The Role of Covariational Reasoning in Pre-Service Teachers’ Meanings for Quadratic and Exponential Relationships

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THE ROLE OF COVARIATIONAL REASONING IN PRE-SERVICE TEACHERS’ MEANINGS FOR QUADRATIC AND EXPONENTIAL RELATIONSHIPS

A DISSERTATION

Submitted to the Faculty of
Montclair State University in partial fulfillment
of the requirements
for the degree of Doctor of Philosophy

by

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Abstract

THE ROLE OF COVARIATIONAL REASONING IN PRE-SERVICE TEACHERS’ MEANINGS FOR QUADRATIC AND EXPONENTIAL RELATIONSHIPS

by Madhavi Vishnubhotla

Researchers have indicated that students have difficulties recognizing quadratic and exponential change and do not maintain productive meanings for these relationships. Other researchers have documented that students are capable of developing productive meanings for mathematical ideas via covariational reasoning. This dissertation reports the results of an investigation into ways in which preservice teachers can leverage covariational reasoning to develop meanings for quadratic and exponential relationships. I collected data by engaging two preservice teachers in semi-structured clinical interviews and a semester long teaching experiment. My analyses reveal that whereas in the pre-interviews, the participants did not have meanings that supported differentiating between quadratic and exponential relationships, engaging in activities that offered opportunities to reason covariationally during the teaching experiment supported in developing productive meanings for quadratic and exponential relationships. By the end of the teaching experiment, the preservice teachers had developed ways to differentiate between these two relationships.

Keywords: Quadratic relationships, exponential relationships, covariational reasoning, teaching experiment, preservice teacher
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Chapter 1: Statement of the Problem

Quadratic functions and exponential functions are greatly emphasized in the secondary mathematics curriculum. In fact, quadratic functions and exponential functions are two of the first families of non-linear functions that students encounter. The National Council of Teachers of Mathematics (NCTM) Standards (2000) have called for connections across classes of functions and advocate that ‘understanding change is fundamental to understanding functions’ (p.40). The NCTM specifically indicates that high school students should be able to identify quantitative relationships in a situation and understand the different kinds of change that occur in different relationships. Students are expected to investigate rates of change, and compare and contrast classes of functions that have non-constant rate of change to build deeper understandings of functions.

Likewise, the Common Core State Standards for Mathematics (CCSSM) highlight that students need to compare properties of functions, distinguish between situations that can be modeled by exponential growth and polynomial growth, and recognize that a quantity increasing exponentially eventually exceeds a quantity increasing quadratically. Despite the emphasis laid by national mathematical organizations on understanding exponential and quadratic relationships, there is limited research on students’ meanings1 for these relationships.

A growing body of research has examined how students reason about two simultaneously varying quantities. Several studies have indicated that covariational reasoning can support students to develop understanding of various mathematical ideas such as rate of change (Johnson, 2012), trigonometric functions (Moore, 2014), and parametric functions (Paoletti &

1 I elaborate on my use of meanings in Chapter 2.
Moore, 2017). Particular to quadratic and exponential growth, although researchers have indicated that middle school students can reason covariationally to construct and reason about quadratic (Ellis, 2011a) and exponential relationships (Confrey, 1994; Confrey & Smith, 1995; Ellis et al., 2015) there have been no reported findings on pre-service teachers’ meanings for these growth patterns.

Researchers (e.g., Copur-Gencturk, 2015; Thompson, 2013, 2016) have argued that students learn mathematical ideas more coherently when teachers understand the ideas coherently. As teachers’ mathematical meanings influence students’ mathematical meanings, it is critical for teacher education programs and math educators to support future teachers’ development of productive meanings (Hill et al., 2008; Silverman & Thompson, 2008; Thompson, 2013, 2016). Therefore, there is a need to examine pre-service teachers’ meanings for quadratic and exponential relationships. The study I describe in this dissertation addresses this research gap by examining how pre-service teachers leverage covariational reasoning to distinguish between exponential and quadratic change.

The research question driving this study is: “How might pre-service teachers reason covariationally to develop meanings for quadratic and exponential relationships?” To support the reader in understanding my operationalization of covariational reasoning, in Chapter 2 I describe the definition of the term meanings and synthesize the definitions and constructs of covariational reasoning and highlight relevant findings from empirical studies that investigated the role of covariational reasoning in promoting the understanding of quadratic and exponential growth. In Chapter 3, I describe the methodology of clinical interviews and teaching experiments as well as the methods of data collection and analysis I used for the study. In Chapter 4, I detail the results of the semester-long teaching experiment in which the students reorganized their meanings for
quadratic and exponential relationships. Finally, in Chapter 5, I discuss the results of my study in relation to the relevant research and theoretical frameworks I leveraged as well as the limitations of the study and directions for future research.
Chapter 2: Theoretical Perspective and Review of the Literature

As I am interested in examining pre-service teachers’ (PSTs’) meanings for quadratic and exponential relationships and how they differentiate between these relationships when engaged in covariational reasoning, this chapter provides a synthesis of the existing literature related to these ideas. In the first two sections, I provide the definitions, constructs, and frameworks researchers have found useful in describing students’ quantitative and covariational reasoning. Subsequently, in the last two sections, I present researchers’ conceptualization of exponential relationships and quadratic relationships from a covariational reasoning perspective. Then I present major findings from empirical studies that investigated the role of covariational reasoning in understanding exponential and quadratic relationships.

Meanings and Understandings

Before describing the various constructs of covariational reasoning, I describe my current understanding of the terminology (e.g., ‘understanding,’ ‘meaning,’ etc.) so that the reader understands my use of these terms. Building on radical constructivist theories of knowing (von Glasersfeld, 1995), Thompson and colleagues (Thompson, 2016; Thompson et al., 2014) described that an understanding refers to a cognitive state resulting from an assimilation and a meaning refers to “the space of implications that the current understanding mobilizes – actions or schemes that the current understanding implies, that the current understanding brings to mind with little effort” (Thompson et al., 2014, p. 13). A student’s meaning, therefore, refers to the space of implications brought forward by the student experiencing an understanding. The student’s space of implications can include a collection of actions, objects, imagery, and/or
schemes\textsuperscript{2} brought forth by an understanding.

Thompson et al. (2014) differentiate between stable and in-the-moment schemes and understandings. They explained that all understandings and meanings are \textit{in-the-moment}. When a student experiences a familiar situation, the in-the-moment understanding may be the result of a stable understanding, which would result in the activation of a stable meaning. For example, when presented with a U-shaped graph in the Cartesian plane (a familiar situation), a PST may construct an in-the-moment understanding that the curve represents a quadratic function. This is likely to be the result of a stable understanding developed through repeated experiences addressing tasks which connect graphical shapes to function classes in Algebra through Calculus courses. The in-the-moment meaning that stemmed from a stable understanding would result in the activation of a stable meaning for quadratic function as a function whose graphical representation is a U-shaped graph.

However, some in-the-moment understandings and schemes are attained through a functional accommodation (Steffe, 1991) which is a modification to a scheme that occurs in the context of using the scheme. I interpret this to mean that in the context of engaging with a novel activity, a student may experience a (potentially momentary) change in her meanings to connect the current situation to her existing meanings. Meanings resulting from a functional accommodation can be momentary and can be lost when the student’s attention changes (Thompson et al., 2014). For example, if a student is learning about the polar coordinate system, she may make a functional accommodation to her meanings for graphs of quadratic functions such that a quadratic need not be represented with a U-shaped graph. If in her later activity the

\textsuperscript{2} A scheme is a mental structure that consists of an action or system of (mental or physical) actions, a state necessary to activate the actions, and an anticipation of outcomes of the actions (Thompson, 1994).
student anticipates that the graph in a polar system is U-shaped then the prior in-the-moment meaning that resulted from the functional accommodation might have been lost.

One way for students to develop stable meanings is to have repeated opportunities to engage in functional accommodations (Thompson, 2016; Thompson et al., 2014). Such repeated experiences may support students in reflecting on their activities in a way that allows them to reorganize their prior meanings into a more coherent whole. For example, Paoletti (2020) reported one PST reorganizing her meanings in the context of inverse functions when she experienced multiple functional accommodations to her in-the moment meanings. The PST reorganized her inverse function meanings to entail that a relation and its inverse relation represent the same relationship by reflecting on activities and connecting her in-the moment meanings to her prior stable meanings that entailed engaging in specific techniques when addressing inverse function prompts in graphical (i.e, reflecting over the line $y = x$) and analytical (switching-and-solving technique) representations.

In the context of this study, when I refer to a student’s meaning or understanding, I mean to imply that I infer the meaning or understanding is stable. When I describe a student constructing in-the moment meanings, I refer to the aspects of the student activity that I inferred were novel relative to their stable meanings I had previously inferred. If the student re-engages in these activities in novel situations and connects her previous in-the-moment meanings to her prior, stable, meanings to form a more coherent whole, then I consider that the student has reorganized her meanings. Lastly, I refer to a meaning the student is consciously aware that they have as an explicit meaning whereas a meaning the student is not consciously aware that they have.

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3 I note that Thompson’s (2016) description of meanings (and my use) relates to Piaget’s notions of reflected and reflective abstractions but a more detailed description is outside the scope of this paper.
have as an implicit meaning.

**Quantitative Reasoning and Covariational Reasoning**

In this study, I adopt Thompson’s (1993, 1994) framing of quantitative reasoning. Thompson (1993) emphasizes that a quantity does not exist to an individual until that individual constructs the quantity for themselves. Hence conception of quantities can vary from one individual to another. For example, let us consider a growing rectangle. In this situation, the rectangle could be the object and some of the possible quantities that one could conceive of in this situation include the length, width, perimeter, and area of the rectangle. When students conceive quantities in a given or imagined situation and construct relationships between these quantities to reason about the situation the student is engaging in quantitative reasoning (Thompson, 1993, 1994).

Building on Thompson’s characterizations of quantitative reasoning, several researchers have proposed different descriptions of covariational reasoning, but they all agree that covariational reasoning supports teachers and students to develop productive conceptions of various mathematical ideas such as rates (Johnson, 2012; Thompson, 1994), exponential functions (Castillo-Garsow, 2012; Ellis et al., 2015), trigonometric functions (Moore, 2014), inverse function (Paoletti, 2020), and graphical representation of relationships (Carlson et al., 2002; Moore & Thompson, 2015). In the following sections I provide descriptions of the different definitions and conceptions of covariational reasoning that informed my study.

**Confrey and Smith’s Conception of Covariational Reasoning**

With an emphasis on coordinating sequences in their meaning of covariation, Confrey and Smith (1995) described, “In a covariation approach, a function is understood as a juxtaposition of two sequences, each of which is generated independently through a pattern of
data values” (p. 67). They focused on tabular representations and distinguished a covariation approach to functions from a correspondence approach to functions. Whereas a correspondence approach to functions entails building a rule (e.g., \( f(x) = 2x + 1 \)) so that for any \( x \)-value one can determine a unique \( y \)-value, Confrey and Smith (1994) explained that “A covariation approach, on the other hand, entails being able to move operationally from \( y_m \) to \( y_{m+1} \) coordinating with movement from \( x_m \) to \( x_{m+1} \)” (p.137). For tables, it involves the coordination of the variation in two or more columns as one moves down (or up) the table” That is a student can describe a relationship between \( x \) and \( y \) if he can identify the pattern of change in \( x \), pattern of change in \( y \), and coordinate the changes in successive values of \( x \) and \( y \). For example, consider the relationship represented in Figure 1. A student might construct a pattern of change in the values of \( x \) (increasing by 1), a pattern of change in the values of \( y \) (increasing by 2, Figure 1 left or by a factor of 3, Figure 1 right) and coordinate these changes. For example, the student may conceive that if \( x \) increases by 1, then the value of \( y \) increases by 2 or by a factor of 3, respectively.

Confrey and Smith (1995) explained that coordinating this repeated action (or variation) in the two quantities (i.e. co-variation) supports students to define a relationship between the \( x \)-values and \( y \)-values.

![Figure 1: Examples of Confrey and Smith’s (1994) conception of covariation.](image-url)
Thompson’s Conception of Covariational Reasoning

Whereas Confrey and Smith (1994, 1995) focused on coordinating sequences of numeric values, Saldanha and Thompson (1998) proposed to extend this notion to include a more continuous perspective of covariational reasoning. They stated that their “notion of covariation is of someone holding in mind a sustained image of two quantities’ values (magnitudes) simultaneously” (p. 298).

According to Saldanha and Thompson (1998), images of covariation are developmental. They argued that in early development of covariational reasoning, one imagines tracking two quantities one after the other for some duration. Later, with an understanding of time as a quantity, the images of covariational reasoning entail coordinating the two quantities with the awareness that both quantities have a value at every moment in time. Thompson’s views on covariational reasoning build on the aforementioned description of quantitative reasoning.

Saldanha and Thompson (1998) emphasized that, “In the case of continuous covariation, one understands that if either quantity has different values at different times, it changed from one to another by assuming all intermediate values” (p. 299). For example, let us consider a growing square situation. Two quantities a student might conceive are the length of the side and area of the square. Upon conceiving the two quantities, at one moment, the student might consider the value of the side length to be $a$ units and the area to be $a^2$ sq. units. As the student’s image of covariational reasoning develops, his understanding may begin with imagining the length of side to increase to $b$ units and the area to $b^2$ sq. units. Then, the student envisions that as time progressed, the side length increased from $a$ to $b$ by taking all intermediate values between $a$ and $b$. Similarly, the student imagines that the area increased continuously from $a^2$ sq. units to $b^2$ sq. units, taking all intermediate values between $a^2$ and $b^2$. The student, thus, attempts to coordinate
quantities that are continuously changing rather than coordinating completed changes in the length and area.

**Carlson’s Conception of Covariational Reasoning**

Drawing upon the work on covariation by Confrey and Smith (1994) and concurring with Saldanha and Thompson’s (1998) view that images of covariational reasoning are developmental, Carlson et al. (2002) offered a framework (Table 1) to analyze students’ reasoning about dynamic situations that involve two simultaneously changing quantities. They proposed five mental actions that characterize students’ covariational reasoning and described the framework as an analytical tool for researchers to analyze students’ covariational reasoning.

*Table 1.* The five mental actions from Carlson et al. (2002, p. 357)

<table>
<thead>
<tr>
<th>Mental Action</th>
<th>Description of Mental Action</th>
<th>Behaviors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mental Action 1 (MA1)</td>
<td>Coordinating the value of one variable with changes in the other</td>
<td>• Labeling the axes with verbal indications of coordinating the two variables (e.g., y changes with changes in x)</td>
</tr>
</tbody>
</table>
| Mental Action 2 (MA2) | Coordinating the direction of change of one variable with changes in the other variable | • Constructing an increasing straight line  
• Verbalizing an awareness of the direction of change of the output while considering changes in the input |
| Mental Action 3 (MA3) | Coordinating the amount of change of one variable with changes in the other variable | • Plotting points/constructing secant lines  
• Verbalizing an awareness of the amount of change of the output while considering changes in the input |
| Mental Action 4 (MA4) | Coordinating the average rate-of-change of the function with uniform increments of change in the input variable. | • Constructing contiguous secant lines for the domain  
• Verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input |
| Mental Action 5 (MA5) | Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function | • Constructing a smooth curve with clear indications of concavity changes  
• Verbalizing an awareness of the instantaneous changes in the rate of change for the entire domain of the function (direction of concavities and inflection points are correct) |
The mental actions, in order, involve coordinating the *change in two variables* (MA1), the *direction of change* (increase/decrease) in the two variables (MA2), specific *amounts of change* (MA3), *average rate of change with uniform increments* (MA4), and *instantaneous rate of change* (MA5). In an attempt to make these mental actions more concrete, I describe the ways students could exhibit these mental actions in the context of a growing square. Let us consider a square that grows; specifically consider the covariation of the length of the side and the area of the square. A student may describe that as the length of the square changes, the area changes (MA1). The student may then identify that as the side length increases the area of the square increases (MA2). A student may then consider specific changes in the side length and determine the corresponding changes in the area. She may identify that for equal changes in the side length, the corresponding successive changes in area increases (MA3). That is the student’s focus is on the amount of change in one quantity with constant changes in the other quantity. A student may then identify that the area is increasing at an increasing rate for uniform increments of change in the side length (MA4) and finally, she may attend to the instantaneous rate of change of area with a continuous change in the side length over the entire domain of values that the side length can take (MA5).

Carlson et al. (2002) described the mental actions framework as an analytic tool that provides researchers ways to analyze students’ covariational reasoning. As I am interested in students coordinating amounts of change (MA3) as well as engaging in MA1-2, I focus on MA1-3 in my data analysis.

**Castillo-Garsow’s Conception of Covariational Reasoning**

Drawing on the results of a teaching experiment with high school students, Castillo-Garsow (2012) classified two different ways a student can think about variation: *Chunky* and
\textit{Smooth. Chunky thinking} is imagining change happening in completed chunks with a focus on the discrete end points of the chunk. For example, in the case of a growing square, a student may imagine the area to change from 16 sq. units to 64 sq. units if the side length changed from 4 units to 8 units without considering values between 4 and 8. Although the student may imagine dividing the interval into sub-intervals, the student attends to what occurs at the end-points of the intervals or chunks and little or no attention is paid to the intermediate values within the chunk (Castillo-Garsow et al., 2013). That is, she may reason that if the side length changed from 4 units to 6 units (a chunk smaller than 4 units to 8 units) or from 4 units to 5 units, the area would change from 16 sq. units to 36 sq. units or 16 sq. units to 25 sq. units respectively, but does not imagine that change occurs within each smaller sub-chunk.

In contrast, \textit{smooth thinking} is imagining a change in progress (Castillo-Garsow, 2012). The change has a start point but no end point. As soon as a student imagines an end point, she imagines that the change is no longer in progress. With reference to the growing square example, a student engaged in smooth thinking would imagine the side length and area increasing and taking on values in a continuum as she imagines time passing. If she attends to the quantities’ magnitudes, then she imagines both the magnitudes passing through all possible measures starting from the initial value (for e.g., 4 units of length and 16 sq. units of area). She would not attempt to slice the lengths or areas into smaller intervals and would not imagine the final values of the two quantities. Essentially, smooth thinking is inherently continuous (Castillo-Garsow et al., 2013).

Furthermore, Castillo-Garsow and colleagues emphasized that because it is difficult to imagine an ongoing change in progress forever, a student could imagine that at some point the change in progress ends. So, the student engaged in smooth thinking would have imagined that a
change has happened in a chunk. Within the change in progress, a student can generate sub-chunks, thereby anticipating smooth change over several chunks. Therefore, the researchers argued that smooth thinking could entail chunky thinking but possibly not vice-versa.

**Graphing and Covariational Reasoning**

Drawing on Thompson’s characterizations of quantitative and covariational reasoning (Saldanha & Thompson, 1998; Thompson, 2011), Moore and Thompson (2015) defined the constructs of *emergent shape thinking* and *static shape thinking* to describe students’ graphing activities. Moore and Thompson (2015) explained that static shape thinking entails “actions based on perceptual cues and the perceptual shape of a graph” (p. 784). Further, they explained that static shape thinking often entails foregrounding the shape of a graph with function names and analytic rules. For example, a student engaged in static shape thinking could interpret a parabolic graph displayed in the rectangular coordinate system as having the shape of a “parabola” and associate the shape with “quadratic function” and the analytic rule \( y = ax^2 + bx + c \). That is, the student focuses on the physical features of the graph rather than the defining property of quadratic relationship. However, such static shape thinking can be problematic when the graph is displayed in a non-Cartesian coordinate system. For example, a U-shaped or a parabola-shaped curve in the Polar coordinate system is not representative of a quadratic relationship. For students to identify a graphical representation of a quadratic relationship in the Polar coordinate system they would have to reason about the constant second differences in the values of one variable corresponding to equal changes in the other variable. Rather than relying on the shape of the curve, students would need to attend to the defining characteristic of a quadratic relationship.

Static shape thinking also can entail thematic and iconic associations (Moore & Thompson, 2015). Thematic associations entail students conceiving an association between the graph and the perceptual features of the situation. For example, a student engaging in thematic
association might draw a smooth curve when she perceives a smooth motion in a situation. Iconic associations entail students conceiving an association between the graph and the visual features of the situation. For example, Stevens et al. (2017) reported on a PST engaging in iconic translation by incorporating a semicircle in her graph to describe the relationship between a car’s total distance travelled and distance from one city (Figure 2). Since both kinds of associations entail students conceiving graphs as having a shape either matching a perceived shape in the situation or a perceived physical property, the student engages in static shape thinking (Moore & Thompson, 2015).

Contrasting with static shape thinking, Moore and Thompson (2015) described emergent shape thinking as conceptualizing a graph as a progressive trace of a point by coordinating two quantities’ values. They explained, “emergent shape thinking entails assimilating a graph as a trace in progress (or envisioning an already produced graph in terms of replaying its emergence), with the trace being a record of the relationship between covarying quantities” (Moore & Thompson, 2015, p. 785). I illustrate emergent shape thinking using the Growing Triangle task. The task shows a growing scalene triangle and students are asked to graph the relationship between the area of the triangle and its base. The actions of a student engaged in emergent reasoning could be as follows:

1) The student conceives that the area and base are two covarying quantities whose magnitudes increase simultaneously.
2) The student constructs a coordinate system with a pair of axes that represents each quantity on an axis.

3) The student conceives a point in the coordinate system that simultaneously represents the magnitudes of the two quantities (Figure 3a).

4) The student imagines how this point moves as the area and base length change so as to make a trace of the point (Figure 3b-c)

![Figure 3. Graphical representation of the Growing Triangle task.](image)

That is, by engaging in smooth reasoning (Castillo-Garsow, 2012) and imagining the side length and area increasing by taking values in a continuum as they imagine time passing, the student anticipates the graph as representing a trace in progress, as opposed to imagining the graph as a curve produced all at once. Because emergent shape thinking entails images of covariation, a student can think emergently when he engages in chunky reasoning (Castillo-Garsow, 2012) by imagining a simultaneous change in side length and area happening in chunks. In this case, the student may plot discrete points because he does not anticipate change to occur within each chunk.

Researchers studying PSTs’ interpretations of graphs reported that PSTs predominantly engage in static shape thinking (e.g., Moore et al., 2013; Moore et al., 2019; Moore et al., 2016). For example, Moore et al. (2019) described PSTs responses when addressing a hypothetical student’s work graphing \( y = 3x \) (see Figure 4 for the hypothetical student’s work). The
researchers asked the PSTs to explain how the student might have been thinking when sketching the graph. Consistent with static shape thinking (Moore & Thompson, 2015), many of the PSTs’ claimed that the student’s graph was incorrect due to visual features of the graph (e.g., the slope was not steep enough, when the \(x\)-axis is horizontal, the graph should not go down and to the left). Moore et al. (2019) reported that 65% of PSTs and 44% of the in-service teachers (ISTs) in their study claimed that the student was incorrect due to the line appearing to have a negative slope when the \(x\)-axis was represented horizontally. Hence, there is a need to find ways to support PSTs in understanding graphs as emergent traces of covarying quantities.

![Graph](image)

*Figure 4. A hypothetical student’s graph representing the relationship \(y=3x\) (Moore & Thompson, 2015).*

**Exponential Growth and Covariational Reasoning**

In the following section, I present researchers’ conceptualization of exponential growth from a covariational reasoning perspective, and then discuss the literature that focuses on role of covariational reasoning in understanding exponential relationships. Both Confrey and Smith (1994, 1995) and Thompson (2008) described different ways that a student might imagine exponential growth based on their characterizations of covariational reasoning.
Confrey and Smith’s Characterization of Exponential Growth

Confrey (1994) described multiplication through an operation called splitting and defined a split as “an action of creating simultaneously multiple versions of an original” (p. 292). For example, let us consider a whole that is split into two. Then each part is split into two to create four parts and repeating this action would create a tree diagram (Figure 5). Consequently, a splitting structure of repeated splits is constructed and we could say that 64 would be the result of six 2-splits.

Confrey (1994) established multiplication as the result of some \( n \)-splits on an original whole and proposed that such splitting structures produce geometric sequences. Confrey explained that, in a \( n \)-splitting world, multiplication by \( n \) is the invariant multiplicative action between a successor and its predecessor, and \( n \) is called a unit of growth. In the example discussed above, 2 would be called the unit of growth and the invariant action is the repeated multiplication by 2.

![Tree diagrams illustrating 2-splitting.](image)

*Figure 5.* Tree diagrams illustrating 2-splitting.

Another operation, a complement to splitting, that described is counting. In the counting world, the constant action is adding 1. The invariant action between a successor and its predecessor in the counting world is the repeated action of adding 1. While splitting structures produce geometric sequences, counting structures produce arithmetic sequences.
Confrey and Smith’s characterization of exponential growth stems from the two operations: splitting and counting. According to Confrey and Smith (1995), “the construction of a counting and a splitting world and their juxtaposition through covariation” (p. 80) provides the basis for exponentiation. Confrey and Smith (1994, 1995) argued that a productive way for understanding exponential growth arises when students coordinate a repeated counting action and a repeated splitting action. They worked with students to examine their understanding of exponential growth and described that rates, which they defined as a unit per unit comparison, where unit is the constant repeated action, could be conceived of additively or multiplicatively. They explained that exponential growth can have either an increasing additive rate of change or a constant multiplicative rate of change. An example to differentiate between the two rates of change is as follows: The number of cells growing in a lab experiment is recorded every hour. The first five entrees are presented in the table in Figure 6.

<table>
<thead>
<tr>
<th>T</th>
<th>C</th>
<th>ΔC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>+6</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
<td>+18</td>
</tr>
<tr>
<td>3</td>
<td>27</td>
<td>+54</td>
</tr>
<tr>
<td>4</td>
<td>81</td>
<td>+162</td>
</tr>
<tr>
<td>5</td>
<td>243</td>
<td></td>
</tr>
</tbody>
</table>

*Figure 6. Growing cells and the differences in the growing number of cells (adapted from Confrey and Smith (1994)).*

A student could calculate the differences (Δc) in successive values of c and note they are 6, 18, 54, and 162. He could find the difference in the successive values of time (Δt) which is always 1. He coordinates these two differences (Δc and Δt) to conceptualize the additive rate of
change. Therefore, additive rate of change = $\Delta c/\Delta t = (c_2 - c_1)/(t_2 - t_1)$. It can be noted from Figure 3 that the additive rate is increasing. Hence, exponential growth has an increasing additive rate of change.

Instead, if the student considers the ratio of succeeding values of $c$ represented by $@c = c_2/c_1$, for the constant unit change in $t$-values, we get a different notion of rate. Naming $@c$ as the multiplicative unit, Confrey and Smith defined multiplicative rate of change as the multiplicative unit per unit of time. Therefore, multiplicative rate of change = $@c/\Delta t$. It can be noted from the table in Figure 7 that the multiplicative rate of change is 3 and remains constant when we consider any set of succeeding values. That is, a student engages in coordinating a repeated multiplication in the $y$-values with additive differences in the $x$-values. Hence, exponential growth has a constant multiplicative rate of change.

<table>
<thead>
<tr>
<th>T</th>
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<td>4</td>
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</tr>
<tr>
<td>5</td>
<td>243</td>
<td>3</td>
</tr>
</tbody>
</table>

*Figure 7. Growing cells and the ratio of succeeding values (adapted from Confrey and Smith (1994)).*

**Thompson’s Characterization of Exponential Growth**

In the model of exponential growth proposed by Confrey and Smith (1995), one compares successive values of quantities as they grow. The constant multiplicative rate of change is evaluated as $@c$ per $\Delta t$, where $@c = c_2/c_1$ and $\Delta t = t_2 - t_1$. However, through this model, Thompson (2008) argued that different values of $\Delta t$ would produce a different constant
multiplicative rate of change for the same underlying relationship; that is the value of $c_2/c_1$ is dependent upon the size of $t_2 - t_1$. Let us consider the example discussed in the previous section. If the underlying relationship between the two quantities in Figure 3 is $3^t$, we notice that the constant multiplicative rate of change is 3 when $\Delta t$ is 1. However, the constant multiplicative rate of change would be $3^2$ for $\Delta t = 2$, and $3^{1/3}$ for $\Delta t = 1/5$ and so on. Hence, Thompson argued that in the context of continuous variation, it is difficult to imagine exponential growth if the underlying idea is that all multiplicative change happens by a split and growth happens by a constant multiplier.

Providing a different way of understanding exponential functions, Thompson (2008) emphasized that “a defining characteristic of exponential functions is that the rate at which an exponential function changes with respect to its argument is proportional to the value of the function at that argument” (p. 39). In other words, the rate of change of a function at a value is proportional to the value. Thompson (2008) also pointed that “the idea that an exponential function’s rate of change is proportional to the value of the function does not arise easily in the splitting approach” (p. 39). He argued that in the context of continuous variation, a productive way to approach exponential reasoning is by introducing simple interest and compound interest.

In a simple interest model, the principal is set at the beginning of the investment and the amount of money in a deposit grows at a constant rate with respect to time and the total amount of the investment over time can be represented by a linear function. The value of the account grows at a rate that is proportional to the principal and the amount paid in interest is proportional to both the time and the initial investment.

Now, let us imagine the interest to be compounded at the end of every year. For example, if a deposit account starts with $P$ dollars and the interest rate is 5% per year, throughout the first
year the value of the investment would be $P$. During the entire second year and until the end of the second year, the value would be $P(1.05)$. That is, for every moment of time during the second year, the value of the account will be $P(1.05)$, and $P(1.05)^2$ during the third year. Therefore, we can imagine that the interest grows as simple interest in each compounding period. Therefore, the accumulated value is a linear function in each compounding period and this linearity in each compounding period would give rise to a step function over the entire time period.

Figure 8. An investment with a Principal of $100 compounded annually (adapted from Thompson (2008)).

If we consider to compound the investment over time periods shorter than a year, that is over months, weeks, days, minutes, and seconds, the interest accrues at a constant rate over every small interval, and the rate of change is proportional to the principal at the beginning of that interval. So, as we compound the interest more frequently, this would result in the piecewise function to gradually changing into a smooth (continuous) function. As the number of compounding periods increase, the time interval decreases and as the size of the interval decreases, the account value at the beginning of each interval is almost equal to the value at every point within the interval. In Figure 8 the red-colored graph is a step function consisting of line segments representing the linearity in each compounding period, and the blue-colored graph
represents a smooth function, showing the value of the principal over time. Thus, Thompson (2008) characterized exponential growth as a rate of change that is always proportional to the function’s value.

**Empirical Studies on Covariational Reasoning and Exponential Relationship**

Existing studies focusing on students’ and teachers’ meanings for exponential functions have documented (a) college students’ difficulties in understanding the rules of exponentiation (Weber, 2002), (b) prospective middle and high school teachers’ difficulties in recognizing exponential growth and properties of exponents (Davis, 2009; Presmeg & Nenduradu, 2005), (c) prospective high school teachers’ difficulties in translating data from a table of values to either recursive representations or the closed-form equation \( y = a^x \) (Davis, 2009), and (d) PSTs’ tendencies to incorrectly graph straight lines to represent exponential data (Alagic & Palenz, 2006; Presmeg & Nenduradu, 2005).

Researchers (e.g., Castillo-Garsow, 2012; Confrey & Smith, 1994, 1995; Thompson, 2008) have argued that students and teachers develop robust meanings for exponential relationships when engaged in covariational reasoning. However, research on students’ and teachers’ meanings for exponential relationships by reasoning with quantities is scant. Castillo-Garsow (2012) conducted teaching experiments with high school Algebra students to create models of students’ understandings of covariation and exponential growth. He found different ways of students’ covariational reasoning, chunkily and smoothly, in the context of interest-bearing bank accounts. Hence, I consider his study as contributing to defining the construct of covariational reasoning, and therefore, discussed it earlier in the chapter.

Ellis and her colleagues (Ellis et al., 2012; Ellis et al., 2015) and Strom (2006) investigated the role of covariational reasoning in middle school students’ and ISTs’
understandings of exponential relationships, respectively. Strom (2006), in her dissertation study, examined how in-service secondary mathematics and science teachers reason about exponential growth. One of the tasks in the task-based clinical interviews required the teachers to think about the mass of a radioactive substance at 1:30 pm if it was decaying in such a way that there were 20 grams at 1:00 p.m. and 10 grams at 2:00 p.m. Teachers could extrapolate the values at 12:00 p.m. and 3:00 p.m. However, interpolating between 1:00 p.m. and 2:00 p.m. seemed to be an obstacle for the teachers. They had difficulty in reasoning through exponential behavior between half-life intervals. In fact, most teachers demonstrated linear reasoning to this task and provided 15 grams as the expected mass of the radioactive substance at 1:30 p.m.

Strom found that while some teachers attempted to coordinate the direction of change in the mass of the substance with changes in time (i.e., the amount of the substance decreases as time increases), they did not exhibit mental actions beyond MA 2 as described by Carlson et al. (2002). Strom concluded that the coordination of the images of two quantities changing together proved to be a weakness for many of the teachers, especially when they grappled with the decay and half-life situations.

In another study, Ellis and colleagues (2012; 2015) over a span of three weeks conducted a teaching experiment in which they studied three eighth graders who explored exponential growth by reasoning with the quantities, height and time, as they investigated the growth of a cactus plant called a Jactus. The task was designed on a dynamic GeoGebra applet in which the students could manipulate the height of the Jactus by sliding its base on a time axis. The applet showed the image of the Jactus with a height that varied exponentially with elapsed time.

Students were presented with a situation in which the initial height of the plant was 1 inch and its height quadrupled every week. Ellis et al. (2015) reported that students initially
envisioned the cactus’s height as a result of repeated quadrupling—the height would be 4, 16, 64 and so on. Students did not take into account the amount of time it took for the plant’s height to change. Gradually, as students encountered more tasks with different growth factors, they began to consider time as a quantity along with the height of the plant. Given that a plant doubles every week, they could explicitly state that the plant doubles from week $n$ to week $n+1$. Eventually, students could imagine the height doubling with an imagined one-week increase. In other words, students began to attend to the corresponding time values while attending to the increase in height. Ellis and her colleagues characterized this generalization as a “conceptual shift” from repeated multiplication to initial coordination of multiplicative growth in height with additive growth in time.

A second conceptual shift noted by the researchers was when they presented students with ordered pairs of height and time. Students began to coordinate the ratio of heights with increases in time for small week time spans followed by larger time spans. For instance, when given two ordered pairs $(0, 0.1)$ and $(2, 1.6)$, students divided 1.6 by 0.1 and claimed that the height of the plant had grown by 16 times in two weeks. Finally, a third conceptual shift was noted when students began to coordinate between the multiplicative growth of height values with the additive growth for time values in which the time span was less than one week. For instance, the researchers asked the students to determine how the plant grew every 0.1 week given that it had a height of 12.513 inches at 2.3 weeks and a height of 13.967 inches at 2.4 weeks. Students stated that the plant would grow 1.116 (ratio of the two heights 13.967 and 12.513) times as tall each tenth of a week.

This study showed that as students worked through the activities, they exhibited key shifts in their thinking and attended to how the two quantities changed together over intervals of
varying size. Grappling with situations of small increments in time played a role in supporting their ability to develop a correspondence relationship between the two quantities.

Ultimately, Ellis and colleagues argued that reasoning with covarying quantities facilitated students in making sense of non-natural exponents (such as 2.3, 2.4), as they envisioned the plant growing continuously over time. Moreover, this approach which is consistent with Confrey and Smith’s (1994, 1995) characterization of covariational reasoning, assisted students transition from understanding exponential growth as repeated multiplication to coordinating multiplicative ratios of height values with additive differences for the time values and thereby build robust understandings of exponential relationships.

To sum up, the findings of the two studies discussed above indicate that students can develop robust meanings for exponential relationships when engaged in covariational reasoning. However, as argued before, there is a dearth of empirical studies focusing on investigating the role of covariational reasoning in building PSTs’ meanings for exponential relationships in the literature and additional research is needed. Given my interest in examining the role of covariational reasoning in PSTs’ differentiating between quadratic and exponential relationships, in the next section, I synthesize the research on quadratic relationships with a focus on covariational reasoning.

**Empirical Studies on Covariational Reasoning and Quadratic Relationships**

Studies focusing on quadratic functions have primarily detailed students’ difficulties related to (a) the solving of quadratic equations (Bossé & Nandakumar, 2005; Kotsopoulos, 2007; Zaslavsky, 1997), (b) the graphical representations of quadratic functions (Zaslavsky, 1997; Zazkis et al., 2003), (c) students’ tendencies to incorrectly generalize from linear situations (Chazan, 2006; Ellis & Grinstead, 2008), and (d) the role of parameters $a$, $b$, and $c$ in the general
form of a quadratic function (e.g., Ellis & Grinstead, 2008; Zaslavsky, 1997). These studies primarily focus on students’ generalizations of the connections between algebraic and graphical representations. Some researchers (Ellis & Grinstead, 2008; Lobato et al., 2012) proposed that providing students with opportunities to reason covariationally helps students in building robust meanings for quadratic relationships. These researchers have emphasized that the big idea foundational to quadratic relationships is that the rate of change of the rate of change is constant.

Ellis (2011a, 2011b) introduced quadratic phenomena to eighth grade students by engaging them in exploring relationships between the lengths, heights, and areas of rectangles that grew while maintaining their length/height ratios. The students worked with a computer simulation of a growing rectangle in Geometer’s Sketchpad to explore changes in the dimensions as it grew and shrank while maintaining the ratio of its length to width. Ellis (2011a, 2011b) described that students imagined what would happen to the area if the length changed by equal amounts and created their own tables of data, graphs, and equations to represent the phenomena they observed.

Ellis (2011a, 2011b) elaborated on how the students conceived of the first difference in area as an increasing rate of growth and the second differences as a constant with uniform increases in the length. Ellis emphasized that students’ images of the changing length and width values supported their ability to make sense of the tables they developed and create correspondence rules. Figure 9 shows the work of one student recording the changes in a growing rectangle. Ellis explained that because the student reasoned directly using the relationships between the quantities height, length, area, and the increase in each of these quantities, he could conclude that each time the rectangle grew by H units in height and L units in length, the difference in the rate of growth (DiRoG) was 2HL.
Figure 9. Table of length/width/area values of a growing rectangle (reproduced from Ellis (2011a)).

Students could frequently shift between the covariation and correspondence views of the quadratic function, which enabled them to make connections between the value of \(a\) in \(y = ax^2\) and the second differences for area. That is, students were able to generalize that the difference in the rate of growth is \(2a\) in \(y = ax^2\) indicating that the rate of change of the rate of change of a quadratic function is constant.

The findings of Ellis’s (2011a, 2011b) studies indicate that exploring the coordinated changes between covarying quantities and relying on the relationships between the quantities helped students develop some sophisticated meanings about quadratic relationships. Nevertheless, the students’ conceptions about quadratic change had their limitations. For example, the students appeared to view quantities as varying in repeated discrete whole-unit increments (changes in which the length or height increased by 1 unit) and did not conceive of the quantities as varying continuously.

Lobato et al. (2012) also studied middle school students’ understandings of quadratic relationships and like Ellis (2011a, 2011b) emphasized that the foundational idea for quadratic relationships is conceiving a constant change in the rate of change. Leveraging velocity and
acceleration context, they provided evidence that at least some eighth grade students could develop such foundational meanings via reasoning covariationally about distances, velocities, acceleration, and time.

To sum up, the studies discussed above suggest that a focus on covariational reasoning and the idea that the constant changes in the rate of change is foundational in constructing robust meanings for quadratic relationships. However, there are no studies that investigated PSTs’ meanings for quadratic relationships. Consequently, this section points to the need to explore PSTs’ meanings for quadratic relationships, especially with a focus on covariation reasoning.

**A Conceptual Analysis of Leveraging Covariational Reasoning to Differentiate Between Quadratic and Exponential Relationships**

I have built my argument for the need of a study devoted to investigating PSTs’ meanings for quadratic and exponential relationships using the construct of covariational reasoning and exploring the possibility of supporting them in differentiating between the two relationships. I leverage one use of conceptual analysis described by Thompson (2008), namely “describing ways of knowing that might be propitious for students’ mathematical learning” (p. 46) to describe ways students may leverage their covariational reasoning in order to reason about quadratic and exponential relationships. The conceptual analysis stems from the work of researchers examining students’ activities and developing meanings for various mathematical ideas including rate of change, polynomial and non-polynomial functions, and inverse functions (Carlson et al., 2002; Ellis, 2011a; Ellis et al., 2015; Johnson, 2012; Moore, 2014; Paoletti, 2020).

Specifically, extending the research (e.g., Moore et al., 2016; Paoletti, 2020; Paoletti & Moore, 2017) showing that PSTs can develop meanings for various relationship classes when
engaging in the mental actions described by Carlson et al. (2002), I used existing tasks in the research literature (e.g., the Power Tower task, Moore et al. (2014)) and designed tasks with the intention of supporting them in identifying amounts of change of one quantity for equal changes of a second quantity (MA3) as a way of representing covariational relationships. Since for both quadratic and exponential growth, for equal amounts of change in one quantity, the amounts of change in the second quantity increase by more, I conjectured that I might be able to support the PSTs in reorganizing their meanings for, and therefore differentiate between, the two growth patterns after they regularly leveraged amounts of change reasoning when constructing and representing relationships between quantities.

Drawing from the literature on quadratic relationships discussed in the previous sections (Ellis, 2011a, 2011b; Lobato et al., 2012), for my study I characterize a productive meaning for quadratic relationships to entail conceiving a covariational relationship by coordinating constant second differences in one quantity with uniform increases in the other quantity. Hence, I designed tasks that I intended to support PSTs in (a) identifying amounts of change as a quantity unto itself and (b) comparing these amounts of change to identify constant differences in the amounts of change. Additionally, as researchers’ have shown, students’ meanings can be constrained by static shape thinking (e.g., Moore et al., 2013; Moore & Thompson, 2015), I included task in non-canonical coordinate systems (i.e. polar coordinates) to provide PSTs opportunities to reorganize their meanings in a way that emphasizes the covariational relationship instead of visual properties of a graph.

With respect to exponential relationships, I designed tasks that I conjectured may support PSTs in leveraging (and possibly connecting) the various forms of reasoning in the extant literature. For instance, as reasoning about amounts of change as described by Carlson et al.
(2002) entails increasing additive differences for exponential growth, I conjectured that students would engage in such reasoning as characterized by Confrey and Smith’s (1994, 1995). Additionally, I designed tasks that had the potential for PSTs to leverage either (or both) of Thompson’s (2008) characterization of exponential growth as entailing the rate of change being proportional to the value of the function and Confrey and Smith’s (1994, 1995) characterization of exponential growth as entailing a constant ratio of consecutive values. As teaching experiments naturally entail the researcher attempting to adopt the participants’ ways of reasoning in order to model their developing understandings, I was prepared to support PSTs if they leveraged any of these ways of reasoning (or other ways of reasoning not addressed in the literature) as they addressed tasks I intended to support them in developing meanings for exponential growth.
Chapter 3: Methodology and Methods

The primary goal of this study was to investigate PSTs’ meanings for quadratic and exponential relationships and examine how they leverage covariational reasoning to distinguish between exponential and quadratic change. Recall that my research question was:

How might pre-service teachers reason covariationally to develop meanings for quadratic and exponential relationships?

In this chapter, I first describe the methodology I used to address my research question which serve as the foundation that underpins the research design and data analysis. Next, I describe my methods, which includes the context, course, and class setting in which the teaching experiment was situated, and the selection procedure of the participants. Then, I provide an overview of my methods of data collection and focus on how these enable me to address my research question.

Methodology

In order to conduct the study, I used a combination of clinical interviews (Clement, 2000; Goldin, 2000; Hunting, 1997) and a teaching experiment (Steffe & Thompson, 2000). In this section, I describe the methodological underpinnings of clinical interviews and teaching experiments.

Clinical Interviews

To gain insights into students’ current meanings for exponential and quadratic relationships, I used clinical interviews before I began the teaching experiment sessions. To examine if there was a shift in their meanings after the teaching experiment sessions, I used clinical interviews as post-interviews. The pre- and post-interviews were one-on-one semi-structured task-based clinical interviews (Clement, 2000; Goldin, 2000; Hunting, 1997).
Designed to elicit naturalistic forms of thinking, clinical interviews involve a researcher, a student, mathematical tasks, and a dialogue between the researcher and student in relation to the mathematical tasks (Clement, 2000; Goldin, 2000; Hunting, 1997). The researcher does not look for a right or wrong answer, but instead encourages the students to explain their actions and ‘think aloud’ as they work through the tasks (Goldin, 2000; Hunting, 1997).

Clinical interviews are aimed to obtain descriptive reports of students’ problem solving and to develop models of students’ meanings where “a model is no more than a plausible explanation of children’s constructive activities” (Cobb & Steffe, 1983, p. 92) that a researcher formulates in the context of her interactions with the students. They allow a researcher “to collect and analyze data on mental processes at the level of a subject's authentic ideas and meanings, and to expose hidden structures and processes in the subject’s thinking that could not be detected by less open-ended techniques” (Clement, 2000, p. 547). A researcher conducting a clinical interview aims at understanding students’ current knowledge and not at provoking shifts in their meanings.

Semi-structured task-based clinical interviews, generally, have a definite opening prompt with provisions for heuristic and follow-up questions leading the students to predict, observe, and explain (Clement, 2000; Hunting, 1997). The researcher analyzes the students’ verbal and non-verbal actions to make inferences about their mathematical meanings in the moment. Thus, the researcher’s goal in a clinical interview is to elicit the activity that provides insights into a student’s meanings and ways of reasoning.

**Teaching Experiment**

Whereas in clinical interviews researchers seek to gain insight into students’ meanings and reasoning without intending to bring a change in their meanings, in a teaching experiment,
researchers strive to “learn what change they can bring forth in their students and how to explain that change” (Steffe & Thompson, 2000, p. 295). My primary methodology was the teaching experiment methodology (Steffe & Thompson, 2000)- an exploratory tool for a researcher intended to explore students’ mathematical learning and reasoning. Teaching experiments provide a way to enact the radical constructivism epistemology (von Glasersfeld, 1995) in research and researchers have used teaching experiments to explore and explain students’ mathematical activity. The goal of a teaching experiment is for the researcher to construct, test, and refine models of students’ mathematics (Steffe & Thompson, 2000).

Consistent with radical constructivist epistemology (von Glasersfeld, 1995), I note that these models are a researcher’s interpretations and explanation of students’ possible cognitive processes that seem viable from the researcher’s perspective (Cobb & Steffe, 1983). Steffe and Thompson (2000) explain that a student’s mathematical reality is constructed by a student as a result of his interaction in his physical and socio-cultural world, and therefore unique to the student. This implies that a researcher cannot describe a student’s mathematical realities in ways that guarantee isomorphism between their descriptions and the student’s reality. Steffe and Thompson (2000) referred to the unknowable mathematical reality of students as students’ mathematics. A researcher attempts to understand students’ mathematical reality by interpreting their words, actions, ways of thinking, and thereby constructs models of students’ mathematics. Steffe and Thompson (2000) defined the researcher’s interpretation of students’ mathematics as the mathematics of students. That is, the terms students’ mathematics and mathematics of students bring out the distinction between students’ meanings which they bring to bear via actions and behaviors, and the researcher’s interpretations of students’ meanings via interpretations of these actions and behaviors.
A teaching experiment consists of a sequence of teaching episodes that include a teaching agent (the researcher), one or more students, and a witness of the teaching episodes (Steffe & Thompson, 2000). Each session is video and audio-recorded which can serve as a record of the sessions and allow for on-going and retrospective analyses. Prior to the teaching episodes, the researcher formulates a hypothesis and a set of tasks or situations to test the hypothesis. However, during a teaching episode, it is possible that students will engage in the mathematical activities in unexpected ways. Steffe and Thompson (2000) described that because of the students’ unanticipated ways of operating with the tasks, the researcher might have to abandon her initial hypotheses, and formulate and test new on-the-spot hypotheses to build models of students’ mathematics.

The researcher builds an initial model based on her interpretations of students’ actions and language in the context of interacting. In a teaching experiment, the researcher strives to examine if her model of the student’s mathematics remains viable. Therefore, she poses different tasks and makes predictions about how students will respond to the tasks designed to test her models. The researcher attempts to see if students’ actions and behavior are compatible with her hypothesized models. Whatever a student says or does when engaged with the mathematical activity serve as data that can support a researchers’ hypothesized models or can allow her to modify her current models based on new information. The researcher generates a new hypothesis and attempts to test her modified model. As the cycle of formulating, testing, re-generating hypotheses continue, the researcher constructs firmer models and refines them to warrant that her developing models are adequate to account for the students’ observable words and actions (Steffe & Thompson, 2000).

Unlike a clinical interview, in which a researcher does not intend to support shifts in a
student’s meanings or reasoning, in a teaching experiment the researcher aims to understand how a student might develop new ways of thinking of a mathematical idea. In my study, along with seeking to understand preservice teachers’ current meanings for quadratic and exponential relationships, I intended to investigate how reasoning covariationally may support them developing more sophisticated meanings for quadratic and exponential relationships. Thus, a combination of task-based semi-structured clinical interviews and teaching episodes comprised an appropriate methodology to address my research question.

**Data Analysis**

Consistent with the teaching experiment methodology (Steffe & Thompson, 2000) in order to analyze the data in this study, I engaged in one of the four uses of conceptual analysis as described by Thompson (2008). He described this use of conceptual analysis is “in building models of what students actually know at some specific time and what they comprehend in specific situations” (Thompson, 2008, p. 45). That is, to analyze the data in a teaching experiment, a researcher uses a conceptual analysis to build models of students’ mathematics by interpreting their observable behaviors and mental actions.

Data analysis has both generative and convergent purposes (Clement, 2000). When analyzing data for generative purposes, a researcher explores actions that are unfamiliar and for which there is little research explaining these actions, and thereby generates ways to describe the students’ actions and responses. On the other hand, analyzing for convergent purposes leads to a refinement of the data. The researcher attempts to identify students’ activities and explain their actions and behaviors within a determined framework.

Steffe and Thompson (2000) described that it is important for the researcher to engage in videotape analysis retrospectively because some of the interactions during the teaching episodes
may happen spontaneously and outside the awareness of the researcher. Analysis of the videos offers opportunity to the researcher to revisit the records of their interactions and the initial models of the students’ mathematics built during the sessions. However, there might be instances when the researcher may not identify an interaction as having been experienced before. The researcher then attempts to modify these initial models by analyzing the data with convergent purposes. Thus, retrospective analysis at the end of the teaching experiment would provide the researcher insight into students’ interactions that were not available during the teaching episode and help modify or stabilize her original models of the students’ mathematics (Steffe & Thompson, 2000).

Methods

In this section I articulate how I leveraged the clinical interview and teaching experiment methodologies in my study. Prior to doing this, I first describe the context, course, and class setting in which the teaching experiment was situated. Additionally, I provide information about the participants of the study and an overview of the data collection.

Context

The teaching experiment was situated in a course designed as part of a secondary mathematics education teacher preparation program at a large state university in the northeast United States. The program consists of four semesters in which courses are taken by mathematics majors who are seeking to obtain a bachelor’s degree in mathematics with a K-12 mathematics teacher certification. Although the certification spans K-12, the intent of the program is to prepare students to teach at the middle and high school level. The study is situated in the first (of two) methods of teaching mathematics course for this program.
The course instructor designed the first part of the methods course to model for the PSTs effective pedagogy that includes the students engaging in small group work, supporting students in productive struggle, and organizing productive classroom discussions. As this part of the course does not have a prescribed content to cover, the course instructor (heretofore referred to as CI) and I were able to design the in-class activities to explore middle and secondary school mathematics topics from a quantitative reasoning and covariational reasoning perspective. While in previous semesters the CI has used a subset of the course materials already designed to engage students in reasoning covariationally and quantitatively, we drew from the tasks in the existing literature on covariational reasoning, quadratic relationships, and exponential relationships (e.g., Ellis, 2011a; Johnson, 2012) to design new tasks specifically to support students’ reasoning about quadratic and exponential relationships. In Chapter 4, I provide a detailed description of the task design.

Nineteen students were enrolled in the course and the class met twice a week for seventy-five minutes per class. Most of the class time was spent with students working on tasks in pairs or small groups sharing 2-foot by 3-foot white-boards to do their work. Students explored tasks presented by the CI on their laptops or phones and all students worked on the same tasks. The CI would walk around the classroom, interact with each group, and ask students to explain their reasoning followed by engaging the whole class in a discussion that entailed having the groups present their work and solutions to the whole class. This setting was conducive to conduct a teaching experiment.

Participants of the Study

The participants of my study were two undergraduates, Rebecca and Josie (pseudonyms), who were enrolled in this course. At the time of the study, Rebecca and Josie had completed a
sequence of calculus courses and were in the second and first semesters of the program, respectively. I chose Rebecca and Josie on a voluntary basis from a pool of four students who volunteered to participate. I conducted pre-interview with all four students. Based on their activities in the pre-clinical interviews, I opted to work with Rebecca and Josie for several reasons. First, both exhibited similar ways of reasoning with respect to quadratic and exponential growth. For example, both students relied on the shape of the curve to identify the relationship represented by a graph. Second, each student showed a willingness and ability to articulate their thinking which would support me in building models of each student's mathematics.

**Data Collection**

Table 2 presents an overview of the data collection efforts and the researcher is noted as “MV”, the class instructor as “CI”, and a graduate assistant as “GS”. In what follows, I outline how I conducted the clinical interviews and teaching experiments in this study.

*Table 2. An overview of data collection*

<table>
<thead>
<tr>
<th>Date</th>
<th>Sessions</th>
<th>Researchers</th>
<th>Tasks</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pre-Clinical Interviews, 1-1 Sessions, Out of Class Sessions</td>
<td></td>
<td></td>
</tr>
<tr>
<td>29-Jan 2019</td>
<td>Josie</td>
<td>MV, GS</td>
<td>Interpretation of graphs</td>
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<td>Interpretation of Tables of Values</td>
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<td>Growing Tree Problem</td>
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<tr>
<td>30-Jan 2019</td>
<td>Rebecca</td>
<td>MV, CI</td>
<td>Interpretation of graphs</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Interpretation of Tables of Values</td>
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<td>Growing Tree Problem</td>
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<tr>
<td></td>
<td>Teaching Experiment, 11 Paired Sessions, In Class</td>
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<td>31-Jan 2019</td>
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<td>MV, CI, GS</td>
<td>Carnival Ride</td>
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</tr>
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<td>07-Feb 2019</td>
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<td>14-Feb 2019</td>
<td>Covariational Reasoning</td>
<td>MV, CI, GS</td>
<td>Power Tower Task</td>
</tr>
<tr>
<td>19-Feb 2019</td>
<td>Quadratic Relationships</td>
<td>MV, CI, GS</td>
<td>Growing Tree problem</td>
</tr>
</tbody>
</table>
Data Corpus

In summary, the data consisted of:

- Pre-clinical interviews with each student (60 minutes session per student)
- In class teaching experiment sessions with a pair of students (11 sessions)
- Out-of-class teaching experiment sessions with each student (60 minutes session per student)
- Post-clinical interviews with each student (60 minutes session per student)

The data I collected from the above mentioned sessions include: Video-taped clinical interview and teaching experiment sessions, pictures of board work of the students, scans of the students’ work in each clinical interview, a written journal documenting my thinking of students’ actions and behaviors at the end of each session, audio recordings and written notes of my discussions.
with the research team. During the clinical interviews and the teaching experiment sessions, I used a laptop computer to present tasks to the pair. In the video recordings the camera mainly focused on the students’ written work but I also captured their hand motions and facial expressions when possible. I scanned all written work from the clinical interviews and took pictures of their white-board work. In addition to the video recording, I used the laptop computer’s inbuilt software to audio record along with screen recording so that I could capture the students’ engagement with dynamic applets on the computer.

**Clinical Interviews in This Study**

First, I conducted pre-clinical interviews with each student individually outside of class time for approximately 60 minutes. I conducted the interviews in the first week of the semester. For each clinical interview an observer was present to video record the interview and to debrief with after the interview to discuss the student’s observable words and actions. Consistent with the clinical interview methodology, during the clinical interviews, I explicitly stated that I was not looking for correct answers and asked the students to explain their thinking aloud as they worked on the tasks.

In collaboration with the CI, I designed pre-interview tasks (Appendix A) to obtain information regarding the students’ meanings for quadratic and exponential relationships, as well as their more general quantitative reasoning and covariational reasoning. Wherever possible, we adapted tasks from the existing literature on covariational reasoning, quadratic relationships, and exponential relationships. We designed questions that involved graphs (e.g., Appendix A, Problem1), unconventional graphs (Appendix A, Problem 1e) and tables of values (e.g., Appendix A, Problem 2). For example, in a prompt on graphs, I presented two graphs and asked the students to identify the relationship represented by each graph (Figure 10).
DISTINCTIONS BETWEEN QUADRATIC AND EXPONENTIAL RELATIONSHIPS

The questions on graphs helped me examine if the students’ meanings for quadratic and exponential relationships were constrained by static shape thinking. For example, a student constrained by static shape thinking might identify a graph representing exponential relationship as a quadratic relationship because the shape of the curve for both relationships is a concave upward curve. On the other hand, a student reasoning covariationally might attempt to reason about the amounts of change in one quantity for equal changes in the other quantity and examine the ways in which the differences were changing. Likewise, a student constrained by shape thinking might claim that the unconventional graph (Appendix A, Problem 1e) represents a linear relationship even after attending to the unconventional labelling of the axes. Presenting them with tables of values helped me understand their ways of identifying a relationship when numeric values were given.

Additionally, we used adaptations of commonly used contextual problems to examine their ways of reasoning to graph a relationship between two covarying quantities (e.g., Appendix A, Problem 3, adapted from Carlson et al. (2002)) and to explore their meanings for exponential growth (e.g., Appendix A Problem 5, adapted from Strom (2006)). My analysis of students’ activities and their ways of reasoning in these tasks in the pre-clinical interviews provided me
insights into their covariational reasoning and meanings for quadratic and exponential relationships entering the study.

For the post-clinical interviews I intended to examine to what extent they would reason covariationally and test my models of the students’ meanings for quadratic and exponential relationships at the conclusion of the teaching experiment, including exploring if there were shifts in the students’ meanings. I presented them with graphs, tables of values, and contextual problems similar to those in the pre-clinical interviews. The post-clinical interviews were 1-1 sessions outside of the classroom for a duration of approximately 60 minutes and occurred at a time convenient to the students and not during class time. An overview of the tasks presented in the pre- and post-clinical interviews is in Table 2. In Chapter 4, I provide a more detailed description of the tasks.

**Teaching Experiment in This Study**

In the second week of the semester, I started the teaching experiment sessions. I conducted 12 teaching experiment sessions. Eleven sessions occurred in the classroom during class time, in which Rebecca and Josie worked as a pair. Additionally, I conducted one 1-1 session with each student that occurred outside the classroom. Each out of class session lasted approximately one hour and occurred at a time convenient to the students and not during class time. Table 2 provides an overview of the sessions, topics, and tasks students engaged with across the 12 sessions. I provide a more detailed description of each task\(^4\) in Chapter 4, where I report the students’ engagement with the tasks. Consistent with the teaching experiment

\(^4\) We adapted tasks from the existing literature on covariational reasoning, quadratic relationships, exponential relationships, and from the tasks used in a pilot study (see Appendix C for details of the study) that I conducted with ISTs.
methodology, the purpose of conducting a 1-1 session was to further investigate each student’s ways of reasoning and developing meanings. Additionally, during the out-of-class sessions I had an opportunity to test my models of the students’ mathematics by predicting how a student would respond to an activity when she works independently without having the other student’s actions and words to which to respond.

During all the in-class sessions, except one session, I worked with the pair of the students and a graduate assistant served as an observer to video record the interview. The CI, when going around the class and interacting with each group, posed probing questions to Rebecca and Josie and asked them to explain their ways of reasoning. At the end of each session, the course instructor, graduate student, and I met and discussed our observations and planned for the next session. Whenever required, the CI and I met in between the sessions to discuss the tasks we designed or modified based on students’ interactions in the previous sessions. When needed, the CI and I met to discuss instances that provided insights into the students’ reasoning. I took notes of our discussions that I used during retrospective analysis.

**Data Analysis in This Study**

I analyzed the data from both clinical interviews and teaching experiment sessions for generative as well as convergent purposes (Clement, 2000). Consistent with the teaching experiment methodology, I analyzed the data with generative purposes in order to build initial models of the participants’ meanings for quadratic and exponential relationships. Next, I analyzed the data for convergent purposes with the goal of testing and refining the models that I built and explain students’ actions. When these models contradicted the initial models, I analyzed the data again for generative purposes. This iterative cycle repeated until I was able to build viable models of students’ mathematics that explained students’ observable words and actions.
During the ongoing analysis, which occurs throughout the teaching experiment, I watched the video recordings of the clinical interviews and teaching experiment sessions to identify instances that might reveal the students’ reasoning covariationally as well as their meanings for quadratic and exponential relationships. This initial analysis of the data from pre-clinical interviews supported the development of preliminary hypotheses for the teaching experiment sessions. Moreover, the analysis of students’ responses provided me a preliminary models of each student’s meanings and their ways of differentiating between change patterns. I compared these models with each student’s activity that emerged through the teaching experiment and in the post-interviews.

I had debriefing sessions with the research team after each clinical interview and teaching experiment session to discuss each student’s reasoning, hypothesized models built during the session, and to plan for the next session. After each session, I watched the videos of the session and took notes of each student’s significant utterances and actions particular to their covariational reasoning and their meanings for quadratic and exponential relationships. At this stage I transcribed important instances; activities which provided me insights into their ways of thinking. During the teaching experiment and while I watched the videos, I maintained a written journal to document my thinking about each student’s actions. Analysis at this stage helped me build tentative models of each student’s thinking that were free to differ from in-the-moment models built during the teaching experiment sessions.

At the end of the teaching experiment, I engaged in retrospective analysis. Consistent with the teaching experiment methodology, during this stage I re-watched the videos to continue to develop and refine models of the student’s mathematics. During this round of analysis, I continued to transcribe important instances in student activity with particular attention to their
activities that provided insight into their covariational reasoning and meanings for exponential and quadratic change. These transcriptions included students’ hand motions and gestures. I also referred to my notes taken during the ongoing analysis to recognize familiar interactions or identify new ones as well as to compare the models I created during retrospective analysis with the models I had created during ongoing analysis. I continued to create, test, modify or reject models of students’ meanings built from the ongoing analysis. Throughout the process of analyses, I collaborated with my advisor and the observer of the teaching episodes to get additional insights regarding my interpretations of the students’ actions and words. The results of this analysis were viable models of each student’s mathematical meanings, including characterizations of shifts in their meanings.
Chapter 4. Results

In this chapter, I present an account of Rebecca’s and Josie’s developing meanings and ways of reasoning over the course of the study. First, I synthesize the results from the pre-interviews to provide a characterization of each student’s covariational reasoning and initial meanings for quadratic and exponential relationships at the outset of the study. In the next three sections, I describe their activities on various tasks from the paired in-class teaching experiment sessions in order to characterize each student’s (i) covariational reasoning, (ii) meanings for quadratic relationships, and (iii) meanings for exponential relationships. I then characterize each student’s meanings and reasoning in the one-on-one out-of-class teaching experiment session. I conclude the chapter with results from the post-interviews including descriptions of shifts in their meanings for quadratic and exponential relationships. In the excerpts that follow in this chapter, Rebecca is noted as “R”, Josie as “J”, the researcher as “MV”, and the class instructor as “CI.” When used in an excerpt, words in parenthesis describe an action, while words in brackets are clarifications of the speaker’s utterances and a … denotes a pause for one or two seconds. Additionally, wherever needed, I provide recreations of the students’ board work that was not captured clearly in the videotapes.

Pre-Clinical Interviews

Results From Rebecca’s Pre-Interview

First, to provide insights into Rebecca’s covariational reasoning, I describe her addressing the Bottle Problem task during which Rebecca explained how two quantities vary in tandem. Next, to provide insights into her meanings for quadratic and exponential relationships, I describe her activities when interpreting graphs and then tables of values that represented these
two relationships from the researchers’ perspective. Finally, I describe her activities when engaged with the Growing Tree problem, which offered additional insights into her meanings for exponential relationships. I conclude this section, with a summary of my inferences about her meanings and ways of reasoning at the outset of the study.

**Rebecca’s Covariational Reasoning**

In the Bottle Problem (Figure 11), when prompted to graph the amount of volume of liquid with the height of liquid in the bottle, Rebecca described the shape of the bottle as “a circle with a cylinder on the top” and stated, “as volume increases, the height increases” which I took to indicate that she reasoned about the directional change in direction of two quantities (MA 2): as liquid entered the bottle, both the volume of the liquid in the bottle and the height of the liquid in the bottle increased.

**Problem 3.** Imagine filling this bottle with a liquid. As the liquid is being poured, graph the relationship between the height of the liquid in the bottle and the volume of liquid in the bottle.

![Figure 11](image-url). The Bottle Problem adapted from Carlson et al., 2002.
When I asked Rebecca to graph the relationship between the volume of the liquid in the bottle and the height of the liquid, she explained, “when you get at the top (pointing to the upper portion of the bottle), since it is such a small amount of volume, it will fill up soon and quick.” Rebecca drew a diagram of the bottle and sketched a graph as seen in Figure 12a/b. Pointing to the graph she sketched, I asked, “Is this [the graph] curved or a straight line?” She explained, “it [the graph] is almost going to be a pretty much straight line, and probably at the end it would curve up a little bit…… because it will fill up pretty soon.” I take this to indicate that she reasoned univariationally about volume of the liquid and time as her description focused on how quickly the bottle would fill (e.g., “fill up soon and quick”). I infer from Rebecca’s activities in this situation that she reasoned about the direction of change in the two quantities (MA2) when interpreting the quantities in the situation. However, when she sketched the graph, she no longer attended to liquid volume and height, instead attended to volume and time.

**Rebecca’s Reasoning About Graphs Representing Quadratic and Exponential Relationships**

When I provided a graph and asked her to make an inference about the relationship, Rebecca relied on static shape thinking. Specifically, when I presented Rebecca with a graph that
represented a quadratic relationship (Figure 13a) and asked her to identify the relationship represented in the graph, the following conversation ensued (Excerpt 1).

![Graphs](image)

*Figure 13. Graphs that represent (a) quadratic relationship and (b) exponential relationship.*

**Excerpt 1.** Rebecca explaining ways to interpret a graph representing a quadratic relationship.

| R | I would say, it’s [graph in Figure 2(a)]… it’s probably an $x$-squared function. |
| MV | Why do you think it is an $x$-squared function? |
| R | Um, just looking at the shape of it. Because I, recently from like tutoring, I know, like the basic shapes of like an $x$-squared or cubed or a..., what’s the other one? Uh, like a square root function. So, that’s what I am thinking. |

I infer that Rebecca was engaging in static shape thinking as she recalled a shape from memory (e.g., “I know like the basic shapes”) to determine the relationship represented by the graph. That is, I infer that Rebecca observed a shape that seemed parabolic to her which led her to the conclusion that the underlying relationship must be defined by a quadratic function.

Rebecca also engaged in static shape thinking when defining a relationship for the graph in Figure 13b which is defined by $y=(1/4)2^x$. Rebecca described, “I would think this could be some kind of cubed graph because just the way I can see it shaping down here (moving her fingers around the point of intersection of the two axes), it looks like… just in this little part, like
near the origin, it looks like it would dip back down…” Similarly, for the graph in Figure 14a, which is defined by \( y = (-1/4)^2 \), Rebecca explained, “This one looks like either a parabola or cubed function but negative. This is in the fourth quadrant and going down. Normally, a regular parabola, a positive one would be like this (making a U shaped curve with her hands, Figure 14b) and is in the first and second quadrant.” It is evident from Rebecca’s explanations that she attempted to identify the relationship represented in the graphs by looking at the shape of the curves, a characteristic of her engaging in static shape thinking. Based on this activity, I infer that Rebecca’s meanings for determining relationships from a graph entail attempting to recall a memorized shape that she associates with a specific function class. From this particular activity, I infer that she associates a parabolic shape with an \( x \)-squared function; this task was not successful in eliciting her meanings for exponential graphs, however, as she did not describe any graph as being defined by exponential relationship.

![Figure 14.](image)

**Figure 14.** (a) Graph representing exponential relationship and (b) Rebecca’s hand gestures to represent a parabola.

**Rebecca’s Reasoning About Tables Representing Quadratic and Exponential Relationships**

When approaching the task with tables of values representing quadratic and exponential growth, Rebecca attempted to establish a correspondence rule between the \( x \)- and \( y \)-values. For example, for the table of values in Figure 15a defined by \( y = 3x^2 \), Rebecca noted that every \( y \)-
value was a multiple of the corresponding \( x \)-value. She explained, “If I am taking the \( x \)-values, I first multiply by 3 to get the \( y \)-value, if I take \( x \) as 2, I multiply by 6 to get 12 and 3 times 9 is 27, the \( y \)-value.” She continued, “I am trying to see if we can multiply or add something to the \( x \) to get the \( y \)’s.” However, she was unable to define any algebraic rule and concluded, “I don’t know what words to describe it as.” Similarly, for the table in Figure 15b, first, Rebecca stated, “I have to play around these numbers. I will start with the \( x \)-values. 1 times 6 is 6, but 2 times 6 is not 18. I will have to figure out how to get to 18.” After two minutes, she noted, “these are all (pointing to the \( y \)-values) if you multiply by 3, you get the next one and... uh... and since each of these [the \( x \)’s] is going up by one, I kinda want to do something... uh... like a number to the \( x \) power, something like that.”

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Figure 15. Tables of values representing (a) exponential and (b) quadratic relationship.

Based on the ways she looked for patterns in the tables of values, I infer that Rebecca’s meanings for determining a relationship from a table entail attempting to find patterns to determine a correspondence rule, which would in turn, provide the type of relationships represented. For the table of values in Figure 15b, I infer, Rebecca identified a repeated multiplication in \( y \)-values and 1-unit changes in \( x \)-values which led her to consider the relationship could be “something to the power of \( x \).” I note that despite making this accurate claim, Rebecca used the word exponential only for graphs; even in her later activities she
identifies a repeated multiplication and rules that entail “the power of $x$,” without referring to such a relationship as exponential.

**Rebecca’s Engagement With the Growing Tree Problem**

Finally, in the *Growing Tree* problem (Figure 16 adapted from Strom, 2006) Rebecca identified repeated multiplication of the height which she connected with a 1-month increase in time.

**Problem 5.** The height of a plant is 10 inches. The height of the plant doubles every month. A student claims that after 2.5 months, the plant will be 60 inches tall. How would you respond to this student?

*Figure 16. The Growing Tree Problem*

Addressing the prompt (Figure 16), Rebecca described the plant’s growth this way: “...uh, kind of planning out month by month, it’s like 10 inches now. Then next month it will be 20 inches or like a 10 times 2, and then next after that, which will be month number 2, it would be 20 times 2 which is 40 and then next month after that, month three, which we are not getting to would be 40 times 2 which would equal 80.” Thus, she coordinated repeated multiplication with additive increases in months.

*Figure 17. Rebecca’s work to determine the height of the tree that doubled each month.*
However, Rebecca did not coordinate the multiplicative growth in height for a change less than one-month. Rebecca shifted to additive reasoning to determine the height of the tree after 2.5 months. As a response to the hypothetical student’s claim that the height of the tree would be 60 inches after 2.5 months, she described, “So, you could do it like that [times 2] for a next whole month and then take off half of a month or go from here (drawing an arrow as seen in Figure 17) to next half ... you could do 40 times 2 minus one half 40. So, for half a month it would only be a half of the double. I would say the student is correct.” I infer, in the moment, Rebecca (implicitly) assumed that in between one month intervals, the relationship would grow linearly (e.g., halfway through the month, we are at half the change across the whole month, a quarter through the month, the change would be a quarter of the change across the whole month).

Next, I asked her, “How tall would the tree be after 100 months?” I designed this prompt to understand students’ ways of reasoning to determine the height for a multiple months gap. In order to find the height after 100 months, Rebecca coordinated repeated multiplication by 2 with 1-month increase in time. She determined the heights of the tree after the first five months correctly as 10, 20, 40, 80, and 160 by doubling the values and claimed,

[the numbers] reminds me a power of 2. 2 to the 0 is 1, 2 to the first is 2, and squared is 4, then 8 and 16, so it’s definitely got to do something to the power of 2... and you could divide all these [the heights of the tree after first five months] numbers by 10. So, maybe it's just a... like a... 2 to the x times 10 and that would give you these [the heights of the tree after first five months]. Yeah, so may be like a 2 to the 100th power times 10.

Rebecca recognized that the power of 2 is the result of repeated multiplication. For instance, 8 is the result of multiplying 2 thrice. I infer Rebecca’s meanings for exponential relationships entailed an analytic rule of the form \( y = ab^x \); she knew how many times she had to multiply the
initial height by 2 for a given number of months. I note that similar to her activity in the tables of values, Rebecca did not explicitly refer either the repeated multiplication or the analytic rule to an exponential relationship.

**Summary of Rebecca’s Pre-Interview**

Rebecca’s activities during the pre-interviews provide evidence of covariational reasoning (MA2) because she reasoned about the direction of change in covarying quantities. On the prompts with graphs, Rebecca engaged in static shape thinking as she focused on the shape of the graph (e.g., “a U shaped curve”, “a parabola”) which she associated with an analytic rule (e.g., $x^2$). Additionally, on the prompts with tables of values, Rebecca illustrated a way to determine the relationship, that is, by defining a rule or a pattern that would have been productive but she was unsure how to leverage this way of reasoning for the particular relationships represented by each table (i.e., look for constant second differences).

Particular to her meanings for exponential relationships and exponential change, Rebecca identified patterns that entailed repeated multiplication and characterized such relationships as having a “power of $x$” rule. I infer Rebecca’s meanings for relationships with exponential growth (from the researcher’s perspective) entailed (i) repeated multiplication and (ii) an analytic rule of the form $y = a^x$ or $y = ab^x$. However, Rebecca only used the term “exponential” when describing a graphical shape; it is unclear the extent which Rebecca associated her meanings for exponential growth (from the researcher’s perspective) with exponential relationships or functions.

**Results From Josie’s Pre-Interview**

At times, Josie exhibited actions similar to Rebecca. I briefly summarize and highlight her actions that differed when engaged with the *Bottle Problem* task, graphs, tables of values, and the *Growing Tree problem*. 
Josie’s Covariational Reasoning

Josie exhibited actions that provide evidence of her limited reasoning covariationally. She described that the bottle “is narrow up here (showing the upper portion of the bottle), so... it is going to fill up lot faster here than it is down here (showing the middle portion of the bottle).” When I asked her to graph the situation, she sketched a graph (Figure 18) and explained, “Uh, yeah... I just think of it as steadily increasing and then kind of fills up a lot faster when it goes to the top.”

Figure 18. Josie’s graphical representation of volume of liquid in the bottle with the height of the liquid.

Based on her descriptions, I infer that Josie initially imagined the volume of the liquid as increasing at a constant rate with respect to time (e.g., “steadily increasing”) and then shifted to reasoning about liquid height and time (e.g., fills up lot faster”). That is, Josie reasoned univariationally either about volume of the liquid (with implicit time) or height of liquid (with implicit time). Josie’s actions of not labelling the axes and sketching a seemingly increasing curve also demonstrate her reasoning univariationally.

Josie’s Reasoning About Graphs Representing Quadratic and Exponential Relationships

Like Rebecca, Josie engaged in static shape thinking when interpreting graphs. For example, for the graph in Figure 13a, she claimed, “It looks like half a parabola, I would say $x^2$” and further explained, “but I would have to see the other half, definitely to clarify. But it looks like it is going to come back up (motions as if making a curve to the left of the graph with her
Similarly, pointing to the graph in Figure 13b, Josie described, “this one looks like an exponential growth… that’s what I am thinking… it started from something very close to zero and then increases very fast. Yeah, that’s what I am thinking, it’s an exponential growth.” I infer that Josie leveraged static shape thinking to (correctly) describe the relationship represented by each graph as she focused on the shape of the graphs. For quadratic relationships, she associated the shape of “half a parabola” with the analytic rule “$x^2$.” Likewise, with respect to exponential relationship, she focused on a physical feature of the graph, “close to zero and then increases very fast” which she associated with “exponential growth.” Grounded in shape thinking in graphical contexts, I infer that Josie’s meanings for quadratic relationship entailed a parabola and an increasing curve close to the intersection of the axes represents exponential growth.

**Josie’s Reasoning About Tables Representing Quadratic and Exponential Relationships**

When initially approaching these tasks Josie used slopes to determine if the relationship represented was linear. For example, for the table of values in Figure 15a, she promptly said, “first I would see if there is a slope” and determined the values of $\frac{y_m - y_{m-1}}{x_m - x_{m-1}}$ to note that the consecutive slopes are not equal and claimed, “I do not have the same slopes, so linear is definitely out of question.” Similarly, for the table of values in Figure 15b, she identified the relationship to be non-linear by determining unequal slopes. I infer that Josie’s meanings for determining a relationship from a table of values supported her in identifying whether the relationship was linear or non-linear.

Josie’s activity is consistent with Rebecca’s in that she attempted to determine analytic rules based on patterns in the table. For example, pointing to the $y$-values in the table of values in Figure 15a, Josie claimed, “All the $y$’s are multiples of three and it is rapidly increasing, maybe it could be an exponential growth.” For the table of values in Figure 15b, she said, “I am thinking
of an equation for \( x \) and \( y \). I am trying to figure out a formula like \( y \) equals to whatever. If it is \( x^2 \), \( x^2 + 1 \ldots \) or if it is something to the power, then it is an exponential growth or decay. If we can get the relation, we can find the family.” Like Rebecca, Josie focused on trying to determine analytic rules to help her determine a relationship. Further, her explanations indicate her meanings for exponential growth are grounded in having an analytic rule of the form \( y = a^x \). Her explanations also indicate that her meanings for exponential growth could entail a pattern of \( y \)-values being multiples of a number.

**Josie’s Engagement With the Growing Tree Problem**

Later during the interview, in the *Growing Tree* problem, like Rebecca, Josie correctly determined the heights of the tree after each month by doubling the previous height. After determining the height as 40 after two months and 80 after three months, she explained, “... to see what is half way in between 40 and 80, I would say 60 minus 40 is 20 and 80 minus 60 is also 20, and that seems to be half way through the month and…. I would say he [the hypothetical student] is right.” Josie’s way of reasoning is similar to Rebecca’s reasoning, that is, while each of them conceptualized repeated multiplication to determine the height of the tree at the end of one month, they reasoned additively to determine the height within one month.

Similar to Rebecca’s approach, to find the height of the tree after 100 months, Josie attempted to find an algebraic rule. She explained, “We know it [the height of the tree] is doubling. We know it is something with \( 2x \), \( x \) being the [height of the tree] previous month.” Based on this reasoning, she determined the height of the tree at the end of the first five months and claimed, “The first few ones [the heights of the tree] are easy, but you can’t find for 100 months if you don’t know the one before.” I took Josie’s activity to indicate she understood that to find the height of the tree after \( n^{th} \) month she needed to know the height after \( (n - 1)^{th} \) month.
Furthermore, I infer that Josie was reasoning recursively to determine the height of the tree after a given month. When she added, “there has to be something that we can plug the 100 into,” I took to mean that Josie coordinated the repeated doubling of the height with a one-month increase and understood that if she could determine a rule for the height of the tree, she then can find the height of the tree at the end of any month.

**Summary of Josie’s Pre-Interview**

Josie, like Rebecca, had relied on static shape thinking when determining if a graphically represented function was quadratic or exponential. She treated graphs as objects and relied on the physical properties of the graph (e.g., “half a parabola”, “close to zero”) which she associated with a rule (e.g, $x^2$) to identify the relationship represented by them. With respect to tables of values, and like Rebecca, the meanings for quadratic functions that she exhibited when addressing the graphing tasks did not support her as she was unable to define a correspondence rule between the $x$ and $y$ to determine the relationship represented. Particular to exponential relationships and exponential change, I infer that Josie’s meanings entailed (i) repeated multiplication and (ii) an analytic rule of the form $y = a^x$.

**Teaching Experiment Sessions- Covariational Reasoning**

Two days after I conducted the one-on-one pre-clinical interviews, I started the teaching experiment sessions. Through the first four sessions, Rebecca and Josie engaged in two tasks: The *Carnival Ride* and the *Power Tower Task*. The goal in implementing these tasks was to provide opportunities for the students to reason quantitatively and covariationally. Particularly, the Carnival task is designed with MA3 in mind (Carlson et al., 2002). Engaging in the Carnival Ride task supported Rebecca and Josie in constructing the sine and cosine relationship in the context of circular motion. In the following paragraphs, I present the students’ activities in one of
the two tasks: the Carnival Ride task. As the goal is to highlight the students developing
covariational reasoning, I do not provide data from their engagement in the Power Tower task as
students’ activity in this task is consistent with their activity in the Carnival Ride task.

The Carnival Ride Task

At the beginning of the second episode⁵, we presented the students with the Carnival
Ride task (an animation of a Ferris wheel rotating counter-clockwise at a constant speed, see
Figure 19 a-c for several screenshots of the ride) and asked them to graph the relationship
between the rider’s vertical distance above the center of the ride and the angle swept by the rider
starting at 3 o’clock position.

![Screenshots of the Carnival Ride Task](image1)

Figure 19. (a)-(c) Screenshots of the Carnival Ride Task and (d) a recreation of Josie’s
representation of the task.

After watching the animation, Josie first drew a picture of the situation (Figure 19d) and
explained, “theta [the angle swept by the rider] is always increasing but the height changes...
height is increasing, decreasing, decreasing and increasing back to three” to which Rebecca
added, “… when we are looking at height relative to theta [the angle swept by the rider], the

⁵ In the first session, we introduced the context of circular motion and the terminology of angle measure, radian,
vertical distance above the horizontal diameter, and horizontal distance to the right of the center of the circle. See
Moore (2014) & Moore et al. (2014) for more information on the task sequence.
height is increasing and once it hits 12 o’clock it’ll be decreasing and decreasing \((motioning her hand on the arcs in the 2^{nd} \text{ then } 3^{rd} \text{ quadrants in an image of the situation})\) and height will start to increase again here \((motioning her hand on the arc in the 4^{th} \text{ quadrant})\) and meanwhile theta will continually increase” (MA1-2). When provided a dynamic animation and asked to reason about two covarying quantities, Rebecca and Josie each reasoned about the direction of change of the two quantities they conceived in the situation. I note that whereas in the pre-clinical interviews Josie did not exhibit mental actions as described by Carlson et al. (2002), engaging with a dynamic task offered opportunities for each student to conceive of and describe directionally of how two quantities vary.

Next, when I asked the students to sketch a graph describing the situation, Rebecca drew a smooth curve (see a recreation in Figure 20a) and Josie explained, “So we are thinking kind of possibly, maybe, would be the sinusoidal graph because it is going up, like as theta is always increasing, it is kind of, like in a sine graph, the height is increasing, decreasing, decreasing, and then increasing.” In order to understand why they drew a smooth curve, I presented a graph that I drew using line segments (Figure 20b) and asked if it was an appropriate representation of the situation. Both of them were unsure if the graph could be made with line segments. Josie claimed, “we only have the max and mins” to which Rebecca added, “if we had more points we could see if they formed a straight line.” I infer that, similar to their activity in the pre-clinical interviews, each of them engaged in shape thinking. They recalled from memory the shape of a sine curve and associated the shape and physical features of the curve (increasing, decreasing, decreasing, and increasing) to the current situation in which the height of the rider increases, decreases, increases, and decreases. Their engaging in static shape thinking is supported by their
response to my questioning if the graph could be composed of straight lines; the students expressed they were unsure how to justify the curvature of their graph.

![Figure 20](image)

Figure 20. (a) Recreation of Rebecca’s sketch to represent the relationship (b) A graph that I presented to the students.

After the students’ graphing activity, I drew a circle and labelled tick marks to get seemingly equal arcs on the first two quarters of the circle (Figure 21a). The goal was to support them in reasoning about the amount of changes in the vertical distances of the rider for equal changes in the angle measure.

![Figure 21](image)

Figure 21. (a) A combination of the pair’s and my work and (b) a recreation of their work in the first quarter of the ride.
I then asked, “Can you show me the vertical distances of the rider at these different positions on the ride?” Josie then drew dashed blue segments to represent the vertical distances and red horizontal segments to help determine the change in the vertical distances. Next, using the cap of a marker pen, they measured the green segments representing the change in the vertical distances (Figure 21b-c) and Josie claimed, “No, it’s not the same, definitely not the same. The vertical distances are increasing by less, yeah, increasing but not at a steady rate” (MA 3). Rebecca agreed with Josie and added, “yeah, they are increasing by less each time.” Because Rebecca said ‘each time’, in the moment I conjectured she was reasoning about time and the vertical distances. So, I asked, “what do you mean by each time?” Both Rebecca and Josie responded to my question (Excerpt 2).

Excerpt 2. Students’ explaining

R  Like this one (showing the arc lengths in Figure 13b) increases by the same amount.

J  Each time you go to the next sub arc length it [the vertical distance] doesn’t increase as much as before.

I took their activity as indicative of each of them reasoning about the change in the vertical distances for equal changes in the arc lengths⁶ (MA 3). They explained that the vertical distances increased by less for equal changes in the arc lengths. I then asked them to graph the situation.

To sketch the graph, when Rebecca described that “it [the vertical distance] definitely increases the most from 3 o’clock to the next, the largest change in the distance is from 3 o’clock

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⁶ Later in the interview, Josie described, “equal arc lengths, uh... make equal angles.” Hence, I consider the students reasoned about changes in the vertical distances of the rider for equal changes in the angle measure.
to the first one”, Josie plotted the point A (Figure 22a). Josie then motioned her pen along imaginary segments AE and E to B saying “here, a little less” and then motioned along imaginary line segments BF and F to C saying, “a little less” and in a similar way plotted D when Rebecca said, “barely anything” (Figure 22a). After plotting the points, Josie joined the points to sketch a smooth curve and drew dashed vertical line segments to represent the vertical distances of the rider (Figure 22b). Furthermore, when I asked how the vertical distances would change if the rider continues the ride into the second quarter, Rebecca said, “it flips, now they are decreasing by more.” Josie placed points and joined them as a smooth curve (Figure 22c).

Until this point, their activity indicates Rebecca and Josie reasoned about the changes in vertical distances for the two quarter turns as the angle measure changes by equal amounts (MA3) and translated the relationship they inferred from the situation to the graphical representation. I infer their actions involved perceiving a graph in terms of how the point representing (angle measure, vertical distance) magnitudes moves as the two quantities covary, as being a strong indication of the students engaging in emergent shape thinking.

![Figure 22](image)

*Figure 22.* (a) Representation of the pair’s work, (b)-(c) The pair’s work in graphing the relationship between the rider’s vertical distance above the center of the ride and the angle swept by the rider starting at 3o’clock position.
As the conversation continued, the CI during his circulation in the room came up to the pair and asked them to justify the curvature of the graph. Josie replied, “because the vertical distance is increasing or decreasing not at a steady rate there’s a curved line” and wrote her explanation on the white board (Figure 23a). In order to assess whether the curve could be concave up instead of concave down, the CI asked, “but why is it curved that way? Why not the other way?” that is, if the curvature could be as though the curve is a concave up curve instead of concave down. Indicative of attending to the changes in vertical distance, Josie explicitly stated, “the vertical distances are increasing, but the change is not.... they are getting smaller.” The pair then worked together to sketch a graph of one complete rotation of the ride. They plotted points by reasoning about the changes in the vertical distances as increasing by decreasing amounts (in magnitude) in the 3rd quarter-rotation and decreasing by increasing amounts (in magnitude) for the 4th quarter-rotation. They sketched the graph by joining the points, drew dashed line segments to represent the vertical distances, and lastly used bold black segments to represent the changes in the vertical distances (Figure 23b). I infer that each student envisioned the graph as an emergent trace of the two quantities-vertical distance and angle measure. Each student understood the graph as representing the relationship between vertical distances and angle measure as the quantities covary for one complete rotation of the rider.
As an example of such reasoning becoming more stable for the students, in the next session, which was the third Sine/Cosine Session, I asked the students to create a graph that illustrates how the rider’s horizontal distance to the right of the center of the ride co-varies with the measure of the angle swept out by the rider (measured in radians). Similar to their previous activity and notably without any prompting from me, Rebecca and Josie considered changes in the horizontal distances for equal changes in the measure of angle swept by arcs of equal length (MA3). Josie drew a circle, marked equal arcs on the first quarter of the circle and drew the radii. Then, she drew dashed line segments to represent the horizontal distances and solid segments to be able to identify the change in the horizontal distances. Josie measured the changes in the horizontal distances with the cap of a marker pen and claimed “the horizontal distances are decreasing by more” (MA 3, Figure 24a). Rebecca indicated her agreement.

When I asked them to represent the situation graphically, Rebecca drew a pair of axes, marked four equal tick marks along the horizontal axis labelled as radians, and marked 2.6 on the vertical axis; the radius of the ride in the task is 2.6 units.
The pair then worked together and plotted five points (as seen in Figure 24b). The points are annotated as C, E, F, G, and D (as seen in Figure 24d) in the recreation for reference purposes. While Rebecca was plotting each point, Josie explained (see Figure 24d, a recreation of their work) “This one [point E] is like almost nothing [the amount of change CH], and that one [point F] decreases by more [amount of change EI is more than CH], that one [point G] decreases by a lot [amount of change FG is more than EI], and that one [point D] decreases by a lot more [amount of change GL is more than FG].” Rebecca joined them with a smooth curve. Josie claimed, “This is the cosine graph and if we keep going it [the graph] will go below.” She
then drew a blue curve below the horizontal axis (Figure 24c). Josie recognized the trace of the curve as representing of the cosine function.

Critically, I note that the students engaged in this activity without prompting, which I take as indicative of each of them developing meanings such that reasoning covariationally was away for them to conceive of and represent a relationship between two quantities. Specifically, coordinating amounts of change in one quantity for equal amounts of change in the other became a way of reasoning for the students to make sense of the dynamic situation and to represent it graphically. As the session continued, they engaged in this way of reasoning to claim that for equal changes in the angle measure, the horizontal distances increase by decreasing amounts and decrease by increasing amounts for the third quarter and fourth quarter of the rotation (Figure 25).

![Figure 25. The pair’s work to represent the horizontal distances and amounts of change in fourth quarters of the ride.](image)

Through the Carnival Ride activity, Rebecca and Josie each attended to the covarying quantities - vertical distances and the angle swept by the rider, and horizontal distances and the angle swept by the rider. For equal amounts of change in the angle measures, they reasoned about the amounts of change in the vertical and horizontal distances both situationally and graphically. By reasoning about the amounts of change in the covarying quantities they anticipated how a point moves and conceived the graph as a trace of the moving point. Thus, each of them reasoned quantitatively and covariationally (MA 3) about the quantities in the
situation, and then engaged in emergent shape thinking as they represented the covariational relationship they conceived (Moore & Thompson, 2015).

In the next session, we engaged the students in the Power Tower task (Moore et al., 2014) with a goal to provide students with further opportunities to reason covariationally and support graphing relationships between covarying quantities in a different context. For brevity’s sake, I do not provide data from their activities, but I note that students attended to the covarying quantities in the situation and were able to represent the situation with an accurate graph.

**Teaching Experiment Sessions- Quadratic Relationship**

After the above sessions that offered students opportunities to develop their quantitative and covariational reasoning with a particular attention to their developing Mental Action 3 described by Carlson et al. (2002), we conducted two sessions focused on quadratic relationships to explore if we could support the students in developing explicit meanings connecting quadratic change with constant amounts of change of amounts of change.

In the first of these sessions, we presented the *Growing Triangle Task* to the students. One goal of this task was for students to construct a relationship between the area of the triangle and its side length via covariational reasoning (i.e. reasoning about amounts of change). A second goal was for the students to visualize and conceive that the second differences of the area of the triangle are constant. The goal was to promote their thinking about quadratic relationships as a relationship between two quantities such that for equal changes in one quantity, the differences in the amounts of change (heretofore referred to as ‘DiAoC’ where AoC refers to amounts of change) are constant, that is the second differences are constant. In the second session, we...

---

7 Students termed the amounts of change in the amounts of change of a quantity as “the differences in the amounts of change” or DiAoC where AoC refers to the amounts of change.
engaged the students with tasks involving graphs and table of values to support them to continue to leverage the idea of having a constant DiAoC. In the following sections, I describe the students’ activities across each of the sessions and characterize each student’s developing meanings for quadratic relationships. I then conclude the section with a summary of the shifts in their meanings.

**Session One of Quadratic Relationships**

**Growing Triangle Task Design**

In this task, we presented a GeoGebra applet (Applet Link) showing an (apparently) smoothly growing isosceles triangle with one of the side lengths colored in pink (Figure 26a). In this applet, we provided two sliders. The longer slider allowed the students to change the length of the pink base of the triangle and the shorter slider allowed them to increase the increment by which the longer slider increased (e.g. have the triangle grow by chunks instead of smoothly). We asked the students to sketch the relationship between the area of the triangle and the pink side length. Additionally, to support the students in constructing the DiAoC in the area of the triangle as a quantity, we provided paper cut-out manipulatives including: (i) a set of five triangles, each with an integer change in the base; and (ii) four trapezoids representing the area added to each triangle to obtain the next triangle (Figure 26b). The set of five triangles were created by taking screen shots of the applet, printing, and cutting them out. By including the trapezoidal cut outs, we intended to support the students in potentially visualizing how the DiAoC in the area increase by an equal amount for consecutive equal changes in the base length.
Students’ Activity Reasoning With the Rule

After watching the animation, Rebecca and Josie described both the side length in pink and the area as increasing (MA 2). When I asked them to graph the relationship between the area and the side length, they spontaneously sought to consider the algebraic formula for the area of triangle in terms of base \( b \) and height \( h \), that is, Area = \( \frac{1}{2} bh \). Rebecca wrote Area = \( \frac{1}{2} bh \), where \( b \) denoted the base and \( h \) denoted the height of the triangle and suggested they measure the base and height of each triangle cut-out. Rebecca and Josie measured the heights and bases of the five triangle cut-outs using a paper clip. Josie explained that “because the base is increasing by the same amount and so is the height, so the area (pointing to the formula Area = \( \frac{1}{2} bh \) that Rebecca wrote) is increasing at the same rate for every skip.” Rebecca nodded her head in agreement to Josie’s explanation. They attempted to reason about AoC in the height and AoC in the bases. Reasoning about these AoC did not enable them to reason about the growth in the area of the triangle.

Conceiving Increased AoC in Area Using the Trapezoid Cut-Outs

As we conjectured that the students were focused on the rule and did not explicitly conceive of area yet, so we turned their attention to the manipulatives and hypothesized they would support students explicitly coordinate area and base length. Hence, the CI prompted “Can you use the trapezoids to make an argument of how the area is increasing?” We also prompted
them to color the triangle cut-outs and show how the area of the triangle increased instead of measuring the bases and heights of the triangles.

The pair then engaged in coloring activity. They placed one triangle on the top of the other and colored the change in area (Figure 27a). After doing so, Rebecca arranged the trapezoids one below the other (Figure 27b) to make a triangle and the following conversation ensued (Excerpt 3).

![Figure 27](image)

Excerpt 3. The pair’s explanation of how the areas of the trapezoids increase.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>MV</td>
<td>What do these strips (pointing to the trapezoids in Figure 27b) represent?</td>
</tr>
<tr>
<td>2</td>
<td>J</td>
<td>Area, the added area, the area added on each time.</td>
</tr>
<tr>
<td>3</td>
<td>R</td>
<td>Yeah, to this one (showing T1 in Figure 27c).</td>
</tr>
<tr>
<td>4</td>
<td>J</td>
<td>Yes, that is a triangle (showing T1). That (showing Trap 1 in Figure 27c) is the area added on to make the next triangle.</td>
</tr>
<tr>
<td>5</td>
<td>MV</td>
<td>And then what does this (showing Trap2) represent?</td>
</tr>
<tr>
<td>6</td>
<td>J</td>
<td>The area added from this triangle (showing triangle AFJ in Figure 27c) to that triangle (showing triangle AGK in Figure 27c).</td>
</tr>
<tr>
<td>7</td>
<td>MV</td>
<td>So, what can you say about the areas that you are adding on?</td>
</tr>
</tbody>
</table>
Both Rebecca (Line 4) and Josie (Lines 2, 4, 6) identified that the areas of the trapezoids represented the AoC in the area of the growing triangle and Rebecca explicitly stated the AoC in the area as increasing (Line 8) to whichJosie indicated her agreement (Line 9). I infer that each of them conceived the trapezoids as representing the AoC in the area of the triangle for unit changes in the side length (MA3). Furthermore, the manipulatives supported the students in conceiving of area in ways compatible with our intentions and they identified that AoC in the area were increasing.

**Conceiving Constant DiAoC**

To further support the students in conceiving constant DiAoC in the area of the triangle, I asked them to explain how the areas of the trapezoids increased. Specifically, pointing to the trapezoids, I asked, “Do these increase by more, less, or by the same amount?” Rebecca and Josie spent five minutes arranging the trapezoids in different ways that did not support them in reasoning about changes in areas of the trapezoids. Eventually Rebecca overlaid the trapezoids as seen in Figure 28a, then pointed to the differences in their areas with her fingers (Figure 28b-c) and explained, “Like that one (A in Fig 17b), that one (B in Fig 17b), that one (C in Fig 17b), that one (D in Fig 17b), they are all the same... they [the trapezoids] are all increasing by the same... the same rate.” Josie nodded her head in agreement.
In the moment, I took Rebecca’s actions and explanation to indicate she was reasoning about the AoC in areas of the trapezoid, and from these actions and explanations I conjectured she conceived the constant second differences in the area of the growing triangle. To test this conjecture and gain further insights into each student’s thinking, I asked them what each trapezoid represented and the following conversation ensued (Excerpt 4).

*Excerpt 4.* The pair’s reasoning about the AoC in the areas of the trapezoids and the area of the triangle.

1. J  ...the added areas, the increase of areas between two triangles.
2. MV OK. What is happening to the amount of areas that you are adding?
3. R Increasing by same.
4. J Yes, they are increasing at the same rate. We put these [the trapezoid strips] on the top of each other and saw that they were increasing at a steady rate. The areas aren’t the same, but the rate at which they are increasing is the same.
5. CI So, for equal changes in the base, what is happening to the area of the triangle?
6. R Increasing by more.
7. J They are increasing by more each time.
I infer that each student leveraged the manipulatives as they claimed that the area was increasing (Rebecca Line 8; Josie Line 9) and that the AoC were increasing by a constant amount (Rebecca Line 3; Josie Lines 4-5). That is, each of them conceived a constant DiAoC in the area of the growing triangle. The students’ response (Lines 8-9) is further evidence that they reasoned covariationally about the change in areas of the triangle for equal changes in the base (MA3).

After the interaction in Excerpt 4 when Josie restated, “for every equal change in the base, the area is increasing by more,” Rebecca drew a concave up curve (Figure 29a) and claimed, “it is definitely exponential.” I infer, similar to her static shape thinking in the pre-interview, that Rebecca associated the name ‘exponential’ to the concave up curve representing the relationship they conceived.

When I asked the students to justify the curve, Josie claimed, “Ah! Like the Ferris wheel graph we did before?” Josie immediately drew dashed blue vertical segments, Rebecca drew red horizontal segments, and then Josie drew green solid vertical segments as representing the change in the blue dashed vertical segments (Figure 29b). The following conversation ensued (Excerpt 5).

![Figure 29](image)

**Figure 29.** (a) Rebecca’s graph after they claimed that the area increases by more (b) color coordinated segments to explain the situation.

**Excerpt 5.** Students’ explanation of the graph representing the relationship between the area and pink side length.

| MV | What do those green segments represent? |
Rebecca claimed that the green segments in the graph represented the change in the area (Line 2) which were increasing (Line 4) (MA3). Further, leveraging the trapezoids situationally, the students described that the differences in the green segments were the same (Lines 6, 9).

Hence, each student accurately described their created graph by reasoning about the AoC in the area of the triangle and the AoC in the areas of the trapezoids both graphically and situationally. The students’ actions of drawing and identifying equal changes along the horizontal axis, drawing horizontal segments, vertical segments, identifying vertical segments that were increasing by same amount provide further evidence that Josie and Rebecca engaged in emergent reasoning when drawing their graph; they understood their graph as representing the emergent relationship between the increasing side length and increasing area of the triangle they had constructed in the situation. However, neither student provided an indication if she related the relationship to a quadratic relationship.
Session Two of Quadratic Relationships

Operationalizing Constant DiAoC

In this session, our goal was to provide the students opportunities to reflect on their activity in the Growing Triangle task such that they can operationalize constant DiAoC in a novel context. Therefore, we presented the students with three graphs with a prompt (Figure 30) and asked them to determine more points and graph the relationship.

For each of the following, we know the differences in the amounts of change of volume are constant with respect to the length of a side. Complete each graph.

Figure 30. Three graphs each having three points such that the differences in the amounts of change in the volume are constant with respect to the length of the side.

For graph (a) in Figure 30, while Josie recreated the graph on their board, Rebecca made a table of values with the three given points. Rebecca then determined the differences in the y values as +1 and +2 (Figure 31). Pointing to the differences in the y-values, Rebecca then explained, “so these [the differences in the y-values] will be plus three, plus four (motioning her fingers as though going down from +2 and then down again) and these (pointing to the values in the y column) will be eight and twelve” and completed the table of values (Figure 31).
Josie agreed with Rebecca, plotted the two new points that Rebecca generated, and explained, “The amount of change of the amount of change is going up by one, like here (pointing to the +1, +2, +3, +4 in the table of values in Figure 31b) it is plus one, plus one, plus one.” Thus, together, Rebecca and Josie operationalized knowing that a constant second difference exists to generate and plot additional more points.

Similarly, for the graph in Figure 30b, the students reasoned about constant second differences in the y-values (Excerpt 6) to determine additional points (Figure 32a).

Excerpt 6. Students’ explanation of constant second differences in a table of values.

R (Pointing to the y values) It [the numeric value of y] is going down by 1 and then going down by 3. So which will be, in between those [-1 and -3 written by Josie] it will be minus 2. So if this one is minus 1 and minus 3 (moving her fingers from 9 to 8 and then 8 to 5 in the table) and then may be minus 5 and then minus 6, uh.. minus 7

J Yes, you are correct (drawing connecting arrows and writing -1, -3, -5, -7 as representing the first differences and -2, -2, -2 as the representing the second differences)
After determining the points accurately, Josie plotted the points and joined them with a smooth curve (Figure 32b). Looking at the graph, Rebecca stated, “It [the graph] is probably like half a parabola without the negative $x$-values” to which Josie added, “Yeah, either half a parabola or exponential function, or it [the graph] could go like that (motioning her hand into the second quadrant from the numeric value 9 on the vertical axis as though she is drawing a smooth concave down curve).” Likewise, for the third graph, they determined two new points, plotted the points on a pair of axes, and joined them with a smooth curve (Figure 33). Then Josie concluded, “So, it’s a parabola” to which Rebecca nodded her head to indicate her agreement.

I noted that during much of their activity, each student reasoned covariationally and she operationalized constant second differences to determine new points. After engaging in this covariational reasoning, and consistent with describing their graph representing the side length and area of the growing triangle (Figure 29a) as exponential, each student made shape-based associations in naming the relationship. After plotting points and drawing a smooth curve, they made claims such as, it is “either half a parabola or exponential function” and “it’s a parabola.” Relevant to their meanings for quadratic functions at this point in the teaching experiment,
Despite identifying the constant second differences, neither student provided an indication she related constant second differences to quadratic relationship.

Figure 33. The students’ activity of making table of values and plotting points for graph (c) in Figure 30.

After students completed this activity, the CI facilitated a whole-class discussion in which groups in the class shared their activity and solutions with the whole class. During his monitoring activity, the CI probed all groups to describe the type of relationship they were representing but no group raised the idea of quadratic relationship as the underlying relationship. Hence during the whole-class conversation, the CI introduced the defining characteristic of quadratic relationship as, “Whenever the amounts of change change constantly, in other words, the second difference is a constant, the relationship is quadratic.” Immediately, Rebecca exclaimed with a surprise in her tone, “Oh!” and Josie pointed to the constant second differences in their work and called out, “Cool!” Based on their actions and words here and in their later activity, I conjecture the combination of engaging in this task sequence in conjunction with the whole class discussion supported each student in reorganizing her meanings for quadratic relationships such that constant second differences became a defining characteristic of quadratic relationships. I provide additional evidence for this claim in the next several sections.

Changing Coordinate Systems to Challenge Shape Thinking

Intending to perturb the student’s shape-based associations (e.g., a parabola means quadratic relationship) we presented them with two graphs in the polar coordinate system and
asked them to identify which of the graphs represented a quadratic relationship (Figure 34). Consistent with their engagement in the previous activity, Rebecca and Josie created a table of values for both the graphs. They then attended to the second differences in the numeric values of theta and noted the second differences in the values of theta for graph (b) as constant. Josie then pointed to the graph in Figure 34 (b) and claimed, “that is quadratic.” Rebecca agreed with her and the following conversation ensued.

Which of these is a quadratic relationship?

Each point is of the form \((r, \theta)\).

**Figure 34.** Two graphs in the Polar coordinate system with graph (b) representing a quadratic relationship.

**Excerpt 7.** Students’ explanation of graphs in Polar Coordinate System.

<table>
<thead>
<tr>
<th></th>
<th>CI</th>
<th>(Pointing to graph (b)) Is it quadratic or is it a parabola?</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>J</td>
<td>Quadratic</td>
</tr>
<tr>
<td>3</td>
<td>R</td>
<td>Quadratic</td>
</tr>
<tr>
<td>4</td>
<td>CI</td>
<td>And the other one?</td>
</tr>
<tr>
<td>5</td>
<td>J</td>
<td>Not quadratic</td>
</tr>
<tr>
<td>6</td>
<td>R</td>
<td>The amounts of amounts of change are all not the same.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>7</td>
<td>CI</td>
<td>Is it [Graph (a)] then a parabola?</td>
</tr>
<tr>
<td>8</td>
<td>R</td>
<td>It looks like it, but then...</td>
</tr>
<tr>
<td>9</td>
<td>J</td>
<td>Just because it looks like a parabola it does not mean it is quadratic... in the polar coordinates</td>
</tr>
<tr>
<td>10</td>
<td>R</td>
<td>Yeah (<em>nodding her head in agreement</em>), ... in the polar coordinates</td>
</tr>
<tr>
<td>11</td>
<td>J</td>
<td>Yes, just because it looks like a parabola it does not mean it is quadratic.</td>
</tr>
</tbody>
</table>

From this interaction, I infer that, both Rebecca and Josie drew on the meanings for quadratic relationships as entailing constant second differences, which may have developed when engaged with the Growing Triangle task, to identify the relationship represented in the graphs in the polar coordinate system. Josie stated (Line 9) and reiterated (Line 11) that a parabolic shape of the graph does not indicate the relationship must be quadratic. Rebecca both verbally and through actions agreed with Josie (Line 10). I take this interaction and each of them identifying constant second differences in the values of theta as suggestive of a shift from each student relying solely on static shape-based associations (i.e. parabola means a quadratic relationship) to leveraging her meaning that constant second differences are a defining characteristic of quadratic relationships.

**Summary of the Two Sessions of Quadratic Relationships**

During the first session, students engaged with the Growing Triangle task and reasoned covariationally (MA3) to coordinate AoC in the areas of the triangle for equal changes in the pink side length. Although, initially, the quantities they considered (height and side length) conflated with the quantities that I intended them to reason about (area and side length), after prompting, they engaged in reasoning about the areas of the trapezoid cut-outs. Specifically, the students identified how the areas of the trapezoids were increasing to conceive that the area of
the triangle was increasing by more. They made claims that the AoC of the areas of the trapezoids was constant to reason that the DiAoC in the area of the triangle were constant. They represented this conceived relationship graphically by leveraging their covariational reasoning. But, to name the relationship between the area and side length, the students continued to rely on static shape thinking and called the relationship as exponential, indicating that the students’ meanings for quadratic relationships did not entail this property.

Building on the first session, in the second session Rebecca and Josie leveraged a meaning for constant second differences to determine new points for relationships that maintained a constant DiAoC. After the CI defined such a relationship as a quadratic relationship, both students were able to leverage this definition to move beyond static shape thinking. Rather than presuming a parabola in the polar coordinate system was quadratic, the students reasoned about constant DiAoC to make determinations about quadratic relationships in this new coordinate system. Both Rebecca and Josie exhibited activities compatible with this constant DiAoC meanings for quadratic relationships later in the teaching experiment.

Teaching Experiment Sessions-Exponential Relationships

In the prior teaching experiment session, after interpreting graphs in the Polar coordinate system, we started with the Growing Quadrilateral task which I consider as the first session of exponential relationship session. For the next five sessions, students worked on different iterations of this task and other tasks which we intended would support them in reasoning covariationally and construct meanings for exponential relationships. I provide a brief overview of these six sessions in Table 3. Specifically, in Sessions One and Two, our goal was to support students to construct meanings compatible with Thompson’s description. In Session Three, we engaged students with tasks that we intended would support them in defining an analytic rule for
exponential growth. Finally, through Sessions Four to Six, students worked on tasks we considered as having the potential to support them reason about unit growth factors and partial growth factors. I structure the results from these six sessions broadly into three sections: Sessions One-Two, Session Three, and Sessions Four-Six with sub-sections in which I describe the students’ activities in each session.

Table 3. An overview of the six sessions focused on exponential relationships

<table>
<thead>
<tr>
<th>Session #</th>
<th>Goal of the session (Support develop meanings for)</th>
<th>Tasks students engaged with</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Exponential relationship</td>
<td>Growing Quadrilateral Task Iteration 1</td>
</tr>
</tbody>
</table>
| 2         | Exponential relationship                          | Growing Quadrilateral Task Iteration 2
|           |                                                   | Growing Quadrilateral Task Iteration 3 |
| 3         | Analytic rule for exponential relationship        | Growing Quadrilateral Task Iteration 1 |
| 4         | 1-unit growth factor                              | Growing Quadrilateral Task Iteration 1
|           |                                                   | Shrinking Quadrilateral Task I |
| 5         | $\frac{1}{2}$- unit partial growth factor         | Two Quadrilaterals Task      |
| 6         | 1/n-unit partial growth                           | Tables of Values Task       |

In each sub-section, I present selected findings that provide insights into each student’s covariational reasoning and the meanings for exponential relationships she developed. As certain sessions provided less insights into each student’s developing understandings of these ideas, not all sessions are described in detail. I explain why certain sessions are only briefly discussed when this is the case.

**Teaching Experiment Sessions One and Two of Exponential Relationships**

In the first two sessions, the students engaged with three iterations of the Growing Quadrilateral task. In the following sub-sections, I first describe the design of the task iteration followed by the students’ activities that provided insights into their meanings.

**Session One of Exponential Relationships**

_Growing Quadrilateral Task Design: Iteration 1_
In the first iteration of the task (Link to Iteration 1 of the Growing Quadrilateral Task), adapted from tasks used by Ellis (2011a) and Johnson (2012), the applet includes one slider and allows the students to animate the quadrilateral (see Figure 35) for several screen shots of the task. We designed the task in such a way that as the slider values increase by one unit (the values increase from 0 to 1, 1 to 2, 2 to 3, 3 to 4, and 4 to 5), the area of the quadrilateral, which starts as a square, increases by one half of the present area.

Figure 35. Screenshots of the Growing Quadrilateral Task Iteration1.
We conjectured that this task had the potential to support students in conceiving of exponential growth consistent with Thompson’s (2008) perspective: the rate of change of area is proportional to the area at that value (e.g. the area added is always $\frac{1}{2}$ of the current area). We were also interested to examine if students would reason in ways compatible with Confrey and Smith’s (1994, 1995) description of exponential growth as having increasing additive rate of change or constant multiplicative rates of change. We asked the students to (i) identify the quantities in the situation and (ii) describe how the area of the quadrilateral was growing with respect to the slider values.

**Reasoning Covariationally and Leveraging Constant DiAoC in Iteration 1**

Watching the animation of the Growing Quadrilateral Task Iteration 1 on the computer screen in front of them, each student identified that the area of the quadrilateral was increasing and the slider values were changing from zero to five (MA1-2). Then, Josie drew a pair of axes, put tick marks on each axis, and numbered and labelled the horizontal axis as ‘slider values’ and the vertical axis as ‘total area.’ Rebecca described, “At first it [the quadrilateral] is a four by four, the area is sixteen, it is like when $a$ [the slider value] is zero.” She continued, “it [the area] goes up by may be a little. Then at two, a little more, at three, a little more again, and then at four it increases little little more, and then the last one is a lot, it looks like.” While Rebecca described how the area was increasing for equal changes in the values of the slider, Josie plotted points and together they sketched a graph with discrete points (Figure 36a). Further, Josie wrote on the white board, “area is increasing by more each time as the slider increases.” From Rebecca’s explanation and Josie’s actions I infer that each of them understood the area was increasing and the AoC in area were increasing for unit changes in the slider values (MA3), which is consistent
with reasoning about increasing additive changes as described by Confrey and Smith (1994, 1995).

Figure 36. (a) The pair’s graph for iteration 1 of the Growing Quadrilateral Task, (b) a recreation of their work, and (c) Josie’s written work of Rebecca’s explanation.

After Josie plotted the points, Rebecca leveraged shape-name associations as she stated, “looks exponential or quadratic.” As the slider values changed from 0 to 4, Rebecca determined the dimensions of the added quadrilateral as the slider values changed from 0 to 4 and the following conversation ensued (Excerpt 8).

Excerpt 8. The pair’s explanation of change in the areas of the quadrilateral

1 R You can see that you are first adding 2 by 4, then 2 by 6, then 3 by 6, and then 3 by 9 and then four and a half by 9 or 10.

2 J Yeah, not the same, we are not adding the same.

3 CI So, what are we thinking?

4 R We are thinking that the area is increasing by more each time, but not like the same amount each time.

5 J Yes, not the same (writes Rebecca’s explanation on the white board, see Figure 36c).

6 R The changes get a lot larger, as the slider value gets bigger.
So, you are saying the amounts of change are getting bigger and not by the same amount?

Yes, correct

Yeah, like from here to here (pointing to E and F in Figure 36b) it is big jump than from here to here (pointing to D and E in Figure 36b)

Which means what?

It is not quadratic.

The students’ explanation here indicates they were reasoning about the AoC in the areas (Line 1) to make claims that the area was increasing by more (Lines 2 and 4). Further, they claimed that the DiAoC are not the same (Lines 4 and 5). I take the pair’s responses (Lines 8 and 9) to the CI’s question as evidence that each student had determined that the relationship did not maintain a constant DiAoC in the areas to identify this relationship as not quadratic. Rebecca’s description provides evidence that she reorganized her meanings for quadratic relationship such that constant DiAoC was a defining characteristic. Rebecca (and Josie) continued to exhibit activities compatible with this meaning for quadratic relationships in the later sessions.

**Session Two of Exponential Relationships**

**Graphing Activity in the Growing Quadrilateral Task Iteration 1**

The previous session ended with the students noting the dimensions of the quadrilaterals, (for example, 4 by 6, 6 by 6) for each slider value to determine the areas of the quadrilaterals and number the vertical axis (Figure 37a). Hence, the next session began with asking the students to graph the relationship between the area of the quadrilateral and the slide values. Their board work from the prior session had a pair of axes with points plotted. Josie quickly joined the points and sketched a smooth curve (Figure 37b) and when I asked her to justify the graph, she replied,
“because the area is increasing by more each time the slider values increase, therefore this graph.” Because Josie connected the points to sketch a smooth curve without attending to the change (or lack thereof) in area in between the points, I conjectured that she was engaging in static shape thinking.

Figure 37. Josie’s work of (a) numbering the vertical axis (b) joining the points to sketch a smooth curve (c) representing the increasing changes in the areas and (d) a recreation of Josie’s work.

To test this conjecture, I asked, “How is it [areas increasing by more] showing in the graph?” As a response to this question, Josie engaged in the following activity. She drew horizontal and vertical segments as seen in Figure 37c and each student identified segments that were increasing for equal changes in the slider values. Showing the red segments (Figure 37d), Josie clearly stated, “these are getting bigger” and Rebecca added, “yeah, ... they are increasing but ...not by the same amount.” Rebecca identified the DiAoC as increasing. Because both Rebecca and Josie constructed and identified segments that were increasing by more for equal
changes in the slider values, their actions exhibit evidence of each of them reasoning about
covariational properties of the two quantities: area and slider values. That is, each of them
reasoned (MA3) and conceived their drawn graph to entail covarying quantities. Hence, when
reasoning about the points they had plotted, the students were engaging in emergent shape
thinking; these points represented the covariation of two quantities’ values. However, because
Josie had drawn (and Rebecca has been using) a smooth curve, I infer that the students were also
using elements of static shape thinking as Josie had constructed the curve via connecting-the-
dots rather than attending to how the quantities were (or were not) varying between integer
values.

*Moving Towards Conceiving Change in Area as Proportional*

Because we designed the task with the hopes of supporting students in reasoning about the
change in area as proportional to the area at that value, the CI asked, “you are saying that the
amounts of change are getting bigger, but can you say something more specifically?” He
conjectured that the students might notice the change in area as one half the current area when
the slider’s values change by one unit. Rebecca suggested determining the numeric values of
AoC in the areas. Together, they determined the numeric values of DiAoC and also the
successive differences in the areas (Figure 38a) and the following conversation ensued (Excerpt
9).
Figure 38. (a) The pair’s work in determining the differences in the areas (b) a recreation of their work.

Excerpt 9. Rebecca and Josie explaining how the area of the quadrilateral increased.

1 MV  (Pointing to the numbers in red in Figure 4b) What are these numbers representing?
2 J Those are the areas that are added on.
3 MV  (Pointing to the numbers in green next to the numbers in red in Figure 4b) What are these?
4 R These (Pointing to the numbers in red in Figure 4b) are the amounts of change and these (Pointing to the numbers in green next to the numbers in red in Figure 4b) are the amounts of amounts of change.
5 J Yes, yeah.
6 MV  (Pointing to the numbers in blue in Figure 4b) and these?
7 R These, uh, are,
8 J The amounts of amounts of amounts of change...uh!
9 R Yeah, and it is not quadratic.
10 MV Why?
Like if this was (Pointing to the numbers in green next to the numbers in red in Figure 4b) like plus 4, plus 4, plus 4 or something like plus 1, plus 1, plus 1, then it would have been quadratic.

Yeah

It is evident that each student reasoned about the successive differences in the amounts of change in the areas. Based on their graphing activity and the conversation (Excerpt 9), I consider that they were reasoning about the increasing additive change in the areas for unit change in the slider values, which is compatible with Confrey and Smith’s (1994, 1995) description of exponential growth. Additionally, Rebecca’s statement that the relationship is not quadratic (Line 9) and Josie’s response (Line 11) as to why the relationship is not quadratic provide further evidence of each student’s meanings for quadratic relationship as entailing constant DiAoC.

I conjectured the students did not envision the situation in terms of the changes of area being proportional to the area, but their actions provide evidence of each student exhibiting MA3 and leveraging their meanings for quadratic relationships to reason about the quantities in the situation. Further, and like the quadratic sessions, despite constructing and reasoning about the relationship between covarying quantities, neither student attempted to provide a name or classification for the type of relationship represented. Whereas we designed the task hoping the students would conceive the change in area as proportional to the area at that value, aligning with Thompson’s (2008) description of exponential relationship, both Rebecca and Josie conceived of the differences in the areas and further differences in the changes of areas. That is, the task did not support them in attending to the proportionality between the area of the quadrilateral and the change in area as we had hoped.
At the end of this activity, the CI facilitated a whole class discussion in which students of all groups shared their work with the class. Towards the end of the discussion, the CI described how the amount of change in area was always one half the current amount. Looking at their board work and pointing to the start area (which is 16) and change in area (which is 8), Rebecca said, “Oh! I see!” to which Josie nodded her head and claimed, “Cool!” Based on their words and actions in this conversation, in that moment I conjectured it was an ‘aha moment’ for Rebecca and Josie and that they might engage in proportional reasoning when presented with the next task.

**Students’ Activity Addressing the Growing Quadrilateral Task Iteration 2**

In the Growing Quadrilateral Task Iteration 2 (Link to the Growing Quadrilateral Task Iteration 2), we tried to further investigate the students’ developing emergent reasoning. We designed this iteration such that the slider values increase continuously from 0 to 5 taking all intermediate values but the area increased only at integer values (e.g. the area was constant from 0 to .99 but then jumped at 1, Figure 39). Whereas in the previous activity the students had connected dots to draw a smooth curve, I conjectured that in the current activity the students would attend more carefully to what is happening between the integral values of the slider. I conjectured that the task might create a minor perturbation for the students in terms of their original graph (i.e., identify they should not have used a smooth curve).

![Figure 39. Several screen shots of Iteration2 of the Growing Quadrilateral task.](image-url)
Describing this new feature of the animation, Josie said, “the slider is not increasing by one, maybe by points and so when the slider is like all the way until one, it is the same area.” Further, Rebecca described that the “area jumps” when the slider takes integral values and Josie added, “yeah, and then from one to two it [the area] is the same.” As they described how they conceived the area changing, Josie drew horizontal line segments in the graph (Figure 40a) and erased the smooth curve to have a graph as seen in Figure 40b.

![Figure 40](image)

(a) ![Figure 40](image) (b)

*Figure 40. The students’ graphing activity for the Growing Quadrilateral Task Iteration 2.*

I asked the pair to comment on the smooth curve they erased. They explained that a graph with discrete points and not a smooth curve would be an appropriate representation of Iteration 1 of the task, which is indicative of their engaging in emergent reasoning. I infer that Iteration2 provided the students an opportunity to re-consider their graph for Iteration1 which did not reflect the relationship they understood Iteration1 as representing (Excerpt 10).

**Excerpt 10.** Rebecca and Josie explaining the graphical representation for Iterations1 and 2 of the Growing Quadrilateral task

| 1 | J | [In Iteration1] Technically the area is increasing, but, it is like.... they are not connected. They are separate rectangles. |
DISTINCTIONS BETWEEN QUADRATIC AND EXPONENTIAL RELATIONSHIPS

Students’ Activity Addressing the Growing Quadrilateral Task Iteration 3.

Next, to further support the students in conceiving a graph as an emergent trace of two covarying quantities, we presented the Growing Quadrilateral Task Iteration 3 in which both the slider values (from 0 to 5) and the area increase continuously (Link to the Growing Quadrilateral...
Task Iteration 3). Josie drew a smooth curve (Figure 41) and claimed “the area of the quadrilateral increases gradually” and the following conversation ensued (Excerpt 11).

![Figure 41. Graphing activity for the Growing Quadrilateral Task Iteration 3.](image)

**Excerpt 11.** Rebecca and Josie reasoning about the increase in the area.

<p>| | |</p>
<table>
<thead>
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<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>J</td>
</tr>
<tr>
<td>2</td>
<td>R</td>
</tr>
<tr>
<td>3</td>
<td>J</td>
</tr>
</tbody>
</table>

As the students moved from Iteration 1 to Iteration 3, I infer, they continued to refine their graphical representations to reflect the relationships represented in the situation via their emergent reasoning. Whereas in Iteration 1 they initially drew a smooth continuous curve, when addressing Iteration 2, they conceived that the correct graph for Iteration 1 should be a series of points. When addressing Iteration 3 they returned to a smooth curve as each student conceived that the area was increasing smoothly as the slider length increased smoothly (e.g. Lines 2-4), which was reflected in the smooth curve.

**Summary of the Students’ Activities in the Three Iterations**
First, I note each student engaged in non-numeric reasoning and correctly identified that the area of the quadrilateral was increasing by more for equal changes in the slider values (MA 3). Next, as they continued to work with the different iterations of the task, each determined the numeric values of the area at integral values of the slider and determined the first, second, and third differences in the amounts of area. Each of them identified that the second differences were not constant and therefore the relationship as not quadratic. Also, Rebecca explained that the consecutive differences would never be the same. Furthermore, I infer that each of them leveraged Confrey and Smiths’ conception of exponential growth. Both situationally and graphically, the students attended to increasing AoC in the area for unit changes in the slider values but neither of them raised the idea that the underlying relationship is exponential relationship.

When graphing the relationship between the increasing area of the quadrilateral and slider values, each of them reasoned emergently. Rebecca and Josie each coordinated the two covarying quantities and conceived the graph in terms of how the slider values and area changed simultaneously. Specifically, the students concluded that (i) for Iteration 1, the graph would consist of discrete points because both area and slider took only integral values, (ii) for Iteration 2, the graph would be in steps because as the slider changed smoothly in an interval of the form \([n, n+1]\), the area stayed the same in that interval, but changed at integral \((n)\) values, and (iii) for Iteration 3, the graph would be a smooth curve because both the area and slider changed continuously.
Session Three of Exponential Relationships

Leveraging AoC to Define an Analytic Rule

In this session, we returned to the Growing Quadrilateral Task Iteration1. The CI reiterated to the whole class that as the slider’s values increase by one unit from 0 to 5, the area of the quadrilateral increases by one half the present area, and presented the notation: for $\Delta x = 1$, $\Delta A = \frac{1}{2}A$ where $\Delta x$ denotes change in the slider values, $\Delta A$ denotes change in the area, and $A$ denotes current area. We returned to this task in order to investigate if the students would leverage reasoning about the AoC to define an analytic rule representing the situation. That is, we asked them to determine $A_5$ in terms of $A_0$, where $A_0$ and $A_5$ represent the areas of the quadrilateral for slider values 0 and 5 respectively. Since, the students did not conceive of an exponential relationship between the area and the slider values up until now, our goal in this session was to help students connect their activity explicitly to exponential functions.

Rebecca and Josie expressed the areas of the quadrilateral at different values of the slider in terms of the notation $A_n$ as seen in Figure 42. Josie determined the additive difference between $A_0$ and $A_1$ as 8 and noted 8 as one half of 16.

When the CI prompted them to consider the given relationship, that is, for $\Delta x = 1$, $\Delta A = \frac{1}{2}A$, Josie wrote $A_1 = A_0 + \frac{1}{2}A_0 = (3/2)A_0$ and the following conversation ensued (Excerpt 12).

Figure 42. The pair’s work in determining the areas of the quadrilateral for different values of the slider.
Excerpt 12. The pair’s explanation to define a rule to determine $A_5$ in terms of $A_0$.

1. CI Can you now determine what $A_2$ is in terms of $A_1$?

2. J It’s going to be the same. It’s going to be three half’s of $A_1$.

3. R Yeah, three half’s $A_1$.

4. J [Writes $A_2 = (3/2) A_1$]

5. R Yeah, makes sense. And so we could do this until for $A_5$ and it goes up three half’s every time.


7. J What if you, what if you...replace $A_2$ in $A_3$.

8. (writes $A_3 = (3/2) (3/2) A_1 = (9/4) A_1$).

9. R Right, so in $A_2$ we plug in what we know of $A_2$ in terms of $A_1$.

10. R Right! And why don’t we keep doing that until $A_5$?

Each student explicitly stated that the area is three halves the previous area (Lines 3, 5, 6) and Josie expressed each area in terms of the preceding area (Lines 4, 7). Josie expressed $A_3$ in terms of $A_1$ (Lines 8-9) after which they expressed each area in terms of $A_1$. Further, Josie claimed that since $A_1 = 3/2 A_0$, they could express each area in terms of $A_0$ instead of $A_1$. So, Josie wrote $A_3 = (3/2) A_2 = (3/2) (3/2) A_1 = (9/4) A_1 = (9/4) (3/2) A_0 = (27/8) A_0$. They continued in this manner to get $A_5 = (243/32) A_0$ (Figure 43).
After they worked together to determine the relationship, Rebecca pointed to 243/32 and claimed, “Oh, my God! Is it like exponents? Because we did like three half’s to the fifth!” Then, Josie added, “So, three halves to whatever this little thing is (showing the subscript 5 in \( A_5 \))” and wrote \( A_5 = \left(\frac{3}{2}\right)^5 A_0 \). Immediately Josie stated, “So basically, the general formula is \( A_{n} \) is three halves to the \( n \) times \( A_0 \)” and we wrote \( A_n = \left(\frac{3}{2}\right)^n A_0 \). Thus, each student described AoC in area as one half the current area then used this to determine the next area as three halves the previous area. Hence, I infer that Rebecca and Josie leveraged both Thompson’s characterization of exponential relationship, change in area is proportional to the current area (e.g. \( A_1 = A_0 + \frac{1}{2} A_0 \)), and Confrey’s characterization of exponential relationship as having a constant multiplicative rate of change to establish the analytic rule in this situation (e.g., \( A_1 = (3/2)^1 A_0 \)). Additionally, for the first time addressing the Quadrilateral Tasks one of the students, Rebecca, explicitly described the underlying relationship as exponential.

To support students in generalizing the idea of exponential change for other situations, the CI then asked if they could determine a rule when \( \Delta A = kA \) where the value of \( k \) is unknown. In response to this question, Josie wrote \( A_1 = A_0 + k A_0 \), performed algebraic factoring and wrote
A_1 = A_0 + kA_0 = (1 + k)A_0. Rebecca explained, “like what we did before, if we continue we get a rule” and wrote \( A_n = (1 + k)^n A_0 \). Thus, I infer that, in determining a generalized analytic rule defining the relationship between the area and slider values, each student’s way of reasoning aligned with Thompson’s characterization and also Confrey and Smith’s characterization of exponential growth.

When I asked the students to identify the relationship between the area and slider value, Rebecca replied, “I think it is exponential” and hesitantly added, “ Uh...(laughs) I think when it is not quadratic, not cubic, not linear, ... yeah, yeah, it is exponential.” I conjecture, based on her in-the-moment laugh that Rebecca was not sure how to justify why the relationship was exponential. Hence, she referred back to her activity with looking for constant differences with quadratic functions to argue that if the relationship never has a constant DiAoC then the relationship would be exponential.

However, before we had the opportunity to question Rebecca further about the ways in which she identified the relationship as exponential, the session ended. Retrospectively, I infer that Rebecca’s meanings for exponential relationship entailed a relationship that was not a polynomial relationship. Rebecca understood that if a relationship would never have a constant DiAoC then the relationship was exponential. Further, since Josie did not explicitly make any claims about the relationship, I am unsure if she was engaging in compatible reasoning with Rebecca.

**Exponential Relationship Teaching Experiment Sessions Four Through Six**

For these sessions, our goal was to support students constructing meanings for exponential relationship by coordinating changes in one quantity with changes in the second quantity to conceive that the value of \( f(x + \Delta x)/f(x) \) is dependent on \( \Delta x \) (Ellis et al., 2015;
Thompson, 2008). That is, the goal was to support students to understand that for an exponential relationship defined by, for example, \( f(x) = 3^x \), the 1-unit growth factor for \( \Delta x = 1 \) would be 3, the \( n \)-unit growth factor for \( \Delta x = n \) would be \( 3^n \), and the partial growth factor for \( \Delta x = 1/n \) would be \( 3^{1/n} \). Therefore, we designed tasks we hoped would support students to coordinate the growth in one quantity with growth in the other quantity for unit intervals (e.g. \( \Delta x = 1 \)) and partial intervals (e.g. \( \Delta x = \frac{1}{2} \)). Below, I describe the students’ actions and my inferences of their actions during these last three in-class teaching experiment sessions.

**Session Four of Exponential Relationships**

In this session, we presented two applets in order to focus the students on developing an understanding of 1-unit growth factor, that is, the number we multiply a \( y \)-value by to get to the next \( y \)-value for \( \Delta x = 1 \). We asked the students to find the growth factor in two related situations:

1. the Growing Quadrilateral Task Iteration1 in which the change in the area is one half the current area (Applet link for the Growing Quadrilateral Task Iteration 1), and
2. a quadrilateral shrinking such that the change in the area is one-third the current area (Applet link for Shrinking Quadrilateral Iteration1). The applets are related as the Shrinking Quadrilateral is the Growing Quadrilateral Iteration1 playing in reverse. One goal was to observe how students would reason about these seemingly similar situations produce different exponential change and the second goal was to offer opportunities for students to conceive of exponential decay. In both the applets, the slider values change by one unit. This task composed the entirety of this session. Most of the time in this session was spent on different groups sharing their work with the rest of the class.

**Conceiving of a 1-Unit Growth Factor**

After watching the Iteration1 of the Growing Quadrilateral task, the students recalled their activity from previous episodes and wrote the numeric values of the areas on the board (\( A_0 \)}
= 16, \( A_1 = 24, A_2 = 36, A_3 = 54, A_4 = 81, A_5 = 121.5 \) and also determined the differences in the areas. Josie correctly claimed, “The change in area is one half the area before.” Rebecca nodded her head in agreement and added, “Yeah, and half of \( A_{\text{sub-zero}} \) plus \( A_{\text{sub-zero}} \) is the new area, and wouldn’t that be three halves?” Josie multiplied each area value, starting from \( A_0 \) to \( A_4 \) by \( \frac{3}{2} \) and confirmed that the successive “area is always \( \frac{3}{2} \) times the previous area” to claim, “so the growth factor is three halves.” Similar to their activity in the previous session, the students engaged in reasoning compatible with both Thompson’s description (“the change in area is one half the area before”) and with Confrey and Smith’s description of exponential growth (e.g. “area is always \( \frac{3}{2} \) times the previous area”).

Next, after watching the Shrinking Quadrilateral I applet, Josie and Rebecca noted that as the slider values change from 0 to 5, the areas decrease from 121.5 to 16 (see Figure 44 for screenshots of the Shrinking Quadrilateral and Table 4 for the values of slider and corresponding areas).
Josie then conjectured, “Previously we added a half, so now we are taking away a half, right?” They calculated the differences in the areas to verify Josie’s claim. For example, they determined 121.50 – 81 to get 40.5 and verified if 40.5 was one half of 121.50. They performed these calculations for the areas at each slider value to realize that the change in area is not one-half the current area. In the moment, I conjectured they were reasoning about the change in area being proportional to the current area for a unit change in the slider value. However, they presumed the amount of change as one-half the current value; the change is actually one-third the current value.
Subsequently, to support them to conceive the change as one-third the current value, the CI sketched a figure (Figure 45), and pointed to the shaded portion of the figure, and asked, “So, if you are adding a half of $A_1$ to $A_1$, what is $A_1$ to the whole quadrilateral (drawing a curly bracket on the left of the whole quadrilateral)?” Josie then noted, “The area [of the whole quadrilateral] is decreasing, is that by a third?” and Rebecca added, “Okay, it is a third we are taking away, so we keep two-thirds of it.” Josie promptly said, “So the growth factor is two-thirds.” After this, Josie performed numeric calculations and verified that as the slider values changed from 0 to 5, the area was two-thirds the current area. The students’ actions provide evidence that each of them identified (a) the change in area as proportional to the current area (e.g., Josie’s claim “It is a third we are taking away”), and (b) a constant multiplicative rate of change (e.g., Rebecca’s claim, “the growth factor is two-thirds”). I infer that the students leveraged both Confrey and Smith’s characterization and Thompson’s description of exponential change as they characterized how area was changing for a unit change in the slider values.

![Figure 45](image)

*Figure 45.* Recreation of the CI’s work to explain the change in area in the shrinking Quadrilateral applet.
Session Five of Exponential Relationships

Moving From 1-Unit Growth Factors to Partial Unit Growth Factors

In this session, our goal was to support students reason about exponential relationships for fractional AoC in the first quantity, that is for $\Delta x < 1$, and make connections between a 1-unit growth factor and a $1/n$ -growth factor. To support students in conceiving partial growth factors, we presented them with the Two Quadrilaterals task (Applet link). The applet has two quadrilaterals and two sliders with one slider longer than the other. The longer slider allows the students to animate the quadrilaterals and the smaller one allows students to change the increment the longer slider changed by. So, the quadrilaterals increased either discretely or continuously. Our motivation in designing this applet was to have the students notice 9 as a 1-unit factor and 3 as a $\frac{1}{2}$ unit factor and support them in potentially conceiving the relationship between the 1-unit factor and $\frac{1}{2}$ unit factor. That is, if the 1-unit factor is $g$, then the $\frac{1}{2}$-unit factor is $g^{\frac{1}{2}}$.

Both Rebecca and Josie explored the applet and chose to work with the slider changing discretely by one half from 0 to 2. They determined the area of the brown quadrilateral for each value of the slider. They noted the area of the brown quadrilateral as 1, 3, 9, 27, and 81 for slider values 0, 0.5, 1, 1.5, and 2 respectively. Rebecca explained “The area is three times every half change in the slider value” to which Josie added, “every half mark, the area is three times the previous area. Each student explicitly attended to both AoC in the slider value and the multiplicative growth in the area; she coordinated tripling the area with every half unit change in the slider value. Further, Josie created a table with the slider values 0 to 4 and the corresponding area values (Figure 46). Although, the slider values in the applet go up to 2, Josie considered
values up to 4 so that they “could see a pattern” and the following conversation ensued (Excerpt 13).

*Figure 46.* (a) Josie’s work to describe the area of the quadrilateral at different values of the slider (b) Rebecca’s work to define a rule representing the situation.

**Excerpt 13.** The students’ explanation of growth in the area of the quadrilateral for 1-unit changes and ½-unit changes. (NOTE: Although the students spoke the numbers aloud, I used numerals for readability purposes.)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>How did you determine the areas?</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>MV</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>R</td>
<td>We are multiplying by 3... uh, by 9...Like, we are multiplying by</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>...okay, for whole one value it is times 9. It’s like from 0.5 to 1.5</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>it’s 3 times 9 is 27. At 1 [the slider value], the area is 9 and at 2 it</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>[the area] is 81... 9 times 9 is 81. 2 to 3 is times 9 <em>(pointing to 81 and 729 in the table of values in Figure 46a).</em></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>J</td>
<td>Correct, 9 times, for every slider value of 1.</td>
</tr>
<tr>
<td>8</td>
<td>R</td>
<td>So for every 1 value we</td>
</tr>
</tbody>
</table>
Rebecca described the growth in area as 9 times the current area for a unit change in the slider value (Lines 2-6). Josie agreed with Rebecca (Line 7) and further explained that the growth factor is 3 times the current area for a ½-unit change in the slider values (Lines 10-12). Each student recognized the 1-unit growth factor as 9 and described 3 as the ½-unit growth factor. Thus, each of them understood the new area is proportional to the previous area for both 1-unit and ½-unit intervals but that the constant of proportionality changed depending on the size of the interval. This conversation led them to determine a rule to describe the situation. Initially, they conjectured the rule as $3^n$ but later concluded $A = 9^n$ as the appropriate rule because they verified that values of $9^n$ matched the table of values in Figure 46a.

Although the students identified the 1-unit growth factor as 9 and ½-unit growth factor as 3, I was not sure if they realized the reciprocal relationship between roots and powers. That is, if $g$ is the growth factor for 1-unit change in the slider value, the area will increase by $g^{1/2}$ for a ½-unit change in the slider value. In an attempt to support the students in conceiving this relationship, the CI asked, “Is nine to the seven and a half halfway between nine to the seven and nine to the eight?” As a response, and consistent with her activity in the pre-interview addressing the Growing Tree problem, Josie initially sought to plug in the values of 7, 7.5, and 8 in $A = 9^n$. Later, prompted by Rebecca, Josie noted the change in slider value from $9^7$ to $9^8$ is 1-unit and a ½-unit from $9^7$ to $9^{7.5}$ and $9^{7.5}$ to $9^8$. The following conversation ensued (Excerpt 14).
Excerpt 14. The students’ response to the question “Is $9^{7.5}$ halfway between $9^7$ and $9^8$?

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>J</td>
<td>So, if we kind of backtrack to our chart here (Figure 46a), 1 to a 1.5</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>is like a....</td>
</tr>
<tr>
<td>3</td>
<td>R</td>
<td>It’s a half jump</td>
</tr>
<tr>
<td>4</td>
<td>J</td>
<td>Yeah</td>
</tr>
<tr>
<td>5</td>
<td>CI</td>
<td>It’s half a jump, but what happens to the area?</td>
</tr>
<tr>
<td>6</td>
<td>J</td>
<td>We multiply by three.</td>
</tr>
<tr>
<td>7</td>
<td>R</td>
<td>But wouldn’t that be 9 for 9 to the 7 and 9 to the 8?</td>
</tr>
<tr>
<td>8</td>
<td>MV</td>
<td>Yeah, so when the jump [the slider value] is one unit</td>
</tr>
<tr>
<td>9</td>
<td>R</td>
<td>We multiply by nine, yeah.</td>
</tr>
<tr>
<td>10</td>
<td>MV</td>
<td>And when the jump is a half unit?</td>
</tr>
<tr>
<td>11</td>
<td>R</td>
<td>Like here (pointing to the x3 and x.5 in Figure 46a), we multiply by three.</td>
</tr>
</tbody>
</table>

During this interaction, Rebecca (Lines 10-11) responded that if the slider value changes by a $\frac{1}{2}$ they would multiply by 3 to get the area value and if the slider value changed by 1, they would multiply by 9 (Lines 7-9). Josie also responded that for a $\frac{1}{2}$-unit change in the slider, she would multiply by 3 to get the area (Lines 5-6). From this interaction, in the moment, I conjectured each of them could extrapolate that to determine the value of $9^{7.5}$, they would multiply $9^7$ by 3 and to determine $9^8$ they would either multiply $9^{7.5}$ by 3 or $9^7$ by 9. However, there is insufficient evidence that they constructed the power and root relationship. Instead, I infer that each coordinated a repeated multiplication meaning as they reasoned about how the area values changed for 1-unit and $\frac{1}{2}$-unit changes in the slider values.
Session Six of Exponential Relationships

In this last in-class teaching experiment session, we specifically wanted to provide opportunities for each student to reflect on their reasoning of partial and unit growth factors and conceive of a root and powers relationship between the growth factors. We began the session by presenting the students with four sets of tables for hypothetical slider (defined by \( a \)) and area values in which they were given that the area increases exponentially (Figure 47). Each table of value had missing values we asked them to determine. Also, we tasked them to determine a rule that defines each relationship. In order to support students construct the relationship between roots and powers of the growth factors, we had \( a \)-values increase by a 1/3, \( \frac{1}{2} \), 1/5, and \( \frac{1}{2} \) in Tables (a), (b), (c), and (d) respectively.

Emphasizing Different Partial Intervals

![Figure 47. Tables of values representing the area growing exponentially.](image)

For each of the following assume the area is growing exponentially. Determine the missing area values.

For Table (a), first, Josie noted the area as 16 times the current area for a change of 2/3 in the slider value, and therefore, reasoned the area would be 8 times the current area for a change of 1/3 in the slider value. I conjecture Josie was attending to the \( \frac{1}{2} \) change in \( a \)-values and
presumed there would be a $\frac{1}{2}$-change in the growth factor, $g$, instead of a $g^{\frac{1}{2}}$ change. I infer that Josie’s in-the-moment meanings from the prior episode did not support her developing stable meanings for growth factors that entailed the relationship between the 1-unit and 1/n-unit growth factors.

Whereas Rebecca initially agreed with Josie’s claim, she quickly then pointed to their work (Figure 46a) and stated:

> if you look at that one (pointing to their work on the earlier activity in Figure 46a) for a half change its times three, but if we jump from three to twenty seven we do nine times... whereas in between them it is three times....and three is the square root of nine. That’s why I think it is four here (moving her finger from 0 to 1/3 and 1/3 to 2/3 in Table (a), Figure 47).

In the moment, I took Rebecca’s explanation to indicate that she conceived the relationship between 1-unit growth factor and $\frac{1}{2}$-unit growth factor, that is, a $\frac{1}{2}$-unit growth factor is the square root of the 1-unit growth factor. Hence, I conjectured that for Table (a) she understood the 1/3-growth factor as 4 which is the cube root of 64. Josie agreed with Rebecca’s explanation, multiplied 48 by 4 to determine area as 192 for $a = 1$ and the following conversation ensued (Excerpt 15).

*Excerpt 15.* Rebecca’s explanation of relationship between unit growth factors and partial growth factor.

```
1 R So, for a change of one (draws an arrow from 0 to 1 as seen in

2 Figure 48), the area is from here to here (draws an arrow from 3 to

3 192) which is times sixty four.

4 MV Yeah, and you have these one-thirds from 0 to 1.
```
Rebecca described the area as 64 times the current area for a 1-unit change in the slider values (Lines 1-3) and explicitly stated the \( \frac{1}{3} \)-partial growth factor as the cube root of 1-unit growth factor (Line 8). As they continued to work on the tables of values, Rebecca generalized the relationship between 1-unit growth factor and \( \frac{1}{n} \)-partial growth factor. She devised a strategy which she would leverage in determining the missing values. That is, first find the 1-unit growth factor and then determine its \( n \)-th root to find the \( \frac{1}{n} \)-partial growth factor.

Figure 48. (a) The pair’s work in determining missing area values (b) A recreation of their work.
Coordination of 1-Unit Growth Factor and 1/n -Partial Growth Factor

For the next three tables of values, Rebecca and Josie first sought to find the 1-unit growth factor. Next, they identified the ‘number of jumps’ in the values of ‘a’ to determine the partial growth factor.

<table>
<thead>
<tr>
<th>a</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>?</td>
</tr>
<tr>
<td>1/2</td>
<td>?</td>
</tr>
<tr>
<td>1</td>
<td>15</td>
</tr>
<tr>
<td>1 1/2</td>
<td>?</td>
</tr>
<tr>
<td>2</td>
<td>75</td>
</tr>
</tbody>
</table>

Figure 49. (a) The Given table of values, and (b) Students’ work to determine the missing areas in the given table.

For example, for the table of values in Figure 49a, Rebecca determined that the area changes “from fifteen to seventy five” which “is times five” when the slider “jump is plus one.” That is, she identified the 1-unit growth factor as 5. When Josie pointed “in the table, the slider jumps by halves” Rebecca claimed, “Yeah, I want to like square root it [the 1-unit growth factor of 5] and do fifteen times square root of five to get here (pointing to the ? against a = 1½ in the table of values in Figure 49a ).” So, for slider value 1½, Rebecca correctly determined the area value as $15\sqrt{5}$. I infer that, Rebecca coordinated the changes in the slider values with the 1-unit growth factor and displayed an understanding of the relationship between roots and powers.

Although Josie noted the 1-unit growth factor as 5 and claimed, “three times five is fifteen and so the area is three” for a value 0, she did not explicitly discuss 1/2 unit growth factors, with Rebecca leading this part of the conversation (Figure 49b).
Similarly, for the third table of values (Figure 50a), first, Josie identified that for one unit change in the value of \( a \), the area is 32 times the current area and stated, “Okay, our growth factor is 32 when the jump is one.”

<table>
<thead>
<tr>
<th>( a )</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>(1/5)</td>
<td>?</td>
</tr>
<tr>
<td>(2/5)</td>
<td>?</td>
</tr>
<tr>
<td>(3/5)</td>
<td>?</td>
</tr>
<tr>
<td>(4/5)</td>
<td>?</td>
</tr>
<tr>
<td>1</td>
<td>96</td>
</tr>
</tbody>
</table>

Figure 50. (a) Third table of values (b) and (c) Students’ work to determine the missing values.

To my question, “How will you determine the missing areas at the one-fifths?” Rebecca replied “I think it will be the fifth root.” Reflecting on her earlier work with the previous two tables of values, Rebecca identified 32 as the 1-unit growth factor and considered the 1/5-partial growth factor as \(5\sqrt[5]{32}\) because she noted five equal jumps in the values of \( a \). To determine the missing area values, Rebecca suggested they ‘multiply by fifth root of thirty two” to find the missing values of area (Figure 50 b & c). Rebecca’s explanation consistently reflects her understanding of the relationship between the roots and power.

Lastly, for the fourth table of values as seen in Figure 51a, Josie noted 36 times 27 gives 972 to which Rebecca added, “and we have three jumps from one to two and half.” Josie promptly claimed, “it’s going to times three, times three, times three.” This was the first instance in this activity when Josie made claims about partial growth factor. I conjectured that Josie, in
the moment, realized that for every $\frac{1}{2}$-unit change in the value of $a$, the area is 3 times the current area. That is she identified the $\frac{1}{2}$-unit partial factor as 3. Responding to my question, “What then is the 1-unit growth factor?”, Rebecca immediately replied, “Nine... uh, for this jump of plus one (drawing an arrow from 1 to 2, Figure 51b)” and Josie wrote “gf:9” on the board (Figure 51b). Rebecca claimed, “so, for each jump of a half, we multiply by three” and filled the missing values in the table.

To better understand if Josie constructed the relationship between 1-unit and partial growth factors, for the fourth table of values (Figure 51a), I asked her to determine the area if the difference in the $a$-values was $\frac{1}{3}$ instead of $\frac{1}{2}$. Josie spent approximately four minutes examining their board work to note the partial factors for each table and the amount of change in the $a$-value. For example she said, “the halfway marks are square roots” and “here (Figure 48b, second table of values) it’s a cube root of 64.” She observed a pattern in all their work and responded to my question for the fourth table of values as, “So if we are cutting in thirds it’ll be the cube root of 9” and Rebecca added “yeah, and if we are cutting in halves we are square rooting the 9.” Josie explicitly stated that if the slider changed from 0 to $\frac{1}{2}$ the area would be 3 times four and if the slider changed from 0 to $\frac{1}{3}$ the area would be $\sqrt[3]{9}$ times 4 (Figure 51c). Josie’s explanation provides evidence of her reasoning about unit and partial growth factors.
After they filled the missing values in all four tables, I asked the students to summarize their activity. Josie articulated, “we first find the growth factor, that is from zero to one, or a jump of one, and then we figure out for the middle stuff. For instance (pointing to their work in Table c), we know that the growth factor is thirty two, so for the fifths we took the fifth root of thirty two and here (pointing to their work in Table b) the growth factor for a jump of one is five and so for half jumps it is square root of five.” Additionally, Rebecca added, “So, we see how many jumps we have and take that root of the growth factor like the $j^{th}$ root ..yeah, like that.” I infer from Rebecca’s claim that she offered a general definition for partial growth factors, that is, if the growth factor is, say $g$, then the $(1/j)$-partial growth factor would be $g^{(1/j)}$. From each student’s actions and explanations, I infer that, they constructed an in-the-moment meaning for 1-unit and partial growth factors in ways consistent with our intentions. Each of them was able to coordinate the growth factor of one quantity with different AoC in the values of the second quantity, that is for $\Delta a = 1, \Delta a < 1$, and $\Delta a > 1$.

**Summary of Exponential Relationship Sessions Four to Six**

In summary, in the last three sessions students reasoned covariationally (MA3). Each student coordinated changes in one quantity (e.g., area) with different-sized intervals of change...
in the second quantity (say, $\Delta a$) which supported them developing in-the-moment meanings for 1-unit growth factor and partial growth factor. I infer that they understood that the growth factor depends on $\Delta a$ and there exists a relationship between 1-unit growth factor and $1/n$-unit partial factor; that is, if the 1-unit factor is ‘$g$’ then the $1/n$-unit partial factor is the $n^{th}$ root of ‘$g$’.

### Out-of-Class Teaching Experiment Session

Six weeks after the last in-class paired teaching experiment session, I conducted one out-of-class session with each student. Because the out-of-class session was a one-on-one session, it helped me better explore each student’s meanings for quadratic and exponential relationship and to what extent she reasoned covariationally. In these sessions, each student addressed two tasks: Growing Trapezoid Task and Shrinking Quadrilateral Task II. Each student watched the tasks on a computer screen in front of them on the table. Additionally, I provided physical manipulatives of the Growing Trapezoid task. In the following sections, I provide my characterizations of Rebecca’s developing meanings for quadratic and exponential relationships. I characterize how Josie’s covariational reasoning supported her in reasoning emergently to construct graphs. Further, I discuss my characterizations of her meanings for quadratic and exponential relationships.

### Growing Trapezoid Task

The applet (Link for the Growing Trapezoid Applet) presents a line which transforms into a growing trapezoid that eventually ends as a triangle. This task is similar to the original Growing Triangle task (Figure 26) in which the relationship between the pink side length and the area is quadratic. However, instead of having a triangle’s area increasing by an increasing amount, this situation has a trapezoid’s area increasing by a decreasing amount. In the applet, we provided the students with two sliders, one longer than the other. The longer slider allowed the
students to animate the trapezoid, while the smaller one allowed them to change the increment by
which the longer slider changed. Thus, the trapezoid changed either discretely or smoothly,
eventually turning into a triangle (see screen shots of the task in Figure 52). This task was
designed in order to gain insights into each students’ covariational reasoning and conceive a
quadratic relationship between the pink line segment and the area of the trapezoid.

Figure 52. Screenshots of the Growing Trapezoid task.

Rebecca’s Activity in the Growing Trapezoid Task

Rebecca explored the applet and claimed, “as the pink line gets bigger, the area
increases” (MA1-2). Rebecca allowed the applet to play for discrete changes, paused the
animation at integral values of the slider and explained

For each increment of slider jumping, uh...like the slider is one, two, three, four, and then
five, the area is increasing, there’s more area... but... it [the area] is increasing by less each
time. So, the first increase is the biggest increase [the slider value is at 1], and then this one
is a little bit less of an increase [the slider value is at 2] and then a little bit less of an
increase [the slider value is at 3] and a little less [the slider value is at 4] and then the final
point is a tiny little bit [the slider value is at 5].

I infer from Rebecca’s explanation that she reasoned covariationally about the changes in the
area for unit changes in the slider (MA3) to make accurate claims that for unit changes in the
slider values, the area increased by smaller amounts. Next, I asked her to graph the relationship
between the area and the slider value. She watched the animation alternating the slider between
integral and continuous values and drew a smooth concave down curve. Then she drew dashed vertical segments and solid blue segments to denote the change in areas at integral values of the slider (Figure 53) and the following conversation ensued (Excerpt 16).

![Figure 53. Rebecca’s graph representing the relationship between the pink segment length and area in the Growing Trapezoid task.](image)

**Excerpt 16.** Rebecca’s justification of smooth curve for the Growing Trapezoid Task

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>MV</td>
<td>Why did you draw a smooth curve?</td>
</tr>
<tr>
<td>2</td>
<td>R</td>
<td>Because, like this one <em>(turns the slider to continuous values)</em> is now taking all values.... it [the applet] is showing you the area at every little piece...uh.. of the pink. Even a tiny piece of area, I would be able to see. I can see how it is increasing.</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>MV</td>
<td>How is the area increasing? What can you say about the change in the areas?</td>
</tr>
<tr>
<td>5</td>
<td>R</td>
<td>The changes <em>(showing the solid blue line segments)</em> are getting smaller.</td>
</tr>
<tr>
<td>6</td>
<td>MV</td>
<td>What is the relationship between the area and the pink side length?</td>
</tr>
<tr>
<td>7</td>
<td>R</td>
<td>It is definitely not linear, not exponential..... uh... it reminds of a square root function.</td>
</tr>
</tbody>
</table>
I infer that Rebecca drew a smooth curve since she imagined a continuum of areas and slider values between 0 and 5 (Lines 2-5). Representing this relationship graphically, Rebecca engaged in emergent reasoning as she constructed a graph that reflected the relationship between the continuum of values she understood the situation to represent. When asked to determine the relationship Rebecca initially made shape-based associations to name the relationship (e.g., square root function Lines 9-11), which is consistent with her activity in the pre-interview. However, as the session progressed, and when further prompted to characterize the relationship, Rebecca was able to move past shape-based associations.

**Rebecca’s Engagement With Physical Manipulatives**

Shortly after this interaction, I provided her with physical manipulatives of four trapezoids and one triangle (Figure 54a). Since the manipulatives are a physical model of the dynamic task, I hoped engagement with the manipulatives would further support her in determining the relationship between the area and the pink line segment. Rebecca quickly determined that each trapezoid represented “the change in area.” She played around with the trapezoids for three to four minutes and then stacked them (Figure 54b) and stated, “They [the trapezoids from the bottom to the top] are getting smaller.” Pointing to the AoC in the trapezoids (Figure 54c-d) she claimed, “these are the differences in the changes in the area.” At this point, I infer Rebecca was identifying the trapezoids as representing the AoC in area and described “the differences” as the DiAoC. However, I was unsure if she conceived of a relationship between the increasing areas of the trapezoids and the line segment. Therefore, pointing to the stacked trapezoids and the applet playing on the computer in front of her I asked her, “What do you think is the relationship between the area and the pink side length?” Rebecca watched the animation again and spent two minutes playing around with the manipulatives and restacking them in the
same fashion. She also referred to the graph she sketched earlier (Figure 53) and the following conversation ensued (Excerpt 17).

![Figure 54](image)

*Figure 54. (a) Physical manipulatives (b) Stacked trapezoids and triangle (c)-(d) Rebecca pointing to the differences in the areas of the trapezoids.*

**Excerpt 17.** Rebecca’s explanation for the relationship as quadratic relationship.

---

1. R I don’t know why I can’t see it here (*pointing to the graph in Figure 53*).
2. Because, when I... like physically see the things, I can see the differences in amounts of change... like these are the differences in the changes of area *(showing each AoC in the trapezoids as seen in Figure 54c-d)*...basically we can see that they are the same and I know that if the differences in those amounts of change are the same, then it would make it quadratic.
3. MV Okay
4. R I know there is some way to see it over here (*pointing to the solid segments in Figure 53*)... It's like I want to take these (*pointing to the blue* sketched in Figure 53)... and put them on the top of the other. Like take this blue one, put it on top of that blue one and see the difference.
5. MV What would those differences tell you?
6. R If I know that the differences in the amounts of change are equal, that is they
Based on Rebecca’s explanation, at this point, it was evident to me that her meanings for quadratic relationship entailed constant DiAoC (Lines 5&6, 13&14). She leveraged this meaning as she compared the AoC of the trapezoid manipulatives to confirm the relationship was quadratic (Lines 2-6). Also, graphically, she realized if she could compare the differences in the lengths of the solid blue segments, which represented the AoC in area, and examine if they were equal, this would confirm the relationship as quadratic (Lines 8-11). I conjecture that due to the lack of numbers available in the task, Rebecca would not confidently say the relationship was quadratic as she could not definitively confirm the constant DiAoC. However, I note that whereas previously she engaged in static shape thinking and considered the relationship to likely represent a square root function, at this point, she was able to unpack this thinking and move towards reasoning about the DiAoC of area to claim the relationship was a quadratic relationship.

**Shrinking Quadrilateral Task II**

We designed this task to offer opportunities to reason about varying areas in order to conceive of an exponential relationship between the area and the slider values. I presented two versions of this task to the students: Shrinking Quadrilateral Task II-Applet 1 (Link to Applet 1) and Applet 2 (Link to Applet 2). In Applet 1, as the slider takes integer values from 0 to 5, the quadrilateral shrinks such that the change in area is always one half the current area. In Applet 2, as the slider values change continuously from 0 to 5, the quadrilateral shrinks such that the change in area is always one half the current area. Applet 1 is similar to the applet used in the in-class teaching experiment sessions, the difference being the numerical values of the areas. For both applets presented in this session, I asked each student to describe how the area varied, graph
the relationship between the area and the slider values, and define and justify the relationship they conceived.

**Rebecca’s Activity in the Shrinking Quadrilateral Task II**

Watching the animation in Applet 1, Rebecca described “the area is getting smaller, it is definitely decreasing.” Next, she noted that the quadrilateral starts as a 4 by 4 square and “it went down by a half and it’s going down by another half, another half, another half, another half” followed by “it’s only going down by a half of the previous area.” Rebecca’s description of the change in area is analogous to Thompson’s description of rate of change proportional to the current value. She also emphasized, “the slider values increase from zero to five” and added “[the area] is decreasing but by smaller amounts each time” (MA3), which is consistent with reasoning about additive differences as described by Confrey and Smith.

Next, I asked Rebecca to graph the relationship between the slider values and the area. She sketched a graph with discrete points by estimating a half of the previous area (Figure 55a). To illustrate this, with reference to Figure 55c, after first plotting A as the ‘max area’, using her fingers she estimated a half of OA to note “at one it is half of the max area” and plotted B. Next she estimated a half of GB, repeated, “it’s a half of that [GB] at two” to plot C. In this manner, she plotted D, E, and finally F. Primarily, for equal changes in the slider value, Rebecca reasoned non-numerically and operationalized her understanding that the AoC was proportional to the current value as she approximated half the previous area by approximating half the vertical

8 To Rebecca, ‘each time’ meant when the slider value changed from one integral value to the other.
distance from the horizontal axis to the current point as she plotted additional points representing this relationship. I take Rebecca’s actions as the first evidence of her engaging in graphing activity by reasoning in ways that align with Thompson’s description of exponential change (e.g., the change in area is one half the current area).

Next, I presented her with Applet 2. After watching the animation, Rebecca explained, “uh... now, “it [the slider] is showing me all the values in between zero and five and I can also see what it [the area] is at all the values and at one, two, three, four, and five.” To represent the situation graphically, Rebecca sketched a smooth curve joining the points (Figure 55b) and claimed, “I think this is precise.” She offered the following explanation as a justification of the smooth curve she sketched:

...if I am thinking of where it [the curve] is ending, that, the line's never going to get to zero, because we'll still be taking half of the previous area, so it'll [the area] get smaller and smaller and smaller, like basically zero, like point o, o, o, o, o, whatever, but we'll still keep taking half of it. So, we'll never be taking away all of it. So, that is why I am...
definite with this curve, because the curve...it's...uh, we are going down and it's just going to keep going. If this was a straight line continuing it would stay, it wouldn't go below and down

Rebecca continued to conceive the area as decreasing by half the current area and attended to a continuum of both slider values and area values. To represent the situation in ways that she envisioned of, I infer, Rebecca reasoned emergently. Whereas for the situation in Applet 1, she plotted a series of points, when graphing the situation in Applet 2, she sketched a smooth curve.

**Defining a Relationship**

In spite of Rebecca demonstrating a key understanding of the situation, the change in area is one-half the previous area, I was not sure if she conceived the relationship as an exponential relationship. Therefore, I asked, “What is the relationship between the area and slider values?” As a response, Rebecca engaged in the process of determining a recursive rule. She leveraged her understanding that area is always ½ the current area, and without much difficulty, wrote \( a_n = \frac{1}{2} a_{n-1} \). Then, recalling the in-class session on expressing the areas in terms of the initial area, Rebecca expressed \( a_4 = \left(\frac{1}{2}\right)^4 a_0 \) (see her work in Figure 56a) and generalized, “area sub \( n \) is one half to the \( n \) times a sub-zero” but wrote Area \( =\left(\frac{1}{2}\right)^n a_0 \) as shown in Figure 56b and concluded, “I think it is exponential.... I can only tell that because going through this (pointing to her work in Figure 56 a-b)... like you have to use one half to the \( n \) power to find the next area.” In the moment, I conjectured that Rebecca’s meanings for exponential relationship implicitly entailed coordinating a repeated multiplication in one quantity with additive increases in the second quantity; a characteristic of exponential relationship described by Confrey and Smith. I note that Rebecca exhibited similar meanings but implicitly in the pre-interview when engaged with the Growing Tree problem and explicitly during the in-class sessions when engaged with different
iterations of the Growing Quadrilateral task. However, Rebecca did not explicitly state this as the defining characteristic of exponential growth which is why I characterize this as part of her implicit meanings.

\[ A(n) = \left(\frac{1}{2}\right)^n \]

(b)

(a)

(c)

Figure 56. (a) Rebecca’s work in (a) determining the area of the quadrilateral for slider value 4, (b) any value \( n \), and (c) finding first and second differences in areas.

After establishing a rule, she stated, “If I had numbers, I would definitely be able to know” if the relationship is exponential. Responding to my question, “How will having numbers be helpful?” Rebecca immediately created a table of values (Figure 56c) and determined the first and second differences in the areas. The following conversation ensued (Excerpt 18).
Excerpt 18. Rebecca’s explanation of why the relationship represented in Applet 2 is exponential

<table>
<thead>
<tr>
<th>Line</th>
<th>MV</th>
<th>R</th>
<th>MV</th>
<th>R</th>
<th>MV</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>MV</td>
<td>How did you determine these numbers (Pointing to the areas and the first differences in Figure 56c)?</td>
<td>R</td>
<td>The area is one half the previous.</td>
<td>MV</td>
</tr>
<tr>
<td>2</td>
<td>R</td>
<td>The change is also one half the previous. (Pointing to the second differences in Figure 56c) I think... for exponential they will never be the same. Which it looks like it is here... even if I go to the next .... they will be two, one, and half</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>MV</td>
<td>(moving her fingers from 4 to 2, 2 to 1, and 1 to 0.5 in the column of second differences in Figure 56c). If I go on, they are never going to be the same. So I will still have to say this is exponential.</td>
<td>R</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Consistent with her earlier descriptions, Rebecca noted the area and change in area as one half the previous area for unit changes in the slider values (Lines 2-4). Consistent with repeated multiplication being implicit in her meanings, she leveraged meanings related to DiAoC in areas to justify the relationship as exponential. She argued that since the successive differences would never be the same (Lines 5-9), the relationship is exponential. Rebecca’s meanings for exponential growth now entailed understanding that non-constant successive DiAoC represents exponential change.

Comparing Activities in the Two Tasks

Finally, Rebecca compared the two situations: Shrinking Quadrilateral and Growing Trapezoid and concluded appropriately that the Shrinking Quadrilateral situation represents an exponential relationship and the Growing Trapezoid situation represents a quadratic relationship. She explained, “Here (showing Figure 54 and referring to the Growing Trapezoid task), we could
see so clearly, I mean if I could make a table of values of \( y \) and find the differences, you would be able to see the differences are getting smaller and their differences would be the same. So, definitely I would say this one (Pointing to Figure 53, graphical representation of the Growing Trapezoid task) is quadratic and this one (Pointing to Figure 55b, graphical representation of the Shrinking Quadrilateral Applet 2) is exponential.”

**Summary of Rebecca’s Out of Class Session**

To summarize, in both the tasks, Rebecca consistently engaged in MA1-3. I note that reasoning covariationally became a way of thinking to Rebecca. She spontaneously reasoned about direction and AoC to represent situations. Additionally her activity is consistent with engaging in emergent reasoning as such constructed graphs representing two covarying quantities she conceived in the situation. In the Growing Trapezoid task, she identified constant DiAoC situationally which had become a defining characteristic of quadratic relationships for her. In the Shrinking Quadrilateral task, she constructed a recursive and an explicit rule by coordinating a repeated multiplication in the area values with unit changes in the slider values which supported her in identifying the relationship as defined by an exponential rule. However, to justify the relationship was exponential, she found the first and second differences in the area values to leverage her meaning that exponential change will never have a constant difference in the area values.

**Josie’s Activity in the Growing Trapezoid task**

Similar to Rebecca, Josie coordinated unit changes in the pink segment with the amounts of changes in the area (MA3). She had the applet play with the slider changing discretely and claimed, “for every equal change of the pink segment, the area is increasing by less amount.” She further explained, “the first increase is a big trapezoid, and then we keep going, the next is
less than the first, which means the area is increasing by less. We are getting a fuller triangle just by starting from a line. At the end we are adding a smaller piece. This means the area is increasing by less each time.” I infer that Josie reasoned covariationally about the area increasing by decreasing amounts (MA1-3).

After she described how the area was increasing, I asked her to graph the situation. Josie drew a pair of axes, plotted green colored points, and drew red colored segments (Figure 57a). Pointing to the red segments she explained, “the red segments are the amounts of change in area and these amounts of change are less each time” (MA3). Immediately after these actions, she changed the slider settings to take continuous values and joined the points in her sketch to draw a smooth curve (Figure 57b). I conjecture that by changing the animation to show the continuous version, Josie was attending to the continuum of values of the slider and area, which prompted her to construct a smooth curve. That is, Josie reasoned emergently as she represented the continuum of changing side length and area values she understood the situation to represent.

Figure 57. Josie’s graph representing the area with respect to the pink side length (a) for unit changes (b) for continuous values (c) Josie’s stacking the trapezoids.

After this activity, I presented the physical manipulatives. I hoped engagement with the manipulatives would support her in conceiving the relationship between the area and pink segment. Josie could easily identify that each trapezoid represented the “added on area.” As Josie continued, she reasoned about the AoC in the areas of the trapezoids. She stacked the trapezoids (Figure 57c) and described, “the difference between the first trapezoid (pointing to the yellow trapezoid) and the next trapezoid (pointing to the pink trapezoid) is the rhombus, between this
(pointing to the pink trapezoid) and this (pointing to the grey trapezoid) is the rhombus and so on....there is a rhombus like looking figure that stays the same.” It is clear from her explanation that she understood the areas of the trapezoids increased by constant amount.

In spite of her reasoning correctly that the AoC in the areas of the trapezoids are the same, it was unclear at this point if she conceived of constant DiAoC of the original growing trapezoid in the applet. With this in mind, I asked her, “What is the relationship between the pink segment and the area?” Josie promptly replied, “as the pink segment is changing by the same amount, the area is increasing but... uh definitely by less.” While Josie correctly described the relationship, and consistently reasoned that the areas of the trapezoid increase by the same amount, it was unclear whether she related the relationship to quadratic relationship. One explanation is that Josie’s way of identifying quadratic change required numeric values; without such values, Josie would not make claims regarding if a relationship is quadratic or not, which is somewhat consistent with Rebecca’s initial wavering on if this relationship was quadratic.

Josie’s Activity in the Shrinking Quadrilateral Task II

Consistent with her covariational reasoning in the Growing Trapezoid task, Josie reasoned about the changes in the area for unit changes in the slider values (MA3). After watching the animation when the slider took integral values, Josie described, “for every equal amount of slider change, the area is half the area that was before. It is decreasing by half of the previous value.” She also determined the numerical values of the area and added, “I also say that it [the area] is decreasing by less each time. I start with sixteen, then eight, then four, and then it is two and then one and the last is a half.” Josie’s description is indicative of her conceiving a constant multiplicative rate of change and the change in area as proportional to the current area.
Next, to graphically represent the situation in Applet 1 in which the slider took integral values, Josie plotted points (Figure 58a) and then drew green dashed line segments to denote the changes in the areas (Figure 58b). Then, for Applet 2 in which the slider took continuous values, she drew a smooth curve (Figure 58c) and justified, “the slider value is increasing not by ones now but by everything in between. So the dots are connected and I... I’m going to say it is going be a curved line. It’s not going to be straight... and that’s because... I know why... because the amounts of change... are... less each time. It’s not a constant amount of change.” She noted that the slider takes a continuum of values and similar to her graphing activity in the Growing Trapezoid task, Josie justified the curvature by emphasizing area decreasing by decreasing AoC. Further, she described, “for every slider value, the area is decreasing by less each time... uh, it it’s not decreasing by the same amount.” Hence, I interpret her explanations indicative of her reasoning covariationally and envisioning the graph as produced because of two covarying quantities, thereby reasoning emergently.

![Figure 58](image)

**Figure 58.** (a) Josie’s graph representing the area with respect to the slider (a) for unit changes in the slider (b) with green dashed segments denoting the differences in the areas (c) for continuous values of the slider.

**Determining Analytic Rule and Comparing Activities in the Two Tasks**

As the session continued, I asked Josie if she could determine a rule that describes the situation. Without much difficulty Josie wrote \( A_n = (\frac{1}{2})A_{n-1} \) and said,” I know the area is one half the previous and so I can write this way, but this one works only when we know the previous [area].” She sought to make a table of values with slider values from 0 to 5 and the
corresponding area values reasoning that the area would be one half the previous area (Figure 59a). Then, similar to Rebecca, she recalled the in-class activities of expressing the areas in terms of the initial area. She leveraged her understanding that the area is one-half the previous area to express \( A_4 = (1/16) A_0 \) (Figure 59b) after which she realized 16 could be expressed as \( 2^4 \) to first write \( A_4 = (\frac{1}{2})^4 A_0 \) and then \( A_n = (\frac{1}{2})^n A_0 \). She claimed, “Now, I can find the area of the quadrilateral for any value of the slider.... I can plug the value for \( n \).” In the moment, I conjectured that Josie coordinated a repeated multiplication in the area for additive differences in the slider values to determine a rule defining the relationship between the area and the slider value. Throughout this activity, Josie consistently reasoned that, (i) the areas decrease and are one half of the previous area, and (ii) the AoC in the area is one half the previous area. However, at no point did she indicate she conceived of an exponential relationship as representing the situation. Consequently, I asked, “what relationship do you see between the area and the slider value?” Josie responded, “as the slider is increasing, the area is decreasing by less each time.” While Josie correctly described the relationship, it was unclear whether Josie related the relationship to exponential relationship.

![Figure 59. Josie’s work to determine a recursive rule to define the relationship between the area and slider values.](image)

Before I concluded the session, I asked Josie if she noticed any similarities or differences in the two tasks. I anticipated that she may reflect on her activity with the manipulatives and begin thinking about constant DiAoC and hoped she would connect that to quadratic
relationship. I conjectured this because she reasoned about constant AoC in areas of the trapezoids but did not connect it to quadratic. In response to my question, Josie replied, “That (pointing to the manipulatives) was increasing by less each time the slider changed and this one (pointing to Shrinking Quadrilateral applet playing on her screen) is decreasing by less each time.” Although Josie did not identify the name of the relationships as quadratic or exponential, I infer that her covariational reasoning supported her conceive the defining characteristics of quadratic and exponential relationships.

**Summary of Josie’s Out-of-Class Session**

In summary, Josie’s actions exhibited her reasoning covariationally (MA3) all through the session. She remained focussed on two quantities to sketch accurate graphs and justified the curvature of the graphs by making correct claims about areas of the trapezoids increasing by decreasing amounts and areas of the quadrilateral decreasing by decreasing amounts. Additionally, she consistently reasoned emergently as she constructed graphs that represented her conceived situations, a clear shift in Josie’s ways of reasoning from the pre-clinical interviews.

In the Growing Trapezoid task, she noted constant AoC in the areas of the trapezoids. In the Shrinking Quadrilateral task, she conceived of a repeated multiplication in area for additive changes in slider value and also the rate of change in area as proportional to the current area. Josie conceived change in areas compatible with the defining characteristics of quadratic and exponential change. However, I was unable to elicit her meanings for these relationships in this session.

**Post Clinical Interviews**

In order to explore what meanings for quadratic and exponential relationships each student had developed during the teaching episodes, and shifts, if any, in their ways of reasoning
and differentiating between the patterns of change, I conducted post clinical interviews. I engaged each student in a one-on-one post clinical interview about two weeks after the sessions described above. I waited two weeks to explore what meanings and ways of reasoning they might have developed over the course of the study. I synthesize the results from the post-interviews to provide a characterization of each student’s covariational reasoning and current meanings for quadratic and exponential relationships at the end of the teaching experiment. In these interviews, I presented (i) a video of a *Growing Trapezoid With Pauses*, (ii) graphs that represented quadratic and exponential relationships, (iii) tables of values that presented quadratic and exponential relationships, and (iv) the *Growing Tree problem*.

The goal in this section is to present my models of the student’s mathematics at the end of the teaching experiment that I inferred from the post interview and previous teaching sessions. As such, rather than giving a play-by-play of the students’ activities in each task, I summarize their meanings in relation to covariational reasoning, quadratic relationships, and exponential relationships. I then provide evidence from these clinical interviews to support my models based on their activity in this interview. Throughout, I make comparisons with their activities in the pre-interviews in order to highlight shifts in their meanings. Before I present my models, I first describe the design of the video of a *Growing Trapezoid With Pauses*. Then, I characterize each student’s (i) covariational reasoning, (ii) meanings for quadratic relationships, and (iii) meanings for exponential relationships, rather than explaining their activities task by task.

**Design of a Video With a Growing Trapezoid With Pauses**

This task, a variation of the Growing Trapezoid task used in the one-one out of class sessions, is a video showing a line which transforms into a growing trapezoid that eventually ends as a triangle. The video pauses for one second for an equal amount of change in the pink
side length (see Figure 60 for screen shots at the start, first pause, third pause, and end of the video). I asked each student to respond to a hypothetical student’s graphical representation (Figure 60e) of the relation between the total area of the shape and the pink side length. Because I conjectured that the students might make claims about the correctness of the graph, I prepared an additional prompt. I intended to ask them to sketch a graph of the relation between the total area of the shape and the pink side length if they claimed the hypothetical student’s graph was incorrect. I designed this task with two goals in mind: (i) to examine the PSTs’ covariational reasoning after providing several opportunities to reason covariationally during the teaching experiment sessions, and (ii) to gain insights into their emergent thinking. Since, I intended for each student to focus on reasoning about covarying quantities (the area and pink side length), I intentionally had straight line segments in the graph. The line segments in the graph is indicative of one attending to one quantity (area) and time versus two quantities. I conjectured that students’ reasoning about the line segments in the graph would offer me opportunities to explore to what extent they conceive a graph as an emergent trace of two covarying quantities. Reasoning emergently to construct a graph representing this situation would entail one imagining a moving point which would stop when the video pauses and continues to move when the video restarts, thus, having no line segments in the graph.

![Graphs](attachment:graphs.png)
Results From Rebecca’s Post-Interview

By the post-interview, Rebecca consistently engaged in reasoning about AoC and focused on how two quantities varied simultaneously (MA3). She engaged in both static and emergent reasoning. Whereas she reasoned emergently in constructing graphs, she engaged in static shape thinking to identify the relationship represented by a graph which she could unpack subsequently by reasoning covariationally. Her meanings for quadratic relationships entailed constant second differences. With regards to exponential relationships, Rebecca’s meanings included coordinating a repeated multiplication with corresponding additive changes, a meaning I infer she had exhibited implicitly since the pre-interview. Her meanings for exponential relationships also included (i) a rule of the form \( y = bx \) and (ii) a conception of non-constant successive differences. In the following sections, I describe her activities that exemplify each of these characterizations.

Rebecca’s Covariational Reasoning

Relative to the pre-interviews, across all tasks Rebecca consistently reasoned about two quantities changing directionally and in terms of amounts of change (MA1-3). Attending to covarying quantities, Rebecca interpreted and constructed graphs as a trace in progress, thus engaging in emergent reasoning.

Figure 60. Screen shots of the video at (a) beginning (b) at the first pause (c) at the second pause and (d) end of the video and (e) graph of a hypothetical student.
Figure 61. Rebecca’s graph to represent the relationship between area and pink side length.

As an example of her covariational reasoning and engaging in emergent thinking, for the Growing Trapezoid with Pauses task, Rebecca noted, “both the area and the length of the pink line is getting bigger” (MA1-2) and the area is “increasing by lesser amounts” (MA3). Indicative of reasoning emergently, Rebecca sketched a smooth curve (Figure 61) by attending to the covarying area and pink segment; she understood the pause in the video did not impact her graph as time was not one of the quantities represented by her graph. She described the line segments in the hypothetical student’s graph (Figure 60e) “show that the area stopped while the pink line kept going and that’s not true.” Further she explained, “since we are showing the relationship between the area and side length, I can’t really show that its [the video] stopping. If I showed where it [the video] paused, then one of the axes would have to be time.” As Rebecca drew the graph to represent her conceived relationship, she engaged in reasoning about AoC; she drew vertical and horizontal line segments as seen in Figure 61 to represent the decreasing AoC in area for equal AoC in the pink line segment.

Rebecca’s activity stood in contrast to her activity in the pre-interview. Whereas in the Bottle Problem in the pre-interview, Rebecca only reasoned about the directional change of the height volume of water varying in tandem to sketch a graph, in the post interview she also reasoned about the amounts of change of area and pink segment (MA1-3) to sketch an appropriate graph of the situation. I infer that repeated experiences in reasoning about and
representing two varying quantities situationally and graphically (e.g., the Ferris wheel task, the Power Tower task) during the teaching experiment sessions supported her in identifying and representing amounts of change of one quantity with respect to the other. Additionally, I infer that all through this activity Rebecca attended to how the two quantities were changing that supported in emergent thinking.

**Rebecca’s Meanings for Quadratic Relationships**

As previously described, having multiple opportunities to reason covariationally supported Rebecca in constructing meanings for quadratic relationships as entailing a constant DiAoC, a shift from her meanings in the pre-interview. Like in the pre-interview, Rebecca engaged in static shape thinking when interpreting graphs that did not have numbers. However, Rebecca leveraged her developed meanings for quadratic relationship as entailing constant second differences when describing the relationship represented by a table of values and across the rest of the tasks in the interview.

As evidence of Rebecca continuing to engage in static shape thinking at times when numeric values were not readily available, when presented with the graph shown in Figure 62a, Rebecca responded, “it could be quadratic or exponential” and stated, “I know that parabolas are
often quadratic shaped, like, like, I could visualize this [the graph] going up on the next side
(moving her hand to make a smooth curve to the left of the graph).” However, when describing
the relationship represented by the other graph, Rebecca created a non-numeric tabular
representation (Figure 62b-d) and argued that if she could determine numeric values and if these
values had constant second differences, then the relationship would be quadratic (Excerpt 19).

Excerpt 19. Rebecca’s explanation of non-numeric reasoning for a quadratic relationship

If these (showing the blue dots in the y column of Figure 62d were 2, 4, 6, 8, the
 amounts of change (showing the red stars in Figure 62d) would be plus 2, plus 2, plus
 2, and then that would make it like linear. If these amounts of amounts of change
 (showing the green timers in Figure 62d) were 2, and 2, and 2, that would make it
 quadratic, but if you get to this point (showing the green timers in Figure 62d and they
 are not the same, it’s not quadratic. But I don’t have like actual points.

Based on this later activity, I infer Rebecca had the ability to unpack her static shape
thinking in such a way that she would be able to determine if a graphed relationship represented
a quadratic if she were to determine numeric values from the graph. Hence, and like her initially
unease with characterizing the Growing Trapezoid in the one-on-one teaching session as
quadratic, Rebecca’s meanings for quadratic relationships entail identifying constant DiAoC, but
she experiences uneasiness identifying such a relationship without a table of values.

As a second example exhibiting a shift in her meanings for quadratic relationships, for
the table of values in Figure 63, Rebecca reasoned about the second differences in y-values to
identify the relationship. Whereas in the pre-interview Rebecca attempted to determine a
correspondence rule, in the post interview she attended to the unit changes in x-values then
claimed, “the amounts of amounts of change are minus three each time, so this is quadratic.”

Collectively, across her activities in the post-interview, I infer that Rebecca’s meanings for quadratic change entailed a relationship with one quantity having constant second differences for unit differences in the second quantity.

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*Figure 63. (a) A table of values representing quadratic relationship (c) Rebecca’s work to identify the relationship.*

**Rebecca’s Meanings for Exponential Relationships**

By the post-interview, Rebecca’s meanings for exponential relationship entailed (i) non-constant successive differences in the value of one quantity for equal changes in another quantity, (ii) a rule of the form \( y = b^x \), and (iii) a coordination of a repeated multiplication in one quantity with additive growth in the other.

Exemplifying her meaning for exponential growth as entailing non-constant second differences is her activity when asked to determine the relationship represented by the table of values in Figure 64a. Rebecca determined the differences in \( x \)-values and successive differences in the \( y \)-values (Figure 64b) and described the relationship was exponential “because no matter how many times we go, until the last one, it won’t be constant, they (motioning her fingers on the differences in her work in Figure 64b) are never going to be constant.” Rebecca’s reasoning is similar to her characterizing non-constant successive differences in the area of the quadrilateral
as exponential change in the one-on-one teaching experiment session (The Shrinking Quadrilateral task, Excerpt 18).

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(a) REBECCA'S WORK TO IDENTIFY THE RELATIONSHIP

Different from reasoning about non-constant successive differences, Rebecca’s engagement with the Growing Tree problem exemplifies her meanings for exponential relationship as a coordination of a repeated multiplication in one quantity with additive growth in the other and as entailing a rule of the form \( y = bx \). Rebecca coordinated a repeated multiplication in the height of the tree with additive differences in time, to identify the relationship between the height of the tree and time as exponential. She explained, “I know that it is [the relationship] exponential because it [the height of the tree] is doubling, we are doubling every month. We are multiplying every time by 2 for each month (Figure 65a). If we are going to continuously multiply by 2, that’s going to prove exponential.” Leveraging this meaning, she successfully determined a rule for the height of the tree after 100 months (Figure 65b).
Figure 65. Rebecca’s work to determine (a) the height of tree for the first three months (b) height after 100 months.

Whereas previously in the teaching experiment and pre-interview, Rebecca implicitly understood repeated multiplication meant exponential growth she explicitly defined a relationship to be exponential only if it had non-constant DiAoC. I conjecture that, reflecting on her experiences, Rebecca reorganized her meanings for exponential growth such that she now explicitly understood both characteristics as defining characteristics of exponential relationships.

Although there is evidence that Rebecca reorganized her meanings for exponential growth generally, as she addressed the student reasoning about the height of the tree after 2.5 months, her activity indicates that her in-the-moment meanings about partial growth factors exhibited in Session 6 did not become a stable part of her meanings for exponential change. Instead of reasoning about partial growth factors, she relied on additive reasoning as she had in the pre-interview. Specifically, she said, “from 40 to 80, we are adding 40. In one month we add 40, so in half a month we add 20” and concluded the hypothetical student’s claim that the tree would be 60 inches tall in two and half months is correct.”

In conclusion, whereas in the pre-interview Rebecca engaged in static shape thinking and her meanings for exponential growth implicitly entailed repeated multiplication, by the end of the study, I infer her meanings for exponential relationships to entail (i) a covariational relationship which entails a repeated multiplication, (ii) non-constant successive differences, and (iii) a rule of the form \( y = b^x \). However, I conjecture that having only three sessions addressing partial growth factors was not sufficient to support her in reorganizing her meanings into a coherent whole.
Results From Josie’s Post-Interview

Josie’s engagement across all tasks reveal that she reasoned covariationally about direction and amounts of change (MA1-3). Josie engaged in static shape thinking to correctly identify the relationship represented by graphs which she could later unpack by reasoning covariationally. Her meanings for quadratic relationship entailed constant second differences. With regards to her meanings for exponential relationships, her meanings included (i) increased additive differences, (ii) non-constant successive differences in y-values, and (iii) repeated multiplication coordinated with additive differences. In the following sections, I describe her activities that exemplify each of these characterizations.

Josie’s Covariational Reasoning

Relative to the pre-interview, across all tasks Josie reasoned about covarying quantities and coordinated AoC in two simultaneously varying quantities (MA3). Consistent with her ways of reasoning in the Growing Trapezoid task in the one-one teaching experiment session in which she described the AoC in the areas of the trapezoid as increasing by the same amounts and sketched an appropriate graph (Figure 57), for the Growing Trapezoid with Pauses, Josie reasoned at MA3 situationally. She described the AoC in area decreases for equal AoC in the pink segment. She created a pictorial representation of the situation by drawing two trapezoids and one triangle (Figure 66a) and explained, “this part (showing the blue trapezoid in Figure 66b) gets smaller than that (showing the red trapezoid in Figure 66b) and this (showing the green trapezoid in Figure 66b) gets smaller than this (showing the blue trapezoid in Figure 66b) and we know that the pink is increasing by the same.” In this and other tasks, Josie leveraged reasoning about amounts of change (MA3) as she conceived of and represented relationships between covarying quantities. I conjecture that the repeated opportunities to reason about and represent
relationships between covarying quantities in the teaching experiment sessions supported Josie’s developing more sophisticated ways of reasoning covariationally.

![Image](a) ![Image](b)

*Figure 66.* (a) Josie’s pictorial representation of the situation (b) a recreation of her work in (c) hypothetical student’s graph and (d) Josie’s graphical representation of the situation.

**Josie’s Meanings for Quadratic Relationships**

Josie maintained meanings for quadratic relationships such that she understood a quadratic change as having constant second differences which indicates a shift in her meanings from the pre-clinical interview. Similar to Rebecca, initially she engaged in static shape thinking, but leveraged her meanings for constant second differences to unpack static shape thinking.

Based on the shape of the curve in Figure 67a which represents a quadratic relationship, like in the pre-interview, Josie first responded, “[the graph] looks quadratic” because “it looks like half a parabola” and making hand motions as if drawing a curve to the left explained, “then it would be a parabola and it would be a quadratic.” Although in the moment I conjectured her engaging in static shape thinking, immediately after this Josie determined two points on the curve, (1,2) and (3,8), without difficulty and explained, “if I knew the next one point, I could check if the amounts of amounts of change are same and it would be quadratic.” Josie’s explanation provides evidence of her unpacking her shape thinking by arguing the relationship would be quadratic if it maintained a constant DiAoC.
As a second example, in the table of values in Figure 67b Josie determined the constant second differences to claim “Okay, it is quadratic, yes, quadratic because the amounts of amounts of change are the same. Quadratic.” Josie’s activity indicates is in contrast to her activities in the pre-interview where she determined the slope of two points and attempted to establish a correspondence rule between the \( x \) and \( y \) values. From her ways of interpreting graphs and table of values, I infer that Josie’s meanings entailed constant second differences in one quantity while coordinating unit changes in the second quantity.

**Josie’s Meanings for Exponential Relationships**

Josie reasoned covariationally across the tasks of interpreting graphs and tables of values, and the Growing Tree problem. Her activities in these tasks indicate her meanings for exponential relationships entailed (i) increasing additive differences, (ii) non-constant successive differences, and (iii) repeated multiplication coordinated with additive differences. I provide evidence for my characterizations below.

Addressing the prompt on interpreting a graph that represented exponential relationship (Figure 68a), although she initially engaged in static shape thinking and claimed, “[the graph] is starting small and then shoots up, I want to say it is exponential,” she could unpack her thinking
by reasoning about AoC, which I consider as a shift in her ways of reasoning from her activities in the pre-interview. Josie imagined points on the horizontal axis and motioned as if drawing vertical segments from these points to the graph, see Figure 68b for a recreation of her hand motions and explained

Like over here, from 1 to 2 (moving her left finger on the horizontal axis from 1 to 2) the change is little, we are not increasing much. The change is from this to this (moving her fingers on the line segments a and b in Figure 68b) only a little, may be a $\frac{1}{2}$, but 2 to 3 we are increasing may be by a 1 (moving her fingers on the line segments b and c), 3 to 4 we are increasing by a 2 (moving her fingers on the line segments c and d ) may be which is a little more and so on and so forth and then it increases by more. So, definitely I am going to say it is exponential.

Josie reasoned covariationally about the increasing differences in the $y$-values for equal changes in the $x$-values (MA3). I infer Josie’s explanation aligns with Confrey and Smith’s description of exponential relationship as having increasing additive differences.
As is evident from her actions in addressing the prompt on a table of values, Josie attended to multiple ways of reasoning to conclude the relationship was exponential which is in contrast to her ways in the pre-clinical interview where she relied on finding the slope between two points. I conjecture that the numerous opportunities to reason covariationally in the teaching sessions supported her in developing these meanings.

For the table of values in Figure 68c, Josie while attending to the change in x values as “increasing by the same,” determined the first, second, and third differences of the y-values to claim “It’s exponential because these amounts of change that we are subtracting aren’t the same.” That is, she attended to the non-constant successive differences in y-values. Further she noted that the ratio of the successive y-values is 3 and stated, “Yeah, it’s something with the number three and it’s exponential, it’s three to the x.” I note that in both ways of reasoning she attended to the unit changes in the x-values. Previously in the pre-clinical interview and teaching sessions, Josie coordinated repeated multiplication of the y-values with additive differences, but did not explicitly define the relation as exponential. I conjecture, reflecting on her experiences in the different sessions, she reorganized her meanings such that her meanings now encompass increasing additive differences, non-constant successive differences in y-values, and also a coordination of repeated multiplication with additive differences.

Josie’s engagement with the Growing Tree problem provides additional evidence of my inferences. She determined the height of the tree after the first four months and claimed, “it [the relationship represented by the table of values in Figure 69] is definitely exponential, because... the amounts of change are increasing by more each time.”
Further, Josie attended to the height doubling and coordinated with unit change in time to determine a rule of the form $y = ab^x$ which she used correctly to calculate the height after two and half months (Figure 69). I take this as a shift in her ways of reasoning from the pre-interview where she reasoned additively and incorrectly claimed the height of the tree after two and half months as 60 inches. However, since Josie substituted the value to determine the height, I am unsure if she experienced shifts in her meanings for exponential change with reference to partial growth.

In comparison with her meanings in the pre-interview which entailed static shape thinking and implicit repeated multiplication, at the end of the study, her meanings entailed an exponential relationship as having (a) non-constant successive differences, (b) increasing additive differences, and (c) a rule of the form $y = ab^x$ by coordinating a repeated multiplication with additive differences.
Chapter 5. Discussion and Conclusions

The goal of this study was to examine and support preservice teachers developing meanings for quadratic and exponential relationships via their covariational reasoning. Recall, the research question I sought to address was:

How might pre-service teachers reason covariationally to develop meanings for quadratic and exponential relationships?

In Chapter 2, I presented a conceptual analysis which theorized the ways in which students may leverage their covariational reasoning, and particularly reasoning about amounts of change, to develop productive meanings for quadratic and exponential relationships. Many features of this conceptual analysis proved useful in supporting shifts in the students’ meanings. In this chapter, I first describe the students’ general covariational reasoning throughout the study. Next, I describe each student’s initial meanings for quadratic relationships and characterize how reasoning covariationally supported her in reorganizing her meanings for quadratic relationships. Then, with regards to exponential relationships, I discuss the multiple meanings each student maintained for exponential relationships and how these meanings interplayed with each other. Throughout, I connect to the extant literature and my theoretical perspective described in Chapter 2. Finally, I conclude by discussing the limitations of this study and areas for future research.

The Students’ Covariational Reasoning

Prior to characterizing shifts in the students’ meanings for quadratic and exponential relationships, I first characterize shifts in their covariational reasoning. During the pre-interviews Rebecca exhibited activities to indicate she reasoned about directional change (MA1-2) (Carlson et al., 2002) and Josie exhibited minimum coordination of two quantities. During the course of the teaching experiment, through repeated engagement in tasks that required them to construct
and reason about relationships between covarying quantities, the students demonstrated different forms of covariational reasoning as described by different researchers.

My analyses illustrate that the students constructed quantities and engaged in continuous reasoning as described by others (Saldanha & Thompson, 1998; Thompson, 1994). For instance, Rebecca understood that as two quantities’ values vary simultaneously (e.g., Excerpt 13), either quantity’s value “changed from one to another by assuming all intermediate values” (Saldanha & Thompson, 1998, p. 299). Further, as the teaching experiment progressed, the students consistently demonstrated actions compatible with (at least) mental actions one through three described by Carlson et al. (2002). By leveraging activities in the extant literature that proved useful in supporting other students in developing their covariational reasoning (e.g., the Power Tower task (Moore et al., 2014)), the students developed reasoning about amounts of change as a stable part of their meanings. By the end of the teaching experiment each student anticipated that identifying amounts of change of one quantity with respect to equal changes in a second would support her in understanding aspects of a covariational relationship as well as in constructing an accurate graph.

Consistent with my conceptual analysis, after developing reasoning about amounts of change, they engaged in this reasoning to further differentiate between quadratic and exponential relationships. I illustrated how students first identified changing AoC in one quantity for equal AoC in the other quantity, a characteristic of both quadratic and exponential change. They then sought to examine successive differences (constant second differences for quadratic change and non-constant successive differences for exponential growth) to identify the growth as either quadratic or exponential. I elaborate on their meanings for quadratic and exponential growth shortly. When engaging in examining AoC, Rebecca and Josie’s activities were often compatible
with Confrey and Smith’s (1994, 1995) description of coordinating variation in two sequences of values. For instance, in the different iterations of the Growing Quadrilateral task, the students introduced numerical values when conceiving of changes in the areas of the quadrilateral explicitly coordinating with variation in the successive values of the slider.

In conclusion, each form of covariational reasoning described in the literature was productive for the students when addressing different tasks. The students often engaged in smooth reasoning (Saldanha & Thompson, 1998; Thompson, 1994) when making sense of smoothly changing dynamic situations but would then begin to break quantities down into chunks (Castillo-Garsow, 2012) according to amounts of change (Carlson et al., 2002) to further understand the relationship between quantities. When diving further into how the AoC changed, the students often found or created numeric values which allowed them to examine coordinated changes in two sequence of values (Confrey & Smith, 1994, 1995). Hence, my results highlight the interplay of the various forms of covariational reasoning characterized by different researchers.

**Reasoning Covariationally to Reorganize Meanings for Quadratic Relationships**

Addressing part of my research question that seeks to investigate PSTs’ meanings for quadratic relationships, I add to the limited literature on students’ meanings for quadratic relationships by providing empirical examples of two PSTs developing meanings for quadratic relationships. In addition, I extend the growing body of research (e.g., Ellis, 2011a, 2011b; Lobato et al., 2012) emphasizing the role of covariational reasoning in supporting students to construct productive meanings for quadratic relationships.

Both Rebecca and Josie entered the study with limited meanings for quadratic relationships. Consistent with PSTs’ static shape thinking reported elsewhere (Moore et al., 2013;
Moore et al., 2016; Moore & Thompson, 2015) prior to the teaching experiment, Rebecca and Josie associated quadratic relationships with a shape (e.g., U-shaped or a parabola) and/or an analytic rule (e.g., \( y = x^2 \)). Compatible with the middle grades students reported by Ellis (2011a, 2011b), during the teaching experiment, each PST leveraged covariational reasoning and conceived of increasing AoC and constant DiAoC in the areas of a triangle in the Growing Triangle task. I note that, despite their mathematics background including up through at least Calculus II, neither student initially associated constant DiAoC with quadratic change. However, by the end of the study each student’s meanings for quadratic relationship included a relationship that entails constant second differences in one quantity for uniform changes in the other quantity.

I note that consistent with Moore et al.’s (2013) findings with regards to PSTs’ use of polar coordinates, using a non-Cartesian coordinate system supported Rebecca and Josie in reflecting on what was critical and not critical to representing quadratic relationships. Rebecca and Josie came to understand that in the polar coordinate system, a parabolic shaped curve is not representative of a quadratic relationship and likewise the graph of a quadratic relationship is not a parabola via their meanings for constant second differences (e.g., Excerpt 9). Such reasoning may have supported the shift from their initial static shape thinking (e.g., a parabola means quadratic).

Because each student was aware of the defining characteristic of a quadratic relationship during the post-interview, she could unpack her static shape thinking by reasoning about constant second differences in \( y \)-values for uniform changes in the \( x \)-values. Hence, my findings add to the literature on preservice teachers’ static and emergent reasoning by providing empirical examples of students’ reorganized meanings for quadratic relationships grounded in static shape.
thinking to meanings that entail constant DiAoC, graphs as emergent traces, and an ability to
unpack shape thinking in terms of covarying quantities.

**Reasoning Covariationally to Reorganize Meanings for Exponential Relationships**

As I described in Chapter 2, researchers (e.g., Confrey & Smith, 1994, 1995; Thompson,
2008) have provided different characterizations of exponential growth. Over the course of the
teaching experiment, when addressing different tasks, students leveraged meanings and
reasoning compatible with each of these characterizations. In what follows, I outline when
students engaged in each type of reasoning.

Consistent with Confrey and Smith’s (1994, 1995) description of exponential growth, the
students calculated both ratios and additive differences between successive $y$-values for constant
changes in $x$-values. Recall, Confrey and Smith (1994, 1995) described exponential growth as
entailing either (1) a constant multiplicative rate of change or (2) an increasing additive rate of
change. Consistent with the description of a constant multiplicative rate of change, Rebecca and
Josie coordinated a repeated multiplication in one quantity (e.g., height of the tree) with additive
growth in the other quantity (e.g., months). For a large gap in time, such as 100 months, the
students could express the height in power notation (e.g. activity in Figures 77 & 81) without
actually carrying out the multiplication. Similar to students’ activities reported elsewhere (Ellis et
al., 2015), both Rebecca and Josie could write expressions of the form $y = ab^x$ representing the
relationship between the height of the tree and the number of months by coordinating a
multiplicative growth with additive growth in two sequences of values.

Consistent with Confrey and Smith’s (1994, 1995) description of increasing additive
differences, when conceiving of growth in the Growing Quadrilateral task, the students regularly
worked with tables of values and attended to the differences in the values of areas (e.g., Excerpt
11, activity in Figure 50) while coordinating unit changes in the slider values. In addition to conceiving the increasing additive differences in the area, the students leveraged their developing meanings for quadratic relationships when reasoning about exponential relationships. Rebecca and Josie were explicitly aware of their meanings for quadratic relationship as entailing constant second differences and I conjecture the students’ meanings for quadratic relationships impacted their meanings for exponential relationship as entailing non-constant successive differences.

Whereas reasoning about non-constant successive differences supported the students in correctly identifying an exponential relationship in this study, I note that non-constant successive differences is not a defining characteristic unique to exponential functions (e.g., for constant additive differences in $x$-values, we get non-constant successive $y$-values for the logarithmic function $y = \log x$). However, such reasoning was productive for the students when tasked to differentiate between quadratic and exponential growth in this study. I return to this in the limitations and directions for future research.

Aligning with Thompson’s description of exponential growth, the students, at times, conceived change as proportional to the current value. Specifically, the students imagined the change in the area of the quadrilateral as proportional to the current area in tasks that included growing and shrinking quadrilaterals (e.g. Excerpt 20, activity near Figure 67). While understanding that the rate of change being proportional to the value of the function did not become part of the students’ defining characteristics of exponential growth, this reasoning was productive when deriving rules for exponential growth (e.g. Excerpt 14, Figure 55). For example, critical to writing a rule such as $A_n = A_{(n-1)} + (k)A_{(n-1)}$, is understanding that the new value is determined by adding $k$ times the previous value to itself. Hence, such reasoning did play an important role in the students’ activity.
Returning to Ellis et al.’s (2015) characterization of partial growth factors, in the moment, imagining growth by coordinating multiplicative rate of change with 1-unit change supported students in conceiving the relationship between the 1-unit growth factor and 1/n-partial growth factor (e.g. Excerpt 17, activity near Figure 62). However, in the post-interview, the students did not leverage this relationship between the roots and powers of the growth factor (e.g., Growing Tree problem). I conjecture having only three sessions on unit and partial growth factors was not sufficient to allow for reorganizing their meanings for partial growth-factors. One possible explanation for this is that the students likely had less prior experience with partial growth factors in their school experiences (as compared to quadratic and exponential growth, generally). Hence, the students may have needed more opportunities to reason about change in sub-intervals in order to be able to reflect on their activities in such a way as to develop (or reorganize) stable meanings for partial growth factors.

In summary, addressing different tasks supported each student in engaging in different forms of reasoning as she constructed multiple meanings for exponential relationships. The students often engaged in coordinating repeated multiplication with unit differences (Confrey & Smith, 1994, 1995) which allowed them to define analytic rules of the form \( y = ab^x \). When interpreting growth in covarying quantities in terms of amounts of change (Carlson et al., 2002), the students made tables of numeric values and reasoned about the increasing additive differences (Confrey & Smith, 1994, 1995) and also the amount of change as proportional to the previous value (Thompson, 2008). The students leveraged several (possibly interconnected) meanings which supported them in translating data from a table of values to either recursive representations or to the closed-form equation \( y = ab^x \), a difficulty in prospective high school teachers as reported by Davis (2009). In addition to exhibiting ways of reasoning compatible
with researchers’ descriptions of exponential growth, the students conceived of a new and different way of reasoning by coordinating non-constant successive differences in one quantity for equal changes in the other quantity. Hence, I add to the limited literature on PSTs’ meanings for exponential change by (a) identifying a novel way of understanding exponential change (e.g., non-constant DiAoC’s) and (b) providing empirical examples of the interplay of the different forms of understanding exponential change characterized by other researchers.

Limitations and Directions for Future Research

In my study, I focused on two PST’s meanings for quadratic and exponential relationships via their covariational reasoning. As noted by Steffe and Thompson (2000),

It does not make sense to demand of teaching experiments that they “generalize” in the way in which one might hope that claims thought to be true about a random sample would be true as well about the population from which the sample was drawn. (p. 300)

The results I presented are my models of each student’s mathematics generated in the context of my interactions with them during the course of this study and will not be transferable to other PSTs. Hence, I call for further research examining PSTs’ meanings for quadratic and exponential relationships in larger settings, including a whole-class teaching experiment, to explore the extent to which these (or similar) activities can promote comparable shifts in their meanings. Particularly, since PSTs are most likely to teach these topics to their future students and the “more coherently teachers understand an idea they teach, the greater are students’ opportunities to learn that idea coherently” (Thompson et al., 2017, p. 95), there is a continued need to support PSTs in reorganizing their meanings for these relationships.

I propose that my models of Rebecca and Josie’s meanings can lay a foundation for the development of curricular materials to mathematics educators and teacher education programs
intended to support other PSTs in developing productive meanings for quadratic and exponential relationships via covariational reasoning. I conjecture that certain features of the tasks supported the students in reasoning covariationally as well as reorganizing their meanings for quadratic and exponential relationships. I attempted to use situations that (i) would be familiar to them (e.g., the Carnival Ride and Power Tower tasks), (ii) used different representational systems (e.g., two graphs represented in the polar coordinate system), and (iii) offered opportunities to reason both numerically and non-numerically (Excerpt 19) about the amounts of change. Also, the physical manipulatives of the Growing Triangle task further supported the students in conceiving the amounts of change in area (i.e., the trapezoids) as a quantity unto itself and reason about their differences. I conjecture that by making quantities available to the students through dynamic and physical representations, the tasks that I used served as mediating artifacts (Greenstein, 2018) that supported the students in reasoning covariationally and in reorganizing their prior meanings for quadratic and exponential relationships. Hence, by describing both the tasks and task design principles, I intend to provide curriculum developers with tools needed to not only support PSTs in reorganizing their meanings for quadratic and exponential growth but also possibly other relationship classes (e.g., cubic, logarithmic).

In addition, researchers may be interested in examining in-service teachers’ meanings and how they might reorganize their meanings for quadratic and exponential relationships via reasoning covariationally as (a) it is uncommon for U.S. teachers to reason covariationally (Thompson et al., 2017) and (b) ISTs often maintain similar meanings with respect to graphically represented relationships as PSTs (Moore et al., 2019). I propose that my models of the two PSTs’ meanings and the tasks and instructional materials implemented in my study can serve as initial resource for others designing intervention programs, for example, professional
development program with in-service teachers, to support in-service teachers in building productive meanings for quadratic and exponential relationships.

For this study, I conducted one teaching experiment over one semester that focused on students’ ways of reasoning as they engaged with tasks designed to support their developing productive meanings for both quadratic and exponential relationships. While the findings of this study inform PSTs’ construction of productive meanings for quadratic relationships, I do not have data that provide evidence of their meanings for other ideas related to quadratic relationships (e.g., translations of quadratic functions, solving a quadratic equation). Hence, the results I presented are not the totality of their meanings for quadratic relationships. Future researchers may be interested in conducting a teaching experiment focusing on supporting PSTs to construct productive meanings for only quadratic relationships via covariational reasoning and examine how PSTs meanings entailing constant second differences might alleviate some of the difficulties students have as described in the current body of literature (e.g., solving a quadratic equation, translations of quadratic functions, and defining an analytic rule representing a quadratic relationship (e.g., Bossé & Nandakumar, 2005; Kotsopoulos, 2007; Zaslavsky, 1997)).

Another limitation of the study is related to the order of the tasks in the study. As part of our task sequence, we first engaged the PSTs with activities focusing on quadratic relationship and next with activities focusing on exponential relationships. I conjecture due to this order of tasks, PSTs’ developing meanings for quadratic relationships as entailing constant second differences might have influenced their meanings for exponential relationships as entailing non-constant successive differences. As mentioned earlier, although this meaning supported the PSTs in correctly identifying an exponential relationship in this study, having non-constant successive differences is a characteristic not unique to exponential relationships (e.g., logarithmic
functions). Hence, one potential line of future research is to engage PSTs with activities first focusing on exponential relationships followed by activities focusing on quadratic relationships. It would be interesting to explore how a reversed order of the tasks may influence PSTs’ developing meanings for exponential relationships. Further, since the PSTs in my study exhibited an interplay of several meanings for exponential relationships, future researchers may be interested in conducting a teaching experiment focused only on PSTs’ meanings for exponential relationships and further explore how their developing meanings via covariational reasoning might alleviate students’ difficulties as reported in the current literature, like understanding the different rules of exponentiation (Davis, 2009; Presmeg & Nenduradu, 2005; Weber, 2002).

In conclusion, I anticipate this study can serve as the foundation for future research examining ways to support PSTs (and possibly ISTs) in developing meanings for exponential and quadratic relationships via their constructing and representing relationships between covarying quantities. This study provides an important first step in such endeavors and I encourage other researchers to continue to investigate how PSTs’ (and ISTs’) covariational reasoning can support them in developing more sophisticated meanings so they can support their future (or current) students in developing similar productive meanings.
References


Appendix A: Pre Clinical Interview Tasks

**Problem 1.** Given the following graphs of a function, identify the class of function.
Problem 2. Identify the relationship from the following table of values.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
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<tbody>
<tr>
<td>1</td>
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<td>4</td>
<td>48</td>
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<td>7</td>
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<td>11</td>
<td>28</td>
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<tr>
<td>14</td>
<td>42</td>
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</table>

<table>
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<th>$x$</th>
<th>$y$</th>
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</thead>
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<td>7</td>
<td>1</td>
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<td>3</td>
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<tr>
<td>16</td>
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<tr>
<td>2</td>
<td>18</td>
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<tr>
<td>3</td>
<td>54</td>
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<tr>
<td>4</td>
<td>162</td>
</tr>
<tr>
<td>5</td>
<td>486</td>
</tr>
</tbody>
</table>

Problem 3. Imagine filling this bottle with a liquid. As the liquid is being poured, graph the relationship between the height of the liquid in the bottle and the volume of liquid in the bottle.
Problem 4. You are working with a student who happens to be graphing the relationship represented by the following table of values.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>14</td>
</tr>
<tr>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
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<tr>
<td>3</td>
<td>4</td>
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<tr>
<td>4</td>
<td>4</td>
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<tr>
<td>5</td>
<td>5</td>
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</tbody>
</table>

He provides the following graph. How might he be thinking about the situation? How would you respond to the student?

Problem 5. The height of a plant is 10 inches. The height of the plant doubles every month. A student claims that after 2.5 months, the plant will be 60 inches tall. How would you respond to this student?

Problem 6. The height of a plant is 10 inches. The height of the plant doubles every month. How tall would the tree be after 100 months?

Problem 7. Two students are arguing. Zareen claims that $y = x^5$ is an exponential function and Basu argues $y = 5^x$ is an exponential function. How would you respond to this discussion?
Appendix B: Post Clinical Interview Tasks

Problem 1. Present the pausing shrinking trap applet. How do you think the student might be reasoning when he sketched the below graph as a representation of the relationship between the area and side length. If they say it is incorrect, ask them to sketch the correct one.

Problem 2. Given the following graphs of a function, identify the class of function.

(a)  (b)
Problem 3: When presented with this graph, Student A responded that the relationship is linear while Student B responded that the relationship is exponential. Who do you think is correct and why?

Problem 4. Identify the relationship from the following table of values.

(a)  
<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>38</td>
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<tr>
<td>2</td>
<td>28</td>
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(b)  
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<tr>
<td>2</td>
<td>324</td>
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<td>4</td>
<td>108</td>
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<td>6</td>
<td>36</td>
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<tr>
<td>8</td>
<td>12</td>
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<td>10</td>
<td>4</td>
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(c)  
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<th>x</th>
<th>y</th>
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<td>-1</td>
<td>1</td>
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<tr>
<td>1</td>
<td>3</td>
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<td>3</td>
<td>29</td>
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<tr>
<td>5</td>
<td>127</td>
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<tr>
<td>7</td>
<td>345</td>
</tr>
</tbody>
</table>

Problem 5. The height of a plant is 10 inches. The height of the plant doubles every month. A student claims that after 2.5 months, the plant will be 60 inches tall. How would you respond to this student? How tall would the tree be after 100 months?
Appendix C: A Pilot Study

I conducted a pilot study exploring in-service teachers’ reasoning about relationships between quantities and differentiating between quadratic and exponential change. My goal for this pilot study was to examine the ways of reasoning the tasks that I co-designed elicited and to test if the tasks would be helpful in building models of PSTs’ and ISTs’ ways of reasoning. Another goal was to improve my interviewing skills and my ability to elicit interviewee’s thinking and reasoning. In the following sections, I first describe the methods of the pilot study, then discuss the results, and finally the implications of the pilot study.

Methods of the Pilot Study

Participants and Setting

During the summer of 2018, three in-service high school mathematics teachers volunteered to participate in this study. They were drawn from a convenience sample of colleagues of my friends. Each of the three of them, David, Rick, and Aman (pseudonyms), had a minimum of ten years teaching experience and had taught a variety of high school math courses that included Geometry, Algebra, Pre-Calculus, Trigonometry, and Business Mathematics.

Data Collection and Analysis Methods

The three teachers participated in an individual semi-structured task-based clinical interview (Clement, 2000; Goldin, 2000; Hunting, 1997) that lasted for 60 minutes. I conducted the sessions on their school premises after school hours and videotaped each session. The video-recordings and the participants’ written materials served as sources for data analysis. At the end of each session, I watched the video tape immediately which helped me refine my interview questions for the next sessions. At the completion of all three interviews, I transcribed significant portions of the data that included the participants’ activities with the tasks, descriptions of their
actions including hand gestures that provided insights into how teachers reasoned covariationally and made distinctions between quadratic and exponential relationships.

**Task Design**

Drawing on Carlson et al.’s covariation framework (2002), my research team and I adapted tasks previously used by researchers to be the initial tasks: *Ferris Wheel Task and Two Quadrilaterals Task*. We grounded the tasks in existing empirical work and adapted the tasks used in those studies (e.g., Ellis, 2011a; Johnson, 2012; Moore, 2014; Moore & Carlson, 2012). We designed each task with an overarching goal to (a) better understand how teachers reason covariationally and (b) distinguish between two growth patterns: exponential and quadratic. Each of the three interviews consisted of both the tasks. In the paragraphs that follow, I briefly explain the two tasks in the order in which they were presented.

**The Ferris Wheel Task**

The goal of this task was to investigate to what extent teachers reason about the quantities that vary in tandem in the given situation. To design this task, we adapted work that Moore and Carlson (2012) had done with undergraduate students to examine students’ reasoning when modeling relationships between covarying quantities.

In this task, I presented the teachers with an animation of a Ferris wheel rotating counterclockwise at a constant speed. The wheel had four cars mounted on the rim of it with one of them named as rider (see screenshot of animation in Figure 70). Prior to the study, when using this task for instructional purposes, we observed that individuals would often focus on the vertical distance of the rider from the ground, and therefore the research team and I decided to include a building with height equal to the radius of the wheel in our adaption of the Ferris Wheel task with the intention of directing the teachers’ attention to the vertical distance above the building (a
directed distance). I asked the teachers to explore the situation and identify potential quantities that could be measured. I prompted each teacher to explain how they conceived of the distance travelled by the rider and his height above the building, followed by asking them to think of ways to represent the two quantities simultaneously.

**Two Quadrilaterals Task**

The goal of the Two Quadrilaterals Task was to examine how teachers may conceptualize and compare the exponential and quadratic growth of two quadrilaterals. In this task, teachers explored a Geogebra animation of two growing quadrilaterals. This task is an adaption of tasks implemented in previous studies that investigated high school students’ reasoning about rate of change (Johnson, 2012) and middle school students’ understandings of quadratic growth (Ellis, 2011a, 2011b). In this applet we provided two sliders, one (pink colored) longer than the other (red colored). While the pink slider allowed teachers to animate the quadrilaterals, the red shorter slider allowed teachers to change the increment by which the longer slider changed. Thus, the two quadrilaterals could grow either continuously or discretely (https://www.geogebra.org/m/uf76nufe). At the start, the two quadrilaterals (one in blue and the other in brown) are overlapping congruent squares. As the pink slider moves by one unit to the right, each side length of the blue quadrilateral increases by one unit, thereby the area of the blue
square represents quadratic growth. As the pink slider moves one unit to the right, the brown quadrilateral doubles in size by first having its width double, then its height double, and so on alternately. Thus, the area of the brown quadrilateral grows exponentially because it doubles (see Figure 71 for the first four jumps of the two quadrilaterals when the red slider is set to allow unit increments in the pink slider). Additionally, when the slider is set to move more smoothly, the brown quadrilateral is designed such that the area of the rectangle grows exponentially, though no teacher set the slider to move continuously. I prompted the teachers to consider how the areas of each quadrilateral covaried with changes in only the longer pink slider.

![Figure 71](image-url)  
*Figure 71. The first four jumps of the Two Quadrilaterals task as the longer pink slider moves to the right.*

**Results**

In this section, I describe each teacher’s work with the two tasks and attempt to explain their ways of reasoning based on my interpretations of their responses, written work, actions and gestures. Following the order in which I conducted the interviews, I first describe David’s ways of reasoning. Next, I describe Rick’s ways of reasoning and finally how Aman engaged with the tasks and his ways of reasoning.

**David: The Ferris Wheel Task**

I presented David with the Ferris wheel task and directed him to consider quantities that were changing in terms of the rider. Promptly, he responded that the rider’s distance from the ground was changing. I encouraged David to explain how the rider’s vertical distance above the
building changed, if the rider starts travelling from the 3’o clock position. To this, David responded saying, “It [the vertical distance above the building] increases, reaches a maximum and decreases. But whether it increases steadily, I am not very sure about”. In the moment, I hypothesized that David attempted to coordinate the direction of change in two quantities, that is the vertical distance above the building first increases and then decreases as the total distance increases (MA2).

To test my conjecture and better understand what he meant by *increasing steadily*, I asked, “Do you think that the vertical distances are increasing by the same amount?” He drew a circle, marked its center and drew a diameter. Concentrating on the first quadrant, he marked off equal horizontal distances by placing dots on the radius between the rider at 3’o clock position and the center. He then drew segments upwards to the circle from each of these points (see Figure 72 a) and explained

Let’s say from here to here is same as from here to here is same as here to here [moving the pen from point A to B, B to C and C to D, in Figure 72b]. This one [the rider] moves [pause] you see what length this is [motioning his finger on BE in Figure 72b], when you move from here to here [point B to C] the increase [hovering the pen on BE and CF] is smaller than what it is here [hovering the pen on BE and A].

![Figure 72.](image)

(a) David’s drawing and (b) a recreation of his drawing with added letters for reference.
The two quantities, different from the ones I intended, that David considered were: (1) horizontal distance to the right of O (point O in Figure 72b) and (2) the rider’s vertical distances above the building. His actions indicate that he attempted to coordinate the amount of change in the horizontal distances with changes in vertical distances. I conjectured in the moment that David was reasoning covariationally with respect to these quantities and his description entailed MA 3.

To test my conjecture, I asked, “Can you show me the increases on the paper”? He promptly drew line segments connecting the vertical distances (red colored line segments in Figure 73), motioned his pen on the differences in the vertical distances (pink colored line segments in Figure 73) and claimed, “the increases are decreasing”. David’s description supported my conjecture that he was coordinating changes in vertical distances for equal changes in horizontal distance (MA 3).

Figure 73. (a) David’s drawing and (b) a recreation of his drawing showing the difference in heights.

I then specifically asked David, “How do you think will the vertical distances change when you consider the total distance travelled?” In response to this, David came up with a diagrammatic representation similar to the earlier one. He marked off equal sized arcs on the rider’s path in the first quadrant and drew line segments from each point to the diameter. He then
drew horizontal segments connecting the vertical segments (*Figure 74*) and claimed, “the increases are more at the beginning”. I infer form David’s actions that for equal changes in the arc length he was attempting to coordinate the changes in vertical distances.

*Figure 74.* (a) Marking equal sized arcs on the rider’s travel path in first quadrant, (b) Vertical distances of the rider and differences of the distances.

As we continued with the interview, I asked David to describe the relationship between the rider’s vertical distance and the total distance travelled, if the rider completed one revolution. Numbering the rider’s positions 1, 2, 3, and 4 as seen in *Figure 75a*, David created a graph (*Figure 75b*) and claimed that it was a sine curve. He represented the rider’s total distance traveled and the vertical distance above the horizontal diameter on the horizontal and vertical axis, respectively, on his graph. When asked to justify the concavity of his graph, David said:

Because I think if you take it in small bits, infinitesimally, you will see that the vertical distance is getting, it is 0 at the beginning [pointing to position 4 in *Figure 75a*] and increasing as he goes along the circumference and then it goes until it gets to the maximum here [pointing to position in *Figure 75a*]. Then it starts to decrease and from here [pointing to position 2], it becomes negative.
DISTINCTIONS BETWEEN QUADRATIC AND EXPONENTIAL RELATIONSHIPS

Figure 75. (a) Labeling the four positions the rider can take and (b) Graphical representation of vertical distances and total distances.

I consider his description as indicative of MA 2. It appeared to me that David was thinking of vertical distances as increasing or decreasing in intervals (from positions 1 to 2, 2 to 3, 3 to 4, and 4 to 1 as seen in Figure 75a) while imagining that within each equal interval the vertical distance varies continuously. Based on his actions and language (e.g. “if you take it in small bits, infinitesimally”), I conjectured, in the moment, that he was imagining covariation as happening in a smooth continuous manner.

In order to test my hypothesis, I drew green colored line segments in his graph (Figure 76a) and asked if that would represent the situation correctly. Promptly David claimed, “It means that you are not thinking of the small, small increments. It means you are thinking of jumping from here to here [moving his fingers from (0,0) to (1,1) on the graph in Figure 76a], here to here [moving his fingers from (1,1) to (3, -1) on the graph in Figure 76a] and here to here [moving his fingers from (3, -1) to (4,0) on the graph in Figure 76a]”. Although David mentioned ‘small-small increments’, he remained quiet for an extended period and did not provide any further explanation about the correctness or incorrectness of the graph I drew. I was unsure if David envisioned change in vertical distances as happening smoothly or in chunks.
Figure 76. Two different representations, (a) in green and (b) in red, offered to David.

As an attempt to draw David’s attention to thinking about how quantities covary between values, I drew a red colored curve (the concavity of this red curve is opposite to the concavity of the black curve David drew) on the graph we were working on (Figure 76b). I asked David, “Could I have drawn the graph this way?” and continued to try to have him discuss the concavity of the graph (Excerpt 20).

Excerpt 20. David’s explanation for the question, ‘Can the red curve represent the situation correctly?’

I: Could I have drawn the graph this [the red curve in Figure 76b] way?

D: If you draw it this way [the red curve], even though it [the vertical distance] is increasing (motioning his fingers on the first piece of the red curve on), it is increasing but it starts very gradually from the beginning and rapidly at the end, which is not the journey. So, I don’t think this red one is…

I: So…?

D: For the red, the increase starts gradually which is not the case when he [the rider] is going that way (motioning his hand counter-clockwise).

David maintained that the increases in vertical distances are decreasing and confirmed that the red curve is not an appropriate representation of the situation. From his explanation, I
infer that he envisioned for equal changes in the total distance travelled, the increases in vertical
distances are decreasing (MA 3).

By the end of this task, David confirmed that the graph he drew (Figure 75b) was an
appropriate representation of the total distance travelled and the vertical distances above the
building in the situation. David was able to reason that the as distance travelled by the rider
increased, his vertical distances from either the ground or top of the building increased and
decreased (MA 2). Moreover, he reasoned that for the first quarter turn, equal changes in the arc
length corresponded to decreasing changes in vertical distances (MA3). David’s actions and
verbalizations provide sufficient evidence that he leveraged MA 1-3 as he constructed a graph to
accurately represent the relationship he conceived in the situation.

David: The Two Quadrilaterals Task

Similar to his activity in the Ferris wheel task, David attempted to coordinate changes in
one quantity for equal changes in the other. For the blue quadrilateral, he reasoned that for equal
changes in the side length, the increase in the areas was increasing (MA3) and sought to draw a
graph to represent the situation. Although he approached the area of the brown quadrilateral
similarly to describe the areas were doubling, he did not comment on what the two growths
represented in terms of function classes.

After describing that the blue quadrilateral remains a square when the slider is dragged,
David drew the graph of $y = x^2$ (Figure 77) where $x$ represents side length and $y$ represents area.
He assigned numerical values to side lengths and the corresponding areas. As he drew the graph,
David claimed, “the areas are increasing faster.” To better understand his claim, I asked David to
explain what he meant by areas increasing faster. David’s explanation allows me to claim that he
leveraged covariational reasoning to make sense of changes in the values of $y$ for changes in the values of $x$. The following excerpt (Excerpt 21) justifies my claim.

**Excerpt 21.** David’s explanation for the increase in areas of the blue square.

I: What do you mean by saying that areas are increasing faster?

D: Ok, you see there are equal changes here *showing the numbers on the x axis*. As you move away from 0 *moving his finger on the x axis towards infinity*, the changes that these equal changes effect on the $y$ become larger.

I: Okay

D: The increase is becoming larger as you move away. For example, let us look at this *showing the graph of $y = x^2$*. From 1 to 2 *moving his finger between 1 and 2 on the x axis* one interval, the interval here will be 3 *moving his finger between 1 and 4 on the y axis*. It moves from 1 *showing the distance between 1 and 2 on axis* to 3 *showing the distance between and 1 and 4 on y axis*. But from 2 to 3 the same interval, it moves from 4 to 9 *moving his fingers on the y axis*. Then from 3 to 4, same interval, it moves from 9 to 16 *on the y axis*. So for equal changes here *showing the x axis*, more changes, bigger changes here *showing the y axis*.

David’s explanation suggests that he coordinated the amounts of change (MA3). He reasoned that for equal changes in $x$, the changes in $y$ were increasing. However, I am unsure if his reasoning was based on his understandings of the curve $y = x^2$ or about the situation. Nevertheless, David envisioned the variation in one variable as changing by interval of fixed size, for example, from 1 to 2, from 2 to 3, from 3 to 4, and so on, simultaneously with changes
in the other, for example, from 1 to 4, from 4 to 9, from 9 to 16, and so on.

![Graph showing the function y = x^2](image)

*Figure 77. David’s representation of the areas of the blue quadrilateral by considering the side length on the horizontal and area on the vertical axis.*

Next, David noted that for every unit increment in the slider, the area of the brown quadrilateral doubles because “first the width doubles and the length stays the same, then the length doubles and the width stays the same, and this is continuing”. Due to time constraints of the interview, I could not further probe to better understand how David would have attempted to differentiate between the growth in the areas of the two quadrilaterals.

**Summary of David**

I infer from David’s collective activity described above that he internalized the idea of using equal changes of one quantity to describe changes in the other. There were instances when he consistently reasoned in ways that were compatible with chunky continuous covariation. Although at times his utterances such as, “as you move away from 0” accompanied by hand motions on the x-axis towards infinity, “if you take it in small bits, infinitesimally” prompted me to conjecture that he could be envisioning smooth variation, I did not get sufficient evidence to make this claim.

**Rick: The Ferris Wheel Task**

Like David, Rick first identified constant quantities such as speed at which the rider is moving and the distance travelled in one revolution. When I asked Rick to find quantities that
were changing, Rick claimed that the height of the rider was going “back and forth” and oscillated between “the min at the ground level and the max at the highest level”. I was not sure if Rick was considering vertical distances and distance travelled by the rider simultaneously.

Hence, my next prompt was, “How would you represent the vertical distance above the building and the total distance travelled together?” In his response, Rick drew a graph (Figure 78) which according to him represented the rider’s one complete revolution.

Figure 78. Rick’s graphical representation of one complete revolution of the rider.

He chose to put the vertical distances on the horizontal axis and the total distance on the vertical axis explained

From here [pointing to the 1 on the horizontal axis of the graph] to here [moving his pen from 1 to 2 on the graph], we are increasing in both [the vertical distances and total distance travelled] and from here [pointing to 2] to all the way till here [moving his pen from 2 to 4 on the graph] you are decreasing in the entire time, but your total distance is increasing as well and then finally back to here [pointing to 1 on the vertical axis of the graph] it is increasing, increasing.

Rick explicitly reasoned about the direction of change in the vertical distances above the building and the total distance (MA 1-2).
In the beginning of the interview, Rick claimed that he identified all the quantities “in the situation as a way of trig, using probably a cosine curve”. This prompted me to examine if Rick was reasoning about the cosine function in a circular motion context. I, therefore, presented a graph (Figure 79a) with line segments and asked if the graph was an appropriate representation of the two quantities. Rick drew another graph (Figure 79b) and replied, “Yours is showing constant while mine is not and I think mine is more accurate”.

Figure 79. (a) A graph with line segments presented to Rick (b) Rick’s drawing as a response to the graph with line segments.

I prompted Rick to provide more explanation of why his graph was more accurate than mine. Rick claimed, “So I am seeing the curve as, uh…[pause] I am just replicating as though it is a sine wave. So, this is what I was thinking about in my head”. Furthermore, Rick wrote three points $A(\pi/6, 1/2)$, $B(\pi/3, \sqrt{2}/2)$, and $C(\pi/3, \sqrt{3}/2)$. He calculated the slopes of $\overline{AB}$ and $\overline{BC}$ and stated, “Based on the three points, I see that the two slopes are not equal. So I am proving that it [showing the graph with line segments] is not linear and mine actually could be closer to what it really is”.

I took Rick’s activity to indicate that he was relying on his understandings of the sine function and the concept of slope of a line. I directed him to describe the meaning of $\pi/2$, $\pi$, $3\pi/2$, and $2\pi$ in relation to the context because he labelled the horizontal axis as total distance and marked $\pi/2$, $\pi$, $3\pi/2$, and $2\pi$ on the axis. Rick replied, “if you think in terms of trig, they are all
π/2 units away from each other”. This led me to infer that Rick maintained a focus on properties of the sine function more than the quantities of the situation as he never mentioned total distance in his description.

**Rick: The Two Quadrilaterals Task**

For the blue quadrilateral, Rick noted that both the side lengths increased by one unit if he moved the slider by one unit and claimed “the area is increasing by whatever that side is squared”. He considered the initial side length to be ‘x’ units and described that the areas of the growing square would be $x^2$, $(x+1)^2$, and $(x+2)^2$. Rick next considered $x$ to take the value of 1, calculated the areas to be 1, 4, 9, 16 and 25, found the differences between these numbers, and also their second differences. Circling the second differences (Figure 80a) Rick explained “this is the rate of the rate. So the rate of the rate is constant. It tells me it is quadratic”. Rick’s work illustrates that he was in fact calculating changes in the area for a unit change in the side length. I infer from Rick’s explanation that he understood quadratic growth in ways compatible with the characterizations of Ellis (2011a) and Lobato et al. (2012); Rick considered quadratic growth to be defined by a relationship such that the rate of change of the rate of change is constant.

![Figure 80. Rick’s work describing the growth of area of (a) the blue quadrilateral and (b) the brown quadrilateral.](image-url)
For the brown quadrilateral, Rick noted that as he moved the slider by one unit, one side length alternately doubles. Similar to his aforementioned work Rick described that the area of the brown quadrilateral would be $x^2$, $2x^2$, $4x^2$, $8x^2$ and $16x^2$ at the first five unit values of the slider, assumed $x$ to take the value 1, and calculated the areas as 1, 2, 4, 8 and 16 (Figure 80b). He identified that each consecutive area is double the previous area and stated, “the first rate is constant multiplication which tells us it is an exponential growth. The rate is constant”. I infer from Rick’s description that he was coordinating the ratio of the area of the quadrilateral with equal changes in the slider to explain exponential growth. His explanation is compatible with Confrey and Smith’s (1994) description of exponential growth as having a constant multiplicative rate.

When I asked Rick to explain what he meant by rate he replied, “with respect to increase in size of the sides, as the size gets bigger, the area gets bigger. Uh, the area is increasing and the rate at which it is increasing by is getting bigger as well. This [showing the +2, +2, +2 in Figure 19 a] is arithmetic while this [showing the x2, x2, x2 in Figure 80b] is, uh, geometric rate”. Rick imagined the variation in the side length as he moved the slider and coordinated these changes with the changes in the areas. However, due to time constraints of the interview, I could not ask Rick to describe his understandings of arithmetic and geometric rate.

**Summary of Rick**

Rick’s activities described above indicate that he consistently attended to the direction of change in quantities. For example, how the areas and the slider (or side length), the vertical distance above the building and total distance, covaried in terms of direction of change (MA2). Also, there were instances where he coordinated the amounts of change in the areas and side lengths (MA3). Further, he made statements regarding the type of growth exhibited by the areas
of the two quadrilaterals as the slider (or side length) increased. Other times, his explanation indicated that he drew meanings that focused on memorized concepts of sine curve.

**Reflection after conducting two interviews.** I first interviewed David followed by Rick. Although, initially, both of them identified constant quantities in the situation and recognized changing quantities only after I prompted them to, David readily considered equal changes in the rider’s total distance and articulated how the vertical distances would change for these equal changes in the rider’s distance. Rick, however, envisioned only change in the direction of both the quantities and not the amount of change in the quantities. He did not exhibit behaviors indicative of reasoning about amounts of change (MA 3). To encourage teachers to think of amounts of change in one quantity while imagining changes in the other, I decided to revise my interview prompt for the Ferris wheel task with Aman, the next participant. Therefore, when Aman similar to Rick described how the vertical distance of the rider from the ground changed, I asked, “For equal changes in the arc length what can you say about the vertical distances above the building?” In what follows, I describe Aman’s ways of reasoning and the behaviors exhibited while he engaged with the tasks.

**Aman: The Ferris Wheel Task**

As a response to my question, Aman drew a circle and focusing on the first quadrant he marked equal sized arcs (Figure 81a) and claimed that “as the rider starts from position 1 and comes back, the vertical distances increase, decrease, and increase” (MA1-2). I infer that he was reasoning about the vertical distance from above the building and the changing position of the rider. Aman promptly constructed a graph in ways compatible with Rick (Figure 81 b) and said “the graph is a sine curve”.
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In order for Aman to justify the concavity of the graph he drew, I asked if it was appropriate to have line segments in the graph. He used the values \( \sin 0^\circ, \sin 30^\circ, \sin 60^\circ \) and claimed that plotting these points and joining them will not produce a straight line. His explanation of why the curve cannot have line segments provides evidence that his understanding of the graph depended on memorized values of the sine function rather than on how the two quantities covaried and how the amounts of change varied.

**Aman: The Two Quadrilaterals Task**

Aman first noted that “for each step the slider takes, areas are increasing.” For the blue quadrilateral, which he determined remains a square always, he considered the side length to be ‘x’ units and generated a list of side lengths and corresponding areas. He described that “the side lengths would be \( x, 2x, 3x, 4x, 5x, 6x, 7x, \) and \( 8x \) and accordingly the areas would be \( 1, 4, 9, 16, 25 \) and so on till 64 times \( x^2 \).” To understand how he conceived the growth in areas, I asked him to explain how the areas were increasing for which Aman established that the differences in the areas of the quadrilateral which are \( 3x^2, 5x^2, 7x^2, 9x^2 \) can be represented by a formula \( (2n-1)x^2 \) where \( n \) is a natural number.
Next, he attended in a similar way to the changing brown quadrilateral. He observed that the quadrilateral “changes from square to rectangle to square to rectangle to square to rectangle”, created a list of length, width, and the corresponding areas, and illustrated that “the areas would be $x^2$, $2x^2$, $4x^2$, $8x^2$, like that till $128x^2$” and established that the areas can be represented by the formula $2^{(n-1)}x^2$, where $n$ is a natural number. When attempting to describe this formula, he claimed, “the brown [quadrilateral] is changing like a geometric progression where the common ratio is 2”. Aman attempted to derive a formula to define the relationship between the side length and corresponding area of both the quadrilaterals. He coordinated the values of one quantity with the values of another quantity, created a collection of pairs, and calculated the change in areas for unit change in the side lengths (MA3).

**Summary of Aman**

Aman’s actions during the interview reveal that a memorized set of values of the sine function guided his construction of the graph (Figure 81 b). Although Aman successfully determined rules to describe patterns in the growth of the areas, he did not elaborate on what these growth patterns meant with respect to specific function classes.

Aman’s ways of describing the change in areas of the quadrilaterals are compatible with Rick’s ways. Both of them attempted to determine the differences in the areas of the blue quadrilateral and ratio of the areas of the brown quadrilaterals. While Rick assigned numerical values to determine the differences and ratios, Aman established a rule in terms of an algebraic expression. Similarly, in the Ferris wheel task, both of them attempted to justify the concavity of the graph based on the values of sine function rather than leveraging their understanding of the amount of changes in the vertical distances for equal changes in the rider’s distance travelled.
Implications

The results of this study had important implications for my dissertation study. I observed that both Rick and Aman used known values of the sine function to justify the concavity of the graph they drew. Similarly, in the Two Quadrilaterals task, they assigned numerical values to the side lengths and successfully produced formulae to explain the growth patterns. These ways of reasoning that relied on memorized values limited my ability to make inferences regarding the extent to which they reasoned covariationally and their understanding of the underlying growth patterns. Hence, it is important to devise tasks that potentially constrain the participants from relying on algorithms or known formulas. Instead, tasks should promote them to reason about how quantities change in tandem so that they could possibly reorganize their meanings for quadratic and exponential relationships in ways compatible with previous researchers’ descriptions of these growth patterns (Confrey & Smith, 1994; Ellis, 2011a; Thompson, 2008).

The pilot study also provided important opportunities for me to hone my skills at posing questions. For instance, in the Two Quadrilaterals task, because the side lengths as well as the area of the quadrilaterals grow when the slider changes, there were times in the interview sessions when teachers attempted to reason about three quantities: the side length, change in the slider, and area. Teachers attempted to reason about quantities that were different from the quantities that I wanted them to reason about. As the interviews progressed, I tried to pay careful attention both to the quantities I was asking the participant to describe as well as the quantities the participants were attending to when describing how quantities covaried. This experience informed my interviewing in my dissertation study, as it is critical that I pose clear and definite questions to the participants and ensure that I understand the quantities with which the students reason.