5-2020

Design of Strips with Geometry Shapes and Mathematical Analysis

Somia Benali

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Abstract

In this research, I investigate different methods to create geometric designs for textile strips and study the geometric properties of the involved shapes. I develop three designs that contain circles, squares, and golden spiral pieces with repeating patterns and certain tangencies. One interesting part of the work is to find the tangent points and to calculate the areas of the regions to which different colors maybe assigned. The main figure for Design I is a circle inscribed in a square and that for Design II is a circle inscribed in an isosceles triangle. The last design integrates Golden Spirals into the image.

The goals for this research are to provide relationships between geometry and the considered textile designs, to examine the mathematics used to characterize the geometrical shapes, and to show how mathematics can be visualized in textile design and how it can help student learners to experience real world applications.

The main results include formulas for the areas of the involved regions in each design and where the tangent points are. In Design III, we focus on certain interesting regions bounded by pieces of circles, squares, and the golden spirals. The sequence of such areas, named as \( \{A_n\}_{n=1}^{\infty} \), follows an interesting pattern. Formulas for \( A_n \) is developed using calculus ideas. The limiting situation of the ratios of two consecutive areas is provided. The last part of the thesis gives an interactive lesson plan, which involves the geometric concepts demonstrated in the textile designs, for high school students to explore real world applications.
MONTCLAIR STATE UNIVERSITY

Design of Strips with Geometry Shapes
and Mathematical Analysis

by

Somia Benali

A Master's Thesis Submitted to the Faculty of
Montclair State University

In Partial Fulfillment of the Requirements
For the Degree of
Master of Science

May 2020

College of Science and Mathematics

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DESIGN OF STRIPS WITH GEOMETRY SHAPES
AND MATHEMATICAL ANALYSIS

A THESIS

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Somia Benali
Montclair State University
Montclair, NJ
2020
I would like to express my special thanks of gratitude to Dr. Aihua Li for her continuous support as a professor and guidance as a mentor. I always feel that she was with me throughout the research. Since my first semester at Montclair State University, she has been a key role in my growth as a student and has been an incentive to pursue this course in my studies. Along the way, I have been introduced to two great professors, Dr. Mark Korlie and Dr. Johnathan Cutler. They have motivated me to do this topic for my thesis. I would also like to thank them for their aid and support as committee members. I am also thankful to my peers at Montclair State University. They have made this journey one of the most enjoyable experiences I have ever taken. Finally, I am deeply grateful to my family. Their constant patience and belief made these efforts possible. Without their unconditional love, prayer and support, I cannot be here.
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1 Introduction

1.1 Importance of Geometry

The word geometry is a Greek word which means “Earth Measurement” and was mainly developed to study the properties of space and figures. Geometry’s origins go back to approximately 3,000 BC. Ancient Egyptians applied geometrical ideas in many ways, such as in the construction of pyramids and the development of astronomy. The famous quote of Plato “Let no man ignorant of geometry enter here” shows the importance of geometry at that time. Euclid turned the study of geometry into an axiomatic form in 300 BC and these axioms are still useful. His famous book, the “Elements”, is one of the most successful and influential textbooks ever written. It gives a collection of concepts in plane and Euclidean geometry and has proven instrumental in the development of logic and modern science. Another important evolution for the science of geometry was created when Rene Descartes was able to establish the concept of analytical geometry that allows us to represent plane figures analytically and to develop calculus.

Today geometry holds an important position in the mathematics curriculum. The knowledge of geometry helps people to visualize and solve a problem in many mathematical topics. In addition, it has many practical applications in real life such as construction and arts. In 2000, the National Council of Teachers of Mathematics (NCTM), the largest organization for teachers of mathematics in the world, highlighted the importance of geometry in school mathematics by stating that “geometry is a natural place for the development of students’ reasoning and justification skills [1].” It is evident that through learning geometry, students can improve their critical thinking and problem solving skills and be better prepared for higher level of mathematics and science courses and for a variety of occupations requiring mathematical skills.

I am interested in topics on geometry textile design because of its fascinating
geometric elements and the beauty of the geometric shapes that can be integrated. Before I came to the United States I studied textile design for two years and I graduated with an associates degree that allowed me to work in textile business and to have a good experience in fashion industries. Currently, I am a mathematics teacher in a public high school. My research goals for this project are to connect textile design to mathematics and mathematics teaching by investigating geometric properties of textile designs.

1.2 Textile Design

Textile design is a process of creating patterns in knitted or woven textiles as well as creating printed designs on fabric surfaces. During this process, designers create the look of fabric by using different geometric shapes and placing them in a given design with desired colors and repeating patterns. A typical way to realize the graphical part of a textile design is to draw the image roughly by hand, scan it into a computer, and then use CAD (computer-assisted design) programs to adjust and finalize it [2]. The construction and modification of a textile design needs deep mathematics knowledge in geometry and calculus.

![Textile designs](image)

Figure 1: Textile designs

Some textile patterns make geometry attractive and abstract that we cannot find in our natural world. Abstract patterns might include circles, triangles, rectangles or zig zags. Some designs have layers of geometric elements and bold colors that work together to create amazing patterns. We use these patterns to produce
strips to decorate our homes or to make our clothes more attractive.

1.3 The GeoGebra Software

Research from many researchers in mathematics education has shown the importance of having students involved interactively in the classroom (NCTM 1989, 2000, 2006). It is evident that using technology has enhanced the learning process and has made concepts come alive through engaging and manipulating. Nowadays most educators support mathematics classrooms where students are involved actively by manipulating technology to clarify mathematical ideas. Using technology in the classroom has many advantages such as providing good learning opportunities for students (Roberts, 2012), and enhancing student engagements (White 2012). In the teaching and learning of Mathematics, specifically geometry, it is important to help students to imagine shapes and understand how to construct them in order to connect their learning to real life. For this reason, a number of technology tools are available such as interactive whiteboards, Graphing calculators, Geometer’s Sketchpad and GeoGebra. GeoGebra is one of the most used math software which is designed for all levels of education involved with geometry, algebra, graphing and calculus. This free software does not require a license (It can be downloaded from www.geogebra.org). This software, created by Markus Hohenwarter, is used by a large international users from almost 190 countries around the world and is translated into 55 languages.[3]

In this paper, I will present some geometric designs using GeoGebra software. I chose to use this attractive software as a tool in this research paper because I utilize it throughout the school year for various activities that are connected to the math curriculum. At the end of this research, I want to create a lesson for my students with interactive presentations of different geometric objects that can be resized or shifted around the plane. They will personalize their work through the adaptation of several interactions such as color, coordinates, shapes, and other
1.4 Definitions and Existing Results

The following basic definitions and well-known results in geometry are useful for this research.

**Proposition 1.** Let \( \triangle ABC \) be an isosceles triangle with \( \angle A = \angle C \). Let \( D \) be a point on the base \( AC \). Regarding the line segment \( BD \), the following are equivalent:

1. \( D \) is the midpoint of \( AC \), that is, \( AD = DC \).
2. \( BD \) is perpendicular to \( AC \).
3. \( BD \) bisects \( \angle ABC \) or \( \angle ABD = \angle DBC \).

Consider a circle in the coordinate plane \( \mathbb{R}^2 \) with center at \( O = (a, b) \) and radius \( r \). Denoted by \( C(O, r) \), it is represented by the quadratic equation \( (x - a)^2 + (y - b)^2 = r^2 \). Let \( P \) be a point on the circle \( C(O, r) \). A line through \( P \) is the tangent line to the circle at \( P \) if \( P \) is the only intersection point between the circle and the line. The point \( P \) then is called the tangent point of the line to the circle. From calculus, the slope of the tangent line to a curve represented by a differentiable function at a point \( P \) can be obtained by evaluating the derivative of
the function at the point $P$. Recall that the derivative function of a differentiable function $f(x)$ is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

**Lemma 1.** The slope of the tangent line to the circle $C((a, b), r)$ at a point $P(x_0, y_0)$, where $y_0 \neq b$, is

$$m = \frac{a - x_0}{y_0 - b}.$$

I am interested in the geometric figures that include circles inscribed in triangles.

**Definition 1.** A circle is inscribed in a triangle if the triangle’s three sides are all tangent to the circle. In this situation, the circle is called an inscribed circle to the triangle and its center is called the inner center, or incenter.

**Lemma 2.** [5] Let $r$ be the radius of an inscribed circle in an isosceles triangle $\triangle ABC$ with $|AB| = |BC| = a$ and $|AC| = b$. Then

$$r = \frac{b}{2} \sqrt{\frac{2a - b}{2a + b}}.$$

The following antiderivative formula is useful for later development of related formulas.

**Lemma 3.** For any fixed real number $a$ we have:

$$\int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a} + C.$$

Many interesting geometric figures have connections with the well known Fibonacci sequence. In my third design of textile strips, a golden spiral is applied which is built from Fibonacci numbers.

**Definition 2.** 1. Fibonacci numbers, denoted by $F_n$, are given by $F_0 = 0$, $F_1 = 1$, and for $n > 1$, $F_n = F_{n-1} + F_{n-2}$. 
2. The Golden Ratio is the number $\varphi = \frac{1 + \sqrt{5}}{2}$, which satisfies the equation

$$x^2 - x - 1 = 0.$$ 

3. A rectangle with sides $a, b$ such that $a, b$ are consecutive Fibonacci numbers is called a golden rectangle.

The following existing results are well known. We skip the proofs.

Lemma 4. [6]

1. The explicit formula for $F_n$ is given by:

$$F_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n.$$ 

2. 

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \varphi.$$ 

1.5 Research Questions and Goals

In this project we plan to provide relationships between geometry and the considered textile designs, to examine the mathematics used to characterize the geometrical shapes, and to show how mathematics can be visualized in textile design and how it helps the student learners to experience real world applications. Specifically, I want to answer the following questions:

1. Refer to the following strip made of geometric shapes including circles and rectangles with certain tangencies and having symmetric and repeating patterns. What geometric properties does the strip have? In particular, what are the relationships between the involved geometric shapes?

2. For each design, if the size of one shape is changed, what adjustments for the other shapes to be made in order to keep the same pattern of the design?
3. How to develop formulas for the areas of certain specific regions of each design?

4. If the golden spiral is involved in a design (e.g., Design III), how to evaluate the involved areas? Are the areas related to the golden ratio?

5. How can I integrate the result obtained from this research into my classroom teaching and high school curriculum?

1.6 Methodology and Main Results

This study aims to create and analyze textile designs for certain strips using geometry concepts, such as the one shown in Figure 3. I present three different designs. The first design is made by a circle inscribed in a square and surrounded by three identical circles (see Figure 4). The second design involves a circle inscribed in an isosceles triangle (see Figure 5). The last design is generated by a part of the golden spiral and related circles (see Figure 9b). We use both college geometry and calculus ideas to compute the sectional areas of each design. Besides, we find the points of tangency involved in each design. For Design III, we are interested in the area of a region, called $A_n$. We prove that the fraction $\frac{A_{n+1}}{A_n}$ approaches $\varphi^2$ as $n$ goes to infinity, where $\varphi$ is the Golden Ratio. Lastly, we develop an interactive lesson plan for high school students that demonstrates the Fibonacci Sequence and the Golden Ratio with different activities in order to show applications of these beautiful mathematics concepts in the real world.
2 Geometric Properties of Design I

Figure 4 represents Design I with the graphical part drawn using GeoGebra. We focus on the square $OABC$ of side $2a$ in the first quadrant, where $O$ is the origin of the coordinate system. The center of $OABC$ is $D = (a, a)$. The four inscribed circles are involved in this part: $C_1(O_1, r_1)$, $C_2(O_2, r_2)$, $C_3(O_2, r_3)$, and $C_4(O_4, r_4)$. Here $C_1(O_1, r_1)$ is inscribed in the square $ODCG$. The circles $C_2(O_2, r_2)$, $C_3(O_3, r_3)$ and $C_4(O_4, r_4)$ are inscribed in the isosceles triangles $\triangle BDC$, $\triangle ADB$, and $\triangle ADO$ respectively. The measurements of the inscribed circles are given in the following theorem:

![Figure 4: Circle $C_1$ inscribed in a square](image)

**Theorem 1.** Refer to Figure 4 and the above notations (the square $OABC$ of side $2a$). Let $O_i(x_i, y_i)$ be the center and $r_i$ be the radius of the circle $C_i(O, r_i)$, $i = 1, 2, 3, 4$.

1. The circle $C_1(O_1, r_1)$ has center $O_1(0, a)$ and radius $r_1 = \frac{a\sqrt{2}}{2}$.

2. The circles $C_2(O_2, r_2)$, $C_3(O_3, r_3)$, and $C_4(O_4, r_4)$ have the same radius: $r_2 = r_3 = r_4 = a(\sqrt{2} - 1)$ and the centers are $O_2 = (a, (3 - \sqrt{2})a)$, $O_3((3 - \sqrt{2})a, a)$, and $O_4(a, (\sqrt{2} - 1)a)$ respectively.

**Proof.**

1. It is obvious.
2. Both circles $C_3(O_3, r_3)$ and $C_4(O_4, r_4)$ are images of reflection of the circle $C_2(O_2, r_2)$, one about the lines $y = x$ and the other about $y = a$. Thus they both have the same radius as that of $C_2(O_2, r_2)$. Note that $|AC| = \sqrt{(2a)^2 + (2a)^2} = 2\sqrt{2}a = 2|CD|$. The circle $C_2(O_2, r_2)$ is inscribed in the isosceles triangle $\triangle BCD$ with base $2a$ and side $\sqrt{2}a$. By Lemma 2,

$$r_2 = r_3 = r_4 = \frac{2a}{2} \left\{ \frac{2\sqrt{2}a - 2a}{2\sqrt{2}a + 2a} = a \sqrt{\frac{2 - 1}{2 + 1}} = (\sqrt{2} - 1)a. \right.$$ 

Let $O_2 = (x_2, y_2)$. Then $x_2 = a$ and

$$y_2 = 2a - r_2 = 2a - (\sqrt{2} - 1)a = (3 - \sqrt{2})a.$$

Thus $O_2 = (a, (3 - \sqrt{2})a)$. Using rules of reflection $O_3 = ((3 - \sqrt{2})a, a)$ and $O_4(a, (\sqrt{2} - 1)a)$ respectively.

We now consider the tangency of each circle $C_i$, $i = 1, 2, 3, 4$, to the line $y = -x + 2a$. It is obvious that $C_3$ and $C_4$ are tangent to the line $y = -x + 2a$ at the same point. Let $T_i(s_i, t_i)$ be the tangent point of the circle $C_i$ to the line $y = -x + 2a$ for $i = 1, 2, 3$. In the following theorem, we find the coordinates of the tangent points $T_1, T_2, T_3$.

**Theorem 2.** Let $T_i = (s_i, t_i)$ be the tangent point of the circle $C_i(O_i, r_i)$ to the straight line $y = -x + 2a$, $i = 1, 2, 3$. Then

(i) $T_1 = \left(\frac{a}{2}, \frac{3a}{2}\right)$.

(ii) $T_2 = \left(\frac{a\sqrt{2}}{2}, \frac{(4 - \sqrt{2})a}{2}\right)$.

(iii) $T_3 = \left(\frac{(4 - \sqrt{2})a}{2}, \frac{a\sqrt{2}}{2}\right)$.

**Proof.** Since all the 3 points $T_1, T_2, T_3$ are on the line $y = -x + 2a$, we have $t_i = -s_i + 2a$ for $i = 1, 2, 3$. 
(i) Obviously, $T_1$ is the midpoint of the two points $C = (0, 2a)$ and $D = (a, a)$.

Thus by the midpoint formula,

$$T_1 = \left( \frac{a}{2}, \frac{3a}{2} \right).$$

(ii) The equation of the circle $C_2 \left( (a, (3 - \sqrt{2})a), (\sqrt{2} - 1)a \right)$ is given by:

$$(x - a)^2 + (y - (3 - \sqrt{2})a)^2 = a^2(\sqrt{2} - 1)^2.$$  

Again by Lemma 1, $T_2 = (s_2, t_2)$ satisfies

$$\frac{a - s_1}{(t_1 - (3 - \sqrt{2})a)} = -1 \implies t_2 = s_2 + (2 - \sqrt{2})a.$$  

Also, $t_2 = -s_2 + 2a$. It implies that

$$t_2 = \frac{(4 - \sqrt{2})a}{2} \quad \text{and} \quad s_2 = \frac{a\sqrt{2}}{2} \quad \implies \quad T_2 = \left( \frac{a\sqrt{2}}{2}, \frac{(4 - \sqrt{2})a}{2} \right).$$

(iii) Note that the center point $D(a, a)$ is the midpoint of the two points $T_2 = (s_2, t_2)$ and $T_3 = (s_3, t_3)$. Thus by the midpoint formula, we have

$$\frac{s_3 + \frac{a\sqrt{2}}{2}}{2} = a = \frac{t_3 + \frac{(4 - \sqrt{2})a}{2}}{2} \implies s_3 = \frac{(4 - \sqrt{2})a}{2} \quad \text{and} \quad t_3 = \frac{a\sqrt{2}}{2}.$$  

The next theorem presents the coordinates of the tangent points $T'_i(s'_i, t'_i)$ of the line $y = x$ to each circle $C_i$ for $i = 1, 2, 3, 4$ (shown in Figure 4). Note that $C_2$ and $C_3$ are tangent to $y = x$ at the same point.

**Theorem 3.** Let $T'_i = (s'_i, t'_i)$ be the tangent points of the circles $C_1$, $C_2$, $C_3$, and $C_4$ to the line $y = x$, where $i = 1, 2, 3, 4$.

1. $T'_1 = \left( \frac{a}{2}, \frac{a}{2} \right).$
2. \( T'_2 = T'_3 = \left(\frac{(4-\sqrt{2})a}{2}, \frac{(4-\sqrt{2})a}{2}\right) \).

3. \( T'_4 = \left(\frac{a\sqrt{2}}{2}, \frac{a\sqrt{2}}{2}\right) \).

**Proof.**

1. The center of \( O_1 \) is \((0, a)\) and the slope of the line \( y = x \) is 1. For the tangent point \( T'_1 = (s'_1, t'_1) \) we apply Lemma 1:

\[
\frac{-s'_1}{t'_1 - a} = 1 \implies t'_1 = -s'_1 + a.
\]

\( T'_1(s'_1, t'_1) \) is also on the line \( y = x \). So \( t'_1 = s'_1 \implies s'_1 = t'_1 = \frac{a}{2} \). Thus, \( T'_1 = \left(\frac{a}{2}, \frac{a}{2}\right) \).

2. The tangent point \( T'_2 = (s'_2, t'_2) = T'_3 = (s'_3, t'_3) \) is the midpoint of the centers \( Q_2 \) and \( O_3 \) of the circles \( C_2 \) and \( C_3 \) respectively. Applying the coordinates of the centers from Theorem 1 we obtain

\[
s'_2 = \frac{a + (3 - \sqrt{2})a}{2} \quad \text{and} \quad t'_2 = \frac{(3 - \sqrt{2})a + a}{2} \implies s'_2 = \frac{(4 - \sqrt{2})a}{2} = t'_2.
\]

3. In this case, the point \( T'_4 = (s'_4, t'_4) \) should have the same \( x \)-coordinate. Thus \( s'_4 = s_2 = \frac{a\sqrt{2}}{2} \) and since \( T'_4 = (s'_4, t'_4) \) is on the line \( y = x \), \( t'_4 = s'_4 \). Therefore \( T'_4 = (s'_4, t'_4) = \left(\frac{a\sqrt{2}}{2}, \frac{a\sqrt{2}}{2}\right) \).

\[\Box\]

To make the design more attractive designers may use different types of materials and colors codes. In order to determine the amount of various materials and color paints needed it is important to know the area of each part of the design. In Design I (refer to Figure 4), each circle is inscribed into a square or a triangle. There are mainly 3 different regions outside a circle and inside the square or triangle where the circle is inscribed. Below we calculate the areas of these regions.

**Theorem 4.** Consider Design I (see Figure 4).
1. Let $A_1$ be the area of the enclosed non-circle region bounded by the arc of the circle $C_1$, the lines $y = x$, and the line $y = -x + 2a$. Then $A_1 = \frac{(4-\pi)a^2}{8} \approx 0.1073a^2$.

2. The area $A_2$ of the region bounded by the circle $C_4$, $y = 0$, and $y = x$ is given by
   
   $$A_2 = (\sqrt{2} - 1)a^2 - \frac{(9 - 6\sqrt{2})\pi a^2}{8} \approx 0.212a^2.$$

3. Let $A_3$ be the area of the enclosed non-circle region bounded by the arc of $C_4$, the line $y = x$, and the line $y = -x + 2a$. Then
   
   $$A_3 = (3 - 2\sqrt{2})a^2 - \frac{(3 - 2\sqrt{2})\pi a^2}{4} \approx 0.037a^2.$$

Proof. 1. The remaining area after removing the circle $C_1$ from the square $ODCG$ is 4 times $A_1$. The side of $ODCG$ is $\sqrt{2}a$. Thus
   
   $$A_1 = \frac{\left(2a^2 - \pi \left(\frac{\pi^2}{2}\right)\right)}{4} = \frac{(4 - \pi)a^2}{8}.$$

2. We calculate the area $A_2$ by evaluating a definite integral from $y = 0$ to $y = t_4 = a\sqrt{2}/2$. The region is bounded by the circle $C_4$, $y = 0$, and $y = x$. The function for the arc of $C_4$ in terms of $y$ is given by
   
   $$x = a - \sqrt{(\sqrt{2} - 1)a^2 - (y - (\sqrt{2} - 1)a)^2} \approx 0.1073a^2.$$

Thus

$$A_2 = \int_0^{a\sqrt{2}/2} a - \sqrt{(\sqrt{2} - 1)a^2 - (y - (\sqrt{2} - 1)a)^2} - y \, dy.$$
Substitute in \( u = y - (\sqrt{2} - 1)^2a \), which implies \( dy = du \). Then

\[
A_2 = \frac{a^2 \sqrt{2}}{2} - \frac{a^2}{4} - \int_{(1-\sqrt{2})a}^{(2-\sqrt{2})a/2} \sqrt{(\sqrt{2} - 1)^2a^2 - u^2} \, du.
\]

\[
= \frac{(2\sqrt{2} - 1)a^2}{4} - \left[ \frac{\sqrt{2}}{2} \right]^2 \left[ \frac{u \sqrt{(\sqrt{2} - 1)a^2 - u^2}}{1-\sqrt{2})a} \right]_{(1-\sqrt{2})a}^{(2-\sqrt{2})a/2}
\]

\[
- \left[ \frac{(\sqrt{2} - 1)^2a^2}{2} \sin^{-1} \left( \frac{u}{\sqrt{2} - 1) \right) \right]_{(1-\sqrt{2})a}^{(2-\sqrt{2})a/2}
\]

\[
= \frac{(2\sqrt{2} - 1)a^2}{4} - \frac{(2 - \sqrt{2})a}{4} \sqrt{(\sqrt{2} - 1)^2a^2}
\]

\[
- \frac{(\sqrt{2} - 1)^2a^2}{2} \left[ \sin^{-1} \left( \frac{2 - \sqrt{2}}{(2\sqrt{2} - 2)} \right) - \sin^{-1} \left( \frac{1 - \sqrt{2}}{(\sqrt{2} - 1)} \right) \right]
\]

\[
= \frac{(2\sqrt{2} - 1)a^2}{4} - \frac{3 - 2\sqrt{2})a^2}{4} \cdot \frac{(\sqrt{2} - 1)a}{\sqrt{2}}
\]

\[
- \frac{(3 - 2\sqrt{2})a^2}{2} \left[ \sin^{-1} \left( \frac{\sqrt{2}}{2} \right) - \sin^{-1}(-1) \right]
\]

\[
= (\sqrt{2} - 1)a^2 - \frac{3 - 2\sqrt{2})a^2}{2} \left( \frac{\pi}{4} + \frac{\pi}{2} \right)
\]

\[
= (\sqrt{2} - 1)a^2 - \frac{(9 - 6\sqrt{2})\pi a^2}{8} \approx 0.212a^2
\]

3. The area of \( \triangle OAD \) is \( a^2 \). The area of the circle \( C_4 \) is \( \pi a^2(\sqrt{2} - 1)^2 = (3 - 2\sqrt{2})\pi a^2 \). The area \( A_3 \) is the area of \( \triangle OAD \) subtract the area of \( C_3 \) and 2 times the area of \( A_2 \). Thus

\[
A_3 = a^2 - (3 - 2\sqrt{2})\pi a^2 - 2 \left( (\sqrt{2} - 1)a^2 - \frac{(9 - 6\sqrt{2})\pi a^2}{8} \right)
\]

\[
= (3 - 2\sqrt{2})a^2 - \frac{(3 - 2\sqrt{2})\pi a^2}{4} \approx 0.037a^2
\]
3 Geometric Properties of Design II

In this chapter I present Design II which is slightly different from Design I. Figure 5 below represents the main feature of Design II whose graphical part is produced using GeoGebra. In the graph, $OCFE$ and $ABCO$ are squares of size $2a$. Similarly as in Design I, it involves 4 circles. Circles $C_2(O_2, r_2)$ and $C_3(O_3, r_3)$ are the same as that in Design I. Circle $C_1(O_1, r_1)$ is inscribed in the isosceles right triangle $\triangle ECA$ and Circle $C_4(O_4, r_4)$ is inscribed in the angle $\angle CAO$. In addition, $C_1(O_1, r_1)$ and $C_4(O_4, r_4)$ share an intersection point $I(x_0, y_0)$. By an easy calculation, we obtain $\angle OAO_4 = \frac{\pi}{8}$.

![Figure 5: Design II](image)

The differences between Design I and II are from circles $C_1$ and $C_4$. Note that $C_1$ is inscribed in a triangle, not in a square, and circle $C_4$ is not inscribed in any triangle. Consequently, the tangent points of $C_4$ to $C_1$, the line $y = -x + 2a$, and $y = x$ are different from those in Design I. We first need to determine the radius of $C_4$ such that the two circles $C_1$ and $C_4$ are tangent to each other. We show that it is possible for the two circles to intersect at exactly one point. We then find the tangent point $I(s_1, t_1)$ and two other tangent points (to the lines $y = 0$ and $y = -x + 2a$.) Finally we calculate the resulting sectional areas. We first focus on the relationship between $C_1$ and $C_4$. Figure 6 shows it in detail.
Theorem 5. Refer to Figure 6 (the square OABC of side 2a). Let \( O_1 = (x_1, y_1) \) and \( O_4(x_4, y_4) \).

1. The center and radius of \( C_1 \) are given by \( x_1 = 0 \), and \( y_1 = 2a(\sqrt{2} - 1) = y_1 = r_1 \).

2. Let \( (s_1, t_1) \) be a point on \( C_1 \) and it is perpendicular to the line \( AO_1 \), then
   \[
s_1 = a\sqrt{2} - \sqrt{2} \approx 0.756a \quad \text{and} \quad t_1 = (\sqrt{2} - 1) \left( 2 - \sqrt{2} - \sqrt{2} \right) a \approx 0.511a.
   \]

3. If the center \( O_4(x_4, y_4) \) of \( C_4 \) satisfies
   \[
x_4 = 4a(3 - 2\sqrt{2}) \left( \sqrt{4 + 2\sqrt{2}} - 1 \right) \approx 1.107a \quad \text{and}
   \]
   \[
y_4 = a(22\sqrt{2} - 30) - 4a(5\sqrt{2} - 7)\sqrt{4 + 2\sqrt{2}} = r_4 \approx 0.37a.
   \]
   then \( C_1 \) and \( C_4 \) are tangent to each other at the point \( I = (s_1, t_1) \).

![Figure 6: Circles \( C_1 \) and \( C_4 \) tangent to each other at \( I(s_1, t_1) \)](image)

Proof. Refer to Figure 6.

1. The sides of the triangle \( \triangle ACE \) are \( 4a \) (base), \( 2\sqrt{2}a \), and \( 2\sqrt{2}a \). Obviously, \( x_1 = 0 \). By Lemma 2, the radius
   \[
r_1 = \frac{4a}{2} \sqrt{\frac{4\sqrt{2}a - 4a}{4\sqrt{2}a + 4a}} = 2a \sqrt{\frac{\sqrt{2} - 1}{\sqrt{2} + 1}} = 2a(\sqrt{2} - 1) = y_1.
   \]
2. The value of the angle $\angle OAO_4$ is $\pi/8$. Thus the slope of the line $AO_1$ is $\tan(\pi/8) = 1 - \sqrt{2}$. The center $O_1$ of $C_1$ is on the line $AO_4$ because $C_1$ is also tangent to $AE$ and $AC$. The equation of the line $AO_4$ is

$$y = -(\sqrt{2} - 1)x + 2a(\sqrt{2} - 1) = (\sqrt{2} - 1)(2a - x).$$

Because $(s_1, t_1)$ is on the line $O_1A$, $t_1 = (\sqrt{2} - 1)(2a - s_1)$. Since $y_1 = 2a(\sqrt{2} - 1)$, $t_1 - y_1 = -(\sqrt{2} - 1)s_1$. The point $(s_1, t_1)$ is also on the circle $C_1$, so

$$s_1^2 + (t_1 - y_1)^2 = y_1^2 \implies s_1^2 \left(1 + (\sqrt{2} - 1)^2\right) = 4a^2(\sqrt{2} - 1)^2$$

$$\implies s_1 = \frac{2a(\sqrt{2} - 1)}{\sqrt{4 - 2\sqrt{2}}} = a\sqrt{2} - \sqrt{2} \approx 0.765a.$$

Consequently,

$$t_1 = y_1 - (\sqrt{2} - 1)s_1 = 2a(\sqrt{2} - 1) - (\sqrt{2} - 1)a\sqrt{2} - \sqrt{2}$$

$$= (\sqrt{2} - 1)\left(2 - \sqrt{2} - \sqrt{2}\right)a \approx 0.511a.$$

3. If $I(s_1, t_1)$ is also on $C_4$, then $(s_1 - x_4)^2 + (t_1 - y_4)^2 = y_4^2$. The point $(x_4, y_4)$ is also on the line $O_1A$, so, $y_4 = (\sqrt{2}-1)(2a-x_4) \implies t_1 - y_4 = (\sqrt{2}-1)(x_4-s_1)$. Thus

$$y_4^2 = (s_1 - x_4)^2 + (t_1 - y_4)^2 = (s_1 - x_4)^2 \left(1 + (\sqrt{2} - 1)^2\right) \implies$$

$$(\sqrt{2} - 1)^2(2a - x_4)^2 = (s_1 - x_4)^2(4 - 2\sqrt{2}).$$
By taking the square root on the above equation, we have

\[ x_4 - s_1 = \frac{(\sqrt{2} - 1)(2a - x_4)}{\sqrt{4 - 2\sqrt{2}}} . \]

Substitute in \( s_1 = a\sqrt{2 - \sqrt{2}} \) from part (2) and solve for \( x_4 \), we obtain

\[ x_4 = 4a(3 - 2\sqrt{2}) \left( \sqrt{4 + 2\sqrt{2} - 1} \right) \approx 1.107a. \]

Finally,

\[ y_4 = (\sqrt{2} - 1)(2a - x_4) \]

\[ = a(22\sqrt{2} - 30) - 4a(5\sqrt{2} - 7)\sqrt{4 + 2\sqrt{2}} = r_4 \approx 0.37a. \]

By Lemma 1, the slope of the tangent line to circle \( C_1 \) at \( I = (s_1, t_1) \) is

\[ m_1 = \frac{-s_1}{t_1 - y_1} = \frac{1}{\sqrt{2} - 1} = 1 + \sqrt{2}. \]

Similarly, the tangent line to circle \( C_4 \) at \( I = (s_1, t_1) \) is given by

\[ m_2 = \frac{x_4 - s_1}{t_1 - y_4} = \frac{x_4 - s_1}{(\sqrt{2} - 1)(x_4 - s_1)} = \frac{1}{\sqrt{2} - 1} = 1 + \sqrt{2} = m_1. \]

Thus the point \( I = (s_1, t_1) \) is a tangent point to both \( C_1 \) and \( C_2 \) and the tangent line is perpendicular to the line \( O_1A \) because \( m_1 = m_2 = -1/(1 - \sqrt{2}) \), the negative reciprocal of the slope of \( O_1A \). 

Below I discuss the areas of regions shown in figure 7 with different colors.

We denote by \( A_1 \) the area of the red region, \( A_2 \) the area of the blue region, \( A_3 \) the area of the purple region, and \( A_4 \) the area of the green region. The color
code is given in figure 7. Recall that \( r_1 = y_1 = 2a(\sqrt{2} - 1) \approx 0.37a \) and \( r_4 = y_4 = a(22\sqrt{2} - 30) - 4a(5\sqrt{2} - 7)\sqrt{4 + 2\sqrt{2}} \approx 0.37a \). The following theorem provides the formulas for these areas.

**Theorem 6.** Let \( A_1, A_2, A_3 \) and \( A_4 \) be the areas of the corresponding color regions assigned above. Then

1. \( A_1 = (3 - 2\sqrt{2})(4 - \pi)a^2 \approx 0.147a^2 \).
2. \( A_2 = 4a^2(\sqrt{2} - 1) - \frac{3\pi a^2(3 - 2\sqrt{2})}{2} \approx 0.848a^2 \).
3. \( A_3 = r_4^2 \left( \sqrt{2} + 1 - \frac{3\pi}{8} \right) \approx 0.169a^2 \).
4. \( A_4 = 4a^2(\sqrt{2} - 1) - \frac{3\pi a^2(3 - 2\sqrt{2})}{2} - r_4^2 \left( \sqrt{2} + 1 - \frac{11}{8}\pi \right) \approx 0.249a^2 \).

**Proof.** Refer to Figure 7. Note that \( \angle CEO_1 = \frac{\pi}{8} \) and \( \tan \frac{\pi}{8} = \sqrt{2} - 1 \).

1. Note that the angle \( \angle ACE \) is a right angle and \( BO_1B'C \) is a square with side \( y_1 = r_1 = 2a(\sqrt{2} - 1) \). Then

\[
A_1 = r_1^2 \left( 1 - \frac{\pi}{4} \right) = 4a^2(\sqrt{2} - 1)^2 \left( 1 - \frac{\pi}{4} \right) = (3 - 2\sqrt{2})(4 - \pi)a^2 \approx 0.147a^2 .
\]

2. The area of \( \triangle OEO_1 \) is \( ar_1 \) and the area of the circular sector \( OO_1B \) is
Thus the area $A_2$ can be calculated:

$$A_2 = 2ar_1 - \frac{3\pi r_1^2}{8} = 4a^2(\sqrt{2} - 1) - \frac{3\pi a^2(\sqrt{2} - 1)^2}{2}$$

$$= 4a^2(\sqrt{2} - 1) - \frac{3\pi a^2(3 - 2\sqrt{2})}{2} \approx 0.848a^2.$$

3. Similarly as before, the area of the triangle $\triangle AO_4 x_4$ is given by

$$\frac{r_4^2}{2(\sqrt{2} - 1)} = \frac{A_3}{2} + \frac{3}{16} r_4^2 \implies A_3 = r_4^2 \left(\sqrt{2} + 1 - \frac{3}{8}\pi\right) \approx 0.169a^2.$$

4. By symmetry, $A_4 = A_2 - A_3 - \pi r_4^2$. Thus

$$A_4 = 4a^2(\sqrt{2} - 1) - \frac{3\pi a^2(3 - 2\sqrt{2})}{2} - r_4^2 \left(\sqrt{2} + 1 - \frac{3}{8}\pi\right) - \pi r_4^2$$

$$= 4a^2(\sqrt{2} - 1) - \frac{3\pi a^2(3 - 2\sqrt{2})}{2} - r_4^2 \left(\sqrt{2} + 1 - \frac{11}{8}\pi\right).$$

Recall that $r_4 \approx 0.37a$. Then $A_4 \approx 0.249a^2$. ■
4 Construction of a Strip Using Golden Ratio

4.1 The Fibonacci Sequence

The Fibonacci numbers are named after a famous Italian mathematician Leonardo Pisano known as Fibonacci which means “Son of Bonacci”. Fibonacci, a son of an Italian businessman, grew up in several trading colonies in North Africa such as Algeria and Egypt. He had been constantly exposed to numbers by calculating product prices, keeping track of their commercial transaction, and converting to different units of money. This interesting experience helped Fibonacci to build a strong background in mathematics and initiated his interest in this subject that became his passion [7]. He wrote several books, including Liber Abaci, published in 1202 and later revised in 1228. Regardless of previous achievements in mathematics, he was not recognized for any of these accomplishments but rather for the sequence of numbers that provided the solution to a depicted problem in Liber Abaci. The solution has forever immortalized him in the mathematical world [8].

The problem, regarding the reproduction of rabbits, is to count the number of rabbit pairs recursively under certain restrictions. It starts with a newly born breeding pair of rabbits at the beginning of the first month. This pair mates in the second month. At the end of the second month, they produce another pair of male-female rabbits (counted at a new born pair in the 3rd month.) Assume each breeding pair mates at the age of one month and at the end of the second month it always produces another pair of rabbits; and rabbits never die, but continue breeding forever. The Fibonacci Puzzle states

**Question 7.** *(Fibonacci Puzzle) Starting with one pair of new born male-female rabbits at Month 1, how many pairs of male-female rabbits are there at the beginning of the \( n^{th} \) month for any positive integer \( n > 1 \)*
The numbers of pairs of rabbits at the beginning of the first, second, third month, etc., are 1, 1, 2, 3, 5, 8, 13, and so on. Fibonacci’s solution provides the recursive pattern: the number of rabbits in month \( n \) is the sum of the numbers from the immediate two previous months [9]. The numbers of the male-female pairs of rabbits in the first 12 months are shown in the table below. Assume in January (month 1) there is one pair of breeding rabbits ready to give birth in March (called “baby”). After one month, all the rabbit pairs are called “Adult”.

<table>
<thead>
<tr>
<th>Month</th>
<th>Ja</th>
<th>Fe</th>
<th>Ma</th>
<th>Ap</th>
<th>Ma</th>
<th>Ju</th>
<th>Au</th>
<th>Se</th>
<th>Oc</th>
<th>No</th>
<th>De</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baby</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>13</td>
<td>21</td>
<td>34</td>
</tr>
<tr>
<td>Adult</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>13</td>
<td>21</td>
<td>34</td>
<td>55</td>
</tr>
<tr>
<td>Total</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>13</td>
<td>21</td>
<td>34</td>
<td>55</td>
<td>89</td>
</tr>
</tbody>
</table>

Table 1: Population of Fibonacci’s Rabbit Pairs

Here is the formal definition.

**Definition 3.** Let \( F_n \) be the total number of the rabbit pairs at the beginning of the \( n^{th} \) month. Then \( F_1 = F_2 = 1 \) and for all integers \( n \geq 3 \),

\[
F_{n+1} = F_n + F_{n-1}.
\]

Fibonacci generated his most famous sequence of numbers, named as Fibonacci numbers:

\[1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots\]

For the completeness, the 0th Fibonacci number is defined as \( F_0 = 0 \). The famous Fibonacci sequence is the sequence of these numbers: \( \{F_n\}_{n=0}^{\infty} \).

As a well-structured sequence, Fibonacci sequence shows many interesting patterns some of which reflected amazing phenomena in nature. It also has a geometric demonstration in the form of the golden spiral. Figure 8 represents a strip made of golden spirals and the attached tangent circles.
4.2 Method of Construction

Refer to Figure 9(a) which shows one base of the strip centered at origin. By symmetry we only need to analyze the quarter of this base in the first quadrant (see Figure 9(b)).

To build up the involved golden spiral we start with one unit square (called Square 1) and attach another unit square by its right side (Square 2). The two squares of size 1 next to each other have vertices at $(3, 2), (3, 3), (4, 2), (4, 3), (5, 2), (5, 3)$ and the shared side is $(4, 2) - (4, 3)$. Right below Square 1 and Square 2, we draw Square 3 of length 3 and the sharing side is $(3, 2) - (5, 2)$. By the left side of Square 1 and Square 3, we drew Square 4 of length 5 at vertices $(0, 0), (3, 0), (3, 3), (0, 3)$. We keep repeating this pattern. Square 5 has the vertices $(0, 3), (5, 3), (5, 8), (0, 8)$. The four points $(0, 5), (0, 13), (13, 8), (0, 8)$ builds the 6th square. Lastly, Square 7’s 4 vertices are $(0, 0), (13, 0), (13, 13), (0, 13)$. The first 7 Fibonacci numbers. Figure 9(b) shows the seven recursively built squares with sides 1, 1, 2, 3, 5, 8, and 13.

After placing all the squares we draw the spiral. In each square, the partial circular arc of the spiral in the $i^{th}$ square connects two opposite vertices of the square. Precisely, the spiral passes through the points $(3, 2), (4, 3), (3, 0), (0, 3), (5, 8), (13, 0)$. In each square the spiral spiral piece is tangent to two interior sides of the square. The other 3 quarters of the base in Figure 9(a) are the reflection of the image in 9(b) about the $x$ axis, $y$ axis and the origin respectively. After putting the 4 pieces together, the whole base in Figure 9(a) is outlined by a square of side 26 centered at origin. A circle of radius 13 is also drawn.
4.3 Calculating Related Areas

Most sectional regions resulting from Figure 9 have basic geometric shapes such as partial circles, squares, and rectangles. One interesting region is the region with the red color. The area of this region cannot be calculated by elementary geometric method. It is bounded by the line $y = 8$, the circle $x^2 + y^2 = 13^2$, and the circle $(x - 5)^2 + y^2 = 8^2$. We further calculate the areas of this region by evaluating a definite integral (see Example 1.) Note that 3 consecutive Fibonacci numbers, 5, 8, 13, appear in the equations of the two boundary circles. It is natural to consider the more general case: the region $S_n$ bounded by $y = F_{n+1}$, the circle $x^2 + y^2 = F_{n+2}^2$, and the circle $(x - F_n)^2 + y^2 = F_{n+1}^2$. Refer to the red region in Figure 12. We calculate the area $A_n$ of the region $S_n$. We then attempt to find patterns among these areas. First we assign the notations.

**Definition 4.** Consider any integer $n \geq 1$ and the $n^{th}$ Fibonacci number $F_n$. Let $S_n$ be the region bounded by the circle $(x - F_n)^2 + y^2 = F_{n+1}^2$, the circle $x^2 + y^2 = F_{n+2}^2$, and the horizontal line $y = F_{n+1}$. We define $A_n$ to be the area of the region $S_n$. 
Example 1. The area $A_5$ of the region, bounded by the following curves

$$(x - 5)^2 + y^2 = 8^2, \quad x^2 + y^2 = 13^2, \quad \text{and} \quad y = 8,$$

is given by:

$$A_5 = \frac{169}{2} \arcsin \left( \frac{8}{13} \right) - 16\pi + 4\sqrt{105} - 40 \approx 6.73.$$

Proof. By the integration formula in Lemma 3,

$$A_5 = \int_0^8 \sqrt{169 - y^2} - \left( \sqrt{64 - y^2} + 5 \right) dy$$

$$= \frac{169}{2} \arcsin \left( \frac{8}{13} \right) - 16\pi + 4\sqrt{105} - 40 \approx 6.73.$$

The areas $A_n$ are listed below for $n = 2$ to $n = 8$. 

Figure 10: Area of the red region between two consecutive arcs
\[ A_1 = \frac{1}{4} \left( 2\sqrt{3} - 4 + \frac{\pi}{3} \right) \approx 0.1278. \]

\[ A_2 = \frac{1}{4} \left( 4\sqrt{5} - 4\pi + 18 \arcsin \left( \frac{2}{3} \right) - 8 \right) \approx 0.3782. \]

\[ A_3 = \frac{1}{4} \left( -9\pi + 50 \arcsin \left( \frac{3}{5} \right) \right) \approx 0.9751. \]

\[ A_4 = \frac{1}{4} \left( 10\sqrt{39} - 25\pi + 128 \arcsin \left( \frac{5}{8} \right) - 30 \right) \approx 2.5817. \]

\[ A_5 = \frac{1}{4} \left( 8\sqrt{105} - 32\pi + 338 \arcsin \left( \frac{8}{13} \right) - 80 \right) \approx 6.7351. \]

\[ A_6 = \frac{1}{4} \left( 104\sqrt{17} - 169\pi + 882 \arcsin \left( \frac{13}{21} \right) - 416 \right) \approx 17.6586. \]

\[ A_7 = \frac{1}{4} \left( 42\sqrt{715} - 441\pi + 2312 \arcsin \left( \frac{21}{34} \right) - 1092 \right) \approx 46.2059. \]

\[ A_8 = \frac{1}{4} \left( 68\sqrt{1869} - 1156\pi + 6050 \arcsin \left( \frac{34}{55} \right) - 2856 \right) \approx 120.9941. \]

**Theorem 8.** For any integer \( n \geq 1 \), the area \( A_n \) defined previously follows the following formula:

\[
A_n = \frac{1}{4} \left( 2F_{n+1} \sqrt{F_{n+2}^2 - F_{n+1}^2} + 2F_{n+2}^2 \arcsin \left( \frac{F_{n+1}}{F_{n+2}} \right) - \pi F_{n+1}^2 - 4F_n F_{n+1} \right).
\]

**Proof.**

\[
A_n = \int_0^{F_{n+1}} \left( \sqrt{F_{n+2}^2 - y^2} - \left( F_n + \sqrt{F_{n+1}^2 - y^2} \right) \right) dy.
\]
Furthermore,

\[
\int_0^{F_{n+1}} \sqrt{F_{n+2}^2 - y^2} \, dy = \frac{1}{2} \left( F_{n+1} \sqrt{F_{n+2}^2 - F_{n+1}^2} + F_{n+2}^2 \arcsin \left( \frac{F_{n+1}}{F_{n+2}} \right) \right);
\]

\[
\int_0^{F_{n+1}} \sqrt{F_{n+1}^2 - y^2} \, dy = \frac{1}{2} F_{n+1}^2 \arcsin(1) = \frac{\pi F_{n+1}^2}{4}.
\]

Thus,

\[
A_n = \frac{1}{4} \left( 2F_{n+1} \sqrt{F_{n+2}^2 - F_{n+1}^2} + 2F_{n+2}^2 \arcsin \left( \frac{F_{n+1}}{F_{n+2}} \right) - \pi F_{n+1}^2 - 4F_n F_{n+1} \right).
\]

Recall that the ratio \( F_{n+1}/F_n \) of two consecutive Fibonacci numbers approaches the golden ratio as \( n \) goes to infinity (see Lemma 4). It is natural to ask: what is the situation for the limit value of the ratio \( A_{n+1}/A_n \)? Let us observe a few of such ratios.

<table>
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<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_{n+1} )/( A_n )</td>
<td>2.959</td>
<td>2.578</td>
<td>2.647</td>
<td>2.608</td>
<td>2.621</td>
<td>2.616</td>
<td>2.618</td>
<td>2.616</td>
</tr>
</tbody>
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<table>
<thead>
<tr>
<th>( n )</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_{n+1} )/( A_n )</td>
<td>2.618</td>
<td>2.618</td>
<td>2.618</td>
<td>2.618</td>
<td>2.618</td>
<td>2.618</td>
<td>2.618</td>
<td>2.618</td>
</tr>
</tbody>
</table>

Table 2: The Ratio Between Two Consecutive Areas

![Figure 11: The Ratio of Two Consecutive Areas](image-url)
It seems that the ratio approaches $\phi^2$, where $\phi = \frac{1 + \sqrt{5}}{2}$ is the Golden Ratio. We show that it is true in the next theorem.

**Theorem 9.** Let $A_n$ be the area defined above. Then

$$\lim_{n \to \infty} \frac{A_{n+1}}{A_n} = \phi^2 \approx 2.618.$$ 

**Proof.**

$$\frac{A_{n+1}}{A_n} = \frac{2F_{n+2}\sqrt{F_{n+3}^2 - F_{n+2}^2} + 2F_{n+3}^2 \arcsin \left( \frac{F_{n+2}}{F_{n+3}} \right) - \pi F_{n+2}^2 - 4F_{n+1}F_{n+2}}{2F_{n+1}\sqrt{F_{n+2}^2 - F_{n+1}^2} + 2F_{n+2}^2 \arcsin \left( \frac{F_{n+1}}{F_{n+2}} \right) - \pi F_{n+1}^2 - 4F_nF_{n+1}}$$

$$= \frac{2\sqrt{\frac{F_{n+2}^2}{F_{n+1}^2}} - 1 + \frac{2F_{n+3}^2}{F_{n+2}^2} \arcsin \left( \frac{F_{n+2}}{F_{n+3}} \right) - \pi - \frac{4F_{n+1}}{F_{n+2}}}{\frac{2F_{n+1}}{F_{n+2}} \sqrt{1 - \frac{F_{n+2}^2}{F_{n+1}^2}} + 2 \arcsin \left( \frac{F_{n+1}}{F_{n+2}} \right) - \pi - \frac{4F_n}{F_{n+1}} \cdot \frac{F_{n+1}}{F_{n+2}}}.$$ 

Note that

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \lim_{n \to \infty} \frac{F_{n+2}}{F_{n+1}} = \phi \quad \text{and} \quad \frac{1}{\phi} = \phi - 1. (\text{see}[10].)$$ 

Then

$$\lim_{n \to \infty} \frac{A_{n+1}}{A_n} = \frac{2\sqrt{\phi^2 - 1} + 2\phi^2 \arcsin \left( \frac{1}{\phi} \right) - \pi - 4(\phi - 1)}{2\phi \sqrt{1 - \frac{1}{\phi^2}} + 2 \arcsin \left( \frac{1}{\phi} \right) - \frac{1}{\phi^2} \pi - \frac{4}{\phi}(1 - \frac{1}{\phi})}$$

$$= \frac{\phi^2(2\sqrt{\phi^2 - 1} + 2\phi^2 \arcsin \left( \frac{1}{\phi} \right) - \pi - 4(\phi - 1))}{2\sqrt{\phi^2 - 1} + 2 \phi^2 \arcsin \left( \frac{1}{\phi} \right) - \pi - 4(\phi - 1)} = \phi^2.$$ 

Design VI involved some specific regions, such as $S_n$, which are bounded by two arcs of circles whose radius are consecutive Fibonacci numbers. Calculating the areas of these regions and arranging them in order allows us to see a new sequence approaching a common ratio. From the work above we can conclude
that $\frac{A_{n+1}}{A_n}$ creates an intriguing pattern that approaches the actual value of $\phi^2$ as the value of $n$ approaches infinity.
5  The Lesson Plan

In this paper we tried to connect geometry with algebra. We achieved this by introducing the Fibonacci numbers in the last design. These famous numbers that we can find in our daily life, but unfortunately they are not normally taught in the K-12 curriculum. The design 3 in my thesis inspired me to think about creating a lesson on this topic to teach to high school students. Creating engaging activities for this lesson will motivate students and encourage them to use their critical thinking abilities. This lesson will be designed to meet the needs of gifted students for extension beyond the standard curriculum in order to challenge them.

This is a lesson plan designed for 9th grade Algebra I Honors classes.

**Time Allowed:** 3 Class Periods

**Teacher/Grade:** Somia Benali/ 9th grade Algebra I Honor  
**Goals:** Students will discover the mathematical constant $\varphi$, the golden ratio, through several activities. They measure dimensions of a nautilus shell and human hand bones and calculate ratios of the measured values, which are close to $\varphi$.

**Learning Objectives:** After Completing the lessons, students will be able to:

1. Explain Fibonacci numbers and their origin.
2. Write the recursive formula of the Fibonacci sequence.
3. Identify $\varphi$ as the limit of the ratio of terms of the Fibonacci sequence.
4. Understand examples of $\varphi$ in nature.
5. Generate the next numbers in the Fibonacci sequence.
6. Create a Fibonacci rectangle and spiral.

**Common Core Standards:** CCSS.Math.Content.HSF-IF.A.3
Materials:
Ruler, the “Golden Ratio Project” worksheet, and the “ϕ in Nature” worksheet.

Fibonacci squares puzzle made from poster board

Procedure:

• Assess students’ prior knowledge
  
  – What is the job of a mathematician?
  
  – How can we calculate the area and the perimeter of a rectangle? Can the area be negative?
  
  – What is the definition of ratio? How do we calculate the ratio?
  
  – Do they know what Fibonacci numbers are?

• Introduce students to Fibonacci numbers.

• Discuss the history of the mathematician Fibonacci and how he discovered his famous sequence.

• Talk about the impact of Mathematics on Fibonacci numbers.
  
  – Why is the study of Fibonacci numbers important to mathematicians?

• Some Terminology
  
  – Fibonacci Sequence.
  
  – Golden Ratio.
  
  – Golden Spiral.

• Have Students work with a partner to complete table 1 of Golden Ratio Project worksheet by finding the area and the perimeter of squares whose sides are Fibonacci numbers. In addition, they will calculate the length of the arc of the quarter circle inscribed in each square.
• Assign homework for the next class: let students write the first thirteen terms of Fibonacci sequence.

• At the beginning of the next class ask two students to provide the class with the first thirteen terms of Fibonacci sequence.

• Let the pairs complete Part 2 of the worksheet together, finding the ratios of successive perimeters, areas and arcs. Start class discussion by asking students the following questions:
  
  – Do they notice any patterns?
  
  – What conjectures can they make?
  
  – Is there any relation between perimeter and area ratios?

Have students complete the squares puzzle. Have you seen the shape of the dotted line in nature?

![Fibonacci puzzle](image)

Figure 12: Fibonacci puzzle

• Have students share their worksheet answers and the squares puzzle with the class. Relate the presence of $\varphi$ in this mathematical activity. Discuss
the relationship of the lengths of the sides of the squares in the puzzle and the Fibonacci sequence.

- Have students write the general form of Fibonacci sequence.
- Assign homework for the next class: students will calculate the ratio of two consecutive terms of Fibonacci sequence.
- At the beginning of the next class ask 3 students to provide the class with the ratio of the first five terms of Fibonacci sequence.
- Introduce the concept of a mathematical constant phi.[11]
- Discuss the constant ratio.
  - Let the pairs complete the $\phi$ in Nature worksheet.
  - Have the class compare their results.
  - Students will watch the following video.
    https://www.youtube.com/watch?v=dREpRHgkjsg
- Assess students’ knowledge of Fibonacci sequence as follows
  - After the lesson is finished, the students are asked the following questions.
    * What have you learned about Fibonacci sequence?
    * How do mathematicians approach the problem?
    * Do you have a new view or new thoughts about mathematics?
6 References


7 Appendices

Appendix A
The Javascript Fibonacci Code

This algorithm is built using Javascript and it allows the user to input a parameter which refers to the term of the Fibonacci sequence the user wants to print to the console. This algorithm takes that parameter and plugs it into a for loop which is a Javascript iterator. The for loop takes two variables num and num2 which both have the value 1 since that is the value of the first two terms of the Fibonacci sequence. The for loop then adds num to num2 and then num2 to num. Modulus is used for this in order to keep track whether to add num to num2 or num2 to num. Modulus is also to identify whether the wanted term is even or odd which is used to know whether num or num2 should be printed to the console. Below the code for this program is shown.

```javascript
function fibonacci(n) {
    var num = 1;
    var num2 = 1;
    for(var i=1; i<n; i++) {
        if(i % 2 ==0) {
            num += num2;
        } else {
            num2 += num;
        }
    }

    if(n % 2 ==0) {
        console.log("The "+ n + "th term is "+ num);
    } else {
        console.log("The "+ n + "th term is "+ num2);
    }
}
```
Appendix B
Worksheet

Name ........................................

Golden Ratio Project

1-Complete the following

Fibonacci sequence: 1, 1, 2, 3, 5, 8, ...., ...., ...., ...., ...., .......

\[ F_n = \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]

<table>
<thead>
<tr>
<th>Side of the square</th>
<th>Perimeter</th>
<th>Area</th>
<th>Arc length ((\frac{3}{4}) circumference)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
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<td></td>
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<tr>
<td>144</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
2-Complete the following table

<table>
<thead>
<tr>
<th>$n$</th>
<th>Ratio $\frac{P(n+1)}{P(n)}$</th>
<th>Ratio $\frac{Arc(n+1)}{Arc(n)}$</th>
<th>Ratio $\frac{A(n+1)}{A(n)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
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<td>144</td>
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</tr>
</tbody>
</table>

$P(n)$ : The perimeter of the square with side $n$.

$A(n)$ : The area of the square with side $n$.

$ARC(n)$ : The arc length of a quarter circle with radius $n$. 

Appendix C
The $\phi$ in Nature Worksheet

We are learning about the amazing number $\phi$. This number appears many places in our world, we are going to find this number and see why it is so special.

1. Looking at the human hand picture, measure the following line segments in centimeters:

\[
\begin{align*}
A & = \\
B & = \frac{B}{A} = \\
C & = \frac{C}{B} = \\
D & = \frac{D}{C} = \\
\end{align*}
\]
2. Looking at the nautilus shell picture, measure the following line segments in centimeters:

\[
\begin{align*}
A & \quad B = \\
B & \quad \overrightarrow{A} = \\
C & \quad \overrightarrow{B} = \\
D & \quad \overrightarrow{C} = 
\end{align*}
\]