Partial Colorings of Graphs

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Abstract

Recently, there has been great interest in counting the number of homomorphisms from a graph $G$ into a fixed image graph $H$. For this thesis, we let $H$ be a complete graph on three vertices with exactly one looped vertex. Homomorphisms from a graph $G$ to this $H$ correspond to partial proper two-colorings of the vertices of $G$. We are mainly interested in finding which graphs maximize the number of partial two-colorings given a graph with $n$ vertices and $m$ edges. The general result is given for all graphs with $m \leq n - 1$ as well as basic enumerative results for some very common graphs.
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Chapter 1

Introduction

1.1 Basic Terminology and Notation

First we present the basic terminology and notation that will be used in this thesis. We define a graph $G$ to be comprised of a vertex set $V(G)$ and an edge set $E(G)$ that consists of two-element subsets of $V(G)$. The standard way to visualize a graph is to think of the vertices as dots and the edges as lines or arcs joining the dots. For example, Figure 1.1 below depicts a graph $G$ with vertex set $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and edge set $E(G) = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_5\}, \{v_4, v_5\}\}$.

![Figure 1.1: A basic visualization of a graph.](image)

If two vertices $u$ and $v$ are joined by an edge $e$ then we say $u$ and $v$ are adjacent and
write $u \sim v$. In addition, we say $u$ and $v$ are incident with $e$. Adjacent vertices are also referred to as neighbors. The set of all vertices that are adjacent to a vertex $v$ is called the neighborhood of that vertex and is denoted $N(v)$. The number of neighbors of $v$, $|N(v)|$, is the degree of $v$ and is written $d(v)$. A graph $G$ is $r$-regular if $d(v) = r$ for all $v \in V(G)$. An independent set is a set of vertices in a graph in which no two are adjacent. The set of independent sets of a graph $G$ is denoted $I(G)$ and $|I(G)| = i(G)$. A subgraph $H$ of a graph $G$ is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

The two vertices incident with an edge $e$ are the endpoints of $e$. A loop is an edge that connects a vertex to itself while multiple edges are a set of edges having common endpoints. We say a graph is simple if it has no loops or multiple edges. For this thesis we will never deal with graphs containing multiple edges, however, graphs containing loops will be of concern.

The Cartesian product of the graphs $G$ and $H$, written $G \square H$, is the graph with vertex set $V(G) \times V(H)$ where $(x_1, y_1) \sim (x_2, y_2)$ if and only if either $x_1 = x_2$ and $y_1y_2 \in E(H)$, or $y_1 = y_2$ and $x_1x_2 \in E(G)$.

There are a variety of "standard" graphs that are mentioned in this thesis. Among some of them are as follows. A path on $n$ vertices, denoted $P_n$, is a graph with $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{v_iv_{i+1} : 1 \leq i \leq n - 1\}$ where $v_1$ and $v_n$ are referred to as the endpoints of the path. Certainly, paths can be subgraphs of graphs. If $u$ and $v$ are the endpoints of a path-subgraph, then we refer to this subgraph as a $u, v$-path. A graph $G$ is connected if there is a $u, v$-path joining every pair of vertices $u$ and $v$ of $G$.

![Figure 1.2: The path $P_5$.](image)

A cycle on $n$ vertices ($n \geq 3$), denoted $C_n$, is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively on the circle.
A complete graph $K_n$ is a graph with $n$ vertices in which there is exactly one edge joining every pair of vertices. Note this implies $|E(K_n)| = \binom{n}{2} = \frac{n(n-1)}{2}$.

If a graph $G$ doesn't contain a cycle as a subgraph then we say $G$ is acyclic. A forest is an acyclic graph and a tree is an acyclic graph that is connected. A leaf is a vertex of degree one.

A graph is bipartite if the vertex set can be partitioned into two sets such that no two vertices within the same set are adjacent. A complete bipartite graph, $K_{m,n}$, is a bipartite graph containing the maximum number of edges where the two vertex parts have sizes $m$ and $n$, respectively.

There is a small variety of specialized graphs that will be of great importance in this
thesis. We’ll define two of them now. Recall that $A \Delta B$ denotes the symmetric difference of the sets $A$ and $B$. In other words, $A \Delta B = (A - B) \cup (B - A)$. Also, recall that $[n] = \{1, 2, \ldots, n\}$. The *lex ordering* on the set $\left(\binom{[n]}{2}\right) = \{A \subseteq [n] : |A| = 2\}$ is an ordering of the elements of $\left(\binom{[n]}{2}\right)$ obeying the following rule: We say a set $A$ is less than a set $B$ in the lex order if $\min(A \Delta B) \in A$ where $A, B \in \left(\binom{[n]}{2}\right)$. For example, the lex ordering of the set $\left(\binom{[n]}{2}\right)$ would look like:

$$\{1, 2\}, \{1, 3\}, \{1, 4\}, \ldots, \{1, n\}, \{2, 3\}, \{2, 4\}, \ldots, \{2, n\}, \{3, 4\}, \ldots$$

We define the *lex graph with $n$ vertices and $m$ edges*, denoted $L(n, m)$, to be the graph
with vertex set $[n]$ and edge set given by the first $m$ elements in the lexicographic ordering on $\binom{[n]}{2}$.

![Figure 1.7: The lex graph $L(6,5)$.](image)

Lastly, we define the graph $H_n$ to be the complete graph on $n + 1$ vertices that contains exactly one loop. We denote the one vertex that is adjacent to itself in $H_n$ as $\ell$.

### 1.2 A Brief History of Proper $q$-colorings

A well-studied combinatorial problem in the subject of graph theory is that of determining the number of proper $q$-colorings given a graph with $n$ vertices and $m$ edges. We define an assignment of colors to the vertices of a graph $G$ as a proper coloring if no two adjacent vertices in $G$ are colored the same, while a proper $q$-coloring of $G$ is a proper coloring using at most $q$ distinct colors.

We consider the following extremal problem that was first posed by Linial [1] and Wilf [2]:

\[ \text{Problem: } \]
Given the fixed values \( n \) and \( m \), is there a unique graph with \( n \) vertices and \( m \) edges that maximizes the number of proper \( q \)-colorings?

There are many different ways of interpreting this problem. One approach that is very handy is through the use of graph homomorphisms. A homomorphism \( \phi \) from a graph \( G \) to a graph \( H \) is a map \( \phi : V(G) \to V(H) \) so that if \( xy \in E(G) \) then \( \phi(x)\phi(y) \in E(H) \). The set of all homomorphisms from \( G \) to \( H \) is denoted by \( \text{Hom}(G, H) \) and we let \( \text{hom}(G, H) = |\text{Hom}(G, H)| \).

Let’s consider the following example to help demonstrate the relationship between graph homomorphisms and proper \( q \)-colorings. Let \( G \) be some simple graph and \( K_q \) be the complete graph on \( q \) vertices. We claim elements of \( \text{Hom}(G, K_q) \) correspond to proper \( q \)-colorings of \( G \). Since a proper \( q \)-coloring uses at most \( q \) colors, we can think of this coloring as a partitioning of \( V(G) \) into at most \( q \) color classes. These color classes correspond to independent sets in \( G \) since a proper coloring forbids adjacent vertices to be colored the same. Looking at the coloring from this point of view, it is a bit easier to describe a map \( \phi \) from \( V(G) \) to \( V(K_q) \) so that if \( xy \in E(G) \) then \( \phi(x)\phi(y) \in E(K_q) \).

Let \( V(K_q) = \{v_1, v_2, \ldots, v_q\} \) and the \( q \) colors used on \( G \) be the first \( q \) nonnegative integers. Define a map \( \phi : V(G) \to V(K_q) \) so that vertices colored \( i \), where \( 1 \leq i \leq q \), are mapped to \( v_i \in V(K_q) \). Let \( xy \) be an edge in \( G \) where the vertex \( x \) is colored \( i \) and the vertex \( y \) is colored \( j \). The homomorphism \( \phi \) maps \( x \) to \( v_i \) and \( y \) to \( v_j \), and gives \( \phi(x)\phi(y) \in E(K_q) \). Thus, \( \phi \) describes a homomorphism from \( G \) to \( K_q \). This implies that any proper \( q \)-coloring of a graph \( G \) can be viewed as a graph homomorphism from \( G \) to \( V(K_q) \).

Interestingly enough, not many exact results for general values of \( q \) are known today. One of the earliest and most notable results was found by Lazebnik [3] who solved the case completely when \( q = 2 \) and also for various cases when \( q = 3 \). We can state the result for \( q = 2 \) in terms of homomorphisms as follows:
Theorem 1.2.1. If $G$ is a graph with $n$ vertices and $m$ edges, then

$$\text{hom}(G, K_2) \leq \begin{cases} 
2^n & \text{if } m = 0 \\
2^{n-\lceil 2\sqrt{m} \rceil + 1} & \text{if } 0 < m \leq \lfloor n^2/4 \rfloor \\
0 & \text{if } m > \lfloor n^2/4 \rfloor
\end{cases}$$

Lazebnik, Pikhurko, and Wolder [4] were able to show a complete bipartite graph is extremal when $q = 3$ and asymptotically extremal for $q = 4$. Loh, Pikhurko, and Sudakov [5] provide a comprehensive summary of some other results related to this problem. Also, the authors in [5] were able to provide an asymptotic answer to the $q$-coloring problem when $q \geq 4$.

1.3 Partial Colorings

The goal of this section is to state the main problem with which this thesis is concerned. Let’s quickly go back to the $\text{Hom}(G, K_q)$ example that was discussed in the previous section. Note that the graph $K_q$ was fixed and the graph $G$ represented any general simple graph. Recently, there has been some interests in studying $\text{hom}(G, H)$ where $H$ is some specific fixed graph and $G$ is from some class of graphs on a fixed number of vertices. Generally speaking, we refer to this fixed graph $H$ as an image graph. Depending on our image graph, our coloring “rules” can become drastically different. There are a handful of results related to fixing $H$ and determining which graph (or maybe graphs) maximizes $\text{hom}(G, H)$ [5–9].

Take for example the graph $H_1$, which is the complete graph on two vertices containing exactly one loop. Given a homomorphism $\phi$ from a graph $G$ to $H_1$, we can view the pre-images of the looped vertex as a vertex subset of $V(G)$, regardless if the subset contains adjacent or non-adjacent vertices. This is since the loop allows a vertex to essentially be
adjacent to itself, hence, allowing adjacent vertices in the pre-image. On the other hand, vertices mapped to the unlooped vertex come from a pairwise non-adjacent vertex subset of $V(G)$. In other words, the vertices of $G$ mapped to the unlooped vertex form an independent set, just like vertices mapped to any vertex of $K_q$ in the Hom$(G, K_q)$ example.

We can also view a proper $q$-coloring as a collection of independent sets. For example, the color classes in the Hom$(G, K_q)$-example that partition $V(G)$ are independent sets since vertices in each color class are not adjacent to one another. From this point of view, many upper bounds on the number of independent sets in a graph, $i(G)$, have been established. For example, Kahn [8] was able to show that if $G$ is an $r$-regular bipartite graph on $n$ vertices, then $i(G) \leq (2^{r+1} - 1)^{\frac{n}{2r}}$. Galvin and Tetali [7] were able to generalize this result for general homomorphisms, establishing the inequality hom$(G, H) \leq \text{hom}(K_{r,r}, H)^{\frac{n}{2r}}$, whenever $G$ is a bipartite $r$-regular graph and $H$ is any image graph.

The fixed image graph that we are concerned with in this thesis is $H_2$. (See Figure 1.8.)

![Figure 1.8: The graph $H_2$.](image)

$H_2$ can also be viewed in the same light as $H_1$, that is, in terms of independent sets. Say $\phi$ is a homomorphism from a graph $G$ to $H_2$. Using the labeling shown in Figure 1.8, we note that the set of vertices mapped to $a$ or $b$ under $\phi$ form two independent sets in $G$ while the vertices mapped to $c$ represent some ordinary vertex subset of $V(G)$. In general, we refer to vertices of $G$ mapped to $a$ under the homomorphism as $a$-colored. Vertices mapped to $b$ or $c$ are defined similarly.
Therefore, when we view an element of $\text{Hom}(G, H_2)$, we can visualize a partitioning of $V(G)$ into three main sets: the two disjoint independent sets and some leftover vertices. If we color the vertices in the disjoint independent sets differently, then what would result is a "partial" proper two-coloring of $G$. We'll refer to this special type of coloring of a graph $G$ as a partial two-coloring of $G$. Generally speaking, we define a partial coloring of a graph as follows:

**Definition.** A partial $k$-coloring of a simple graph $G$ is an element from the set $\text{Hom}(G, H_k)$. For convenience, we denote the set $\text{Hom}(G, H_k)$ as $\text{PC}_k(G)$ and $\text{hom}(G, H_k)$ as $\text{pc}_k(G)$.

Note we may also define a partial $k$-coloring as a map $\phi : V(G) \to \{1, 2, \ldots, k\} \cup \{u\}$ where vertices mapped to $u$ are left "uncolored." We can now state precisely the main problem we are concerned with in this thesis: Given the positive integers $n$ and $m$ with $0 < m < \binom{n}{2}$, which graph with $n$ vertices and $m$ edges has the maximum number of partial two-colorings?

Before we begin an attempt to answer this question, we need to first give a brief summary of how the sections in this thesis are laid out.

Chapter 2 deals primarily with establishing some basic enumerative results. It is here where we depend on some basic combinatorics to help answer the question: How many partial two-colorings are there of a graph $G$? The basic graphs that are looked at in this section are $K_n$, $P_n$, and $C_n$. It turns out, enumerating $\text{pc}_2(C_n)$ is less straightforward than one may think. To help find this value, we look into a surprising relationship between the number of independent sets in the prism graph $Y_n$ and the value $\text{pc}_2(C_n)$. This section concludes with a more general result for paths. Through the use of chromatic polynomials, we find a closed form for $\text{pc}_k(P_n)$.

Chapter 3 gives a complete solution to our question when we restrict our maximization problem to forests on $n$ vertices. We show the lex graph $L(n, m)$ maximizes $\text{pc}_2(T)$ for any tree on $n$ vertices and $m$ edges. An even more general result is stated for forests: If $G$
is a forest with \( n \) vertices and \( m \leq n - 1 \) edges then the graph that maximizes \( \text{pc}_2(G) \) is 
\[ L(m + 1, m) \cup E_{n-m-1}, \]
where \( E_{n-m-1} \) denotes the empty graph (the graph with no edges) with \( n - m - 1 \) vertices. Many tools are used to help derive and prove this result. Perhaps the most important is the idea of a compression of a graph. This is a way we can “transform” a graph into a new graph with the same number of vertices and edges. For trees and forests, it turns out that compressions help increase the number of partial two-colorings.

In addition, this section includes a more general result for all graphs \( G \) with \( n \) vertices and \( m \) edges where \( m \leq n - 1 \). Once again, we show that the lex graph maximizes \( \text{pc}_2(G) \) for these types of graphs. There are a few ingredients needed to help prove this result. In particular, we take cases on the sizes of the cycles that (possibly) exist in such a graph \( G \).

Finally, Chapter 4 gives some possible future directions that can be taken.
Chapter 2

A Few Basic Results

2.1 Enumerating \(pc_2(G)\) for some Basic Graphs

The goal of this section is to enumerate \(pc_2(G)\) where \(G\) is some basic graph. This task becomes a bit simpler to do if we take cases on the colors of the vertices in the partial two-coloring of \(G\). Our first result, Theorem 2.1.1, depends on a case by case breakdown on which vertices are \(a\)- or \(b\)-colored.

**Theorem 2.1.1.** If \(K_n\) is the complete graph on \(n\) vertices, then

\[
pc_2(K_n) = n^2 + n + 1.
\]

**Proof.** We take cases on the number of vertices that are \(a\) or \(b\)-colored. Since every vertex of \(K_n\) is adjacent to one another, there can only be a maximum of one \(a\) or \(b\)-colored vertex in each coloring. There are a total of \(2n\) colorings containing exactly one \(a\) or one \(b\)-colored vertex, \(n(n - 1)\) colorings containing exactly one \(a\)-colored and one \(b\)-colored vertex, and exactly one coloring consisting of neither. This yields a grand total of \(n^2 + n + 1\) such colorings. Therefore, \(pc_2(K_n) = n^2 + n + 1\).  

\(\square\)
On a different note, enumerating $pc_2(P_n)$ as well as $pc_2(C_n)$ will require us to take cases on which vertices are $c$-colored.

**Theorem 2.1.2.** If $P_n$ is a path on $n$ vertices, then

$$pc_2(P_n) = \frac{(1 + \sqrt{2})^{n+1} + (1 - \sqrt{2})^{n+1}}{2}.$$

**Proof.** We'll verify this result by first developing a recurrence relation and then proceed by solving the recurrence via the use of generating functions. Let $f_n$ denote the number of partial two-colorings of a path $P_n$ and $V(P_n) = \{v_1, v_2, ..., v_n\}$. We develop the recurrence by taking cases on the smallest value of $i$ where $1 \leq i \leq n$ for which $v_i$ is $c$-colored.

If $v_1$ is the first $c$-colored vertex, then there are a total of $f_{n-1}$ partial two-colorings. If $v_2$ is the first, then there are a total of $2f_{n-2}$ partial two-colorings since $v_1$ can be $a$-colored or $b$-colored. Following this procedure, we note that the set vertices preceding the first $c$-colored vertex can be colored in exactly two ways, in particular, by alternating between the colors $a$ and $b$. This also implies we obtain a total of 2 partial two-colorings in the case where none of our vertices are $c$-colored. Summing up the totals for each case yields

$$pc_2(P_n) = f_n = f_{n-1} + 2f_{n-2} + \cdots + 2f_1 + 2f_0 + 2.$$

We should note that this recursion looks slightly different for "small" values of $n$. For example, if $n = 0$, we take $pc_2(P_0) = f_0 = 1$ and when $n = 1$, we clearly have that $pc_2(P_1) = f_1 = 3$. As soon as we reach $n = 2$, the above recursion kicks in: $pc_2(P_2) = f_2 = f_1 + 2f_0 + 2 = 3 + 2(1) + 2 = 7$. We know this is true since $pc_2(K_2) = 7$ by our formula from Theorem 2.1.1.

To simplify things a bit, we can proceed by induction in order to condense the above
recurrence into:

\[ f_n = f_{n-1} + 2f_{n-2} + \ldots + 2f_1 + 2f_0 + 2 = f_{n-1} + f_{n-2} + [f_{n-2} + 2f_{n-3} + \cdots + 2f_1 + 2f_0 + 2] \]
\[ = f_{n-1} + f_{n-2} + f_{n-1} \]
\[ = 2f_{n-1} + f_{n-2}. \]

From this we obtain the new cleaner equality \( f_n = 2f_{n-1} + f_{n-2} \), which for convenience we'll write as \( f_{n+2} = 2f_{n+1} + f_n \) where \( f_1 = 3 \) and \( f_2 = 7 \). We claim this recursion will lead us to the desired result.

Let \( f(x) = \sum_{n \geq 0} f_n x^n \). We multiply both sides of \( f_{n+2} = 2f_{n+1} + f_n \) by \( x^{n+2} \), and sum over all nonnegative integers \( n \) to obtain

\[ \sum_{n \geq 0} f_{n+2} x^{n+2} = 2 \sum_{n \geq 0} f_{n+1} x^{n+2} + \sum_{n \geq 0} f_n x^{n+2}, \]

which is equivalent to

\[ f(x) - 3x - 1 = 2x[f(x) - 1] + x^2 f(x). \]

Note here we are taking advantage of the fact that \( pc_2(P_0) = 1 \) and \( pc_2(P_1) = 3 \). This is not difficult to verify since \( P_0 \) is a graph with no vertices and \( P_1 \) is a path consisting of a single vertex. When solving for \( f(x) \), we obtain

\[ f(x) = \frac{x + 1}{1 - 2x - x^2}. \]

The denominator is a quadratic polynomial with roots \( \alpha = 1 + \sqrt{2} \) and \( \beta = 1 - \sqrt{2} \). With this in mind, we can rewrite \( f(x) \) as
where the last line in the equality follows from the fact that \( \alpha \cdot \beta = -1 \). Now, we can make the substitution
\[
f(x) = \frac{1}{2} \cdot \frac{\beta}{1 - \beta x} + \frac{1}{2} \cdot \frac{\alpha}{1 - \alpha x} = \frac{\beta}{2} \sum_{n=0}^{\infty} (\beta x)^n + \frac{\alpha}{2} \sum_{n=0}^{\infty} (\alpha x)^n = \sum_{n=0}^{\infty} \frac{\alpha^{n+1} + \beta^{n+1}}{2} x^n.
\]
This tells us the coefficient of the \( x^n \) term of \( f(x) \) is \( \frac{\alpha^{n+1} + \beta^{n+1}}{2} \) and therefore,
\[
\text{PC}_2(P_n) = \frac{(1 + \sqrt{2})^{n+1} + (1 - \sqrt{2})^{n+1}}{2}.
\]

The proof of Theorem 2.1.3 attacks the problem for cycles in a manner that’s vaguely similar to that of Theorem 2.1.2. However, this time our recurrence relation isn’t as nice to work with. This leads to the conclusion that other measures may need to be taken in order to find a closed form for \( \text{PC}_2(C_n) \).

**Theorem 2.1.3.** If \( C_n \) is a cycle on \( n \) vertices and \( f_n = \text{PC}_2(P_n) \), then
\[
\text{PC}_2(C_n) = f_{n-1} + 4 \left\lfloor \frac{n}{2} \right\rfloor + 2 \sum_{i=0}^{n-3} (n - 2 - i) f_i.
\]

**Proof.** We’ll take cases based on the parity of \( n \). For convenience, we label the vertices of the cycle \( C_n : v_1, v_2, ..., v_n \) where \( v_1 \sim v_2, v_2 \sim v_3, ..., v_n \sim v_1 \).
Let's first assume $n$ is even and take cases on the color of $v_1$. If $v_1$ is $c$-colored, then we obtain with a total of $f_{n-1}$ partial two-colorings. If $v_1$ is $a$-colored, then there is either none, one, or at least two $c$-colored vertices on $C_n$. If there are no $c$-colored vertices, then we easily obtain a total of one partial two-coloring, formed by alternating between the colors $a$ and $b$. If there is exactly one $c$-colored vertex, then it must be one of $v_2, v_3, ..., v_n$. Regardless of the choice, we partition the vertices of $C_n$ into a $c$-colored vertex and a path of length $n - 1$ that alternates between $a$- and $b$-colored vertices. Hence, there are $n - 1$ total such partial two-colorings. If we account for the case where $v_1$ is $b$-colored, then we double our total to obtain $2 + f_{n-1} + 2(n - 1)$ partial two-colorings.

We still need to account for the case where $v_1$ is $a$ or $b$-colored and there are at least two $c$-colored vertices on $C_n$. So, we first fix $v_2$ to be $c$-colored, and then take cases on the smallest $i$ where $3 \leq i \leq n$ for which $v_i$ is also $c$-colored. If $v_3$ is the "smallest" $c$-colored vertex, then we obtain $f_0$ such colorings. If $v_4$ is the smallest, we obtain a total of $f_1$ such colorings. Continuing this process, we can eventually see that when $v_n$ is the smallest, we obtain a total of $f_{n-3}$ such colorings, giving a total of $f_{n-3} + f_{n-4} + \cdots + f_1 + f_0$ partial two-colorings.

Next, we fix $v_3$ to be $c$-colored and take cases on the smallest $i$ where $4 \leq i \leq n$ for which $v_i$ is $c$-colored. By applying the same process as in the preceding paragraph, we obtain a total of $f_{n-4} + f_{n-5} + \cdots + f_1 + f_0$ partial two-colorings.

Continuing onwards, we proceed in a similar manner until we reach the final case when $v_{n-1}$ and $v_n$ are the only two $c$-colored vertices on $C_n$. This yields a total of $f_0 = 1$ partial two-coloring. Of course, we fixed $v_1$ to be $a$-colored, hence to account for the case where $v_1$ is $b$-colored, we double our total sum. Accounting for every possible case, we can see that when $n$ is even:

$$pc_2(C_n) = f_{n-1} + 2 + 2(n - 1) + 2 \sum_{i=0}^{n-3} (n - 2 - i)f_i = f_{n-1} + 2n + 2 \sum_{i=0}^{n-3} (n - 2 - i)f_i.$$
The argument for when \( n \) is odd is essentially identical, except we have to realize there is no way to proper two-color an odd cycle. Thus, there must be at least one \( c \)-colored vertex in a partial two-coloring of an odd cycle. Therefore, when \( n \) is odd:

\[
\text{pc}_2(C_n) = f_{n-1} + 2(n - 1) + 2 \sum_{i=0}^{n-3} (n - 2 - i)f_i.
\]

Since

\[
4 \left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} 
2n & \text{if } n \text{ is even} \\
2(n - 1) & \text{if } n \text{ is odd}
\end{cases},
\]

we write the general result for any positive integer \( n \):

\[
\text{pc}_2(C_n) = f_{n-1} + 4 \left\lfloor \frac{n}{2} \right\rfloor + 2 \sum_{i=0}^{n-3} (n - 2 - i)f_i.
\]

□

It would be nice to derive a closed form for \( \text{pc}_2(C_n) \). It turns out we can do this by establishing a relationship between the number of independent sets of a particular type of graph called a prism graph and the value \( \text{pc}_2(C_n) \). We claim these two values are the same and will use this fact to derive an exact result.

**Definition.** The *prism graph*, denoted \( Y_n \), is defined as the Cartesian product: \( K_2 \Box C_n \). We may think of constructing \( Y_n \) by taking two isomorphic copies of \( C_n \) and placing edges between vertices that correspond under the isomorphism.

To help describe the correspondence, we consider some partial two-coloring of \( C_n \). We can associate the \( a \)-colored vertices of our given \( C_n \) with the vertices of the independent set in one of the two cycles of the prism graph and associate the \( b \)-colored vertices with those in the independent set in the other cycle. Any vertices not chosen from either cycle could then
in turn correspond to \( c \)-colored vertices.

In fact, we can prove a stronger result in which the correspondence we are looking for easily follows. Theorem 2.1.4 shows that partial two-colorings of any simple graph \( G \) correspond to the independent sets in \( K_2 \square G \).

**Theorem 2.1.4.** Independent sets in \( K_2 \square G \) are in 1-1 correspondence with partial two-colorings of \( G \).

**Proof.** It suffices to find a bijection \( f : I(K_2 \square G) \rightarrow \text{PC}_2(G) \). Let \( V(G) = \{v_1, v_2, \ldots, v_n\} \) and \( V(K_2) = \{0, 1\} \) so that \( V(G \square K_2) = \{(v_i, x) : i = 1, 2, \ldots, n; x = 0, 1\} \).

Define \( f : I(K_2 \square G) \rightarrow \text{PC}_2(G) \) so that if \( I \in I(G) \), we let \( f(I) = \phi \) where \( \phi \in \text{PC}_2(G) \) via:

\[
\phi(v_i) = \begin{cases} 
  a & \text{if } (v_i, 0) \in I \\
  b & \text{if } (v_i, 1) \in I \\
  c & \text{otherwise}
\end{cases}
\]

Note that this function is well defined since for any fixed vertex \( v_i \in V(G) \), we know \((v_i, 0)\) and \((v_i, 1)\) cannot be both in \( I \) since they are adjacent.

It is not hard to see that this function is invertible. We can define \( f^{-1}(\phi) = I \) where
There are two corollaries that follow immediately from Theorem 2.1.4. Both will contribute to the derivation of the closed form we are looking for. Before we can state them, we need to define a special type of graph called a ladder graph.

**Definition.** The ladder graph, denoted $L_n$, is the graph obtained via the cartesian product $K_2 \square P_n$. The visual depiction of the ladder graph $L_7$ is shown in the figure below.

![Figure 2.2: The ladder graph $L_7$.](image)

**Corollary 2.1.5.** $i(L_n) = pc_2(P_n)$ for all $n \geq 0$.

*Proof.* This follows exactly from Theorem 2.1.4 where $G = P_n$. 

Recall from Theorem 2.1.2 that if $f_n = pc_2(P_n)$ then $f_n$ satisfies the recursion $f_n = 2f_{n-1} + f_{n-2}$. This same recursion now holds true for independent sets in $L_n$. So, for the remainder of this section, we would like to think of $f_n$ as the number of independent sets in $L_n$, that is, $i(L_n) = f_n$.

**Corollary 2.1.6.** $i(Y_n) = pc_2(C_n)$ for all $n \geq 0$.

*Proof.* This follows exactly from Theorem 2.1.4 where $G = C_n$. 

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If we can find a closed form for \( i(Y_n) \) then we would automatically know a closed form for \( p_{c_2}(C_n) \). To do this, we will first cook up a recursion for \( i(Y_n) \) and then solve the recursion using generating functions. Deriving the recursion is not the most straightforward thing to do, so we proceed slowly by first stating various relationships between independent sets in certain subgraphs of \( Y_n \).

The following 3 subgraphs of \( Y_n \) will be of great importance:

**Definition.** The graph \( S^1_n \) is obtained by removing two non-adjacent corner vertices belonging to the same "path" in \( K_2 \Box P_{n+2} \). A visual depiction of \( S^1_n \) is shown in the figure below. We refer to the number of independent sets of \( S^1_n \) as \( k_n \). That is, \( i(S^1_n) = k_n \).

![Figure 2.3: The graph \( S^1_n \).](image1)

**Definition.** The graph \( S^2_n \) is obtained by removing one of the corner vertices of the ladder graph \( K_2 \Box P_n \). We refer to the number of independent sets of \( S^2_n \) as \( g_n \). That is, \( i(S^2_n) = g_n \).

![Figure 2.4: The graph \( S^2_n \).](image2)

**Definition.** The graph \( S^3_n \) is obtained by removing two non-adjacent corner vertices of \( K_2 \Box P_{n+2} \). We refer to the number of independent sets of \( S^3_n \) as \( z_n \). That is, \( i(S^3_n) = z_n \).

Also, we refer to the number of independent sets in \( Y_n \) as \( h_n \). That is, \( i(Y_n) = h_n \).
Figure 2.5: The graph $S^3_1$.

**Lemma 2.1.7.** For $n \geq 3$, $h_n = f_{n-1} + 2k_{n-3}$.

*Proof.* The left-hand side counts the number of independent sets in the prism graph $Y_n$. The right-hand side does the same, but counts independent sets based on which ones contain particular vertices of $Y_n$. Let $v_1$ and $v_2$ be two adjacent vertices belonging to separate "cycles" in the prism graph. Clearly, $v_1$ and $v_2$ cannot both belong to an independent set since they are adjacent. If both $v_1$ and $v_2$ are not contained in the independent set, then we are really just looking at independent sets of $L_{n-1}$. Hence, $f_{n-1}$ counts the number of independent sets not containing $v_1$ and $v_2$.

If $v_1$ is in an independent set and $v_2$ is not, then the three vertices adjacent to $v_1$ cannot be in the independent set. If we don't account for these vertices, we are looking at independent sets of the graph $S_{n-3}^1$. By symmetry, we can see that the number of independent sets containing either $v_1$ or $v_2$ is given by $2k_{n-3}$. □

**Lemma 2.1.8.** For $n \geq 1$, $k_n = g_n + z_{n-1}$.

*Proof.* The left-hand side counts the number of independent sets of the graph $S_n^1$. The right hand side does the same, but takes cases on which ones contain a particular vertex. Let $v_1$ be one of the degree one vertices of $S_n^1$. Note that $g_n$ counts the number of independent sets not containing $v_1$ and $z_{n-1}$ counts the number of independent sets containing $v_1$. □

**Lemma 2.1.9.** For $n \geq 2$, $z_{n-1} = g_{n-1} + k_{n-2}$.

*Proof.* The left-hand side counts the number of independent sets of $S_{n-1}^3$. That is, it counts the number of independent sets of the graph obtained by removing two non-adjacent corner
vertices of $K_2 \Box P_{n+1}$. Let $v_1$ be one of the degree 1 vertices of this graph. Note $g_{n-1}$ counts the number of independent sets not containing $v_1$ and $k_{n-2}$ counts the independent sets containing $v_1$.

**Lemma 2.1.10.** For $n \geq 2$, $g_n = k_{n-1} + k_{n-2}$.

**Proof.** The left-hand side counts the number of independent sets in the graph $S^2_1$. The right-hand side does the same but takes cases on which ones contain a particular vertex. Let $v_1$ be one of the "corner" vertices of the graph $S^2_1$. Note $k_{n-1}$ counts the number of independent sets of $S^2_1$ not containing $v_1$ and $k_{n-2}$ counts the number of independent sets containing $v_1$.

**Lemma 2.1.11.** If $h_n$ denotes the number of independent sets in the prism graph $Y_n$ then $h_n = h_{n-1} + 3h_{n-2} + h_{n-3}$.

**Proof.** Since we know that $f_n$ satisfies the recurrence $f_n = 2f_{n-1} + f_{n-2}$, it follows via induction that:

$$f_{n-1} = 2f_{n-2} + f_{n-3}$$

$$= f_{n-2} + f_{n-3} + f_{n-2}$$

$$= f_{n-2} + f_{n-3} + [2f_{n-3} + f_{n-4}]$$

$$= f_{n-2} + 3f_{n-3} + f_{n-4}.$$  

Hence,

$$f_{n-1} = f_{n-2} + 3f_{n-3} + f_{n-4}. \tag{2.1}$$

By Lemmas 2.1.8 and 2.1.9 we know that $k_n = g_n + z_{n-1}$ and $z_{n-1} = g_{n-1} + k_{n-2}$. Substituting the latter into the former gives us

$$k_n = g_n + g_{n-1} + k_{n-2}.$$
By Lemma 2.1.10 we know $g_n = k_{n-1} + k_{n-2}$. Hence it follows that

$$k_n = g_n + g_{n-1} + k_{n-2} = [k_{n-1} + k_{n-2}] + [k_{n-2} + k_{n-3}] + k_{n-2} = k_{n-1} + 3k_{n-2} + k_{n-3}. \quad (2.2)$$

Using (2.1) and (2.2), we obtain the desired result.

$$h_n = f_{n-1} + 2k_{n-3}$$

$$= [f_{n-2} + 3f_{n-3} + f_{n-4}] + 2[k_{n-4} + 3k_{n-5} + k_{n-6}]$$

$$= [f_{n-2} + 2k_{n-4}] + 3[f_{n-3} + 2k_{n-5}] + [f_{n-4} + 2k_{n-6}]$$

$$= h_{n-1} + 3h_{n-2} + h_{n-3}.$$  

Note the last step follows from Lemma 2.1.7, that is, $h_n = f_{n-1} + 2k_{n-3}$.

\[\square\]

**Theorem 2.1.12.** If $C_n$ is a cycle on $n$ vertices, then

$$pc_2(C_n) = (1 + \sqrt{2})^n + (1 - \sqrt{2})^n + (-1)^n.$$  

**Proof.** By Lemma 2.1.11, it suffices to solve the recurrence $h_n = h_{n-1} + 3h_{n-2} + h_{n-3}$. For convenience, we will rewrite this as $h_{n+3} = h_{n+2} + 3h_{n+1} + h_n$. Let $h(x) = \sum_{n\geq0} h_n x^n$. We multiply both sides of $h_{n+3} = h_{n+2} + 3h_{n+1} + h_n$ by $x^{n+3}$, and sum over all nonnegative integers $n$ to obtain

$$\sum_{n\geq0} h_{n+3} x^{n+3} = \sum_{n\geq0} h_{n+2} x^{n+2} + 3 \sum_{n\geq0} h_{n+1} x^{n+1} + \sum_{n\geq0} h_n x^n,$$

which is equivalent to

$$h(x) - 7x^2 - x - 3 = x[h(x) - x - 3] + 3x^2[h(x) - 3] + x^3h(x).$$

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When solving for $h(x)$, we obtain

$$h(x) = \frac{-3x^2 - 2x + 3}{1 - x - 3x^2 - x^3} = \frac{-3x^2 - 2x + 3}{(1 + x)(1 - 2x - x^2)}.$$ 

As we saw in Theorem 2.1.2, the polynomial $1 - 2x - x^2$ has roots $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$.

Using partial fraction decomposition we can rewrite $h(x)$ as

$$h(x) = \frac{1}{1 - (-x)} - \frac{\alpha}{-\alpha - x} + \frac{\beta}{\beta + x}.$$ 

Taking advantage of the fact $\alpha \cdot \beta = -1$ allows to rewrite $h(x)$ again as

$$h(x) = \frac{1}{1 - (-x)} + \frac{1}{1 - \beta x} + \frac{1}{1 - \alpha x}$$

$$= \sum_{n=0}^{\infty} (-1)^n x^n + \sum_{n=0}^{\infty} (\alpha x)^n + \sum_{n=0}^{\infty} (\beta x)^n$$

$$= \sum_{n=0}^{\infty} [\alpha^n + \beta^n + (-1)^n] x^n.$$ 

This tells us the coefficient of the $x^n$ term of $f(x)$ is $\alpha^n + \beta^n + (-1)^n$ and therefore,

$$pc_2(C_n) = (1 + \sqrt{2})^n + (1 - \sqrt{2})^n + (-1)^n.$$ 

Table 2.1 displays $pc_2(K_n)$, $pc_2(P_n)$, and $pc_2(C_n)$ for small values of $n$. Note that values for $pc_2(C_1)$ and $pc_2(C_2)$ in the table above can be interpreted as partial two-colorings of the graph consisting of a single looped vertex and the graph consisting of two vertices sharing two distinct edges.
<table>
<thead>
<tr>
<th>$n$</th>
<th>$pc_2(K_n)$</th>
<th>$pc_2(P_n)$</th>
<th>$pc_2(C_n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>13</td>
<td>17</td>
<td>13</td>
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<td>21</td>
<td>41</td>
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<td>31</td>
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</tr>
<tr>
<td>9</td>
<td>91</td>
<td>3363</td>
<td>2785</td>
</tr>
<tr>
<td>10</td>
<td>111</td>
<td>8119</td>
<td>6727</td>
</tr>
</tbody>
</table>

Table 2.1: The number of partial two-colorings of small $K_n$, $P_n$, and $C_n$.

### 2.2 Enumerating $pc_k(P_n)$

Using similar methods to that of the proof of Theorem 2.1.2, it’s possible to obtain a recursive formula for $pc_k(P_n)$. This general result will depend on a quick application of chromatic polynomials.

**Definition.** The *chromatic polynomial* of a graph $G$ is a polynomial that counts the number of proper $k$-colorings of $G$ when evaluated at $k$. We denote this polynomial as $\chi(G; k)$.

The next lemma serves as an example and an important tool that will be utilized in the proof of the main result. It is a standard example of the chromatic polynomial, but we include it’s proof for completeness.

**Lemma 2.2.1.** If $T$ is a tree on $n$ vertices, then $\chi(T; k) = k(k - 1)^{n-1}$.

**Proof.** Choose some vertex $v$ of $T$ as a root. Note we can color $v$ in $k$ ways and vertices in $N(v)$ a total of $k - 1$ ways. Neighbors of vertices in the set $N(v) - v$ can be colored a total of $k - 1$ ways since none are adjacent to $v$. We can proceed in an inductive manner and easily show that all other vertices can be colored in $k - 1$ possible ways. Hence, $\chi(T; k) = k(k - 1)^{n-1}$. \qed
To obtain a recursive formula for $pc_k(P_n)$, we need to think about the graph $H_k$ and the one self-adjacent vertex $\ell$. Let's refer to the vertices of a graph that are mapped to this vertex under some homomorphism as $\ell$-colored.

**Theorem 2.2.2.** If $P_n$ is a path on $n$ vertices and $f_n$ denotes the number of partial $k$-colorings of $P_n$, then

$$pc_k(P_n) = k(k - 1)^{n-1} + f_{n-1} + \sum_{i=0}^{n-2} k(k - 1)^i f_{n-2-i}.$$ 

**Proof.** Define $V(P_n) = \{v_1, v_2, ..., v_n\}$. We proceed by taking cases on the smallest value of $i$ where $1 \leq i \leq n$ for which $v_i$ is $\ell$-colored.

First note that if there are no vertices that are $\ell$-colored, then our partial $k$-coloring is actually a proper $k$-coloring of $P_n$. By Lemma 2.2.1, the total number of colorings we would obtain is $k(k - 1)^{n-1}$.

If $v_1$ is the first $\ell$-colored vertex in our path, then we just need to partially $k$-color the other $n - 1$ vertices in the path, which can be done in $f_{n-1}$ ways. If $v_2$ is first, then we need to properly $k$-color the first vertex and partially $k$-color the other $n - 2$. Hence, the total number of resulting colorings would be $k(k - 1)^{1-1} \cdot f_{n-2} = kf_{n-2}$. It is not hard to see that we can continue this procedure for all the other vertices in the path. That is, we find the total number of partial two-colorings for each case by properly $k$-coloring vertices before the $v_i$ in question and partially $k$-color the vertices after it. Summing up all of these other cases helps derive the sum in the result. Hence,

$$pc_k(P_n) = k(k - 1)^{n-1} + f_{n-1} + \sum_{i=0}^{n-2} k(k - 1)^i f_{n-2-i}.$$ 

$\square$
Chapter 3

Partial Two-Colorings of Forests and Trees

3.1 An Observation of Double Stars

For this part of the thesis, we show that the lex graph maximizes $pc_2(G)$ where $G$ is either a forest or a tree. This is quite easy to see for a graph $G$ where $|V(G)| = n$ and $|E(G)| = 0$ or $|E(G)| = 1$ since there is only one graph to consider for both cases. These graphs also happen to be lex.

Once $|E(G)| \geq 2$, enumeration yields multiple graphs to consider as candidates for this extremal problem. In fact, as $|E(G)|$ increases, finding such a graph via enumeration becomes more and more tedious. To help pave the way for a more efficient technique, we'll first focus on a special type of graph known as a double star.

**Definition.** The *double star graph*, denoted $S_{r,s}$, is the graph with a vertex of degree $r$ adjacent to a vertex of degree $s$, and all other vertices of degree one.

Let's consider the double star $S_{k+1,n-k-1}$ where $x$ and $y$ represent the two non degree-one
vertices. Without loss of generality, assume $d(x) \geq d(y)$. Note here if $|N(x) \setminus \{y\}| = k$ then $|N(y) \setminus \{x\}| = n - k - 2$ and $k \geq n - k - 2$. For the following result, we can show that we can increase the number of partial two-colorings by removing one of the $n - k - 2$ "hanging" edges incident with vertex $y$ and pairing it up with the $k$ "hanging" edges incident with vertex $x$. For example, this action transforms $S_{k+1,n-k-1}$ into $S_{k+2,n-k-2}$.

**Lemma 3.1.1.** $p_{c2}(S_{k+1,n-k-1}) < p_{c2}(S_{k+2,n-k-2})$.

**Proof.** A general counting argument shows that

$$p_{c2}(S_{k+2,n-k-2}) = 2^{n-1} + 2^{k+2}3^{n-k-3} + 2^{n-k-2}3^{k+1} + 3^{n-2},$$

and

$$p_{c2}(S_{k+1,n-k-1}) = 2^{n-1} + 2^{k+1}3^{n-k-2} + 2^{n-k-1}3^k + 3^{n-2}.$$
So,

\[
\begin{align*}
\text{pc}_2(S_{k+2,n-k-2}) - \text{pc}_2(S_{k+1,n-k-1}) &= 2^{k+2}3^{n-k-3} + 2^{n-k-2}3^{k+1} - 2^{k+1}3^{n-k-2} - 2^{n-k-1}3^k \\
&= 2^{n-k-2}3^{n-k-3} \\
&\quad \cdot (2^{2k-n+4} + 3^{2k-n+4} - 2^{2k-n+3} \cdot 3 - 2 \cdot 3^{2k-n+3}) \\
&= 2^{n-k-2}3^{n-k-3} \\
&\quad \cdot \left(2^{2k-n+4} - \frac{3}{2} \cdot 2^{2k-n+4} + 3^{2k-n+4} - \frac{2}{3} \cdot 3^{2k-n+4}\right) \\
&= 2^{n-k-2}3^{n-k-3}(3^{2k-n+3} - 2^{2k-n+3}).
\end{align*}
\]

Note the assumption \( k \geq n - k - 2 \) is equivalent to \( 2k - n + 2 \geq 0 \), so it follows \( 2k - n + 3 > 0 \). This tells us the expression \((3^{2k-n+3} - 2^{2k-n+3})\) found in the last line of the equality is positive which subsequently implies \( 2^{n-k-2}3^{n-k-3}(3^{2k-n+3} - 2^{2k-n+3}) > 0 \). The result follows. \( \square \)

The motivation behind the following lemma is to gain some insight on how we can go about increasing the number of partial two-colorings in general for any tree. One important observation to note, is that if we apply Lemma 3.1.1 a total of \( n - k - 2 \) times to our double star, we would produce \( K_{1,n-1} \). This is since all of the original \( n - k - 1 \) edges that were incident with vertex \( y \) would be removed one at a time by use of the lemma, leaving us with a graph where \( d(y) = 1 \) and \( d(x) = (k + 1) + (n - k - 2) = n - 1 \). This describes \( K_{1,n-1} \) precisely.

The next section views this edge-swapping idea in a more general sense.
3.2 Compressing Trees

So far we have observed that moving around the edges in trees that are double stars has an effect on the number of partial two-colorings. In this section, we explore this edge-swapping technique in a more general context to help maximize $pc_2(T)$ for any tree $T$ on $n$ vertices.

Consider an edge $xy \in E(T)$ where $T$ is any tree. It is not hard to see that this choice defines a natural partition of the set $V(T \setminus \{x, y\})$ into the following four components.

We write:

$$A = \{v \in (T \setminus \{x, y\}) : v \sim x, v \sim y\}$$

and

$$B = \{v \in (T \setminus \{x, y\}) : v \sim x, v \sim y\}.$$

We can view the set $A$ as the set of vertices of $V(T \setminus \{x, y\})$ adjacent to $x$ and the set $B$ as the set of vertices of $V(T \setminus \{x, y\})$ adjacent to $y$. Also, we define $T_A$ as the component of $T - x - N(x)$ not containing vertex $y$. Similarly, we define $T_B$ as the component of $T - y - N(y)$ not containing vertex $x$.

**Definition.** The *compression of $T$ from $x$ to $y$*, denoted $T_{x \rightarrow y}$, is the graph obtained from $T$ by deleting all edges between $x$ and $A$ and adding all edges from $y$ to $A$.

Note that if $T$ is a tree, then $T_{x \rightarrow y}$ is also a tree. Figure 3.2 gives us a visual rendering of how $T_{x \rightarrow y}$ acts on the components of $T$. For example, if $T = S_{k+1,n-k-1}$ where $x$ and $y$ are the vertices described earlier, then $T_{x \rightarrow y} = K_{1,n-1}$.

The following lemma shows that compressions can be applied to any edge of a tree to produce a new tree which has at least as many partial two-colorings of the original tree.

**Lemma 3.2.1.** If $T$ is a tree with $x \sim y$, then $pc_2(T_{x \rightarrow y}) \geq pc_2(T)$. 

Proof. It suffices to find an injection \( \phi \mapsto \phi' \) from \( PC_2(T) \setminus PC_2(T_{x \rightarrow y}) \) to \( PC_2(T_{x \rightarrow y}) \setminus PC_2(T) \). Suppose that \( \phi \in PC_2(T) \setminus PC_2(T_{x \rightarrow y}) \). Since the only edges that are in \( PC_2(T_{x \rightarrow y}) \) but not \( PC_2(T) \) are between \( y \) and \( A \), it must be the case that there exists a \( z \in A \) such that either \( \phi(y) = a \) and \( \phi(z) = a \), or \( \phi(y) = b \) and \( \phi(z) = b \). Without loss of generality assume the former case holds. It follows that \( \phi(x) = b \) or \( \phi(x) = c \).

If \( \phi(x) = b \) then we define the mapping \( \phi' \) as follows:

\[
\phi'(\mu) = \begin{cases} 
\phi(x) & \text{if } \mu = y \\
\phi(y) & \text{if } \mu = x \\
\phi(\mu) & \text{if } \phi(\mu) = c, \mu \in A, \mu \in T_A \\
a & \text{if } \mu \in T_B \cup B \text{ and } \phi(\mu) = b \\
b & \text{if } \mu \in T_B \text{ or } \mu \in B \text{ and } \phi(\mu) = a.
\end{cases}
\]

This mapping swaps the colors of the vertices \( x \) and \( y \) and adjusts the colors of all other vertices in the other sets accordingly. For example, when \( \phi(x) = b \) and \( \phi(y) = a \), compressing would create edges between \( a \)-colored vertices and other potential \( a \)-colored vertices in \( A \). This is not allowed in a partial two-coloring. We avoid such an issue via color-swapping.
However, we need to be careful about the colors of vertices in $B$ and in $T_B$, since the color of $y$ has been changed. By switching $b$-colored and $a$-colored vertices to $a$- and $b$-colored (respectively) in $B$ and in $T_B$, we account for any "bad" adjacency issues.

On the other hand, if $\phi(x) = c$ then we define the mapping $\phi'$ as follows:

$$\phi'(\mu) = \begin{cases} 
\phi(\mu) & \text{if } \mu \neq x, y \\
\phi(x) & \text{if } \mu = y \\
\phi(y) & \text{if } \mu = x 
\end{cases}$$

This mapping is a bit easier to understand since the vertex $x$ in this case is $c$-colored.

When we compress, the only new edges we gain are edges joining vertices in $A$ to $y$. Since $y$ becomes $c$-colored under the mapping, we don’t encounter any adjacency issues.

Based on the way $\phi'$ is defined above, it is quite easy to verify that $\phi' \in \text{PC}_2(T_{x \rightarrow y}) \setminus \text{PC}_2(T)$. Since $\phi'(x) = a = \phi'(z)$ in both of the previous cases, the statement must hold. Otherwise, $\phi(x) = a = \phi(z)$ which would violate the definition of a partial two-coloring.

Next, we verify that $\phi \mapsto \phi'$ is an injection. If $\phi'(y) = c$, then $\phi(x) = c, \phi(y) = \phi'(x)$, and $\phi = \phi'$ everywhere else.

If $\phi'(y) = b$, then $\phi(x) = \phi'(y), \phi(y) = \phi'(x)$, and $\phi = \phi'$ for all vertices in both $A$ and $T_A$. In addition, if $v$ is a vertex in $B$ or $T_B$ and $\phi'(v) = b$, then $\phi(v) = a$. On the other hand, if $\phi'(v) = a$, then $\phi(v) = b$.

This assures us that any $\phi' \in \text{PC}_2(T_{x \rightarrow y}) \setminus \text{PC}_2(T)$ can be traced back to a unique pre-image $\phi \in \text{PC}_2(T) \setminus \text{PC}_2(T_{x \rightarrow y})$. Hence, $\phi \mapsto \phi'$ is an injection and the result follows.

We now have enough under our belt to prove the main result of this section.
Theorem 3.2.2. If $T$ is a tree on $n$ vertices, then

$$pc_2(T) \leq pc_2(K_{1,n-1}) = 3^{n-1} + 2^n.$$  

Proof. Let $T$ be a tree with $n$ vertices and $y \in V(T)$. Arrange $T$ in such a way so that it resembles a rooted tree with $y$ acting as the single vertex in the topmost row. We apply the compression $T_{x \rightarrow y}$ for each vertex $x \in N(y)$. This process reduces the degree of each vertex $x \in N(y)$ to one (since each $x$ and $y$ are still adjacent after the compression). In addition, this process forms adjacencies between $y$ and $N(x)$ for each $x \in N(y)$. We repeat this procedure on $y$ until all the neighbors of $y$ are leaves. The graph we are left with when this process is done is precisely $K_{1,n-1}$. By Lemma 3.2.1 we know applying compressions to $T$ cannot possibly reduce the number of partial two-colorings. Thus, it follows that $pc_2(T) \leq pc_2(K_{1,n-1})$.

We verify $pc_2(K_{1,n-1}) = 3^{n-1} + 2^n$ by taking cases on the color of the vertex of degree $n - 1$. Let's call this vertex $v$. If $v$ is $c$-colored then we have 3 choices for the remaining $n - 1$ vertices, yielding a total of $3^{n-1}$ partial two-colorings. If $v$ is $a$- or $b$-colored, then for both cases there are two possible choices for the remaining $n - 1$ vertices, yielding a total of $2 \cdot 2^{n-1} = 2^n$. Summing all the totals yields the equality $pc_2(K_{1,n-1}) = 3^{n-1} + 2^n$. 

3.3 An Exact Result for Forests

We now explore the case where $G$ is a graph with $n$ vertices, $m$ edges, and $m \leq n - 1$. Note this could imply that $G$ has more than one component. For this section, we only consider the case where $G$ is a forest (the case where each component of $G$ is a tree). The main result for this section can be thought of as a generalization of Theorem 3.2.2. We claim the lex graph maximizes $pc_2(G)$. To prove this, we need to make use of the following two lemmas.

Lemma 3.3.1. Let $G$ and $H$ be two graphs. If $G \cup H$ denotes the disjoint union of $G$ and
Let \( L(n_1, m_1) \) and \( L(n_2, m_2) \) be lex graphs where \( m_1 + 1 = n_1 \) and \( m_2 + 1 = n_2 \). If \( n_1 + n_2 = n \), \( m_1 + m_2 = m \), and \( m + 1 < n \) then \( \text{pc}_2(L(n_1, m_1) \cup L(n_2, m_2)) < \text{pc}_2(L(n, m)) \).

Proof. Note \( \text{pc}_2(L(n, m)) \) consists of two main components: an isolate vertex and \( K_{m+1,m} \). There are 3 ways to color the isolate and \( 2^{m+1} + 3^m \) ways to color the star (here we are using Theorem 3.2.2). We calculate \( \text{pc}_2(L(n_1, m_1) \cup L(n_2, m_2)) \) in a very similar manner. Therefore, by Lemma 3.3.1 we have:

\[
\text{pc}_2(L(n, m)) = 3 \cdot (2^{m+1} + 3^m)
\]

and

\[
\text{pc}_2(L(n_1, m_1) \cup L(n_2, m_2)) = (2^{m_1+1} + 3^{m_1})(2^{m_2+1} + 3^{m_2}).
\]
So, since

\[ \text{pc}_2(L(n, m)) - \text{pc}_2(L(n_1, m_1) \cup L(n_2, m_2)) = 3 \cdot (2^{m_1+m_2+1} + 3^{m_1+m_2}) - (2^{m_1+1} + 3^{m_1})(2^{m_2+1} + 3^{m_2}) \]

\[ = \frac{3}{2} \cdot 2^{m_1+m_2+2} - 2^{m_1+m_2+2} + 3^{m_1+m_2+1} - 3^{m_1+m_2} - 2^{m_1+1} \cdot 3^{m_2} - 2^{m_2+1} \cdot 3^{m_1} \]

\[ = 2^{m_1+m_2+1} + 2 \cdot 3^{m_1+m_2} - 2^{m_1+1} \cdot 3^{m_2} - 2^{m_2+1} \cdot 3^{m_1} \]

\[ = 2 \cdot 3^{m_1}(3^{m_2} - 2^{m_2}) - 2^{m_1+1}(3^{m_2} - 2^{m_2}) \]

\[ = 2(3^{m_2} - 2^{m_2})(3^{m_1} - 2^{m_1}) > 0, \]

the result follows.

\[ \square \]

We have just shown that a tiny forest consisting of two lex graphs can essentially be replaced with one "larger" lex graph that has more partial two-colorings. This replacement idea is what will be utilized to maximize the number of partial two-colorings and help yield the general result. Note that the graph we obtain for \( L(n_1, m_1) \cup L(n_2, m_2) \) in Lemma 3.3.2

![Figure 3.3: A visual rendering of Lemma 3.3.2.](image)
is precisely \( L(m + 1, m) \cup K_1 \). In fact, any time we replace two lex-components in the forest, we are left with exactly one isolate vertex. This is since \( (m_1 + 1) + (m_2 + 1) = (n_1 + n_2) \) is equivalent to \( m + 2 = n \). In other words, the star in \( L(n, m) \) uses up all the edges and forces one vertex to be an isolate.

If we were to keep replacing pairs of lex components in our forest, what will the graph look like once we can’t go any further? It’s clear that the structure of our graph will be one giant star and a collection of isolates. To see how many isolates we are left with, all we need to do is find the difference between the total number vertices in the graph, and the total number of vertices in the big star. This is clearly \( n - (m + 1) = n - m - 1 \).

We can now state and prove the main result.

**Theorem 3.3.3.** Let \( G \) be a graph with \( n \) vertices and \( m \) edges where \( m \leq n - 1 \). If every component of \( G \) is a tree, then the graph that maximizes \( pc_2(G) \) is \( L(n, m) \). Furthermore, \( pc_2(L(n, m)) = pc_2(K_{1,m} \cup E_{n-m-1}) = 3^{n-1} + 2^{m+1} \cdot 3^{n-m-1} \).

**Proof.** If \( m = n - 1 \), then \( G \) is a tree. So, by Theorem 3.2.2, we know the graph that maximizes \( pc_2(G) \) is \( K_{1,n-1} = L(n, m) \).

If \( m < n - 1 \) and each component is a tree, then we apply Theorem 3.2.2 to turn each component into a lex graph. By Lemma 3.3.2 we can systematically replace any pair of components with a larger lex graph without decreasing the number of partial two-colorings. That is, we can replace two lex components \( L(n_1, m_1) \) and \( L(n_2, m_2) \) with \( L(n_1 + n_2, m_1 + m_2) \). Repeat this procedure until we are left with exactly one non-isolate component. The resulting graph is precisely \( L(n, m) = K_{1,m} \cup E_{n-m-1} \).

To enumerate \( pc_2(K_{1,m} \cup E_{n-m-1}) \), we note that each of the \( n - m - 1 \) isolates can be colored in 3 ways while \( K_{1,m} \) can be colored \( 3^m + 2^{m+1} \) ways. Therefore, by Lemma 3.3.1 we
have:

\[ \text{pc}_2(L(n, m)) = \text{pc}_2(K_{1,m} \cup E_{n-m-1}) = 3^{n-m-1} \cdot (3^m + 2^{m+1}) \]

\[ = 3^{n-1} + 2^{m+1} \cdot 3^{n-m-1}. \]

\[ \Box \]

3.4 The General Result for all Graphs with \( m \leq n - 1 \)

For this section, we'd like to look at all graphs \( G \) with \( n \) vertices and \( m \) edges, where \( m \leq n - 1 \). Being that we took care of the acyclic cases in the previous section, we focus on developing a result for graphs containing a cycle. Once again, it can be shown the lex graph \( L(n, m) \) maximizes the number of partial two-colorings for these types of graphs.

**Theorem 3.4.1.** Let \( G \) be a graph with \( n \) vertices and \( m \) edges, where \( m \leq n - 1 \). Then, the graph that maximizes \( \text{pc}_2(G) \) is \( L(n, m) \).

To verify this, we will take cases on the sizes of the cycles that exist in \( G \). We begin with the case where \( G \) contains a cycle of size five or larger. This can be done by fixing a cycle \( C_k \) in \( G \) with \( k \geq 5 \) and then deleting all other edges in \( G \) not on this cycle (this includes edges that could possibly exist between non-adjacent vertices on \( C_k \)). This will leave us with a graph \( G' = C_k \cup E_{n-k} \), the disjoint union of a cycle of size \( k \) together with \( n - k \) isolate vertices.

This process of deleting edges from \( G \) to form \( G' \) does not decrease the value \( \text{pc}_2(G) \). In other words, \( \text{pc}_2(G) \leq \text{pc}_2(G') \). Theorem 3.4.2 will show that \( \text{pc}_2(G') \) is strictly smaller that \( \text{pc}_2(L(n, m)) \), establishing the result for this first case.

**Lemma 3.4.2.** Let \( G \) be a graph with \( n \) vertices and \( m \) edges with \( m \leq n - 1 \). If \( G \) contains \( C_k \) as a subgraph with \( k \geq 5 \) then \( \text{pc}_2(G) < \text{pc}_2(L(n, m)) \).
Proof. We fix a cycle $C_k$ in $G$ with $k \geq 5$ and delete all edges of $G$ not on this cycle, leaving us with this graph $G' = C_k \cup E_{n-k}$. Clearly, $pc_2(G') \geq pc_2(G)$, so it suffices to show that $pc_2(L(n,m)) - pc_2(G') > 0$.

Since $G'$ is just the disjoint union of $C_k$ and $n - k$ isolates, it follows by Lemma 3.3.1 that

$$pc_2(G') = pc_2(C_k) \cdot 3^{n-k} = [(1 + \sqrt{2})^k + (1 - \sqrt{2})^k + (-1)^k] \cdot 3^{n-k}.$$ 

Note we are using the formula derived in Theorem 2.1.12 to rewrite $pc_2(C_k)$ in the previous equality. By Theorem 3.3.3, we know

$$pc_2(L(n,m)) = 3^{n-1} + 2^{m+1} \cdot 3^{n-m-1}.$$ 

Now, taking the difference gives us:

$$pc_2(L(n,m)) - pc_2(G') = 3^{n-1} + 2^{m+1} \cdot 3^{n-m-1}$$

$$- [(1 + \sqrt{2})^k + (1 - \sqrt{2})^k + (-1)^k] \cdot 3^{n-k}.$$ 

When $k = 5$, we see that $(1 + \sqrt{2})^5 + (1 - \sqrt{2})^5 + (-1)^5 \cdot 3^{n-5} = 81 \cdot 3^{n-5} = 3^{n-1}$. Making this substitution into the difference allows all terms to cancel out except for $2^{m+1} \cdot 3^{n-m-1}$, which is clearly positive since $n \geq m + 1$ and $m > 0$. Thus, the claim holds for $k = 5$.

To see why the claim holds when $k > 5$, we will need to take advantage of the inequality

$$(1 + \sqrt{2})^k + (1 - \sqrt{2})^k + (-1)^k \leq 2(1 + \sqrt{2})^k + 1.$$ 

With this bound, it’s a bit easier to see which values of $k$ does the lex graph “beat out” $G'$. We will solely depend on the $3^{n-1}$ term from our lex formula to bound this approximation.
That is, we seek the values of $k$ that satisfy the following inequality:

$$3^{n-1} > [2(1 + \sqrt{2})^k + 1] \cdot 3^{n-k}.$$ 

Using some algebra, (as well as taking natural logarithms of both sides) we can easily find the values of $k$ that do the job:

$$3^{n-1} > [2(1 + \sqrt{2})^k + 1] \cdot 3^{n-k}$$

$$3^{k-1} > 2(1 + \sqrt{2})^k + 1$$

$$\frac{3^k}{1 + \sqrt{2}} > 6$$

$$k > \frac{\ln 6}{\ln\left(\frac{3}{1+\sqrt{2}}\right)}.$$

Since $\frac{\ln 6}{\ln\left(\frac{3}{1+\sqrt{2}}\right)} \approx 8.248$, the claim holds for $k \geq 9$. We’ll quickly check the remaining cases $k = 6, k = 7, \text{and } k = 8$ by comparing the $3^{n-1}$ term from the lex formula with $pc_2(C_k) \cdot 3^{n-k}$.

For $k = 6$, we see that $pc_2(C_6) \cdot 3^{n-6} = 199 \cdot 3^{n-6} < 3^5 \cdot 3^{n-6} = 3^{n-1}$. For $k = 7$, we see that $pc_2(C_7) \cdot 3^{n-7} = 472 \cdot 3^{n-7} < 3^6 \cdot 3^{n-7} = 3^{n-1}$. Finally, for $k = 8$, we see that $pc_2(C_8) \cdot 3^{n-8} = 1155 \cdot 3^{n-8} < 3^7 \cdot 3^{n-8} = 3^{n-1}$. So, the claim holds for $k = 6, 7, 8$. This completes the proof. □

We now move onto the case where $G$ does not contain $C_k$ as a subgraph for $k \geq 5$. So, we will be dealing with the cases where $G$ only contains cycles of size three, four, or both. The proof for this case is much different than that of Lemma 3.4.2. This is mainly due to the fact that deleting all the edges off of some fixed cycle of size three or four doesn’t help us in establishing the inequality $pc_2(G) < pc_2(L(n, m))$. Other measures need to be taken.

Given such a graph $G$, we again fix a cycle $C_k$ with $k = 3$ or $k = 4$. This time around, our strategy will involve deleting any other edges incident with the vertices of the fixed cycle, besides the edges forming the cycle, to form a graph $G''$. Once again, we note that
this deletion of edges does not decrease the number of partial two-colorings, and therefore\( \text{pc}_2(G) \leq \text{pc}_2(G''). \) If what remains is the disjoint union of \( C_k \) and some acyclic graph, we can proceed by applying Theorem 3.3.3 to help show the lex graph wins.

It's certainly possible that other cycles of size three or four can be present in \( G'' \). To our luck, it's easy to check that the lex graph wins against these types of graphs. Lemma 3.4.3 provides the proof for this.

**Lemma 3.4.3.** Let \( G \) be a graph with \( n \) vertices and \( m \) edges with \( m \leq n - 1 \). Also, assume \( G \) only contains cycles of size three or four. If the graph \( G'' \) contains a disjoint pair of cycles then \( \text{pc}_2(G) \leq \text{pc}_2(L(n,m)) \).

**Proof.** There are three scenarios for us to check. Either \( G'' \) must contain two disjoint cycles of size four, a cycle of size three disjoint from one of size four, or two disjoint cycles of size three.

If \( G'' \) has two disjoint cycles of size four, we can find an upper bound for \( \text{pc}_2(G'') \) by deleting all other edges in \( G'' \) not on the two four cycles. As before, we note this cannot possibly decrease the total number of partial two-colorings. This will leave us with the graph \( C_4 \cup C_4 \cup E_{n-8} \). By applying Lemma 3.3.1, we can easily see that

\[
\text{pc}_2(G'') \leq \text{pc}_2(C_4 \cup C_4 \cup E_{n-8}) = 35 \cdot 35 \cdot 3^{n-8} < 3^7 \cdot 3^{n-8} = 3^{n-1}.
\]

Since \( \text{pc}_2(L(n,m)) = 3^{n-1} + 2^{m+1} \cdot 3^{n-m-1} \), we immediately can see

\[
\text{pc}_2(G) \leq \text{pc}_2(G'') < \text{pc}_2(L(n,m)).
\]

If \( G'' \) has a cycle of size three and one of size four, we can once again find an upper bound by deleting all the other edges in \( G'' \) not on these two cycles. This leaves us with the graph \( C_4 \cup C_3 \cup E_{n-7} \). Proceeding as before, it's easy to see that
\[ \text{pc}_2(G') \leq \text{pc}_2(C_4 \cup C_3 \cup E_{n-7}) = 35 \cdot 13 \cdot 3^{n-7} < 3^6 \cdot 3^{n-7} = 3^{n-1}. \]

Since \( \text{pc}_2(L(n,m)) = 3^{n-1} + 2^{m+1} \cdot 3^{n-m-1} \), we immediately can see

\[ \text{pc}_2(G) \leq \text{pc}_2(G') < \text{pc}_2(L(n,m)). \]

Lastly, if \( G'' \) has two disjoint cycles of size three, we proceed as before by finding an upper bound on \( \text{pc}_2(G'') \) by deleting all the other edges in \( G'' \) not on these two cycles. This leaves us with the graph \( C_3 \cup C_3 \cup E_{n-6} \). With this in mind, we have

\[ \text{pc}_2(G'') \leq \text{pc}_2(C_3 \cup C_3 \cup E_{n-6}) = 13 \cdot 13 \cdot 3^{n-6} < 3^5 \cdot 3^{n-6} = 3^{n-1}. \]

Since \( \text{pc}_2(L(n,m)) = 3^{n-1} + 2^{m+1} \cdot 3^{n-m-1} \), we immediately can see

\[ \text{pc}_2(G) \leq \text{pc}_2(G'') < \text{pc}_2(L(n,m)). \]

\( \square \)

The only case that needs to be checked is when \( G \) only contains cycles of size three or four and \( G'' \) is the graph consisting of the disjoint union of \( C_k \) \((k = 3, 4)\) together with an acyclic graph on \( n - k \) vertices. We note that the case where \( G \) contains a vertex \( v \) belonging to two or more cycles does not need to be checked separately. This is since when forming \( G'' \), we delete edges on all the other cycles that contain \( v \) besides one fixed cycle of size three or four.

Let's define the value \( m^* \) to represent the number of edges in the acyclic component of \( G'' \). Note since this component is acyclic, by definition it is a forest. As mentioned earlier, we
would like to apply Theorem 3.3.3 in order to achieve the following upper bound on $\text{pc}_2(G)$:

$$\text{pc}_2(G) \leq \text{pc}_2(G'') = \text{pc}_2(C_k) \cdot L(n - k, m^*).$$

This follows from the fact that the lex graph $L(n - k, m^*)$ beats out any other forests on $n - k$ vertices and $m^*$ edges. Unfortunately, the inequality

$$\text{pc}_2(C_k) \cdot L(n - k, m^*) < \text{pc}_2(L(n, m))$$

only holds for $m^* \geq 2$. Lemmas 3.4.4 and 3.4.5 will prove this. Some other machinery will later be needed to check the cases when $m^* = 0$ and $m^* = 1$.

**Lemma 3.4.4.** Let $G$ be a graph with $n$ vertices and $m$ edges with $m \leq n - 1$. If $G$ contains $C_3$ as a subgraph, then $\text{pc}_2(G) < \text{pc}_2(L(n, m))$ provided $m^* \geq 2$.

**Proof.** It suffices to show $\text{pc}_2(L(n, m)) - \text{pc}_2(G) > 0$ when $m^* \geq 2$. Note this inequality immediately follows from Lemma 3.4.2 if $G$ contains any other cycle of size five or larger. If not, then we fix a subgraph $C_3$ of $G$ and delete any edges incident with vertices on this cycle (except for the edges on the cycle) to obtain a new graph $G''$. Once again, since deleting edges does not decrease the number of partial two-colorings, it follows that $\text{pc}_2(G) \leq \text{pc}_2(G'')$.

If $G''$ contains two or more cycles, then the result follows from Lemma 3.4.3. Otherwise, $G''$ is the disjoint union of $C_3$ together an acyclic graph on $n - 3$ vertices and $m^*$ edges. Since this acyclic component is a forest, we can apply Theorem 3.3.3 and Lemma 3.3.1 to achieve the following upper bound on $\text{pc}_2(G)$:

$$\text{pc}_2(G) \leq \text{pc}_2(G'') \leq \text{pc}_2(C_3) \cdot \text{pc}_2(L(n - 3, m^*)).$$

At this point, it suffices to show that $\text{pc}_2(L(n, m)) - \text{pc}_2(C_3) \cdot \text{pc}_2(L(n - 3, m^*)) > 0$. 

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Using the fact $pc_2(C_3) = 13$ and the result of Theorem 3.3.3, we can see that

$$pc_2(C_3) \cdot pc_2(L(n - 3, m^*)) = 13 \cdot (3^{n-4} + 2^{m^*+1} \cdot 3^{n-m^*-4}).$$

Hence,

$$pc_2(L(n, m)) - pc_2(C_3) \cdot pc_2(L(n - 3, m^*)) =
\begin{align*}
&= \left[3^{n-1} + 2^{m+1} \cdot 3^{n-m-1}\right] \\
&\quad - \left[13 \cdot (3^{n-4} + 2^{m^*+1} \cdot 3^{n-m^*-4})\right] \\
&= [27 \cdot 3^{n-4} - 13 \cdot 3^{n-4}] + 2^{m+1} \cdot 3^{n-m-1} \\
&\quad - 13 \cdot 2^{m^*+1} \cdot 3^{n-m^*-4} \\
&= 14 \cdot 3^{n-4} + 2^{m+1} \cdot 3^{n-m-1} \\
&\quad - 13 \cdot 2^{m^*+1} \cdot 3^{n-m^*-4}.
\end{align*}$$

We investigate the cases for $m^* = 2$ and $m^* = 3$ to help describe the general scenario for larger values of $m^*$.

When $m^* = 2$ the sum of the first and last terms in the last line of the equality is

$$14 \cdot 3^{n-4} - 13 \cdot 2^3 \cdot 3^{n-6} = 3^2 \cdot 14 \cdot 3^{n-6} - 13 \cdot 2^3 \cdot 3^{n-6} > 0.$$ 

We observe here the powers of 3 in the first term beat out the powers of two in the second term as well as the fact that $14 > 13$.

When $m^* = 3$ the sum of the first and last terms in the last line of the equality is

$$14 \cdot 3^{n-4} - 13 \cdot 2^4 \cdot 3^{n-7} = 3^3 \cdot 14 \cdot 3^{n-7} - 13 \cdot 2^4 \cdot 3^{n-7} > 0.$$
Once again, the result is due to the fact that the powers of three beat out the powers of two as well as the fact $14 > 13$. In general, as $m^*$ increases, we are comparing terms of the form $14 \cdot 3^j$ and $13 \cdot 2^{j+1}$, where $j$ is a non-negative integer. Using a bit of algebra, we can see for which values of $j$ give $14 \cdot 3^j > 13 \cdot 2^{j+1}$:

\[
14 \cdot 3^j > 13 \cdot 2^{j+1}
\]

\[
\frac{7}{13} > \left(\frac{2}{3}\right)^j
\]

\[
\frac{\ln(7/13)}{\ln(2/3)} < j.
\]

The last line implies $j$ must be greater than (approximately) 1.526. This tells us in general, the lex graph wins once $m^* \geq 2$. □

The argument is essentially the same for the case where $G$ only contains cycles of length four and $m^* \geq 2$. Here, we will use the fact that $pc_2(C_4) = 35$.

**Lemma 3.4.5.** Let $G$ be a graph with $n$ vertices and $m$ edges with $m \leq n - 1$. If $G$ contains $C_4$ as a subgraph, then $pc_2(G) \leq pc_2(L(n, m))$ provided $m^* \geq 2$.

**Proof.** We mimic the first few steps of Lemma 3.4.4. Hence, it suffices to show that $pc_2(L(n, m)) - pc_2(L(n - 4, m^*)) > 0$ when $m^* \geq 2$. Once again, using the formula from Theorem 3.3.3 as well as Lemma 3.3.1 we see that
\[ pc_2(L(n, m)) - pc_2(L(n - 4, m^*) = \]
\[ = [3^{n-1} + 2^{m+1} \cdot 3^{n-m-1}] \]
\[ - [35 \cdot (3^{n-5} + 2^{m^*+1} \cdot 3^{n-m^*-5})] \]
\[ = [81 \cdot 3^{n-5} - 35 \cdot 3^{n-5}] + 2^{m+1} \cdot 3^{n-m-1} \]
\[ - 35 \cdot 2^{m^*+1} \cdot 3^{n-m^*-5} \]
\[ = 46 \cdot 3^{n-5} + 2^{m+1} \cdot 3^{n-m-1} \]
\[ - 35 \cdot 2^{m^*+1} \cdot 3^{n-m^*-5}. \]

When checking for values of \( m^* \) that satisfy the previous equality, we are essentially comparing terms of the form \( 46 \cdot 3^j \) and \( 35 \cdot 2^{j+1} \) where \( j \) is a nonnegative integer. Some algebra can tell us exactly for which values of \( j \) give \( 46 \cdot 3^j > 35 \cdot 2^{j+1} \):

\[
46 \cdot 3^j > 35 \cdot 2^{j+1}
\]
\[
\frac{23}{35} > \left(\frac{2}{3}\right)^j
\]
\[
\frac{\ln(23/35)}{\ln(2/3)} < j.
\]

The last line implies that \( j \) must be greater than (approximately) 1.035. This tells us in general, the lex graph wins once \( m^* \geq 2 \).

\[ \square \]

The rest of this section is devoted to verifying the result for graphs \( G \) that only contain a cycle of size at most four with \( m^* = 0 \) or \( m^* = 1 \). We’d like to provide a visual of what these types of graphs look like. When \( m^* = 0 \), the graph \( G \) looks like the disjoint union of a
cycle with edges incident to the vertices of the cycle together with isolate vertices. We will refer to these types of cycles as “hairy cycles.” Since the cycles in $G$ are only of size three or four, there are really just two types of graphs for us to consider for $m^* = 0$. Examples of these two types of graphs are shown in Figures 3.4 and 3.5.

![Figure 3.4: An example of a graph $G$ containing a hairy $C_3$ with $m^* = 0.$](image)

![Figure 3.5: An example of a graph $G$ containing a hairy $C_4$ with $m^* = 0.$](image)

To find an upper bound on the number of partial two-colorings for these types of graphs, we proceed as before by deleting edges incident with vertices on the cycle. However, we don't want to delete too many. Deleting too many edges makes it difficult for us to make any sort of comparison with the value $pc_2(L(n, m))$.

If we are given a graph $G$ that looks like the one in Figure 3.4, we proceed by deleting all edges incident with the cycle except for possibly two edges incident with a single vertex on the cycle. Let’s call this graph $G^0_3$. Keep note that after deleting edges, we obtain a total of $n - 5$ isolate vertices. This new graph we obtain can provide an obvious upper bound on
It is a straightforward task to check that this upper bound is beat by $L(n, m)$.

If all the vertices on the cycle have degree at most three, then we cannot obtain the graph $G^0_3$ by deleting edges. It must be the case that each vertex on the $C_3$ has degree at most three. This leaves us with four types of graphs to check: a $C_3$ where each vertex is of degree three together with $n - 6$ isolate vertices, a $C_3$ with exactly two vertices of degree three together with $n - 5$ isolate vertices, a $C_3$ with exactly one vertex of degree three together with $n - 4$ isolate vertices, and $C_3 \cup E_{n-3}$. We deal with these cases by making a direct comparison between each graph and its corresponding lex graph.

On the other hand, if $G$ looks like the graph in Figure 3.5, we delete all edges incident with the cycle except for possibly one of them. Call this resulting graph $G^0_4$ (note this graph also has $n - 5$ isolate vertices). The graph $G^0_4$ provides an obvious upper bound on $pc_2(G)$ and it can easily be shown that the lex graph beats out this upper bound.

If this cannot be done, then our graph must be a $C_4 \cup E_{n-4}$. We deal with this case by making a comparison to the lex graph.

We can now state and prove the result of the case when $m^* = 0$.

**Lemma 3.4.6.** Let $G$ be a graph with $n$ vertices and $m$ edges with $m \leq n - 1$. If $G$ only contains cycles of size at most 4 and $m^* = 0$, then $pc_2(G) \leq pc_2(L(n, m))$.  

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Proof. Let's assume that $G$ looks like the graph in Figure 3.4 and we can obtain the graph $G_3^0$ by deleting edges. It is a straightforward task to check that the number of partial two colorings of the $C_3$ with the two incident edges is equal to 49. Applying Lemma 3.3.1 it's easy to see that:

$$pc_2(G) \leq 49 \cdot 3^{n-5} < 3^4 \cdot 3^{n-5} = 3^{n-1} < pc_2(L(n, m)).$$

If we cannot obtain the graph $G_3^0$ by deleting edges, then it must be the case that the degree of each vertex on the $C_3$ is at most three. If this is the case, then either one, two, or all three of the vertices of the $C_3$ are of degree three or there are no incident edges with the cycle. We check each case separately.

If there is exactly one vertex of degree three on the $C_3$ then there are $n - 4$ isolate vertices. A routine calculation tells us there us there are $33 \cdot 3^{n-4}$ total partial two-colorings of this type of graph. It is a straightforward task to check that this value is smaller than the corresponding lex graph on $n$ vertices and 4 edges. That is, we need to verify that

$$33 \cdot 3^{n-4} < pc_2(L(n, 4)).$$
Some algebra justifies that this inequality holds:

\[ 33 \cdot 3^{n-4} < pc_2(L(n, 4)) \]
\[ 33 \cdot 3^{n-4} < 3^{n-5} \cdot [3^4 + 2^5] \]
\[ \frac{33}{113} < \frac{3^{n-5}}{3^{n-4}} = \frac{1}{3}. \]

If there is exactly two vertices of degree three on the \( C_3 \) then we are left \( n - 5 \) isolate vertices. A routine calculation tells us there us there are \( 84 \cdot 3^{n-5} \) total partial two-colorings of the cycle of this type of graph. It is a straightforward task to check that this value is smaller than the corresponding lex graph on \( n \) vertices and 5 edges. That is, we need to verify that

\[ 84 \cdot 3^{n-5} < pc_2(L(n, 5)). \]

Some algebra justifies that this inequality holds:

\[ 84 \cdot 3^{n-5} < pc_2(L(n, 5)) \]
\[ 84 \cdot 3^{n-5} < 3^{n-6} \cdot [3^5 + 2^6] \]
\[ \frac{84}{307} < \frac{3^{n-6}}{3^{n-5}} = \frac{1}{3}. \]

If all three of the vertices on the \( C_3 \) are of degree three then we are left \( n - 6 \) isolate vertices. A routine calculation tells us there us there are \( 285 \cdot 3^{n-6} \) total partial two-colorings of the cycle of this type of graph. It is a straightforward task to check that this value is smaller than the corresponding lex graph on \( n \) vertices and 6 edges. That is, we need to verify that

\[ 285 \cdot 3^{n-6} < pc_2(L(n, 6)). \]
Some algebra justifies that this inequality holds:

\[ 285 \cdot 3^{n-6} < pc_2(L(n, 6)) \]
\[ 285 \cdot 3^{n-6} < 3^{n-7} \cdot [3^6 + 2^7] \]
\[ \frac{285}{857} < \frac{3^{n-7}}{3^{n-6}} = \frac{1}{3}. \]

If the \( C_3 \) contains no incident edges, then there are \( n - 3 \) isolate vertices. Using the fact that \( pc_2(C_3) = 13 \), it suffices to verify that the following inequality holds:

\[ pc_2(C_3) \cdot 3^{n-3} < pc_2(L(n, 3)). \]

Algebra verifies the result:

\[ 13 \cdot 3^{n-3} < pc_2(L(n, 3)) \]
\[ 13 \cdot 3^{n-3} < 3^{n-4} [3^3 + 2^4] \]
\[ \frac{13}{43} < \frac{3^{n-4}}{3^{n-3}} = \frac{1}{3}. \]

Now, let's assume our graph \( G \) looks like the graph in Figure 3.5 and we can obtain the graph \( G_4^0 \) by deleting edges. It is a straightforward task to check that \( pc_2(G_4^0) = 77 \cdot 3^{n-5} \). Thus, it suffices to verify that

\[ 77 \cdot 3^{n-5} < pc_2(L(n, 5)). \]
Some algebra easily verifies that this inequality holds:

\[ 77 \cdot 3^{n-5} < pc_2(L(n, 5)) \]
\[ 77 \cdot 3^{n-5} < 3^{n-6}[3^5 + 2^6] \]
\[ \frac{77}{307} < \frac{3^{n-6}}{3^{n-5}} = \frac{1}{3}. \]

If the \( C_4 \) contains no incident edges, then it suffices to verify that

\[ pc_2(C_4) \cdot 3^{n-4} < pc_2(L(n, 4)). \]

Using the fact that \( pc_2(C_4) = 35 \), we can easily verify this inequality holds:

\[ 35 \cdot 3^{n-4} < pc_2(L(n, 4)) \]
\[ 35 \cdot 3^{n-4} < 3^{n-5}[3^4 + 2^5] \]
\[ \frac{35}{113} < \frac{3^{n-5}}{3^{n-4}} = \frac{1}{3}. \]

This completes the proof. \( \square \)

What remains to be proved is the case where \( m^* = 1 \). Graphs of this form look like the graphs in Figures 3.4 and 3.5 along with an edge not incident to a vertex on the cycle. To get a better visual of what these types of graphs look like, we reference Figures 3.8 - 3.12. Note here we have deal with some other “special cases.” As we will see in the proof, we handle each case a little differently.
Figure 3.8: A graph containing a hairy $C_3$ with $m^* = 1$.

Figure 3.9: A graph containing $C_3$ with $m^* = 1$.

Lemma 3.4.7. Let $G$ be a graph with $n$ vertices and $m$ edges with $m \leq n - 1$. If $G$ only contains cycles of size at most 4 and $m^* = 1$, then $pc_2(G) \leq pc_2(L(n, m))$.

Proof. Let's suppose $G$ looks like the graph in Figure 3.8. We can find an upper bound on $pc_2(G)$ by deleting all edges incident with vertices on the cycle except for two edges incident to a common vertex. If we can perform such a deletion, then the graph we obtain is a $C_3$ with 2 edges incident to a common vertex together with an isolated edge and $n - 7$ isolate vertices. It is not hard to check that the number of partial two-colorings of the component containing the $C_3$ is equal to 49. Hence, by Lemma 3.3.1 and the fact that $pc_2(K_2) = 7$, we see that

$$pc_2(G) \leq 49 \cdot 7 \cdot 3^{n-7} < 3^{n-1} < pc_2(L(n, m)).$$

If we cannot perform such a deletion, then we delete all other edges of $G$ incident with
$C_3$ except for possibly one. Then, one our graph will look like is a $C_3$ with one incident edge together with an isolated edge and $n - 6$ isolate vertices. It is not hard to check that the number of partial two-colorings of the component containing the $C_3$ is equal to 33. Proceeding as before, we can see that

$$pc_2(G) \leq 33 \cdot 7 \cdot 3^{n-6} < 3^{n-1} < pc_2(L(n,m)).$$

Another possibility as to what our graph will look like is a $C_3$ together with an isolated edge and $n - 5$ isolated vertices. Here we will make a direct comparison to the lex graph $L(n,4)$. Using our formula from Theorem 3.3.3, we can check that $pc_2(L(n,4)) = 106 \cdot 3^{n-5}$. Since $pc_2(C_3) = 13$ and $pc_2(K_2) = 7$, we see that

$$pc_2(G) = 13 \cdot 7 \cdot 3^{n-5} < 3^{n-1} < 106 \cdot 3^{n-5} = pc_2(L(n,4)).$$
Hence, the claim holds for graphs that look like the graph in Figure 3.8.

If $G$ looks like the graph in Figure 3.9, we can attain an upper bound for $pc_2(G)$ by deleting all the edges in $G$ besides the edges on the two cycles. This leaves us with a graph containing $n - 5$ isolate vertices. It is not hard to check that the number of partial two-colorings of this graph is $67 \cdot 3^{n-5}$. We make a direct comparison to the lex graph $L(n, 6)$ to verify the result for this case. Proceeding as before, we see that

$$pc_2(G) = 67 \cdot 3^{n-5} < 3^{n-1} < 857 \cdot 3^{n-6} = L(n, 6).$$

Hence, the claim holds for graphs that look like the graph in Figure 3.9.

If $G$ looks like the graph in Figure 3.10, we can attain an upper bound for $pc_2(G)$ by deleting the edge not incident with vertices on the cycle (this cannot possible decrease $pc_2(G)$). This will leave us with the graph that we've previously encountered in Lemma 3.4.6 which we know loses to the lex graph. Hence, the claim holds for these types of graphs.

Let us now assume $G$ looks like the graph shown Figure 3.11. We can find an upper bound on $pc_2(G)$ by deleting all edges incident with vertices on the cycle except for one, provided there are edges incident with the $C_4$.

Assuming we can perform such a deletion, we would be left with the graph that is a $C_4$ with one incident edge together with an isolated edge and $n - 7$ isolate vertices. It is not
hard to calculate that the number of partial two-colorings of a $C_4$ together with one incident edge is equal to 35. Proceeding as before, we see that

$$pc_2(G) \leq 35 \cdot 7 \cdot 3^{n-7} < 3^{n-1} < pc_2(L(n,m)).$$

If such a deletion cannot be performed, then we must be dealing with the graph $C_4 \cup K_2 \cup E_{n-6}$. For this case, we make a direct comparison to the lex graph $L(n, 5)$. Using the formula from Theorem 3.3.3, it is not hard to see that $pc_2(L(n, 5)) = 307 \cdot 3^{n-6}$. Since

$$pc_2(C_4 \cup K_2 \cup E_{n-6}) = 35 \cdot 7 \cdot 3^{n-6} < 307 \cdot 3^{n-6} = pc_2(L(n, 5)),$$

we have that this case also loses to the lex graph.

Lastly, we assume our graph looks like the graph in Figure 3.12. We can attain an upper bound for $pc_2(G)$ by deleting the edge not incident with vertices on the cycle (this cannot possibly decrease $pc_2(G)$). This will leave us with the graph that we've previously encountered in Lemma 3.4.6 which we already know loses to the lex graph. Hence, the claim holds for these types of graphs.

$\square$
Chapter 4

Conclusions and Future Work

We were only able to partially answer the maximization problem that this thesis is concerned with. The structure of the extremal graph is currently unknown for graphs with $n$ vertices and $m$ edges where $m > n - 1$. From a very broad point of view, it would appear the extremal graphs that maximize $pc_2(G)$ transition between “bipartite” and “lex-like” graphs.

Although the exact result for a fixed $n$ and $m$ may be difficult to obtain, the results of Loh, Pikhurko, and Sudakov [5] can be used to help determine certain structural properties of the desired extremal graphs as well as where this transition seems to occur. They show via Szemerédi’s Regularity Lemma, that the asymptotic solution to the proper $q$-coloring problem reduces to a certain quadratically-constrained linear program. This program can be adapted to asymptotically maximize the number of graph homomorphisms to our fixed image graph $H_2$.

The question of how to adapt the techniques of Loh, Pikhurko, and Sudakov is currently being worked on. However, based on some preliminary testing, we propose a conjecture. This conjecture is written in terms of the edge density $\gamma$ of a graph. The edge density of a graph is given by $\gamma = e(G) / \binom{n}{2}$ where $n = |V(G)|$.

**Conjecture 1.** For graphs of relatively small edge density $\gamma$, that extremal graph that max-
imizes $pc_2(G)$ is bipartite. However, once $\gamma \approx .300964$, the extremal graph that maximizes $pc_2(G)$ is a "lex-like" graph.

A natural extension of this problem would be to generalize the result for partial $n$-colorings. For this problem, we could think of the fixed image graph $H_{n-1}$ as the complete graph on $n$ vertices, $K_n$, with exactly one looped vertex. Unfortunately, the quadratically constrained linear program becomes more and more computationally rigorous as $n$ increases. In fact, even working with relatively small values of $n$ becomes quite a task.
Bibliography


