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Interlace Polynomials of Certain Graphs

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Abstract

In this research, we investigated the interlace polynomials of a shell graph as well as other related graphs. A shell graph, T_n is constructed by adding edges to a cycle graph such that all vertices are adjacent to one vertex. The main results of this thesis include iterative and explicit formulas for the interlace polynomial of a shell graph, denoted $q(T_n, x)$. A linear algebra application using the adjacency matrices of the chosen graphs is also explored.

MONTCLAIR STATE UNIVERSITY

Interlace Polynomials of Certain Graphs

by

Cheyenne Petzold

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Chapter 1

Introduction

1.1 History

Sequencing by hybridization is a method of reconstructing a long DNA string in order to determine its nucleotide sequence. Arratia, Bollobás, and Sorkin constructed interlace polynomials motivated by a problem relating to DNA sequencing by hybridization, more specifically, to find the number of possible reconstructions of a random string. The number of reconstructions is the number of Euler circuits in a 2-in, 2-out digraph. The problem was converted to counting the number of 2-in, 2-out digraphs having a given number of Euler circuits.

The interlace polynomial of a graph is generated from a toggling process on the graph. Information about a graph G can be given by its interlace polynomial $q(G, x)$, such as, the number of Euler circuits in a 2-in,2-out digraph, the number of k-component circuit partitions, and structural properties of the graph through special values. For example, $q(G, 2)$ gives the number of vertices in the graph G. Interlace polynomials for some well-known simple graphs like paths, cycles, stars, complete graphs, and certain trees have been studied.

The graph I am mostly interested in is called a shell graph. A shell graph on n vertices, denoted T_n, is constructed by adding $n-3$ edges to a cycle graph on n vertices such that all vertices are adjacent to one vertex. The main goal of this research is to develop formulas for these types of graphs and study the properties of them.

1.2 Graph Theory Basics

In this section, we list some basic definitions and well known theorems about graphs as well as provide descriptions of well known graphs.

Definition 1.2.1. *A graph G is an ordered pair of sets denoted $G = (V(G), E(G))$ where*

1. *$V(G)$ is the vertex set of G and $E(G)$ is the edge set of G which is a set of 2-element subsets of $V(G)$ where $E(G) \subseteq \{\{u, v\} : u, v \in V(G)\}$. When $\{u, v\} \in E(G)$, we use the notation uv.*

2. *$\forall u \in V(G)$ the neighborhood of u is the set $N(u) = \{v \in V(G) \,|\, uv \in E(G)\}$.*

3. *The degree of a vertex, v, is the number of edges that are incident to that vertex, denoted $d(v)$.*

4. *A loop is an edge that connects a vertex to itself.*

A set of special graphs are well known and well studied.

Definition 1.2.2. *(Special Graphs)*

1. *A simple graph is a graph containing neither loops nor multiple edges.*

2. *A path with n edges, denoted by P_n, is a sequence of vertices such that each vertex in the sequence is adjacent to the vertex next to it. For vertices v_i in graph G, a graph with n edges can be represented as $v_1v_2 \ldots v_nv_{n+1}$.*

3. *A cycles with $n \geq 3$ vertices, denoted as C_n, is a path G with an added edge $v_n v_1$.*

4. *A star with n edges, denoted S_n, is a tree with one vertex degree n and the other vertices are leaves.*

5. *A complete graph on n vertices, denoted K_n, is a simple graph where every vertex is adjacent to every other vertex.*

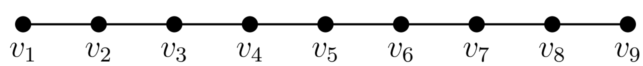
6. *A tree is a connected graph without cycle. A vertex with degree 1 in a tree is called a leaf.*

Some graph theory terms are defined below for later reference.

Definition 1.2.3. 1. *a cut vertex is a vertex that, when removed from a graph, results in a graph with more components than the original graph*

2. *A matching in a graph is a set of edges without common vertices*

Example 1.2.4. *A path with 8 edges, P_8 is shown below.*



P_8

From definition 1.2.2, the graph P_8 is also a tree. From definition 1.2.3, $v_2, v_3, \ldots, v_7, v_8$ are all cut vertices.

Some well-known properties for these special graphs are listed below.

Theorem 1.2.5. *Let $G = (|V(G)|, E(G)|)$ be any graph*

1. $\sum_{v \in V(G)} d(v) = 2|E(G)|$;

2. *If G is a tree then $|E(G)| = |V(G)| - 1$;*

3. $|E(K_n)| = \frac{n(n-1)}{2}$;

4. $|E(K_{m,n})| = nm$.

For example, in example 1.2.4, the number of edges in P_8, $|E(P_8)|$, is 8. Since P_8 is a tree, from theorem 1.2.5,

1. $\sum_{v \in V(G)} d(v) = 2|E(P_8)| = 16$;

2. $|E(P_8)| = |V(P_8)| - 1$ is true.

In this research we only consider undirected graphs.

1.3 Defining the Interlace Polynomial

The definition of the interlace polynomial is described recursively and was defined by Arratia, Bollobás, and Sorkin in [2]. From now on, all of our graphs are simple graphs. For an edge ab from a graph G, we denote V_a to be the neighborhood of a excluding b, V_b to be the neighborhood of b excluding a, and $V_{a,b}$ to be in the neighborhood of both a and b. That is, $V_a = N(a) \setminus (N(b) \cup \{b\})$, $V_b = N(b) \setminus (N(a) \cup \{a\})$, and $V_{a,b} = N(a) \cap N(b)$.

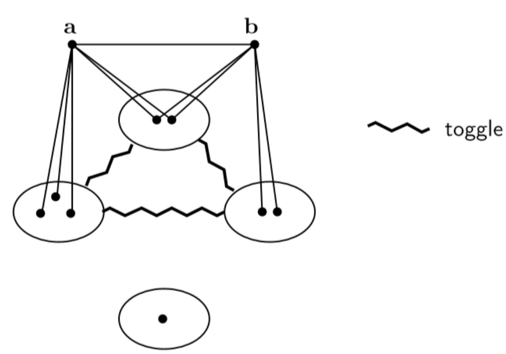


Figure 1.1: The Toggle Operation at the Edge ab

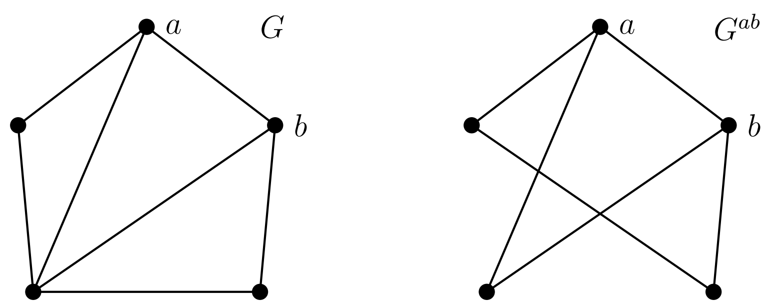
Definition 1.3.1 (Toggling Process). *Let G be a graph and ab be an edge of G. The toggling process of G on ab means to create a new graph $G^{ab} = (V(G), E(G^{ab}))$ and for every pair of* *vertices u, v belonging to different neighborhoods V_a, V_b, V_{ab}, uv is an edge of G^{ab} if and only if uv is not an edge of G. The resulting graph G^{ab} is called the pivot of G at ab.*

The interlace polynomial of G is defined by fixing one edge and considering the interlace polynomial of the smaller graphs $G-a$ and $G^{ab}-b$. If G is a union of disconnected graphs, it is known that if G_1, G_2 are 2 disconnected components of a graph, the interlace polynomial of $G_1 \cup G_2$ is the product of the interlace polynomials of G_1 and G_2. For the smallest graph K_1, the interlace polynomial is x. Following this rule, the interlace polynomial of the empty graph E_n (no edge) is then x^n. We adopt the definition from [2].

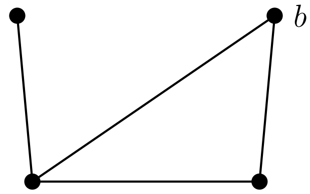
Definition 1.3.2 (Interlace Polynomial)**.** *[2] Let G be any undirected graph with n vertices and ab be an edge of G. The interlace polynomial $q(G,x)$ of G is defined by*

$$q(G,x) = \begin{cases} x^n & \text{if } E(G) = \emptyset; \\ q(G-a,x) + q(G^{ab}-b,x) & \text{if } ab \in E(G) \\ q(G_1,x)q(G_2,x) & \text{if } G = G_1 \cup G_2 \text{ disjoint union} \end{cases} .$$

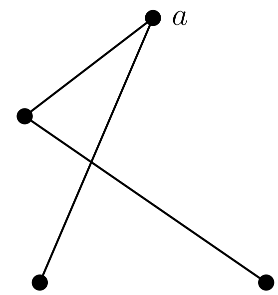
Example 1.3.3. *Consider the graph G made up of a cycle C_5 with 2 additional cords. The toggling process for the graph G is shown below on edge ab.*



After removing the vertices a and b, from their corresponding graphs G and G^{ab}, the following graphs are shown below. The graph $G^{ab} - b = P_3$ is a path of length 3, and the graph $G - a$ is made of C_3 with a leaf attached to one of the vertices.



$G - a$



$G^{ab} - b$

By this toggling process and by definition 1.3.2, the polynomial of G *is* $q(G, x) = q(G - a, x) + q(G^{ab} - b, x)$ *where* $G - a$ *and* $G^{ab} - b$ *are smaller graphs.*

1.4 Existing Results

Research has been done on properties of interlace polynomial of well-known graphs. The following properties show that the interlace polynomial of a graph G can describe the ground graph in certain ways.

Theorem 1.4.1. *[2] Let* G *be a graph.*

1. *The degree of the lowest-degree term of* $q(G, x)$ *is the number of components of* G*.*
2. *If* G *is a forest with* n *vertices, then* $\deg(q(G, x)) = n - \mu(G)$*, where* $\mu(G)$ *denotes the size of a maximum matching in* G*.*
3. *For any graph* G *of order* n*,* $q(G, 2) = 2^n$*.*
4. *If* G *is connected, then the constant is 0.*

The interlace polynomials of some well known graphs are given below. After a graph has been toggled, the well known graphs, $P_n, C_n, K_n, K_{m,n}$, and S_n can be found.

Theorem 1.4.2. *Consider the special graphs* $P_n, C_n, K_n, K_{m,n}$*, and* S_n*.*

1. $q(P_0,x) = x, q(P_1,x) = 2x,\ q(P_2,x) = x^2+2x$ *and for* $n \geq 2,\ q(P_n,x) = q(P_{n-1},x) + xq(P_{n-2},x)$*;*

2. $q(C_3,x) = 4x,\ q(C_4,x) = 3x^2+2x,\ q(C_5,x) = 5x^2+6x$ *and for* $n \geq 4,\ q(C_n,x) = q(P_{n-2},x) + xq(P_{n-4},x) + q(C_{n-2},x)$*.*

3. $q(K_n,x) = 2^{n-1}x$*;*

4. $q(K_{m,n},x) = (1+x+\ldots+x^{m-1})(1+x+\ldots+x^{n-1}) + x^m + x^n - 1$*;*

5. $q(S_n,x) = x^n + q(S_{n-1},x) = x^n + x^{n-1} + \ldots + x^2 + 2x$*.*

Comparing the above two theorems, we note that

1. All of the graphs in Theorem 1.4.2 are connected and their interlace polynomials all have 0 constant. This confirms Theorem 1.4.1(1);

2. The maximum matching for S_n is 1 and S_n has $n+1$ vertices. From theorem 1.4.1(2), $\deg(q(S_n,x)) = n+1-1 = n$. From theorem 1.4.2(7), the degree is n;

3. The graph K_n has n vertices. By Theorem 1.4.1, $q(K_n,2) = 2^n$. By Theorem 1.4.2, $q(K_n,2) = 2^{n-1} \cdot 2 = 2^2$.

Some theorems involving special values of $q(G,x)$ already exist. These values have a connection to properties of the graph G.

Theorem 1.4.3. *[1] Let G be a graph on n vertices, A_n be the $n \times n$ adjacency matrix of a graph G and also let $r_n = rank(A_n + I_n) \pmod 2$ where I_n is an $n \times n$ identity matrix. Then*

$$q(G,-1) = (-1)^n(-2)^{n-r_n}.$$

The value of $q(P_n,x)$ at $x=-1$ is described below.

From theorem 1.4.2(2), $q(P_0,x) = x, q(P_1,x) = 2x$, and $q(P_2,x) = x^2+2x$. It would follow that $q(P_0,-1) = -1, q(P_1,-1) = -2$, and $q(P_2,-1) = -1$.

1.5 Results on $q(P_n, x)$

A recursive formula for the interlace polynomial of the path P_n is given in Theorem 1.4.2. Other useful results on $q(P_n, x)$ are given below.

Theorem 1.5.1. *For any positive integer n, The polynomial $q(P_n, x)$ is of degree $\left\lfloor \frac{n+2}{2} \right\rfloor$ and can be described explicitly as*

$$q(P_n, x) = \sum_{r=0}^{\lfloor n/2 \rfloor} \left[\binom{n-r}{r} + \binom{n-r-1}{r} \right] x^{r+1}.$$

The value of $q(P_n, -1)$ is given below.

Proposition 1.5.2. *For any integer $n \geq 0$*

1. *If $n \equiv r \pmod 6$, where $0 \leq r < 6$, then $q(P_n, -1) = q(P_r, -1)$;*

2. *$q(P_n, -1) = -\sigma^{n-1} - \tau^{n-1}$, where $\sigma = \frac{1+i\sqrt{3}}{2}$ and $\tau = \frac{1-i\sqrt{3}}{2}$.*

Definition 1.5.3. *The interlace polynomial of a path $q(P_n, x)$ is denoted by*

$$q(P_n, x) = b_{n,j_n} x^{j_n} + b_{n,j_n-1} x^{j_n-1} + \ldots + b_{n,1} x,$$

where $j_n = \deg(q(P_n, x))$ and $b_{n,i}$ is the coefficient of the x^i-term of $q(P_n, x)$.

Lemma 1.5.4. *Consider a path P_n with $n \geq 0$, then*

1. *$j_n = \lfloor \frac{n}{2} \rfloor + 1$*

2. *the leading coefficient of $q(P_n, x)$, b_{n,j_n}, is*

$$b_{n,j_n} = \begin{cases} 1 & \text{if } n \text{ is even} \\ \frac{n+3}{2} & \text{if } n \text{ is odd} \end{cases};$$

3. *the second leading coefficient of* $q(P_n, x)$, b_{n,j_n-1}, *is*

$$b_{n,j_n-1} = \begin{cases} \frac{n^2+6n}{8} & \text{if } n \text{ is even} \\ \frac{(n^2-1)(n+9)}{48} & \text{if } n \text{ is odd} \end{cases};$$

4. *the third leading coefficient of* $q(P_n, x)$, b_{n,j_n-2}, *is*

$$b_{n,j_n-2} = \begin{cases} \frac{n^4+12n^3-4n^2-48n}{384} & \text{if } n \text{ is even} \\ \frac{n^5+15n^4-10n^3-150n^2+9n+135}{3840} & \text{if } n \text{ is odd} \end{cases};$$

5. *the* x *coefficient of* $q(P_n, x)$ *is* $b_{n,1} = 2$.

6. *The* i*th coefficient of* $q(P_n, x)$ *is given by* $b_{n,i} = \binom{n-i+1}{i-1} + \binom{n-i}{i-1}$. *This result comes directly from Theorem 1.5.1.*

7. *The value of* $q(P_n, x) \mod 6$ *when* $x = -1$,

$$q(P_n, -1) = \begin{cases} 1 & \text{if } n \equiv 3, 5 \pmod 6 \\ -2 & \text{if } n \equiv 1 \pmod 6 \\ -1 & \text{if } n \equiv 0, 2 \pmod 6 \\ 2 & \text{if } n \equiv 4 \pmod 6 \end{cases};$$

1.6 Graphs of Interest

Our main graphs of interest is called a "shell graph" which is built from a cycle. The shell graph is defined below.

Definition 1.6.1. *Let* n *be a positive integer at least 3. Consider the cycle* $C_n = v_1v_2 \dots v_nv_1$ *with* n *edges. Define* T_n *to be the resulting graph by adding* $n-3$ *edges, all adjacent to* v_n, $v_nv_2, v_nv_3, \dots v_nv_{n-2}$, *to* C_n. *We call this graph the shell graph with* n *vertices. Precisely,*

$T_n = (V(T_n), E(T_n))$, *where* $V(T_n) = \{v_1, v_2, \ldots, v_n\}$ *and*

$$E(T_n) = \{v_i v_{i+1}, v_n v_i, \text{ for } i = 2, 3, \ldots, n-2, \ v_1 v_2, v_1 v_n, v_{n-1} v_n\}.$$

This particular graph is given the name shell graph because its appearance is similar to a scallop shell. All the lines(threads) of the scallop start from a point(beak) and end at the margin. Below are examples of shell graphs.

Example 1.6.2. *Shell graphs on 7 and 8 vertices,* T_7 *and* T_8*.*

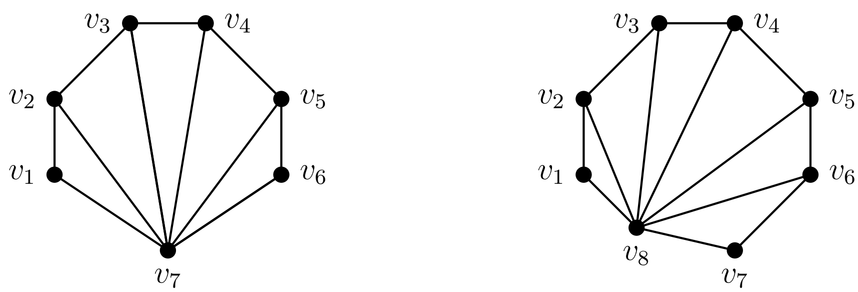


Figure 1.2: The graphs T_7 and T_8.

Cycles with a tail were developed in order to study the interlace polynomial of wheel graph.

Definition 1.6.3. *Let* r, s *be two integers with* $r \geq 3$ *and* $s \geq 0$*. Let* $D_{r,s}$ *be the graph obtained by gluing the cycle* C_r *and the path* P_s *at one vertex of* C_r *and one end vertex of* P_s*.*

Example 1.6.4. *Cycles* C_3*,* C_4*, and* C_5 *with respective tails* P_1*,* P_2*, and* P_3*. These are labeled respectively* $D_{3,1}$*,* $D_{4,2}$*, and* $D_{5,3}$*.*

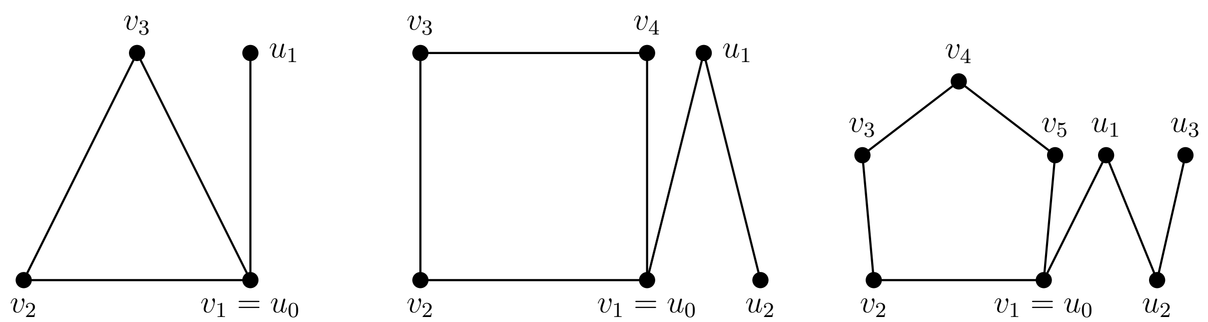


Figure 1.3: The graphs $D_{3,1}$, $D_{4,2}$, and $D_{5,3}$.

A labeled graph $D_{r,s}$ is given in Figure 1.4.

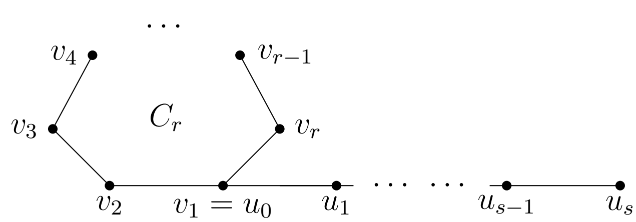


Figure 1.4: The Labeled Graph $D_{r,s}$.

Note that $s \geq 0$ and $D_{r,0} = C_r$.

A wheel graph on n vertices is denoted by W_n. The graph is constructed by adding one vertex, v_n to the cycle C_{n-1} and adding $n-1$ edges all adjacent to v_n, $v_nv_1, v_nv_2, \ldots, v_nv_{n-1}$ to C_{n-1}.

Definition 1.6.5. *Let n be a positive integer at least 4. Consider the cycle with n vertices, $C_{n-1} = v_1v_2\ldots v_{n-1}v_1$. Define W_n to be the resulting graph by adding a vertex v_n and $n-1$ edges all adjacent to v_n, $v_nv_1, v_nv_2, v_nv_3, \ldots v_nv_{n-2}, v_nv_{n-1}$, to C_{n-1}. We call this graph the wheel graph with n vertices. Precisely, $W_n = (V(W_n), E(W_n))$, where $V(W_n) = \{v_1, v_2, \ldots, v_n\}$ and*

$$E(W_n) = \{v_iv_{i+1}, v_nv_i, \text{ for } i = 1, 2, \ldots, n-2, \; v_{n-1}v_n\}.$$

Example 1.6.6. *Wheel graphs on 4, 5, and 6 vertices, W_4, W_5, and W_6.*

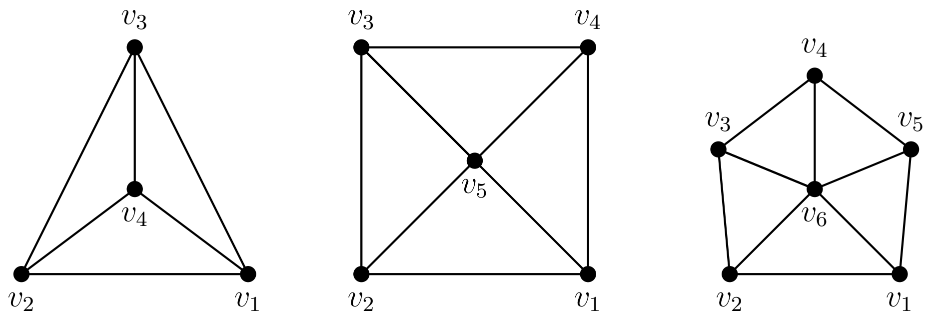


Figure 1.5: The graphs W_4, W_5, and W_6.

Chapter 2

The Interlace Polynomial of the Shell Graph T_n

Before discussing the interlace polynomial of a shell graph, we discuss basic graph properties of it.

2.1 Properties of T_n

Some basic graph theory properties about T_n are obvious from the structure of the graph described in definition 1.6.1.

Proposition 2.1.1. *Let T_n be a graph with n vertices,*

1. $|V(T_n)| = n$ *and* $|E(T_n)| = 2n - 3$;

2. *(Maximal/minimal degree)* $\Delta(T_n) = n - 1$ *and* $\delta(T_n) = 2$. *In particular, the degree sequence for T_n is* $\{n-1, 3, 3, \cdots, 3, 2, 2\}$.

3. *(Diameter and radius) diam*$(T_n) = 2$, *rad*$(T_n) = 1$;

4. $\omega(T_n) = 3$ *(clique number)*;

5. *The independent number of* T_n *is* $\alpha_n = \lfloor \frac{n}{2} \rfloor$;

6. *The chromatic number of* T_n *is* $\chi(T_n) = 3$;

7. *The connectivity of* T_n *is* $\lambda(T_n) = 2$.

Proof. (1) -(3) are obvious.

For (4), the maximum clique is the cycle $C_3 = v_1 v_n v_{n-1}$.

For (5), a maximum independent set is $S_n = \{v_1, v_3, \ldots, v_{n-1}\}$ if n is even and $S_n = \{v_1, v_3, \ldots, v_{n-2}\}$ if n is odd. It implies that $|S_n| = n/2$ if n is even and $(n-1)/2$ if n is odd. Thus, $\alpha_n = \lfloor n/2 \rfloor$.

(6): The vertices of T_n can be colored by 3 colors. We can assign color 1 to v_n, color 2 to the vertices with odd indices excluding v_n if n is odd. Then the rest of the vertices take color 3.

(7): Since T_n has a Hamiltonian circuit, the connectivity is at least 2. If we remove the vertices v_n and v_2, the graph becomes disconnected. Thus, $\lambda(T_n) = 2$. □

The above properties are verified for T_8 in the next example.

Example 2.1.2. *Consider* T_8. *Refer to Example 1.6.2, The following can be easily calculated.*

$$
\begin{aligned}
|V(T_8)| &= 8, \quad |E(T_8)| = 13; \\
\Delta(T_8) &= 7, \quad \delta(T_8) = 8, \quad diam(T_8) = 2 \quad rad(T_8) = 1; \\
\omega(T_8) &= 3 \quad \alpha(T_8) = 4, \quad \chi(T_8) = 3, \quad \lambda(T_8) = 2.
\end{aligned}
$$

2.2 Recursive Formulas for $q(T_n, x)$

Let us first examine the toggling process of T_8.

Example 2.2.1. *Consider the graph T_8. We start the toggling process at the edge v_1v_8. The decomposition of T_8 is as follows:*

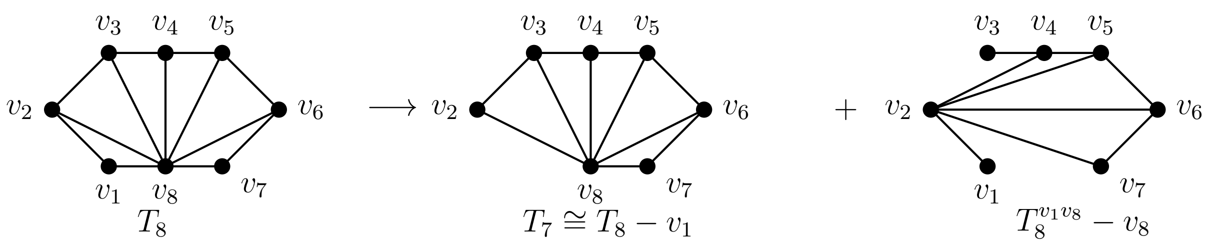


Figure 2.1: Toggling of T_8 on v_1v_8.

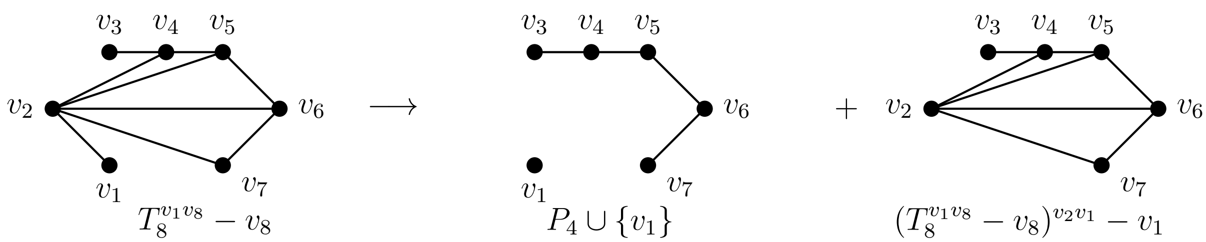


Figure 2.2: Toggling of $T_8^{v_1v_8}$ on v_2v_1.

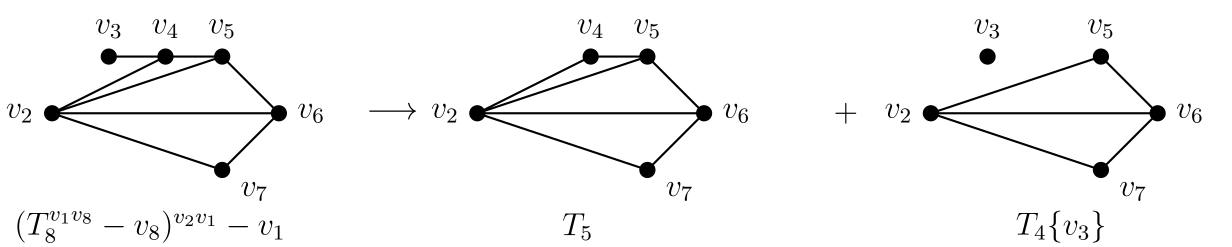


Figure 2.3: Toggling Process of of $(T_8^{v_1v_8} - v_8)^{v_2v_1} - v_1$ on v_3v_4.

After the above toggling process, we obtain a recursive formula for $q(T_8, x)$:

$$q(T_8, x) = q(T_7, x) + xq(P_4, x) + q(T_5, x) + xq(T_4, x).$$

By similar procedures, we can obtain explicit formulas for the interlace polynomials of T_n for small values of n.

Lemma 2.2.2. *Formulas for T_n for small n, $3 \leq n \leq 11$.*

1. $q(T_3, x) = 4x$;

2. $q(T_4, x) = 2x^2 + 4x$;

3. $q(T_5, x) = 5x^2 + 6x$;

4. $q(T_6, x) = x^3 + 9x^2 + 10x$;

5. $q(T_7, x)) = 4x^3 + 17x^2 + 14x$;

6. $q(T_8, x) = x^4 + 11x^3 + 28x^2 + 20x$;

7. $q(T_9, x) = 5x^4 + 24x^3 + 45x^2 + 30x$;

8. $q(T_{10}, x) = x^5 + 15x^4 + 46x^3 + 74x^2 + 44x$;

9. $q(T_{11}, x) = 6x^5 + 36x^4 + 85x^3 + 118x^2 + 64x$.

Proof. We focus on $q(T_8, x)$. Recursively, we obtain

$$\begin{aligned}
q(T_8, x) &= q(T_7, x) + xq(P_4, x) + q(T_5, x) + xq(T_4, x) \\
&= (4x^3 + 17x^2 + 14x) + x(x^3 + 5x^2 + 2x) + (5x^2 + 6x) + x(2x^2 + 4x) \\
&= x^4 + 11x^3 + 28x^2 + 20x.
\end{aligned}$$

□

An existing result for any graph G with n vertices, $q(G, 2) = 2^n$. We confirm it with our graph T_n.

Proposition 2.2.3. $q(T_n, 2) = 2^n$

Proof. We prove this by mathematical induction. By Lemma 2.2.2, one can check easily that $q(T_n, 2) = 2^4$ for $n = 3, 4, 5, 6$. Assume that the statement is true for all integers m with $k \geq 6$. Then by the recursive formula given in Theorem 2.2.4,

$$\begin{aligned} q(T_{k+1}, 2) &= q(T_k, 2) + q(T_{k-2}, 2) + 2q(P_{k-3}, 2) + 2q(T_{k-3}, 2) \\ &= 2^k + 2^{k-2} + (2)2^{k-2} + (2)2^{k-3} = 2^k + 2^{k-2} + 2^{k-1} + 2^{k-2} = 2^{k+1}. \end{aligned}$$

Thus the statement if true for all $n \geq 3$. $\square$

Refer to the pivoting process in Example 2.2.1, a recursive formula for $q(T_n, x)$ is given below.

Theorem 2.2.4. *For* $n \geq 7$,

$$q(T_n, x) = q(T_{n-1}, x) + q(T_{n-3}, x) + xq(P_{n-4}, x) + xq(T_{n-4}, x).$$

Proof. We begin to perform the toggling process starting at v_1v_n of T_n. For $n \geq 7$, the decomposition of T_n is as follows:

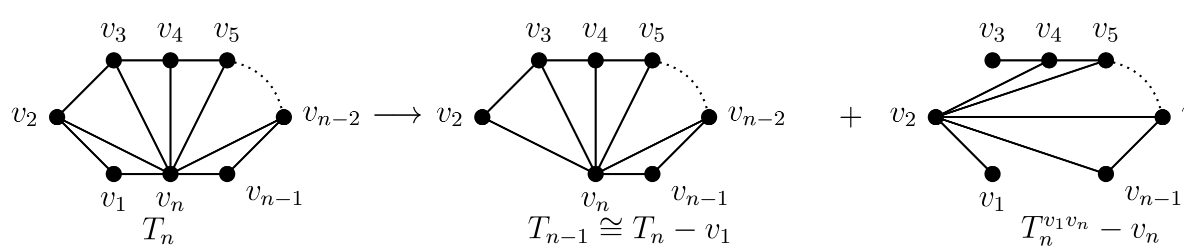


Figure 2.4: Toggling of T_n on v_1v_n.

The toggling process decomposes T_n into four disjoint graphs, T_{n-1}, $P_{n-4} \cup \{v_1\}$, T_{n-3}, and $T_{n-4} \cup \{v_3\}$. Here the two unions are disjoint unions. The corresponding interlace polynomials are $q(T_{n-1}, x)$, $xq(P_{n-4}, x)$, $q(T_{n-3}, x)$, and $q(T_n, x)$. Thus the formula is true.

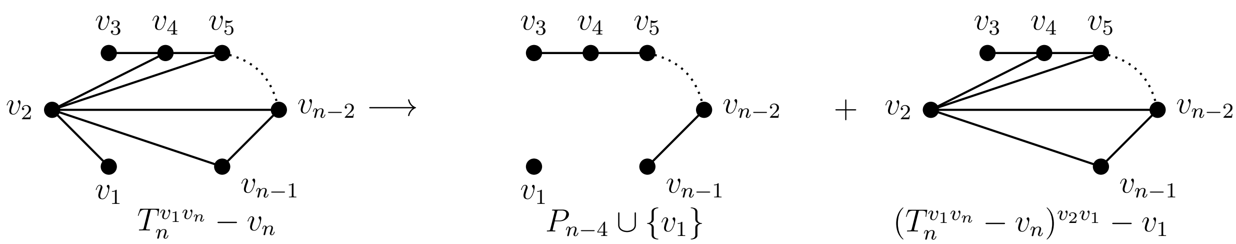


Figure 2.5: Toggling of $T_n^{v_1v_n}$ on v_2v_1.

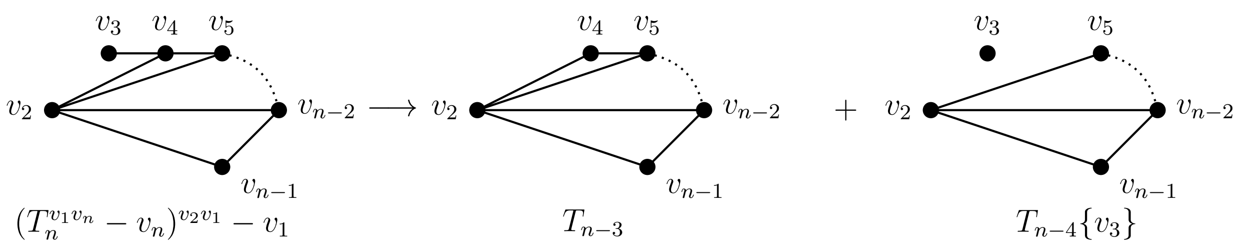


Figure 2.6: Toggling $(T_n^{v_1v_n} - v_n)^{v_2v_1} - v_1$ on v_3v_4.

□

Now we define the explicit form of the polynomial $q(T_n, x)$.

Definition 2.2.5. *The interlace polynomial of the shell graph T_n is denoted by*

$$q(T_n, x) = a_{n,k_n}x^{k_n} + a_{n,k_n-1}x^{k_n-1} + \ldots + a_{n,1}x$$

where $k_n = \deg(q(T_n, x))$ and $a_{n,i}$ is the coefficient of the x^i-term of $q(T_n, x)$.

From Lemma 2.2.2, we observe that when n is even the degree of $q(T_n, x)$ is $n/2$ and the leading coefficient is 1 if $n = 4, 6,$ or 8. While, the degree is $(n-1)/2$ and the leading coefficient is $(n+1)/2$ for $n = 7$ or 9. The following proposition shows that it is true in general.

Proposition 2.2.6. *Consider the shell graph T_n with $n \geq 3$. Then*

1. $k_n = \lfloor \frac{n}{2} \rfloor$*;*

2. *For $n \geq 6$, the leading coefficient a_{n,k_n} of $q(T_n, x)$ is*

$$a_{n,k_n} = \begin{cases} 1 & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}.$$

Proof. 1. We apply the recursive relation, Theorem 2.2.4 and prove it by mathematical induction.

From Lemma 2.2.2, both (1) and (2) are true for n with $3 \leq n \leq 9$. For $n \geq 9$, assume $\deg(q(T_n, x)) = \left\lfloor \frac{n}{2} \right\rfloor$. By Lemma 1.5.4, $\deg(q(P_n, x)) = \left\lfloor \frac{n+2}{2} \right\rfloor$. By the recursive formula given in Lemma 2.2, $\deg(q(T_{n+1}, x))$ is the maximum of $\deg(q(T_n, x))$, $\deg(q(T_{n-2}, x)), \deg(xq(T_{n-3}, x))$, and $\deg(xq(P_{n-3}, x))$. That is,

$$\begin{aligned} \deg(q(T_{n+1}, x)) &= \max\left(\left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{n-2}{2} \right\rfloor, \left(\left\lfloor \frac{n-3}{2} \right\rfloor + 1\right), \left(\left\lfloor \frac{n-1}{2} \right\rfloor + 1\right)\right) \\ &= \left\lfloor \frac{n+1}{2} \right\rfloor. \end{aligned}$$

2. By the analysis in the proof of (1), only the leading term(s) of $q(T_{n-1}, x)$ or $xq(P_{n-4}, x)$ may contribute to the leading term of $q(T_n, x)$. Furthermore, if n is even, $\lfloor n/2 \rfloor > \lfloor (n-1)/2 \rfloor$, so the leading term of of $q(T_n, x)$ is the same as the leading term of $xq(P_{n-4})$, which is $x^{n/2}$ with leading coefficient 1. When n is odd, $\lfloor n/2 \rfloor = \lfloor (n-1)/2 \rfloor$. Then the leading term of $q(T_n, x)$ is the leading term of $q(T_{n-1}, x)$ + the leading term of $xq(P_{n-4}, x)$. Recall that the leading coefficient of $q(P_n, x)$ is $\frac{n+3}{2}$ if n is odd (1.5.4). We know $n-1$ is even. Then we have

$$a_{n,k_n} = a_{n-1,k_{n-1}} + \frac{n-1}{2} = 1 + \frac{n-1}{2} = \frac{n+1}{2}.$$

□

Next we develop the formulas for the second and third leading coefficients and the coefficient for the x-term (the last coefficient) of the polynomial $q(T_n, x)$ denoted $a_{n,(k_{n-1})}$, $a_{n,(k_{n-2})}$, and $a_{n,1}$.

Proposition 2.2.7. *The coefficients for the $x^{k_{n-1}}$-term, the $x^{k_{n-2}}$-term, and the x-term are given below.*

1. *The second leading coefficient is given by*

$$a_{n,k_n-1} = \begin{cases} \frac{n^2+2n}{8} & \text{if } n \text{ is even and } n \geq 10 \\ \frac{n^3+3n^2-n+45}{48} & \text{if } n \text{ is odd and } n \geq 11 \end{cases}.$$

2. *The third leading coefficient is given by*

$$a_{n,+k_n-2} = \begin{cases} \frac{n^4+4n^3-4n^2+176n}{384} & \text{if } n \text{ is even and } n \geq 14 \\ \frac{n^5+5n^4-10n^3+430n^2+9n+3405}{3840} & \text{if } n \text{ is odd and } n \geq 17 \end{cases}.$$

3. *The last coefficient for $n > 3$ is given by*

$$a_{n,1} = 2\sum_{k=0}^{n-2} \binom{n-2-k}{\lfloor \frac{k}{2} \rfloor}.$$

Proof. By Proposition 2.2.4, for $n \geq 8$,

$$q(T_{n+1}, x) = q(T_n, x) + q(T_{n-2}, x) + xq(T_{n-3}, x) + xq(P_{n-3}, x).$$

We apply the mathematical induction idea for the proof using the above recursive formula. It is straightforward to check that the above theorem is true for all initial values needed for the initial conditions of the inductive proof.

1. There are two cases.

Case 1: n is even. Then $n+1$ is odd. In the proof of Proposition 2.2.6, it shows that the leading term of $q(T_{n+1},x)$ is from the leading terms of $q(T_n,x)$ and $xq(P_{n-3},x)$, which are both of degree $\frac{n}{2}$. Thus, the second leading coefficients of $q(T_n,x)$ and $q(P_{n-3},x)$ contribute to the second leading coefficient of $q(T_{n+1},x)$. From Lemma 1.5.4 and Proposition 2.2.6, $\deg(q(T_{n-2},x)) = \frac{n-2}{2} = \deg(xq(T_{n-3},x))$. So the leading coefficients of $q(T_{n-2},x))$ and $q(T_{n-3},x))$ also contribute to the second leading term of $q(T_{n+1},x)$. Note that from Lemma 1.5.4(3), the second leading term of $q(P_{n-3},x)$ is $b_{n-3,\frac{n}{2}-1} = \frac{((n-3)^2-1)((n-3)+9))}{48} = \frac{n^3-28n+48}{48}$. Thus, inductively, for n being even and $n \geq 7$ we have

$$\begin{aligned} a_{n+1,k_{n+1}-1} &= a_{n,k_n-1} + a_{n-2,k_{n-2}} + a_{n-3,k_{n-3}} + b_{n-3,j_{n-3}-1} \\ &= \frac{n^2+2n}{8} + 1 + \frac{n-2}{2} + \frac{n^3-28n+48}{48} \\ &= \frac{(n+1)^3+3(n+1)^2-(n+1)+45}{48}. \end{aligned}$$

Case 2: n is odd ($n+1$ is even.) Similarly to the previous case, the proof of Proposition 2.2.6 shows that the leading term of $q(T_{n+1},x)$ is from the leading term of that of $xq(P_{n-3},x)$ of degree $\frac{n+1}{2}$. Thus, the second leading coefficient of $q(P_{n-3})$ contributes to the second leading coefficient of $q(T_n,x)$. From Lemma 1.5.4 and Proposition 2.2.6, $\deg(q(T_n,x)) = \frac{n-1}{2} = \deg(xq(T_{n-3},x))$. So the leading coefficients of $q(T_n,x)$ and $q(T_{n-3},x)$ and the second leading coefficient of $q(P_{n-3},x)$ make up the second leading coefficient of $q(T_{n+1},x)$. This implies that

$$
\begin{aligned}
a_{n+1,k_{n+1}-1} &= a_{n,k_n} + a_{n-3,k_{n-3}} + b_{n-3,j_{n-3}-1} \\
&= \frac{n+1}{2} + 1 + \frac{(n-3)^2 + 6(n-3)}{8} \\
&= \frac{n^2+4n+3}{8} = \frac{(n+1)^2 + 2(n+1)}{8}.
\end{aligned}
$$

2. Similar as in the proof of Part 1 above, by the recursive formula in Proposition 2.2.4, when n is odd,

$$
\begin{aligned}
a_{n+1,k_{n+1}-2} &= a_{n,k_n-1} + a_{n-2,k_{n-2}} + a_{n-3,k_{n-3}-1} + b_{n-3,j_{n-3}-2} \\
&= \frac{n^3+3n^2-n+45}{48} + \frac{n-1}{2} + \frac{n^2-4n+3}{8} \\
&+ \frac{n^4-58n^2+192n-135}{384} \\
&= \frac{n^4+8n^3+14n^2+184n+177}{384} \\
&= \frac{(n+1)^4+4(n+1)^3-4(n+1)^2+176(n+1)}{384}
\end{aligned}
$$

If n is even,

$$
\begin{aligned}
a_{n+1,k_{n+1}-2} &= a_{n,k_n-2} + a_{n-2,k_{n-2}-1} + a_{n-3,k_{n-3}-1} + b_{n-3,j_{n-3}-2} \\
&= \frac{n^4+4n^3-4n^2+176n}{384} + \frac{n^2-1}{8} + \frac{n^3-6n^2+8n+48}{48} \\
&+ \frac{n^5-100n^3+480n^2-576n}{3840} \\
&= \frac{n^5+10n^4+20n^3+440n^2+864n+384}{3840} \\
&= \frac{(n+1)^5+5(n+1)^4-10(n+1)^3+430(n+1)^2+9(n+1)+3405}{3840}.
\end{aligned}
$$

3. We prove it by mathematical induction.

$$
\begin{aligned}
&\text{for } n=4, \qquad 2\sum_{k=0}^{2}\binom{2-k}{\lfloor \frac{k}{2}\rfloor} = 2(2) = 4 = a_{4,1};\\
&\text{for } n=5, \qquad 2\sum_{k=0}^{3}\binom{3-k}{\lfloor \frac{k}{2}\rfloor} = 2(3) = 6 = a_{5,1};\\
&\text{for } n=6, \qquad 2\sum_{k=0}^{4}\binom{4-k}{\lfloor \frac{k}{2}\rfloor} = 2(5) = 10 = a_{6,1};\\
&\text{for } n=7, \qquad 2\sum_{k=0}^{1}\binom{5-k}{\lfloor \frac{k}{2}\rfloor} = 2(7) = 14 = a_{7,1}.
\end{aligned}
$$

Assume that the statement is true for all integers $n \geq 7$. Then by the recursive relationship $a_{n,1} = a_{n-1,1} + a_{n-3,1}$,

$$
\begin{aligned}
a_{n+1,1} &= a_{n,1} + a_{n-2,1} = 2\sum_{k=0}^{n-2}\binom{n-2-k}{\lfloor \frac{k}{2}\rfloor} + 2\sum_{k=0}^{n-4}\binom{n-4-k}{\lfloor \frac{k}{2}\rfloor}\\
&= 2\left[\binom{n-2}{0} + \binom{n-3}{0} + \sum_{k=2}^{n-2}\binom{n-2-k}{\lfloor \frac{k}{2}\rfloor} + \sum_{k=2}^{n-2}\binom{n-2-k}{\lfloor \frac{k-2}{2}\rfloor}\right]\\
&= 2\left[\binom{n-1}{0} + \binom{n-2}{0} + \sum_{k=2}^{n-2}\left[\binom{n-2-k}{\lfloor \frac{k}{2}\rfloor} + \binom{n-k-2}{\lfloor \frac{k}{2}\rfloor - 1}\right]\right]\\
&= 2\left[\binom{n-1}{0} + \binom{n-2}{0} + \sum_{k=2}^{n-2}\left[\binom{n-1-k}{\lfloor \frac{k}{2}\rfloor} + \binom{0}{\lfloor \frac{n-1}{2}\rfloor}\right]\right]\\
&= \sum_{k=0}^{(n+1)-2}\binom{(n+1)-2-k}{\lfloor \frac{k}{2}\rfloor}.
\end{aligned}
$$

In the above proof, we applied the following known formulas:

$$
\binom{0}{j} = 0 \ \text{ if } j > 0 \ \text{ and } \quad \binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k} \quad (n \geq k).
$$

$\square$

Chapter 3

Other Related Graphs

During the decomposition process of the graph T_n, two related graphs are exposed, called $D_{r,s}$ and W_n. There are defined in Definition 1.6.3. In this chapter I study these two graphs.

3.1 Formulas for $D_{r,s}$

Refer to the labeled graph $D_{r,s}$ ($r \geq 3, s \geq 0$) shown in Figure 1.4. When $s = 0$, $D_{r,0} = C_r$. Some basic graph theory properties about $D_{r,s}$ are obvious from the structure of the graph.

Proposition 3.1.1. *Consider the graph labeled $D_{r,s} = (V(D_{r,s}), E(D_{r,s}))$ with $s > 0$.*

1. $|V(D_{r,s})| = r + s = |E(D_{r,s})|$;

2. $\Delta(D_{r,s}) = 3$, $\delta(D_{r,s}) = 1$;

3. $diam(D_{r,s}) = \lfloor \frac{r}{2} \rfloor + s$, $rad(D_{r,s}) = \left\lfloor \frac{\lfloor \frac{r}{2} \rfloor + s + 1}{2} \right\rfloor$;

4. $\omega(D_{r,s}) = 3$ *if* $r = 3$ *and* $\omega(D_{r,s}) = 2$ *otherwise;*

5. $\alpha(D_{r,s}) = \lfloor \frac{r+s+1}{2} \rfloor$;

6. $\chi(D_{r,s}) = 2$ *if* r *is even and* $\chi(D_{r,s}) = 3$ *when* r *is odd;*

7. The connectivity of $D_{r,s}$ is $\lambda(D_{r,s}) = 1$.

Proof. Refer to the labeled graph $D_{r,s}$ shown in Definition 1.6.3. (1), (2), (4), (6), (7), and (8) are obvious.

For (3), $\mathrm{diam}(D_{r,s}) = \mathrm{diam}(C_r) + s = \lfloor r/2 \rfloor + s$. The vertex with the minimum eccentricity is the midpoint in between v_s and v_k where $k = \lfloor r/2 \rfloor$. Thus,

$$\mathrm{rad}(D_{r,s}) = \left\lfloor \frac{\lfloor \frac{r}{2} \rfloor + s + 1}{2} \right\rfloor.$$

For (5), $\alpha(D_{r,s}) = \alpha(C_r) + \alpha(P_s) = \lfloor r/2 \rfloor + \lfloor (s+1)/2 \rfloor$. Then the result holds.

□

We next investigate the interlace polynomials for the graph $D_{r,s}$. It is obvious that $q(D_{r,0}, x) = q(C_r, x)$. We examine a few graphs of small sizes.

Example 3.1.2. *1. $q(D_{3,1}, x) = 2x^2 + 4x$;*

2. $q(D_{4,1}, x) = 3x^3 + 7x^2 + 6x$;

3. $q(D_{3,2}, x) = q(D_{3,1}, x) + xq(C_3, x) = 6x^2 + 4x$.

The interlace polynomial of $D_{r,s}$, $q(D_{r,s}, x)$, can be described recursively as follows. The proof is straightforward by togging the graph at the end leaf. We skip the proof.

Lemma 3.1.3. *Let $r \geq 3$ and $s \geq 0$. Then*

1. $q(D_{r,1}, x) = q(C_r, x) + xq(P_{r-2}, x)$;

2. $q(D_{r,2}, x) = (1 + x)q(C_r, x) + xq(P_{r-2}, x)$;

3. $q(D_{r,s}, x) = q(D_{r,s-1}, x) + xq(D_{r,s-2}, x)$ for $x \geq 2$.

From the above Lemma 3.1.3(2), $q(D_{r,2}, x)$ is expressed as a combination of $q(C_r, x)$ and $q(P_{r-2}, x)$, with $x + 1$ and x in front of them respectively.

Definition 3.1.4. *For any integer $s \geq -1$, define a sequence of functions $f_s(x)$ as follows.*

$$f_{-1}(x) \equiv 0, \quad f_0(x) = f_1(x) \equiv 1, \quad f_s(x) = f_{s-1}(x) + x f_{s-2}(x) \ \text{ for } s \geq 2.$$

An explicit formula for $f_s(x)$ is given below.

Proposition 3.1.5. *For any integer $s \geq 0$,*

1. *Let $y(x) = \sqrt{1+4x}$. Then*

$$f_s(x) = \frac{1}{y(x)}\left(\left(\frac{1+y(x)}{2}\right)^{s+1} - \left(\frac{1-y(x)}{2}\right)^{s+1}\right) = \frac{(1+y(x))^{s+1} - (1-y(x))^{s+1}}{2^{s+1}y(x)}.$$

2. $\deg(f_s(x)) = \left\lfloor \frac{s}{2} \right\rfloor$;

3. *The leading coefficient of $f_s(x)$ is*

$$\begin{cases} 1 & \text{if } s \text{ is even} \\ \frac{s+1}{2} & \text{if } s \text{ is odd} \end{cases};$$

4. $f_s(0) = 1$.

5. $f_s(-1) = f_{s-6}(-1) = -f_{s-3}(-1)$ *and the value of $f_s(-1)$ is given by*

$$f_s(-1) = \begin{cases} 1 & \text{if } s \equiv 0,\ 1 \pmod 6 \\ 0 & \text{if } s \equiv 2,\ 5 \pmod 6 \\ -1 & \text{if } s \equiv 3,\ 4 \pmod 6 \end{cases};$$

Proof. 1. We prove it by mathematical induction on s. Obviously the formula is true for $s = 0$ and 1. Note that $y(x)^2 = 1 + 4x$ and so $(1 \pm y(x))^2 = 1 \pm 2y(x) + y(x)^2 = 2(1 \pm y(x) + 2x)$. Assume the induction hypothesis. For $s \geq 2$, apply the recursive formula given in definition 3.1.4, we obtain

$$\begin{aligned}
f_{s+1}(x) = \quad & f_s(x) + x f_{s-1}(x) \\
= \quad & \frac{(1+y(x))^{s+1} - (1-y(x))^{s+1}}{2^{s+1}y(x)} + \frac{x\left((1+y(x))^s - (1-y(x))^s\right)}{2^s y(x)} \\
= \quad & \frac{(1+y(x))^s[1+y(x)+2x] - (1-y(x))^s[1-y(x)+2x]}{2^{s+1}y(x)} \\
= \quad & \frac{(1+y(x))^s[1+y(x)]^2/2 - (1-y(x))^s[1-y(x)]^2/2}{2^{s+1}y(x)} \\
= \quad & \frac{(1+y(x))^{s+2} - (1-y(x))^{s+2}}{2^{s+2}y(x)}.
\end{aligned}$$

Thus the formula is true for all $s \geq 0$.

2. We prove it by mathematical induction on s. The formula is true for $f_0(x) = 1$ and $f_1(x) = 1$. Assume the induction hypothesis and apply the recursive formula given in definition 3.1.4. Then for $f_{s+1}(x)$,

$$\begin{aligned}
\deg(f_{s+1}(x)) \quad & = \max\left(\deg(f_s(x)), \deg(x f_{s-1}(x))\right) \\
& = \max\left(\left\lfloor \frac{s}{2} \right\rfloor, \left\lfloor \frac{s-1}{2} \right\rfloor + 1\right) \\
& = \max\left(\left\lfloor \frac{s}{2} \right\rfloor, \left\lfloor \frac{s+1}{2} \right\rfloor\right) = \left\lfloor \frac{s+1}{2} \right\rfloor
\end{aligned}$$

Thus the formula is true for all $s \geq 0$.

3. By definition 1, $f_0(x) = f_1(x) \equiv 1$. Similar to the analysis in the above proof, when s is even and $s > 1$, the leading coefficient of $f_s(x)$ is that of $f_{s-2}(x)$. Since $f_0(x) = 1$, we have $f_s(x) = 1$ for all $s \geq 2$. When s is odd, the leading coefficient of $f_s(x)$ is the sum of $f_{s-1}(x)$ and $f_{s-2}(x)$. Since $f_1(x) = 1$, and $f_{s-1}(x)$'s leading coefficient is 1, inductively, $f_s(x)$ has leading coefficient $(s+1)/2$.

4. By the recursive formula, $f_s(0) = f_{s-1}(0)$ for all $s > 0$. Then $f_{s-1}(0) = 1$ implies

$f_s(0) = 1$ for all $s > 0$.

5. By the recursive formula given in Definition 3.1.4, for $s > 6$, $f_s(-1) = f_{s-1}(-1) - f_{s-2}(-1) = (f_{s-2}(-1) - f_{s-3}(-1)) - f_{s-2}(-1) = -f_{s-3}(-1) = -(-f_{s-6}(-1)) = f_{s-6}(-1)$. It is straightforward to check that $f_1(-1) = f_6(-1) = 1$, $f_2(-1) = f_5(-1) = 0$, and $f_3(-1) = f_4(-1) = 1$. The result follows.

$\square$

From Definition 3.1.4, $f_2(x) = 1 + x$. The formula in Lemma 3.1.3(2) can be changed to $q(D_{r,2}, x) = f_2(x)q(C_r, x) + xf_1(x)q(P_{r-2}, x)$. This result can be generalized to $q(D_{r,s}, x)$.

Theorem 3.1.6. *For any integers $r \geq 3$ and $s \geq 0$,*

$$q(D_{r,s}, x) = f_s(x)q(C_r, x) + xf_{s-1}(x)q(P_{r-2}, x).$$

Proof. The above formula is true by Lemma 3.1.3(1)(2). We prove the rest by mathematical induction on s. By the recursive relation shown in Lemma 3.1.3 (3), Definition 3.1.4, and the induction hypothesis, for $s \geq 2$

$$\begin{aligned} q(D_{r,s+1}, x) &= q(D_{r,s}, x) + xq(D_{r,s-1}, x) \\ &= f_s(x)q(C_r, x) + xf_{s-1}(x)q(P_{r-2}, x) \\ &\quad + x[f_{s-1}(x)q(C_r, x) + xf_{s-2}(x)q(P_{r-2}, x)] \\ &= q(C_r, x)[f_s(x) + xf_{s-1}(x)] + xq(P_{r-2}, x)[f_{s-1}(x) + xf_{s-2}(x)] \\ &= f_{s+1}(x)q(C_r, x) + f_s(x)q(P_{r-2}, x). \end{aligned}$$

$\square$

The next example confirms Lemma 3.1.3 and Theorem 3.1.6

Example 3.1.7. *1.* $q(D_{3,3}, x) = 2x^3 + 10x^2 + 4x$;

2. $q(D_{4,2}, x) = 4x^3 + 7x^2 + 2x$;

3. $q(D_{4,3}, x) = x^4 + 9x^3 + 9x^2 + 2x$.

Here for (1), we use the formula in Theorem 3.1.6 for $r = s = 2$ and the formula given in Example 3.1.2(3).

$$q(D_{3,3}, x) = f_3(x)q(C_3, x) + xf_2(x)q(P_1, x) = (1 + 2x)(4x) + x(1 + x)2x = 2x^3 + 10x^2 + 4x.$$

For (2), from Lemma 3.1.3,

$$(D_{4,2}, x) = (1 + x)(C_4, x) + x(P_2, x) = (1 + x)(3x^2 + 2x) + x(x^2 + 2x) = 4x^3 + 7x^2 + 2x.$$

For (3),

$$\begin{aligned} q(D_{4,3}, x) &= f_3(x)q(C_4, x) + xf_2(x)q(P_2, x) = (1 + 2x)(3x^2 + 2x) + x(1 + x)(x^2 + 2x) \\ &= x^4 + 9x^3 + 9x^2 + 2x. \end{aligned}$$

Results for $q(C_n, x)$ as well as $q(P_n, x)$ are needed to show other properties of the interlace polynomial of $D_{r,s}$. The recursive formulas for $q(C_n, x)$ and $q(P_n, x)$ are provided in Theorem 1.4.2. Below are other useful properites for $q(C_n, x)$.

Lemma 3.1.8. *Consider a cycle C_n with $n \geq 3$, then*

1. $\deg(q(C_n, x)) = \left\lfloor \frac{n}{2} \right\rfloor$;

2. *the leading coefficient of* $q(C_n, x)$ *is*

$$\begin{cases} 2 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases};$$

3. *the* x *coefficient of* $q(C_n, x)$ *is*

$$\begin{cases} n-2 & \text{if } n \text{ is even} \\ n+1 & \text{if } n \text{ is odd} \end{cases};$$

Theorem 3.1.9. *Consider the interlace polynomial of* $D_{r,s}$, $q(D_{r,s}, x)$ *with* $r \geq 3$, $s \geq 0$. *Then*

1. $\deg(q(D_{r,s}, x)) = \left\lfloor \frac{r+s+1}{2} \right\rfloor$;

2. *the leading coefficient of* $q(D_{r,s}, x)$ *is given by*

$$\begin{cases} \frac{s+4}{2} & \text{if } r, s \text{ are even} \\ \frac{r+1}{2} & \text{if } r, s \text{ are odd} \\ 1 & \text{if } r \text{ is even and } s \text{ is odd} \\ \frac{r(s+4)+s}{4} & \text{if } r \text{ is odd and } s \text{ is even} \end{cases};$$

3. *the* x*-coefficient of* $q(D_{r,s}, x)$ *is*

$$\begin{cases} n-2 & \text{if } n \text{ is even} \\ n+1 & \text{if } n \text{ is odd} \end{cases}.$$

Proof. 1. From Theorem 3.1.6, $q(D_{r,s}, x) = f_s(x)q(C_r, x) + xf_{s-1}(x)q(P_{r-2}, x)$.

$$
\begin{aligned}
\deg(q(D_{r,s},x)) \quad &= \max\left(\deg(f_s(x)q(C_r,x)), \deg(xf_{s-1}(x)q(P_{r-2},x))\right) \\
&= \max\left(\left\lfloor\frac{s}{2}\right\rfloor + \left\lfloor\frac{r}{2}\right\rfloor, \left\lfloor\frac{s-1}{2}\right\rfloor + \left\lfloor\frac{r}{2}\right\rfloor + 1\right) \\
&= \max\left(\left\lfloor\frac{s+r}{2}\right\rfloor, \left\lfloor\frac{s+r+1}{2}\right\rfloor\right) = \left\lfloor\frac{s+r+1}{2}\right\rfloor.
\end{aligned}
$$

2. The proof is split into 4 cases. For each fixed r, we apply mathematical induction on s using the recursive formula from Lemma 3.1.3, $q(D_{r,s},x) = q(D_{r,s-1},x) + xq(D_{r,s-2},x)$ and the above degree result. We first check the initial conditions ($s = 1, 2$).

    (1) For $s = 1$, by Lemma 3.1.3(1), $q(D_{r,1},x) = q(C_r,x) + xq(P_{r-2},x)$. Since $\deg((q(C_r,x))) = 1 + \deg(q(P_{r-2},x))$, the leading coefficient of $q(D_{r,1},x)$ is that of $q(P_{r-2},x)$, denoted as $\mathrm{lc}(q(D_{r,1},x))$. By Theorem 1.5.4(2), $\mathrm{lc}(q(D_{r,1},x)) = 1$ if r is even; If r is odd, $\mathrm{lc}(q(D_{r,1},x)) = \mathrm{lc}(q(P_{r-2},x)) = \lfloor\frac{r+1}{2}\rfloor$. Thus the result is true for $\mathrm{lc}(q(D_{r,1},x))$ $(r \geq 3)$.

    (2) For $s = 2$, by Lemma 3.1.3(2), $q(D_{r,2},x) = (1+x)q(C_r,x) + xq(P_{r-2},x)$. Since $\deg(q(C_r,x)) = 1+\deg(q(P_{r-2},x))$, the leading coefficient $\mathrm{lc}(q(D_{r,2},x))$ is the sum of $\mathrm{lc}(q(C_r,x))$ and $\mathrm{lc}(q(P_{r-2},x))$. By Lemma 3.1.8(2) and Theorem 1.5.4(2), if r is even, $\mathrm{lc}(q(D_{r,2},x)) = 2+1 = 3 = \lfloor\frac{2+4}{2}\rfloor$. When r is odd, $\mathrm{lc}(q(D_{r,2},x)) = r + \frac{r+1}{2} = \frac{3r+1}{2}$, which also equal to $\frac{r(2+4)+2}{4}$. Thus the formula holds for $\mathrm{lc}(q(D_{r,2},x))$.

Case 1: Both r, s even. The degree of the interlace polynomial is $\deg(q(D_{r,s+1},x)) = \deg(xq(D_{r,s-1},x)) = \frac{r+s+2}{2}$. The leading coefficient of $q(D_{r,s+1},x)$ directly comes from the leading coefficient of $xq(D_{r,s-2},x)$ which is 1. We first check the initial conditions ($s = 1, 2.$)

Case 2: r, s are both odd. The leading coefficient of $q(D_{r,s+1},x)$ is the sum of the leading coefficients of $q(D_{r,s}, x$ and $xq(D_{r,s-1}, x)$ which is $\frac{r+1}{2} + \frac{r(s+3)+s-1}{4} = \frac{r(s+5)+s+1}{4}$.

Case 3: r is even and s is odd. The leading coefficient of $q(D_{r,s+1}, x)$ is the sum of the leading coefficients of $q(D_{r,s}, x)$ and $xq(D_{r,s-1}, x)$ which is $1 + \frac{s+3}{2} = \frac{s+5}{2}$.

Case 4: r is odd and s is even. For $q(D_{r,s+1}, x)$, the degree of the interlace polynomial is $\deg(q(D_{r,s+1}, x)) = \deg(xq(D_{r,s-1}, x)) = \frac{r+s+1}{2}$. The leading coefficient of $q(D_{r,s+1}, x)$ directly comes from the leading coefficient of $xq(D_{r,s-1}, x)$ which is $\frac{r+1}{2}$.

3. From Theorem 3.1.6, $q(D_{r,s}, x) = f_s(x)q(C_r, x)+xf_{s-1}(x)q(P_{r-2}, x)$. Because $f_s(0) = 1$ from Proposition 3.1.4, the x term is from the x-erm of $q(C_r, x)$, which is given in Lemma 3.1.8.

□

3.2 Formulas for $q(W_n, x)$

Some basic graph theory properties about W_n are obvious from the structure of the graph.

Proposition 3.2.1. *Consider the graph $W_n = (V(W_n), E(W_n))$ with $n \geq 4$ as described in definition 1.6.5.*

1. *$|V(W_n)| = n$ and $|E(W_n)| = 2(n-1)$;*

2. *$\Delta(W_n) = n - 1$, $\delta(W_n) = 3$;*

3. *diam$(W_n) = 2$ if $n > 4$ and diam$(W_n) = 1$ if $n = 4$. rad$(W_n) = 1$;*

4. *$\omega(W_n) = 4$ if $n = 4$ and $\omega(W_n) = 4$ otherwise;*

5. *$\alpha(W_n) = \left\lfloor \frac{n-1}{2} \right\rfloor$;*

6. *for $n \geq 6$, $\chi(W_n) = 4$ if n is even and $\chi(W_n) = 3$ if n is odd;*

7. $\lambda(W_n) = 3.$

Proof. (1) -(3) are obvious.

For (4), the maximum clique is the cycle $C_3 = v_1 v_n v_{n-1}$ for $n > 4$. For $n = 4$, the maximum clique is the complete graph K_4.

For (5), if n is odd the maximum independent set is $S_n = \{v_1, v_3, \ldots, v_{n-1}\}$. If n is even the maximum independent set is $S_n = \{v_1, v_3, \ldots, v_{n-2}\}$. This implies that $|S_n| = (n-1)/2$ if n is odd and $(n-2)/2$ if n is even. Thus, $\alpha(W_n) = \lfloor (n-1)/2 \rfloor$.

(6): The vertices of W_n can be colored by 3 colors if n is odd. We can assign color 1 to v_n, color 2 to the vertices with odd indices excluding v_n. Then the rest of the vertices take color 3. For even n, assign color 1 to v_n, color 2 to v_1 and color 3 to vertices with even indices excluding v_n. Then the rest of the vertices take color 4.

(7): Since W_n has a Hamiltonian circuit, the connectivity is at least 2. If we remove the vertices v_n, v_2, and v_{n-1}, the graph becomes disconnected. Thus, $\lambda(W_n) = 3$. □

Lemma 3.2.2. *Formulas for W_n for small n, $4 \leq n \leq 13$.*

1. $q(W_4, x) = 2q(T_3, x) = 8x;$

2. $q(W_5, x) = q(T_4, x) + xq(P_2, x) = x^3 + 4x^2 + 4x;$

3. $q(W_6, x) = q(T_5, x) + q(P_3, x) + q(T_3, x) + xq(P_1, x) = 10x^2 + 12x;$

4. $q(W_7, x) = q(T_6, x) + q(P_3, x) + 2q(T_3, x) + xq(P_2, x) + 2xq(P_1, x) = 2x^3 + 18x^2 + 20x;$

5. $q(W_8, x) = q(T_7, x) + q(W_4, x) + 2q(T_4, x) + xq(P_3, x) + 3xq(T_3, x) = 7x^3 + 35x^2 + 30x;$

6. $q(W_9, x) = q(T_8, x) + q(W_5, x) + 2q(T_5, x) + xq(P_4, x) + 3xq(T_4, x) = 2x^4 + 23x^3 + 56x^2 + 36x;$

7. $q(W_{10}, x) = q(T_9, x) + q(W_6, x) + 2q(T_6, x) + xq(P_5, x) + 3xq(T_5, x) = 9x^4 + 48x^3 + 93x^2 + 62x;$

8. $q(W_{11},x) = q(T_{10},x) + q(W_7,x) + 2q(T_7,x) + xq(P_6,x) + 3xq(T_6,x) = 2x^5 + 27x^4 + 92x^3 + 158x^2 + 92x$

9. $q(W_{12},x) = q(T_{11},x) + q(W_8,x) + 2q(T_8,x) + xq(P_7,x) + 3xq(T_7,x) = 11x^5 + 66x^4 + 176x^3 + 253x^2 + 134x$

10. $q(W_{13},x) = q(T_{12},x) + q(W_9,x) + 2q(T_9,x) + xq(P_8,x) + 3xq(T_8,x) = 2x^6 + 38x^5 + 147x^4 + 318x^3 + 393x^2 + 190x$

A recursive formula for $q(W_n,x)$ is given below.

Lemma 3.2.3. *For $n \geq 9$,*

$$q(W_n,x) = q(T_{n-1},x) + q(W_{n-4},x) + 2q(T_{n-4},x) + xq(P_{n-5},x) + 3xq(T_{n-5},x).$$

Proof. We begin by performing the toggling process starting at v_1v_{n-1} of W_n. The toggling process decomposes W_n into 8 disjoint graphs, $T_{n-1}, W_{n-4}, T_{n-4}, T_{n-4}, \{v_1\} \cup P_{n-5}, \{v_{n-2}\} \cup T_{n-5}, \{v_2\} \cup T_{n-5}$, and $\{v_3\} \cup T_{n-5}$. Here the four unions are disjoint unions. For $n \geq 9$ the toggling process is as follows:

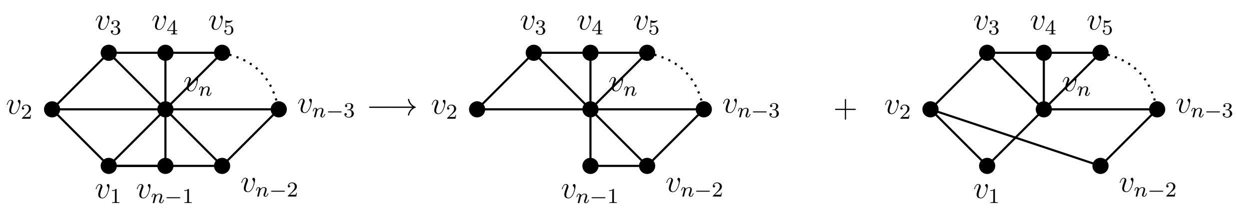


Figure 3.1: Toggle of W_n on v_1v_{n-1}.

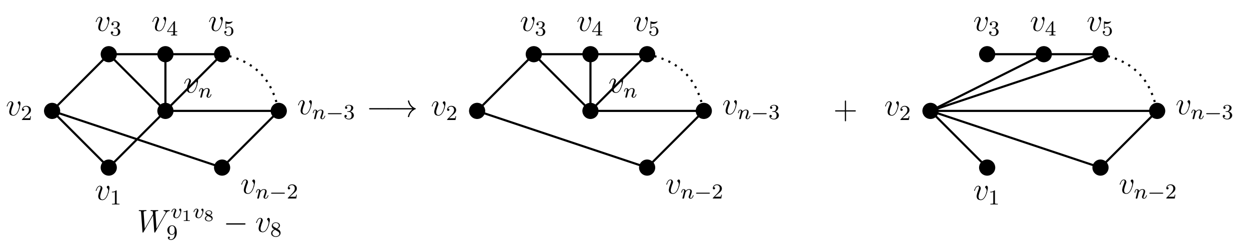


Figure 3.2: Toggle of $W_n^{v_1v_{n-1}} - v_{n-1}$ on v_1v_n.

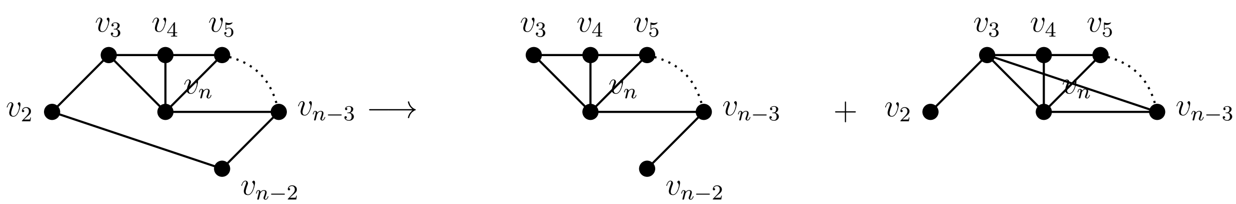


Figure 3.3: Toggle of $W_n^{v_1v_{n-1}} - v_{n-1} - v_1$ on v_2v_1.

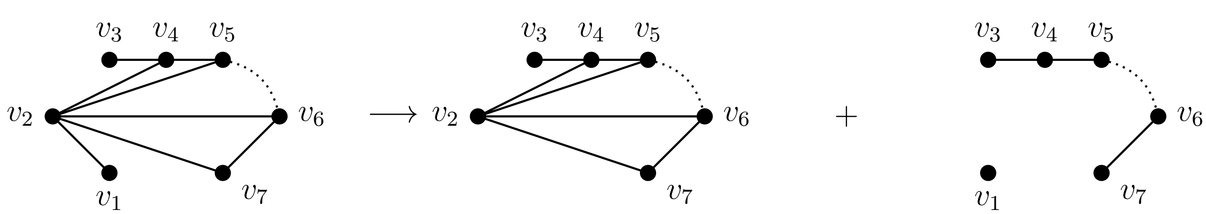


Figure 3.4: Toggle of $(W_n^{v_1v_{n-1}} - v_{n-1})^{v_1v_n} - v_n$ on v_1v_2.

Thus the recursive formula is true.

$\square$

Proposition 3.2.4. *Consider the wheel graph W_n with $n \geq 4$. Then*

1. *for $n \geq 6$, $\deg(q(W_n, x)) = \left\lfloor \frac{n-1}{2} \right\rfloor$;*

2. The leading coefficient of $q(W_n, x)$ *is given by*

$$\begin{cases} n-1 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd} \end{cases}.$$

Proof. 1. From Lemma 3.2.2, (1) is true for $6 \leq n \leq 13$. For $n > 13$, assume that $\deg(q(W_n, x)) = \lfloor \frac{n-1}{2} \rfloor$. By Proposition 2.2.6 and Lemma 1.5.4, $\deg(q(T_n, x)) = \lfloor \frac{n}{2} \rfloor$ and $\deg(q(P_n, x)) = \lfloor \frac{n+2}{2} \rfloor$. By the recursive formula $q(W_n, x)$ given in Lemma 3.2.3, the $\deg(q(W_{n+1}, x))$ is the maximum value shown below:

$$\begin{aligned} \deg(q(W_{n+1}, x)) &= \max\{\deg(q(T_n, x)), \deg(q(W_{n-3}, x)), \\ &\quad \deg(q(T_{n-3}, x)), \deg(xq(P_{n-4}, x)), \deg(xq(T_{n-4}, x))\}. \end{aligned}$$

That is,

$$\deg(q(W_{n+1}, x)) = \max\left(\left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{n-4}{2} \right\rfloor, \left\lfloor \frac{n-3}{2} \right\rfloor, \left\lfloor \frac{n-2}{2} \right\rfloor \right) = \left\lfloor \frac{n}{2} \right\rfloor.$$

2. From Lemma 3.2.2, (2) is true for $9 \leq n \leq 13$. The result from (1) showed that only the leading terms of $\deg(q(T_{n-1}, x))$ and $\deg(xq(P_{n-5}, x))$, since $\deg(q(W_n, x)) = \lfloor \frac{n-1}{2} \rfloor = \deg(q(T_{n-1}, x)) = \deg(xq(P_{n-5}, x))$. We apply the mathematical induction idea for the proof using the recursive relationship given in Lemma 3.2.3 and the results for the leading coefficients of $q(T_n, x)$ and $q(P_n, x)$ from Proposition 2.2.6 and Lemma 1.5.4 respectively.

Case: 1 n is even. $n+1$ is odd. The leading coefficient of $q(W_{n+1}, x)$ is given by

$$a_{n,k_n} + b_{n-4,j_{n-4}} = 1 + 1 = 2.$$

Case: 2 n is odd. $n+1$ is even. The leading coefficient of $q(W_{n+1}, x)$ is given by

$$a_{n,k_n} + b_{n-4,j_{n-4}} = \frac{n+1}{2} + \frac{n-1}{2} = n.$$

$\square$

Chapter 4

Related Matrices

In this chapter we discuss some results related to the adjacency matrix of T_n. We show how to generate an explicit formula for the interlace polynomial of T_n using adjacency matrices of the subgraphs of T_n. We also discuss the rank of some related matrices over the field $\mathbb{Z}_2$. For any graph G, we denote $A[G]$ as the adjacency matrix of G.

4.1 The Adjacency Matrix of T_n

The following example gives $A[T_3], A[T_4]$, $A[T_5]$, $A[T_6]$, and a general form of $A[T_n]$ for any $n \geq 4$.

Example 4.1.1. *The matrices* $A[T_n]$ *for* $n = 3, 4, 5,$ *and* 6.

$$A[T_3] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad A[T_4] = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix},$$

$$A[T_5] = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}, \quad A[T_6] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

For $n \geq 4$, $A[T_n]$ has the following general form:

$$A[T_n] = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & 1 & \ddots & 0 & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 0 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 1 & 0 \end{bmatrix}.$$

The matrix $A[T_n]$ can be easily constructed from the smaller matrix $A[T_{n-1}]$.

Lemma 4.1.2. *The adjacency matrix of T_n, $A[T_n]$, for $n \geq 5$, can be constructed iteratively as below:*

$$A[T_n] = \begin{bmatrix} 0 & \mathbf{v} \\ \mathbf{v}^T & A[T_{n-1}] \end{bmatrix},$$

where $\mathbf{v}^T = [1, 0, \ldots, 0, 1]$ is a row vector with $n-1$ components.

Next we investigate the rank of a related matrix, $A[T_n] + I_n$, where I_n is the $n \times n$ identity matrix.

4.2 The Rank of Matrix $A[T_n] + I_n$ Modulo 2

We focus on the rank of $A[T_n] + I_n \pmod 2$. A few matrices $A[T_n] + I_n$, for $n = 3, 4, 5, 6$ are shown below.

Example 4.2.1.

$$A[T_3] + I_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad A[T_4] + I_4 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

$$A[T_5] + I_5 = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad A[T_6] + I_6 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Applying Lemma 4.1.2, the general form for $n \geq 5$ is given below:

$$A[T_n] + I_n = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 1 & 1 & \ddots & 0 & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{v} \\ \mathbf{v}^T & A[T_{n-1}] + I_n \end{bmatrix},$$

where $\mathbf{v}^T = [1, 0, \ldots, 0, 1]$ *is a row vector with* $n-1$ *components.*

By Theorem 1.4.3, the value of the interlace polynomial of a graph G at -1 is well related the rank of of $A[G]+I$ modulo 2. We describe the value $q(T_n, -1)$ below.

Theorem 4.2.2. *For* $n \geq 9$, $q(T_n, -1) = q(T_{n-6}, -1)$ *and for all* $n \geq 3$,

$$q(T_n, -1) = \begin{cases} -2 & \text{if } n \equiv 0, 2, 4 \pmod 6 \\ -1 & \text{if } n \equiv 1, 5 \pmod 6 \\ -4 & \text{if } n \equiv 3 \pmod 6 \end{cases} .$$

.

Proof. We first calculate $q(T_n, -1)$ for $3 \leq n \leq 14$.

$$\begin{gathered} q(T_3, -1) = -4, \quad q(T_4, -1) = -2, \quad q(T_5, -1) = -1, \\ q(T_6, -1) = -2, \quad q(T_7, -1) = -1, \quad q(T_8, -1) = -2, \\ q(T_9, -1) = -4, \quad q(T_{10}, -1) = -2, \quad q(T_{11}, -1) = -1, \\ q(T_{12}, -1) = -2, \quad q(T_{13}, -1) = -1, \quad q(T_{14}, -1) = -2. \end{gathered}$$

Thus $q(T_n, -1) = q(T_{n-6}, -1)$ is true for $n = 9$ to $n = 14$.

We apply mathematical induction on $n \geq 9$ and assume the induction hypothesis. Note that $q(P_n, -1) = q(P_{n-6}, -1)$ for $n \geq 6$ from Proposition 1.5.2. By the recursive formula given in Theorem 2.2.4,

$$\begin{aligned} q(T_n, -1) &= q(T_{n-1}, -1) + q(T_{n-3}, -1) - q(T_{n-4}, -1) - q(P_{n-4}, -1) \\ &= q(T_{n-7}, -1) + q(T_{n-9}, -1) - q(T_{n-10}, -1) - q(P_{n-10}, -1) \\ &= q(T_{n-6}, -1). \end{aligned}$$

The result holds for the first six values: $q(T_n, -1)$ for $n = 3, 4, 5, 6, 7, 8$. Thus it holds for all

$n \geq 3$. □

Theorem 1.4.3 states that $q(T_n, -1) = (-1)^n(-2)^{n-r_n}$, where $r_n = \text{rank}(A[T_n] + I_n)$ modulo 2. We use this formula to calculate the rank r_n.

Theorem 4.2.3. *For $n \geq 3$, the rank r_n of $A[T_n] + I_n$ mod 2 is given by*

$$r_n = \begin{cases} n-1 & \text{if } n \equiv 0, 2, 4 \pmod 6 \\ n & \text{if } n \equiv 1, 5 \pmod 6 \\ n-2 & \text{if } n \equiv 3 \pmod 6 \end{cases} .$$

.

Proof. Refer to the values $q(T_n, -1)$ given in Theorem 4.2.2. If $n \equiv 0, 2, \text{ or } 4 \pmod 6$, n is even and $q(T_n, -1) = -2$. Then

$$q(T_n, -1) = -2 = (-1)^n(-2)^{n-r_n} = (-2)^{n-r_n} \implies n - r_n = 1 \implies r_n = n - 1.$$

Similarly, if $n \equiv 1 \text{ or } 5 \pmod 6$, n is odd and

$$q(T_n, -1) = -1 = (-1)^n(-2)^{n-r_n} = (-1)(-2)^{n-r_n} \implies n - r_n = 0 \implies r_n = n.$$

Lastly, if $n \equiv 3 \pmod 6$, n is odd and

$$q(T_n, -1) = -4 = (-1)(-2)^{n-r_n} = -(-2)^{n-r_n} \implies n - r_n = 2 \implies r_n = n - 2.$$

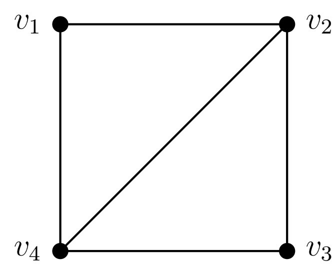
□

One can easily check that the ranks (mod 2) of the matrices given in Example 4.4.1 are: $\text{rank}(A[T_3] + I_3) = 1 = 3 - 2$, $\text{rank}(A[T_4] + I_4) = 3 = 4 - 1$, $\text{rank}(A[T_5] + I_5) = 5$, and $\text{rank}(A[T_6] + I_6) = 5 = 6 - 1$ (all $\pmod 2$). It confirms the result of Theorem 4.2.3.

4.3 An Explicit Formula for $q(T_n, x)$

Definition 4.3.1. *Consider any graph G. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S. Let $m(G[S])$ and $r(G[S])$ denote the nullity and rank of the adjacency matrix $A[G[S]]$ respectively. Also, $m(G[\emptyset]) = 0$.*

Example 4.3.2. *The graph of T_4 is below, while $A[T_4]$ is given in Example 4.1.1.*



T_4

The subgraphs of T_4 may have one vertex, two vertices, three vertices, T_4, and the null graph $\emptyset$.

Singleton subgraphs are P_0: $\quad \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\},$

The subgraphs with two vertices are P_1 or E_2: $\quad v_1v_2, \{v_1\} \cup \{v_3\}, v_1v_4, v_2v_3, v_2v_4, v_3v_4,$

The subgraphs with three vertices are P_2 or C_3: $\quad v_1v_2v_3, v_1v_4v_3, v_1v_2v_4v_1, v_2v_3v_4v_2.$

By Definition 4.3.1, for each singleton subgraph $\{v_i\}$, $m(A[\{v_i\}]) = 1$ and $r(m(A[\{v_i\}]) = 0$. For subgraphs of two vertices, $r(A[P_1]) = 2$ and $m(A[P_1]) = 0$, $r(A[E_2]) = 0$ and $m(A[E_2]) = 2$. For subgraphs of 3 vertices, $r(A[P_2]) = 2$ and $m(A[P_2]) = 1$; $r(A[C_3]) = 2$, $m(A[C_3]) = 1$ Lastly, $r(A[T_4]) = 2$, $m(A[T_4]) = 2$.

Theorem 4.3.3. *[1] Let G be a simple graph. then*

$$q(G, x) = \sum_{S \subseteq V(G)} (x-1)^{m(G[S])}.$$

Example 4.3.4. *Using the formula from Theorem 4.3.3 the interlace polynomial $q(T_4, x)$ can be described explicitly as*

$$\begin{aligned} q(T_4, x) &= \sum_{T \subseteq V(T_4)} (x-1)^{m(T_4[S])} \\ &= 6(x-1)^0 + 8(x-1) + 2(x-1)^2 \\ &= 2x^2 + 4x. \end{aligned}$$

From Example 4.3.2, there are 6 sugbraphs of T_4 having nullity 0 for the adjacency matrix: five P_1 graphs of two vertices and the null graph. Thus the coefficient for the $(x-1)^0$-term is 6. There are 8 subgraphs whose adjacency matrices have nullity 1: the 4 subgraphs of 3 vertices two P_2 graphs and two C_3 graphs and the four singleton subgraphs. It gives 8 for the coefficient of $(x-1)^1$.

It is straightforward to check that any maximum independent set of a graph G also admits the maximum nullity of adjacency matrices among all the subgraphs of G. It implies the following: Recall that $\alpha(G)$ is the independence number of G.

Lemma 4.3.5. *For any simple graph G, $\deg(q(G, x)) = \alpha(G)$ and the leading coefficient of $q(T_n, x)$ is the number of maximum independent sets of T_n.*

By applying our previous results about the polynomial $q(T_n, x)$, we obtain the following results related to the independence subsets of T_n. It shows a connection between the interlace polynomial and its underlying graph.

Theorem 4.3.6. *Assume $n \geq 6$.*

1. *When n is even, T_n has exactly one maximum independent subset and the independence number is $\alpha(T_n) = \frac{n}{2}$.*

2. *When n is odd, there are $(n+1)/2$ maximum independent subsets of $V(T_n)$ with the independence number $\alpha(T_n) = \frac{n-1}{2}$.*

3. The value of $q(T_n, 1)$ is the number of subgraphs of T_n whose adjacency matrices are of full rank (mod 2).

The following example confirms the above theorem.

Example 4.3.7. *Refer to the graphs T_7 and T_8 shown in Example 1.6.2. The 4 maximum independent subsets of T_7 are*

$$\{v_1, v_3, v_5\}, \{v_1, v_3, v_6\}, \{v_1, v_4, v_6\}, \{v_2, v_4, v_6\}.$$

Refer to Lemma 2.2.2, $\deg(q(T_7, x)) = 3 = \frac{7-1}{2} = \alpha(T_4)$ *and the leading coefficient of* $q(T_7, x)$ *is* $\frac{7+1}{2} = 4$.

Obviously, the graph T_8 has one maximum independent set, $\{v_1, v_3, v_5, v_7\}$, of size 4. So $\alpha(T_8) = 4$. *Lemma 2.2.2 shows* $\deg(q(T_8, x)) = 4\frac{8}{2}$ *and the leading coefficient of $q(T_7, x)$ is 1.*

Corollary 4.3.8. *Assume $n \geq 10$. If n is even, then T_n has exactly $\frac{n^2+6n}{8}$ subsets of $V(T_n)$ with nullity $\frac{n}{2} - 1$. If n is odd, T_n has exactly $\frac{n^3+15n^2-n+33}{48}$ subsets of $V(T_n)$ with nullity $\frac{n-1}{2} - 1$.*

Proof. We first write $q(T_n, x)$ in terms of $(x-1)$ by setting $x = (x-1) + 1$:

$$\begin{aligned} q(T_n, x) &= a_{n,k_n} x^{k_n} + a_{n,k_n-1} x^{k_n-1} + \text{lower terms} \\ &= a_{n,k_n}(x-1)^{k_n} + (k_n a_{n,k_n} + a_{n,k_n-1})(x-1)^{k_n-1} + \text{lower terms in } (x-1), \end{aligned}$$

where $k_n = \lfloor \frac{n}{2} \rfloor$. The number of the independent sets with the second largest size $(\alpha(T_n) - 1)$ is the second leading coefficient of $q(T_n, x)$ in terms of $(x-1)$, that is, the number $k_n a_{n,k_n} + a_{n,k_n-1}$. Then by Proposition 2.2.6 and Proposition 2.2.7(1), when n is even,

$$k_n a_{n,k_n} + a_{n,k_n-1} = \frac{n}{2} \cdot 1 + \frac{n^2 + 2n}{2} = \frac{n^2 + 6n}{8}.$$

When n is odd,

$$k_n a_{n,k_n} + a_{n,k_n-1} = \frac{n-1}{2} \cdot \frac{n+1}{2} + \frac{n^3+3n^2-n+45}{48} = \frac{n^3+15n^2-n+33}{48}.$$

$\square$

4.4 Related Matrices for W_n

A few matrices $A[W_n] + I_n$, for $n = 4, 5, 6, 7$ are shown below.

Example 4.4.1.

$$A[W_4] + I_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \qquad A[W_5] + I_5 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$A[T_6] + I_6 = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad A[W_7] + I_7 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The general form for $n \geq 7$ is given below:

$$A[W_n] + I_n = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 1 & 1 \\ 1 & 1 & 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 1 & 1 & \ddots & 0 & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 & 1 & 1 \\ 1 & 0 & \cdots & 0 & 1 & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 1 & 1 \end{bmatrix}$$

We next calculate the value of $q(W_n, x)$ at $x = -1$.

Theorem 4.4.2. *Consider the graph W_n for $n \geq 3$.*

$$q(W_n, -1) = \begin{cases} -2 & \textit{if } n \equiv 0, 2 \pmod 6 \\ -1 & \textit{if } n \equiv 1, 3, 5 \pmod 6 \\ -8 & \textit{if } n \equiv 4 \pmod 6 \end{cases}.$$

.

Proof. We first calculate $q(W_n, -1)$ for $4 \leq n \leq 15$.

$$q(W_4, -1) = -8, \quad q(W_5, -1) = -1, \quad q(W_6, -1) = -2,$$
$$q(W_7, -1) = -4, \quad q(W_8, -1) = -2, \quad q(W_9, -1) = -1,$$
$$q(W_{10}, -1) = -8, \quad q(W_{11}, -1) = -1, \quad q(W_{12}, -1) = -2,$$
$$q(W_{13}, -1) = -4, \quad q(W_{14}, -1) = -2, \quad q(W_{15}, -1) = -1.$$

Thus $q(W_n, -1) = q(W_{n-6}, -1)$ is true for $n = 10$ to $n = 15$.

We apply mathematical induction on $n \geq 10$ and assume the induction hypothesis. Note that $q(P_n, -1) = q(P_{n-6}, -1)$ (Theorem 1.5.2 (7)) for $n \geq 6$ and $q(T_n, -1) = q(T_{n-6}, -1)$ for $n \geq 9$ (Theorem 4.2.2). By the recursive formula given in Theorem 3.2.3,

$$\begin{aligned}
q(W_n, -1) &= q(T_{n-1}, -1) + q(W_{n-4}, -1) + 2q(T_{n-4}, -1) - q(P_{n-5}, -1) - 3q(T_{n-5}, -1) \\
&= q(T_{n-7}, -1) + q(W_{n-10}, -1) + 2q(T_{n-10}, -1) - q(P_{n-11}, -1) - 3q(T_{n-11}, -1) \\
&= q(W_{n-6}, -1).
\end{aligned}$$

The result holds for the first six values: $q(W_n, -1)$ for $n = 4, 5, 6, 7, 8, 9$. Thus it holds for all $n \geq 10$. $\square$

From Theorem 1.4.3, $q(W_n, -1) = (-1)^n(-2)^{n-r(W_n)}$. Where $r(W_n)$ is the rank of $A[W_n] + I_n$ over $\mathbb{Z}_2$. We use $q(W_n, -1)$ from Theorem 4.4 to calculate $r(W_n)$.

Theorem 4.4.3. *For $n \geq 3$, the rank $r_n = r(W_n)$ of $A[W_n] + I_n \pmod 2$ is given by*

$$r_n = \begin{cases} n-1 & \text{if } n \equiv 0, 2 \pmod 6 \\ n & \text{if } n \equiv 1, 3, 5 \pmod 6 \\ n-3 & \text{if } n \equiv 4 \pmod 6 \end{cases}.$$

.

Proof. Refer to the values $q(T_n, -1)$ given in the proof of Theorem 4.4. If $n \equiv 0$ or $2 \pmod 6$, n is even and $q(W_n, -1) = -2$. Then

$$q(W_n, -1) = -2 = (-1)^n(-2)^{n-r_n} = (-2)^{n-r_n} \implies n - r_n = 1 \implies r_n = n - 1.$$

Similarly, if $n \equiv 1, 3$ or $5 \pmod 6$, n is odd and

$$q(W_n, -1) = -1 = (-1)^n(-2)^{n-r_n} = (-1)(-2)^{n-r_n} \implies n - r_n = 0 \implies r_n = n.$$

Lastly, if $n \equiv 4 \pmod 6$, n is even and

$$q(T_n, -1) = -8 = (-1)^n(-2)^{n-r_n} = (-2)^{n-r_n} \implies n - r_n = 3 \implies r_n = n - 3.$$

$\square$

Chapter 5

Appendix

Python software was used to generate the formulas provided here.

5.1 Interlace Polynomials of P_n for $0 \leq n \leq 22$

$$q(P_0, x) = x$$

$$q(P_1, x) = 2x$$

$$q(P_2, x) = x^2 + 2x$$

$$q(P_3, x) = 3x^2 + 2x$$

$$q(P_4, x) = x^3 + 5x^2 + 2x$$

$$q(P_5, x) = 4x^3 + 7x^2 + 2x$$

$$q(P_6, x) = x^4 + 9x^3 + 9x^2 + 2x$$

$$q(P_7, x) = 5x^4 + 16x^3 + 11x^2 + 2x$$

$$q(P_8, x) = x^5 + 14x^4 + 25x^3 + 13x^2 + 2x$$

$$q(P_9, x) = 6x^5 + 30x^4 + 36x^3 + 15x^2 + 2x$$

$$q(P_{10}, x) = x^6 + 20x^5 + 55x^4 + 49x^3 + 17x^2 + 2x$$

$$q(P_{11}, x) = 7x^6 + 50x^5 + 91x^4 + 64x^3 + 19x^2 + 2x$$

$$q(P_{12}, x) = x^7 + 27x^6 + 105x^5 + 140x^4 + 81x^3 + 21x^2 + 2x$$

5.2 Interlace Polynomials of C_n for $3 \leq n \leq 25$

$$
\begin{aligned}
q(C_3, x) &= 4x \\
q(C_4, x) &= 3x^2 + 2x \\
q(C_5, x) &= 5x^2 + 6x \\
q(C_6, x) &= 2x^3 + 10x^2 + 4x \\
q(C_7, x) &= 7x^3 + 14x^2 + 8x \\
q(C_8, x) &= 2x^4 + 16x^3 + 21x^2 + 6x \\
q(C_9, x) &= 9x^4 + 30x^3 + 27x^2 + 10x \\
q(C_{10}, x) &= 2x^5 + 25x^4 + 50x^3 + 36x^2 + 8x \\
q(C_{11}, x) &= 11x^5 + 55x^4 + 77x^3 + 44x^2 + 12x \\
q(C_{12}, x) &= 2x^6 + 36x^5 + 105x^4 + 112x^3 + 55x^2 + 10x \\
q(C_{13}, x) &= 13x^6 + 91x^5 + 182x^4 + 156x^3 + 65x^2 + 14x \\
q(C_{14}, x) &= 2x^7 + 49x^6 + 196x^5 + 294x^4 + 210x^3 + 78x^2 + 12x \\
q(C_{15}, x) &= 15x^7 + 140x^6 + 378x^5 + 450x^4 + 275x^3 + 90x^2 + 16x \\
q(C_{16}, x) &= 2x^8 + 64x^7 + 336x^6 + 672x^5 + 660x^4 + 352 * x^3 + 105x^2 + 14x \\
q(C_{17}, x) &= 17x^8 + 204x^7 + 714x^6 + 1122x^5 + 935x^4 + 442x^3 + 119x^2 + 18x \\
q(C_{18}, x) &= 2x^9 + 81x^8 + 540x^7 + 1386x^6 + 1782x^5 + 1287x^4 + 546x^3 + 136x^2 + 16x \\
q(C_{19}, x) &= 19x^9 + 285x^8 + 1254x^7 + 2508x^6 + 2717x^5 + 1729x^4 + 665x^3 + 152x^2 \\
&\quad + 20x \\
q(C_{20}, x) &= 2x^{10} + 100x^9 + 825x^8 + 2640x^7 + 4290x^6 + 4004x^5 + 2275x^4 + 800x^3 \\
&\quad + 171x^2 + 18x \\
q(C_{21}, x) &= 21x^{10} + 385x^9 + 2079x^8 + 5148x^7 + 7007x^6 + 5733x^5 + 2940x^4 + 952x^3 \\
&\quad + 189x^2 + 22x
\end{aligned}
$$

5.3 Interlace Polynomials of T_n for $3 \leq n \leq 20$

$$q(T_3, x) = 4x$$

$$q(T_4, x) = 2x^2 + 4x$$

$$q(T_5, x) = 5x^2 + 6x$$

$$q(T_6, x) = x^3 + 9x^2 + 10x$$

$$q(T_7, x) = 4x^3 + 17x^2 + 14x$$

$$q(T_8, x) = x^4 + 11x^3 + 28x^2 + 20x$$

$$q(T_9, x) = 5x^4 + 24x^3 + 45x^2 + 30x$$

$$q(T_{10}, x) = x^5 + 15x^4 + 46x^3 + 74x^2 + 44x$$

$$q(T_{11}, x) = 6x^5 + 36x^4 + 85x^3 + 118x^2 + 64x$$

$$q(T_{12}, x) = x^6 + 21x^5 + 77x^4 + 150x^3 + 185x^2 + 94x$$

$$q(T_{13}, x) = 7x^6 + 57x^5 + 152x^4 + 256x^3 + 291x^2 + 138x$$

$$q(T_{14}, x) = x^7 + 28x^6 + 133x^5 + 283x^4 + 432x^3 + 455x^2 + 202x$$

$$q(T_{15}, x) = 8x^7 + 85x^6 + 281x^5 + 509x^4 + 719x^3 + 706x^2 + 296x$$

$$q(T_{16}, x) = x^8 + 36x^7 + 218x^6 + 555x^5 + 892x^4 + 1181x^3 + 1093x^2 + 434x$$

$$q(T_{17}, x) = 9x^8 + 121x^7 + 499x^6 + 1044x^5 + 1531x^4 + 1927x^3 + 1688x^2 + 636x$$

$$\begin{aligned} q(T_{18}, x) &= x^9 + 45x^8 + 339x^7 + 1053x^6 + 1893x^5 + 2593x^4 + 3126x^3 + 2598x^2 \\ &\quad + 932x \end{aligned}$$

$$\begin{aligned} q(T_{19}, x) &= 10x^9 + 166x^8 + 838x^7 + 2092x^6 + 3342x^5 + 4348x^4 + 5040x^3 \\ &\quad + 3989x^2 + 1366x \end{aligned}$$

$$\begin{aligned} q(T_{20}, x) &= x^{10} + 55x^9 + 505x^8 + 1891x^7 + 3971x^6 + 5784x^5 + 7229x^4 + 8089x^3 \\ &\quad + 6113x^2 + 2002x \end{aligned}$$

5.4 Interlace Polynomials of $D_{r,s}$ $(3 \leq r \leq 10, 1 \leq s \leq 3)$

$$q(D_{3,1},x) = 2x^2 + 4x$$

$$q(D_{4,1},x) = x^3 + 5x^2 + 2x$$

$$q(D_{5,1},x) = 3x^3 + 7x^2 + 6x$$

$$q(D_{6,1},x) = x^4 + 7x^3 + 12x^2 + 4x$$

$$q(D_{7,1},x) = 4x^4 + 14x^3 + 16x^2 + 8x$$

$$q(D_{8,1},x) = x^5 + 11x^4 + 25x^3 + 23x^2 + 6x$$

$$q(D_{9,1},x) = 5x^5 + 25x^4 + 41x^3 + 29x^2 + 10x$$

$$q(D_{10,1},x) = x^6 + 16x^5 + 50x^4 + 63x^3 + 38x^2 + 8x$$

$$q(D_{3,2},x) = 6x^2 + 4x$$

$$q(D_{4,2},x) = 4x^3 + 7x^2 + 2x$$

$$q(D_{5,2},x) = 8x^3 + 13x^2 + 6x$$

$$q(D_{6,2},x) = 3x^4 + 17x^3 + 16x^2 + 4x$$

$$q(D_{7,2},x) = 11x^4 + 28x^3 + 24x^2 + 8x$$

$$q(D_{8,2},x) = 3x^5 + 27x^4 + 46x^3 + 29x^2 + 6x$$

$$q(D_{9,2},x) = 14x^5 + 55x^4 + 68x^3 + 39x^2 + 10x$$

$$q(D_{10,2},x) = 3x^6 + 41x^5 + 100x^4 + 99x^3 + 46x^2 + 8x$$

$$q(D_{3,3},x) = 2x^3 + 10x^2 + 4x$$

$$q(D_{4,3},x) = x^4 + 9x^3 + 9x^2 + 2x$$

$$q(D_{5,3},x) = 3x^4 + 15x^3 + 19x^2 + 6x$$

$$q(D_{6,3},x) = x^5 + 10x^4 + 29x^3 + 20x^2 + 4x$$

$$q(D_{7,3},x) = 4x^5 + 25x^4 + 44x^3 + 32x^2 + 8x$$

5.5 Interlace Polynomials of $D_{r,s}$ $(3 \leq r \leq 10, 3 \leq s \leq 5)$

$$q(D_{8,3}, x) = x^6 + 14x^5 + 52x^4 + 69x^3 + 35x^2 + 6x$$

$$q(D_{9,3}, x) = 5x^6 + 39x^5 + 96x^4 + 97x^3 + 49x^2 + 10x$$

$$q(D_{10,3}, x) = x^7 + 19x^6 + 91x^5 + 163x^4 + 137x^3 + 54x^2 + 8x$$

$$q(D_{3,4}, x) = 8x^3 + 14x^2 + 4x$$

$$q(D_{4,4}, x) = 5x^4 + 16x^3 + 11x^2 + 2x$$

$$q(D_{5,4}, x) = 11x^4 + 28x^3 + 25x^2 + 6x$$

$$q(D_{6,4}, x) = 4x^5 + 27x^4 + 45x^3 + 24x^2 + 4x$$

$$q(D_{7,4}, x) = 15x^5 + 53x^4 + 68x^3 + 40x^2 + 8x$$

$$q(D_{8,4}, x) = 4x^6 + 41x^5 + 98x^4 + 98x^3 + 41x^2 + 6x$$

$$q(D_{9,4}, x) = 19x^6 + 94x^5 + 164x^4 + 136x^3 + 59x^2 + 10x$$

$$q(D_{10,4}, x) = 4x^7 + 60x^6 + 191x^5 + 262x^4 + 183x^3 + 62x^2 + 8x$$

$$q(D_{3,5}, x) = 2x^4 + 18x^3 + 18x^2 + 4x$$

$$q(D_{4,5}, x) = x^5 + 14x^4 + 25x^3 + 13x^2 + 2x$$

$$q(D_{5,5}, x) = 3x^5 + 26x^4 + 47x^3 + 31x^2 + 6x$$

$$q(D_{6,5}, x) = x^6 + 14x^5 + 56x^4 + 65x^3 + 28x^2 + 4x$$

$$q(D_{7,5}, x) = 4x^6 + 40x^5 + 97x^4 + 100x^3 + 48x^2 + 8x$$

$$q(D_{8,5}, x) = x^7 + 18x^6 + 93x^5 + 167x^4 + 133x^3 + 47x^2 + 6x$$

$$q(D_{9,5}, x) = 5x^7 + 58x^6 + 190x^5 + 261x^4 + 185x^3 + 69x^2 + 10x$$

$$q(D_{10,5}, x) = x^8 + 23x^7 + 151x^6 + 354x^5 + 399x^4 + 237x^3 + 70x^2 + 8x$$

5.6 Explicit Formulas for $f_s(x)$ $(0 \leq s \leq 20)$

$$f_0(x) = 1$$

$$f_1(x) = 1$$

$$f_2(x) = x + 1$$

$$f_3(x) = 2x + 1$$

$$f_4(x) = x^2 + 3x + 1$$

$$f_5(x) = 3x^2 + 4x + 1$$

$$f_6(x) = x^3 + 6x^2 + 5x + 1$$

$$f_7(x) = 4x^3 + 10x^2 + 6x + 1$$

$$f_8(x) = x^4 + 10x^3 + 15x^2 + 7x + 1$$

$$f_9(x) = 5x^4 + 20x^3 + 21x^2 + 8x + 1$$

$$f_{10}(x) = x^5 + 15x^4 + 35x^3 + 28x^2 + 9x + 1$$

$$f_{11}(x) = 6x^5 + 35x^4 + 56x^3 + 36x^2 + 10x + 1$$

$$f_{12}(x) = x^6 + 21x^5 + 70x^4 + 84x^3 + 45x^2 + 11x + 1$$

$$f_{13}(x) = 7x^6 + 56x^5 + 126x^4 + 120x^3 + 55x^2 + 12x + 1$$

$$f_{14}(x) = x^7 + 28x^6 + 126x^5 + 210x^4 + 165x^3 + 66x^2 + 13x + 1$$

$$f_{15}(x) = 8x^7 + 84x^6 + 252x^5 + 330x^4 + 220x^3 + 78x^2 + 14x + 1$$

$$f_{16}(x) = x^8 + 36x^7 + 210x^6 + 462x^5 + 495x^4 + 286x^3 + 91x^2 + 15x + 1$$

$$f_{17}(x) = 9x^8 + 120x^7 + 462x^6 + 792x^5 + 715x^4 + 364x^3 + 105x^2 + 16x + 1$$

$$f_{18}(x) = x^9 + 45x^8 + 330x^7 + 924x^6 + 1287x^5 + 1001x^4 + 455x^3 + 120x^2 + 17x + 1$$

$$f_{19}(x) = 10x^9 + 165x^8 + 792x^7 + 1716x^6 + 2002x^5 + 1365x^4 + 560x^3 + 136x^2 + 18x + 1$$

$$f_{20}(x) = x^{10} + 55x^9 + 495x^8 + 1716x^7 + 3003x^6 + 3003x^5 + 1820x^4 + 680x^3 + 153x^2 + 19x + 1$$

5.7 Interlace Polynomials of W_n for $4 \leq n \leq 21$

$$
\begin{aligned}
q(W_4, x) &= 8x \\
q(W_5, x) &= x^3 + 4x^2 + 4x \\
q(W_6, x) &= 10x^2 + 12x \\
q(W_7, x) &= 2x^3 + 18x^2 + 20x \\
q(W_8, x) &= 7x^3 + 35x^2 + 30x \\
q(W_9, x) &= 2x^4 + 23x^3 + 56x^2 + 36x \\
q(W_{10}, x) &= 9x^4 + 48x^3 + 93x^2 + 62x \\
q(W_{11}, x) &= 2x^5 + 27x^4 + 92x^3 + 158x^2 + 92x \\
q(W_{12}, x) &= 11x^5 + 66x^4 + 176x^3 + 253x^2 + 134x \\
q(W_{13}, x) &= 2x^6 + 38x^5 + 147x^4 + 318x^3 + 393x^2 + 190x \\
q(W_{14}, x) &= 13x^6 + 104x^5 + 299x^4 + 546x^3 + 624x^2 + 288x \\
q(W_{15}, x) &= 2x^7 + 51x^6 + 247x^5 + 569x^4 + 933x^3 + 983x^2 + 422x \\
q(W_{16}, x) &= 15x^7 + 155x^6 + 533x^5 + 1048x^4 + 1568x^3 + 1523x^2 + 618x \\
q(W_{17}, x) &= 2x^8 + 66x^7 + 402x^6 + 1078x^5 + 1874x^4 + 2587x^3 + 2352x^2 + 900x \\
q(W_{18}, x) &= 17x^8 + 221x^7 + 935x^6 + 2074x^5 + 3264x^4 + 4233x^3 + 3638x^2 + 1328x \\
q(W_{19}, x) &= 2x^9 + 83x^8 + 623x^7 + 2009x^6 + 3836x^5 + 5597x^4 + 6887x^3 + 5601x^2 + 1946x \\
q(W_{20}, x) &= 19x^9 + 304x^8 + 1558x^7 + 4066x^6 + 6897x^5 + 9481x^4 + 11115x^3 + 8588x^2 \\
&\quad + 2852x \\
q(W_{21}, x) &= 2x^{1}0 + 102x^9 + 927x^8 + 3567x^7 + 7861x^6 + 12132x^5 + 15877x^4 + 17838x^3 \\
&\quad + 13145x^2 + 4174x
\end{aligned}
$$

5.8 Explicit Form of $q(T_n, x)$ in Terms of $x - 1$ for $6 \leq n \leq 16$

$$
\begin{aligned}
q(T_6, x) &= (x-1)^3 + 12(x-1)^2 + 31(x-1) + 20(x-1)^0 \\
q(T_7, x) &= 4(x-1)^3 + 29(x-1)^2 + 60(x-1) + 35(x-1)^0 \\
q(T_8, x) &= (x-1)^4 + 15(x-1)^3 + 67(x-1)^2 + 113(x-1) + 60(x-1)^0 \\
q(T_9, x) &= 5(x-1)^4 + 44(x-1)^3 + 147(x-1)^2 + 212(x-1) + 104(x-1)^0 \\
q(T_{10}, x) &= (x-1)^5 + 20(x-1)^4 + 116(x-1)^3 + 312(x-1)^2 + 395(x-1) + 180(x-1)^0 \\
q(T_{11}, x) &= 6(x-1)^5 + 66(x-1)^4 + 289(x-1)^3 + 649(x-1)^2 + 729(x-1) + 309(x-1)^0 \\
q(T_{12}, x) &= (x-1)^6 + 27(x-1)^5 + 197(x-1)^4 + 688(x-1)^3 + 1322(x-1)^2 + 1333(x-1) \\
&\quad + 528(x-1)^0 \\
q(T_{13}, x) &= 7(x-1)^6 + 99(x-1)^5 + 542(x-1)^4 + 1574(x-1)^3 + 2626(x-1)^2 + 2423(x-1) \\
&\quad + 901(x-1)^0 \\
q(T_{14}, x) &= (x-1)^7 + 35(x-1)^6 + 322(x-1)^5 + 1403(x-1)^4 + 3489(x-1)^3 + 5220(x-1)^2 \\
&\quad + 4380(x-1) + 1534(x-1)^0 \\
q(T_{15}, x) &= 8(x-1)^7 + 141(x-1)^6 + 959(x-1)^5 + 3469(x-1)^4 + 7545(x-1)^3 + 10170(x-1)^2 \\
&\quad + 7872(x-1) + 2604(x-1)^0 \\
q(T_{16}, x) &= (x-1)^8 + 44(x-1)^7 + 498(x-1)^6 + 2675(x-1)^5 + 8267(x-1)^4 + 15975(x-1)^3 \\
&\quad + 19592(x-1)^2 + 14074(x-1) + 4410(x-1)^{\cdot}
\end{aligned}
$$

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