_n,x) = x^2+6x$,

$$Q(M_1,-1) = (-1)^2 + 6(-1) = -5 = -1-4(1).$$

Now assume $Q(M_{n-1}, -1) = -1-4(n-1)$ is true. Recall that $Q(\Lambda_n, x) = -3-4n$.

$$Q(M_n, -1) = (-1)Q(M_{n-1}, -1) + 2Q(\Lambda_{n-1});$$

$$= -1(-1-4(n-1)) + 2(-3-4(n-1)) = -1-4n.$$

Thus $Q(M_n, -1) = -1-4n$ for all $n \geq 1$.

$\square$

We now develop a formula for $\sum_{i=0}^{n} x^k Q(\Lambda_{n-k})$.

Lemma 8.9. *For all $n \geq 0$,*

$$\sum_{i=0}^{n} x^{n-k} Q(\Lambda_k) = \frac{1}{4}\left[Q(\Lambda_{n+1}) - xQ(\Lambda_n) - 2x^{n+1}\right].$$

Proof. By the recursive formula $Q(M_n) - xQ(M_{n-1}) = 2Q(\Lambda_{n-1})$, for $n \geq 0$,

$$\begin{aligned}
Q(M_{n+1}) - xQ(M_n) &= 2Q(\Lambda_n) \\
xQ(M_n) - x^2Q(M_{n-1}) &= 2xQ(\Lambda_{n-1}) \\
x^2Q(M_{n-1}) - x^3Q(M_{n-2}) &= 2x^2Q(\Lambda_{n-2}) \\
&\vdots \\
x^nQ(M_1) - x^{n+1}Q(M_0) &= 2x^nQ(\Lambda_0)
\end{aligned}$$

Add the above equations we obtain:

$$Q(M_{n+1}) - x^{n+1}Q(M_0) = 2\sum_{i=0}^{n} x^{n-k}Q(\Lambda_k) \Longrightarrow Q(M_{n+1}) = x^{n+2} + 2\sum_{i=0}^{n} x^kQ(\Lambda_{n-k}).$$

But from equation 11, $Q(M_{n+1}) = \frac{Q(\Lambda_{n+1}) - xQ(\Lambda_n)}{2} \Longrightarrow$

$$\frac{Q(\Lambda_{n+1}) - xQ(\Lambda_n)}{2} = x^{n+2} + 2\sum_{i=0}^{n} x^{n-k} Q(\Lambda_k),$$

Which derives the result.

□

The recursive formula for $Q(\Delta_n)$ is given by the following lemma.

Lemma 8.10. *The recursive formula for $Q(\Delta_n, x)$ with $n \geq 1$ is:*

$$Q(\Delta_n, x) = xQ(\Delta_{n-1}, x) + 2Q(\Lambda_n).$$

Define $\Delta_0 = x^2 + 6x$. Recall the recursive formulas: $Q(M_n) = xQ(M_{n-1}) + 2Q(\Lambda_{n-1})$ for $n \geq 1$, and $2Q(\Lambda_n) = Q(M_{n+1}) - xQ(M_n)$. We claim:

Theorem 8.11. *For $n \geq 0$,*

1. $Q(\Delta_n) = Q(M_{n+1}) = \frac{Q(\Lambda_{n+1}) - xQ(\Lambda_n)}{2}$ *for $x \neq -1$;*

2. $Q(\Delta_n, (-1)) = -5 - 4n$.

Proof. 1. By mathematical induction.

Obviously $Q(\Delta_0) = Q(M_1)$. Assume $Q(\Delta_{n-1}) = Q(M_n)$ for $n > 1$. Then

$$Q(\Delta_n) = xQ(\Delta_{n-1}) + 2Q(\Lambda_n) = xQ(M_n) + Q(M_{n+1}) - xQ(M_n) = Q(M_{n+1}).$$

□

Proof. 2. By mathematical induction. Assume $x = -1$.

Our initial value for $n = 1$ holds true.

Since $Q(\Delta_n, x) = x^3 + 16x^2 + 24x$,

$$Q(\Delta_1, -1) = (-1)^3 + 16(-1)^2 + 24(-1) = -9 = -5 - 4(1).$$

Now assume $Q(\Delta_n, -1)$ is true for $n - 1$.

$$Q(\Delta_n, -1) = (-1)Q(\Delta_{n-1}, -1) + 2Q(\Lambda_n, -1);$$

$$= -1(-5 - 4(n-1)) + 2(-1 - 4(n)) = -5 - 4n.$$

$\therefore \quad Q(\Delta_n, -1) = -5 - 4n$ for all $n \geq 1$.

$\square$

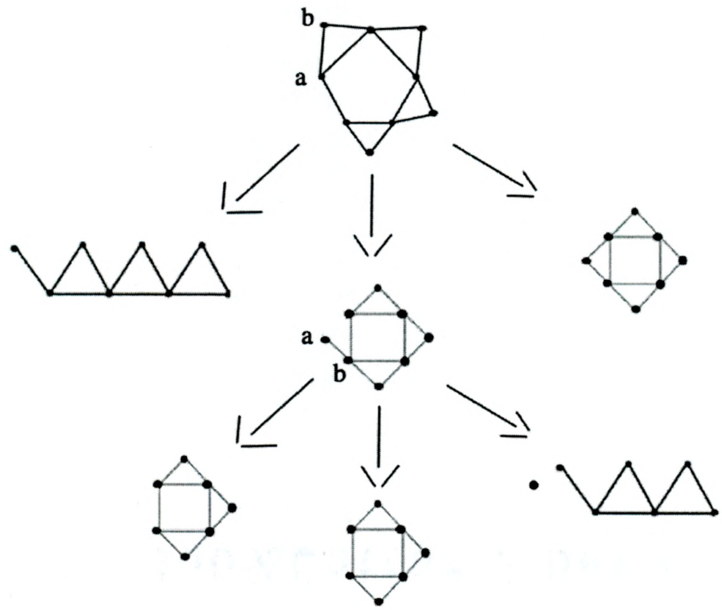
Now we look at our last recursive relation in order to form an explicit formula for $Q(\Gamma_n, x)$.

Lemma 8.12. *The recursive formula for $Q(\Omega_n, x)$, with $n \geq 2$, is:*

$$Q(\Omega_n) = Q(\Gamma_n) + Q(\Lambda_{n-1}) + 2Q(\Omega_{n-1}) + xQ(\Lambda_{n-2}) \tag{12}$$

The idea of the proof can be expressed by looking at $Q(\Omega_4)$ shown below.

Example 9. *Breaking down $Q(\Omega_4)$.*



After one step, from left to right, we have Λ_3, Ω_3, Ω_3, $x\Lambda_2$, and Γ_4, respectively.

Similar to the recursive relation between $Q(\Lambda_n)$ and $Q(M_n)$, we use the same technique for $Q(\Omega_n)$ and $Q(\Gamma_n)$.

Define

$$H_n(x) = Q(\Lambda_n) + (x-2)Q(\Lambda_{n-1}) - 2xQ(\Lambda_{n-2}) + Q(M_n) + xQ(\Delta_{n-2}).$$

We simplify $H_n(x)$ using the previous formulas.

$$H_n = 2(x+1)\left[Q(\Lambda_{n-1}) - xQ(\Lambda_{n-2})\right], \quad n \geq 3. \tag{13}$$

Then we have

$$Q(\Omega_{n+1}) - Q(\Omega_n) = 4(Q(\Omega_n) - Q(\Omega_{n-1})) + H_n(x). \tag{14}$$

In order to achieve an explicit formula for $Q(\Omega_n)$, we define the following:

$$V_n(x) = Q(\Omega_{n+1}) - Q(\Omega_n), \quad n \geq 2 \tag{15}$$

We substitute equation (15) into equation (14) to result in the following.

$$V_n(x) = 4V_{n-1}(x) + H_n(x), \quad n \geq 2 \tag{16}$$

We develop an explicit formula for $Y_n(x)$ using similar techniques from the previous polynomials. Note that the smallest value of n for $Y_n(x)$ is 2 since Ω_2 is the smallest graph of its kind.

$$V_2(x) = 14x^3 + 118x^2 + 168x; \tag{17}$$

$$V_n(x) = 4^{n-2}V_2(x) + \sum_{j=3}^{n} 4^{n-j}H_j(x), \ \ n \geq 3. \tag{18}$$

Now we use equation (15) to develop an explicit function for $Q(\Omega_n)$. Again we use similar techniques and we also use equation (18) to substitute into the function to simplify further and achieve the final function. Some steps are shown below.

$$\begin{aligned}
Q(\Omega_n) &= Q(\Omega_2) + V_2(x) + \sum_{i=3}^{n-1} V_i(x); \\
&= 14x^3 + 133x^2 + 204x + \sum_{i=3}^{n-1}\left(4^{i-2}V_2(x) + \sum_{j=3}^{i} 4^{i-j}H_j(x)\right); \\
&= 14x^3 + 133x^2 + 204x + \tfrac{4^{n-2}-4}{3}(14x^3 + 118x^2 + 168x) + \sum_{i=3}^{n-1}\sum_{j=3}^{i} 4^{i-j}H_j(x); \\
&= 14x^3 + 133x^2 + 204x + \tfrac{4^{n-2}-4}{3}(14x^3 + 118x^2 + 168x) + \frac{1}{3}\sum_{i=3}^{n-1}(4^{n-i}-1)H_i(x).
\end{aligned}$$

Theorem 8.13. *The explicit function $Q(\Omega_n, x)$, where $H_n(x)$ is given by equation (13) is:*

1. $Q(\Omega_n, x)$
   $= 14x^3 + 133x^2 + 204x + \tfrac{4^{n-2}-4}{3}(14x^3 + 118x^2 + 168x) + \frac{1}{3}\sum_{i=3}^{n-1}(4^{n-i}-1)H_i(x),$
   for $n \geq 4$ and $x \neq -1$;

2. $Q(\Omega_n, -1) = \frac{1}{3}(1 - (4)^{n+1})$ *for $n \geq 2$.*

Proof. 1. By mathematical induction. Assume $x \neq -1$.

Our initial conditions for $n = 4$ and $n = 5$ can be confirmed by *Mathematica.*

$Q(\Omega_4) = 18x^4 + 240x^3 + 949x^2 + 1068x;$

$Q(\Omega_5) = 22x^5 + 440x^4 + 2464x^3 + 5983x^2 + 5292x.$

Now assume $Q(\Omega_n)$ and $Q(\Omega_{n-1})$ are true.

$$\begin{aligned}
Q(\Omega_{n+1}) &= 5Q(\Omega_n) - 4Q(\Omega_{n-1}) + H_n(x) \\
&= (5-4)(14x^3 + 133x^2 + 204x) \\
&\quad + \left(5\left(\tfrac{4^{n-2}-4}{3}\right) - 4\left(\tfrac{4^{n-3}-4}{3}\right)\right)(14x^3 + 118x^2 + 168x) \\
&\quad + \tfrac{5}{3}\left(\sum_{i=3}^{n-1}(4^{n-i}-1)H_i(x)\right) - \tfrac{4}{3}\left(\sum_{i=3}^{n-2}(4^{n-1-i}-1)H_i(x)\right) + H_n(x) \\
&= 14x^3 + 133x^2 + 204x + \tfrac{4^{n-1}-4}{3}(14x^3 + 118x^2 + 168x) \\
&\quad + \tfrac{1}{3}\left(\sum_{i=3}^{n-2}\left[5(4^{n-i}-1) - 4(4^{n-1-i}-1)\right]H_i(x)\right) + \tfrac{5}{3}(4-1)H_{n-1}(x) \\
&\quad + H_n(x) \\
&= 14x^3 + 133x^2 + 204x + \tfrac{4^{n-1}-4}{3}(14x^3 + 118x^2 + 168x) \\
&\quad + \tfrac{1}{3}\left(\sum_{i=3}^{n-2}(4^{n+1-i}-1)H_i(x)\right) + 5H_{n-1}(x) + H_n(x).
\end{aligned}$$

For this to be true for $Q(\Omega_{n+1})$, we need $5H_{n-1} + H_n$ to be equivalent to $\frac{1}{3}\sum_{i=n-1}^{n}(4^{n+1-i}-1)H_i$.

$$\begin{aligned}
\tfrac{1}{3}\sum_{i=n-1}^{n}(4^{n+1-i}-1)H_i(x) &= \tfrac{1}{3}(4^{n+1-(n-1)}-1)H_{n-1} + \tfrac{1}{3}(4^{n+1-n}-1)H_n \\
&= \tfrac{1}{3}(4^2-1)H_{n-1}(x) + \tfrac{1}{3}(4-1)H_n(x) \\
&= 5H_{n-1}(x) + H_n(x).
\end{aligned}$$

$\therefore \quad Q(\Omega_n, x)$ is true for all $n \geq 4$.

□

Proof. 2. By mathematical induction.

Our initial case for $n = 2$ holds true.

$Q(\Omega_2, -1) = \frac{1}{3}(1 - (4)^3) = -21.$

Now assume $Q(\Omega_{n-1}, -1)$ is true, and from equations (15) and (18) we have,

$Q(\Omega_n, -1) = Q(\Omega_{n-1}, -1) + V_{n-1}(-1)$ and $V_n(x) = 4^{n-2}V_2(x) + \sum_{j=3}^{n} 4^{n-j}H_j(x).$

Note that for $H_j(x) = 2(x+1)\,[2Q(\Lambda_{j-1} - xQ(\Lambda_{j-2})]$. Clearly $H_j(-1) = 0$.

$Q(\Omega_n, -1) = \frac{1}{3}(1 - (4)^n) + 4^{n-3}(-64) = \frac{1}{3}(1 - (4)^{n+1}).$

$\therefore \quad Q(\Omega_n, -1)$ holds true for all $n \geq 2$.

□

Finally, we go back to the recursive function from Lemma 8.2 and define one more function in order to finalize an explicit function for $Q(\Gamma_n, x)$.

$$R_n(x) = Q(\Omega_{n-1}) + Q(M_{n-1}) + xQ(\Delta_{n-3}).$$

We use our previous formulas to simplify R_n.

$$R_n(x) = Q(\Omega_{n-2}) + \frac{Q(\Lambda_{n-1}) - x^2 Q(\Lambda_{n-3})}{2} \tag{19}$$

Using the function from Lemma 8.2 and equation (19), we express $Q(\Gamma_n)$ as shown below.

$$Q(\Gamma_n) = 2Q(\Gamma_{n-1}) + R_n(x), \quad \text{for } n \geq 4. \tag{20}$$

We apply similar techniques used previously to achieve the explicit function.

$$Q(\Gamma_n) = 2^{n-4}Q(\Gamma_4) + \sum_{i=5}^{n} 2^{n-i}R_i(x), \quad n \geq 5.$$

Theorem 8.14. *The interlace polynomial* $Q(\Gamma_n, x)$*, with* $R_n(x)$ *given by equation (19), for* $n \geq 5$ *is:*

1. $Q(\Gamma_n, x) = 2^{n-4}(2x^4 + 64x^3 + 363x^2 + 468x) + \sum_{i=5}^{n} 2^{n-i} R_i(x)$*;*

2. $Q(\Gamma_n, -1) = \frac{1}{3}(11 - 2^{2n+1})$

Proof. 1. By mathematical induction. When $x \neq -1$.

A confirmation with *Mathematica* shows our initial condition for $n = 5$ holds true.

$$Q(\Gamma_5, x) = 2x^5 + 100x^4 + 800x^3 + 2411x^2 + 2388x.$$

Now assume $Q(\Gamma_{n-1}, x)$ is true.

$$\begin{aligned}
Q(\Gamma_n, x) &= 2Q(\Gamma_{n-1}) + R_n(x);\\
&= 2\left[2^{n-5}(2x^4 + 64x^3 + 363x^2 + 468x) + \sum_{i=5}^{n-1} 2^{n-1-i} R_i(x)\right] + R_n(x);\\
&= 2^{n-4}(2x^4 + 64x^3 + 363x^2 + 468x) + \sum_{i=5}^{n-1} 2^{n-i} R_i(x) + R_n(x);\\
&= 2^{n-4}(2x^4 + 64x^3 + 363x^2 + 468x) + \sum_{i=5}^{n} 2^{n-i} R_i(x).
\end{aligned}$$

Therefore, $Q(\Gamma_n, x)$ is true for all $n \geq 5$.

□

Proof. 2. By mathematical induction.

The initial case for $n = 5$ is confirmed below.

$Q(\Gamma_5, -1) = \frac{1}{3}(11 - 2^{11}) = -679.$

Now assume $Q(\Gamma_{n-1}, -1)$ is true, and from equation (20) we have,

$Q(\Gamma_n, -1) = 2Q(\Gamma_{n-1}, -1) + R_n(-1).$

We use equation (19) to determine $R_n(-1)$.

$$
\begin{aligned}
Q(\Gamma_n, -1) &= 2(\tfrac{1}{3}(11 - 2^{2n-1}) + \tfrac{1}{3}(1 - 4^n) - 4; \\
&= \frac{22 - 2^{2n} + 1 - 2^{2n} - 12}{3}; \\
&= \tfrac{1}{3}(11 - 2^{2n+1}).
\end{aligned}
$$

$\therefore \quad Q(\Gamma_n, -1) = \frac{1}{3}(11 - 2^{2n+1})$ for all $n \geq 5$.

□

8.2 Applications of $Q(\Gamma_n, x)$

Study has been done specifically on x values of 2 and 4 for the interlace polynomial $Q(G, x)$. $Q(G, 2)$ equals the number of general induced subgraphs of G (with possible loops attached to the vertices) with an odd number of general perfect matchings [1]. $Q(G, 4)$ equals $2n$ times the number of induced Eulerian subgraphs of G [1]. Below we will relate these specific values to our graph Γ_n.

Corollary 8.15. *The number of general induced subgraphs for a few Γ_n (with possible loops attached to the vertices), with an odd number of general perfect matchings are defined below:*

1. *For Γ_3, we have* 372 *general induced subgraphs with an odd number of general perfect matchings;*

2. *For Γ_4, we have* 2932 *general induced subgraphs with an odd number of general perfect matchings;*

3. *For Γ_5 we have* 22484 *general induced subgraphs with an odd number of general perfect matchings. general induced subgraphs.*

Corollary 8.16. *The number of induced Eulerian subgraphs of Γ_4 is 1536.*

9 Future Directions

Below I discuss the direction I intend to continue with the interlace polynomial of $\Gamma_n(x)$ as well as $Q(\Gamma_n, x)$. I wish to show more direct relationships within the coefficients of $\Gamma_n(x)$ and $Q(\Gamma_n, x)$.

9.1 More Properties on $\Gamma_n(x)$

The adjacency matrix can give us specific information about our graph. Let us look at one theorem that can tell us how many walks of length k are bewteen two specific vertices.

Theorem 9.1. *(see, eg., [7].) Let G be a graph on labeled vertices, let A be its adjacency matrix, and let k be a positive integer. Then $A^k_{i,j}$ is equal to the number of walks from i to j that are of lenth k.*

Let us use 9.1 and take a look at the adjacency matrix for Γ_5 and concentrate on walks of size 4 and 5.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}_{10\times 10},$$

$$A^4 = \begin{bmatrix} 10 & 11 & 7 & 10 & 4 & 6 & 4 & 10 & 7 & 11 \\ 11 & 24 & 11 & 14 & 10 & 14 & 6 & 14 & 10 & 14 \\ 7 & 11 & 10 & 11 & 7 & 10 & 4 & 6 & 4 & 10 \\ 10 & 14 & 11 & 24 & 11 & 14 & 10 & 14 & 6 & 14 \\ 4 & 10 & 7 & 11 & 10 & 11 & 7 & 10 & 4 & 6 \\ 6 & 14 & 10 & 14 & 11 & 24 & 11 & 14 & 10 & 14 \\ 4 & 6 & 4 & 10 & 7 & 11 & 10 & 11 & 7 & 10 \\ 10 & 14 & 6 & 14 & 10 & 14 & 11 & 24 & 11 & 14 \\ 7 & 10 & 4 & 6 & 4 & 10 & 7 & 11 & 10 & 11 \\ 11 & 14 & 10 & 14 & 6 & 14 & 10 & 14 & 11 & 24 \end{bmatrix}_{10\times 10},$$

$$A^5 = \begin{bmatrix} 22 & 38 & 21 & 28 & 16 & 28 & 16 & 28 & 21 & 38 \\ 38 & 50 & 38 & 59 & 28 & 44 & 28 & 44 & 28 & 59 \\ 21 & 38 & 22 & 38 & 21 & 28 & 16 & 28 & 16 & 28 \\ 28 & 59 & 38 & 50 & 38 & 59 & 28 & 44 & 28 & 44 \\ 16 & 28 & 21 & 38 & 22 & 38 & 21 & 28 & 16 & 28 \\ 28 & 44 & 28 & 59 & 38 & 50 & 38 & 59 & 28 & 44 \\ 16 & 28 & 16 & 28 & 21 & 38 & 22 & 38 & 21 & 28 \\ 28 & 44 & 28 & 44 & 28 & 59 & 38 & 50 & 38 & 59 \\ 21 & 28 & 16 & 28 & 16 & 28 & 21 & 38 & 22 & 38 \\ 38 & 59 & 28 & 44 & 28 & 44 & 28 & 59 & 38 & 50 \end{bmatrix}_{10\times 10}$$

Each of the entries, A_{ij}, in the matrices for A^4 and A^5 represent the number of walks, respectively, of size 4 and 5, between the two vertices i and j. My goal here is to be able to show the number of walks of size k, correlate to the coefficients in my interlace polynomial for Γ_n at a specific n. I will concentrate on specific vertices and the patterns associated in the matrix to try and see if in fact there is a correlation between a walk from two specific vertices and my coefficients in the polynomial.

Another interesting theorem that can be used for further investigation deals with the number of spanning trees within a graph.

Theorem 9.2. *(see, eg., [7].) Let U be a simple undirected graph. Let $\{v_1, v_2, \cdots, v_n\}$ be the vertices of U. Define the $(n-1) \times (n-1)$ matrix L_0 by*

$$l_{i,j} = \begin{cases} \textit{the degree of } v_i \textit{ if } i = j, \\ -1 \textit{ if } i \neq j \textit{ and } v_i \textit{ and } v_j \textit{ are connected, and,} \\ 0 \textit{ otherwise.} \end{cases}$$

where $1 \leq i, j \leq n-1$. Then U has exactly $det(L_0)$ spanning trees.

Research has been done on the interlace polynomial for arbitrary trees [2]. I will use Theorem 9.2 to show how many spanning trees Γ_n has and will try to determine any significance for the interlace polynomial of arbitrary trees and the graph polynomial $\Gamma_n(x)$. Let us take a look at L_0 for Γ_5.

$$L_0 = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 4 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 4 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}_{9\times 9}$$

Using Maple Software, I was able to determine the determinant.

$$det(L_0) = 810.$$

This tells us, from Theorem 9.2 that Γ_5 has exactly 810 spanning trees. Now the question is, how can we relate the interlace polynomial from [2], for arbitrary trees for the spanning trees in my graph, and do they relate to my graph polynomial for $\Gamma_5(x)$?

9.2 More Properties on $Q(\Gamma_n, x)$

The interlace polynomial $Q(\Gamma_n, x)$ is a lot more complicated from the original graph polynomial we created. I would like to further study the change in coefficients for this graph polynomial and try to relate them as I did for the coefficients of $\Gamma_n(x)$. Hopefully I can lead myself into a better understanding of the graph polynomial itself and in turn more properties about the graph itself.

References

[1] Aigner, M., and Holst, H. (2004). Interlace polynomials. Linear Algebra and Its Applications, 377(0), 11-30.

[2] Anderson, C., Cutler, J., Radcliffe, A.J., and Traldi, L. On the interlace polynomials of forests, Discrete Mathematics (2009).

[3] Arratia, Richard, Bela Bollobas, and Gregory B. Sorkin. "The Interlace Polynomial Of A Graph." (2002): arXiv.

[4] Arratia, R., Bollobas, B., Coppersmith, D., and Sorkin, B. G. "Euler Circuits And DNA Sequencing By Hybridization." Discrete Applied Mathematics 104.(2000): 63-96. ScienceDirect.

[5] Arratia, R., Bollobas, B., and Sorkin, G. The interlace polnomial: a new graph polynomial, manuscript.

[6] Austin, Andrea. The Circuit Partition Polynomial with Applications and Relation to the Tutte and Interlace Polynomials (2007). Rose-Hulman Undergraduate Mathematics Journal.

[7] Balister P.N., Bollobas B., Cutler J. and Pebody L., (2002). The Interlace Polynomial of Graphs at -1, Europ. J. Combinatorics, 23, 761-767.

[8] Bona, Miklos. A Walk through Combinatorics: An Introduction to Enumeration and Graph Theory. Third ed. Hackensack, NJ: World Scientific Pub., 2006. Print.

[9] Ellis-Monaghan, Joanna, and Criel Merino. "Graph Polynomials And Their Applications I: The Tutte Polynomial." (2008): arXiv.

[10] Ellis-Monaghan, J. A. and Merino, C. "Graph Polynomials and Their Applications II: Interrelations and Interpretations." 28 Jun 2008.

[11] Ellis-Monaghan, Joanna. Properties of the Interlace Polynomial via Isotropic Systems.

[12] Friedberg, Stephen H., Arnold J. Insel, and Lawrence E. Spence. Linear Algebra. Fourth ed. Englewood Cliffs, NJ: Prentice-Hall, 1979. Print.

[13] Godlin, Benny, Emilia Katz, and Johann A. Makowsky. "Graph Polynomials: From Recursive Definitions To Subset Expansion Formulas." (2008): arXiv. Web. 9 Oct. 2014.

[14] Paoletti, Teo. "Leonard Euler's Solution to the Konigsberg Bridge Problem - The Fate of Konigsberg," Loci (May 2011).