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Polynomial Solutions to the Diophantine Equation $x^2 + y^3 = 6912z^2$

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Polynomial Solutions to the Diophantine Equation $x^2 + y^3 = 6912z^2$

by

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POLYNOMIAL SOLUTIONS TO THE DIOPHANTINE EQUATION: $X^2 + Y^3 = 6912Z^2$

MASTER'S THESIS

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Abstract

In this paper, I investigate polynomial solutions to the Diophantine equation, $X^2 + Y^3 = 6912Z^2$, where $X = g(x, y)$, $Y = h(x, y)$ and $Z = f(x, y)$ are polynomials with integer coefficients. The focus is on the greatest common divisors for the integer values of these polynomials when the polynomials $f(x, y)$, $g(x, y)$ and $h(x, y)$ are relatively prime in $\mathbb{Q}[x, y]$. However, for a fixed integer pair x_0, y_0, the integer values $f(x_0, y_0)$, $g(x_0, y_0)$ and $h(x_0, y_0)$ are not necessarily relatively prime in $\mathbb{Z}$. I investigate the greatest common divisors (GCDs) of these three polynomial values for specific integer pairs x_0 and y_0. First, I study the cases where $y_0 = 1$ and $y_0 = 2$. For these cases, a complete distribution of the GCDs is given. Furthermore, I use the Euclidean Algorithm and Gröbner Basis techniques to determine the GCDs for $f(x_0, y_0)$, $g(x_0, y_0)$ and $h(x_0, y_0)$ in $\mathbb{Z}$ by obtaining multiples of the GCDs of the polynomials. Then, the results from the cases $y_0 = 1$ and $y_0 = 2$ are generalized to obtain similar properties of the GCDs for all possible integer values of x and y. For the cases where the integer values are not relatively prime, the possible prime divisors of the GCDs and integer bounds for the powers of prime divisors are determined. Finally, polynomial solutions to new Diophantine equations are derived from the original Diophantine equation.

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Contents

2 Introduction to Diophantine Equations

In this paper, I investigate a certain Diophantine equation and a set of polynomials that satisfy this equation. Let me first define a Diophantine equation.

Definition 2.1 *A Diophantine equation is a polynomial equation with integer coefficients to which the only concerned solutions are integers.*

There are different types of Diophantine equations, often of the form $Ax^p + By^q = Cz^r$, where A, B, C are non-zero integers. Recent research including [6,7,8] focus on Diophantine equations of this form. Some of the famous ones of this type include Fermat's Equation, $x^n + y^n = z^n$; the equation $x^2 + y^2 = z^2$, whose solutions are Pythagorean Triples; and, Pell's Equation $x^2 - ny^2 = \pm 1$. Mathematicians who have worked on Diophantine equations have focused on obtaining the number of solutions to such equations. In [6], Beukers showed that there are at least 25 integer solution triples to $X^5 + Y^3 = Z^2$. In [8], Kraus focuses on relatively prime solution triples to the Diophantine equation $X^p + Y^q = Z^r$. In particular, he investigates solutions to $X(p,q,r) > 0$, $X(p,q,r) = 0$ and $X(p,q,r) < 0$, where $X(p,q,r) = p^{-1} + q^{-1} + r^{-1} - 1$. In [8], Darmon and Granville investigate integer solutions to the equation $z^m = F(x,y)$ and the Diophantine equation $Ax^p + By^q = Cz^r$, where F is a homogeneous polynomial in $\mathbb{Z}[x,y]$ and A, B, C are non-zero integers. They propose that in certain cases, these equations have finitely many solutions such that $gcd(x,y,z) = 1$.

The particular Diophantine equation I am interested in is

$$X^2 + Y^3 = 6912Z^2, \tag{1}$$

whose coefficients and power triple were obtained by Cihan Karabulut and Aihua Li in [4]. The procedure to find a set of polynomial solutions in $\mathbb{Z}[x,y]$ to this equation is explained in their paper. In [4], Karabulut and Li showed that if (X, Y, Z) is a polynomial solution triple of the equation $X^p + Y^m = CZ^q$, where C, p, m, q are nonzero integers and $p, m, q > 1$, then the degree of the polynomial Z is either $3, 4, 6$ or 12.

Algorithm 2.2 *This algorithm (from [4]) describes a procedure to find the polynomials that are relatively prime in $\mathbb{Q}[x,y]$ that satisfy a Diophantine equation $X^p + Y^m = CZ^q$, where $X = g(x,y)$, $Y = h(x,y)$, and $Z = f(x,y)$.*

1. *Choose a positive integer $n = 3, 4, 6$ or 12 for the total degree of polynomial $f(x,y) = a_n x^n + a_{n-1}x^{n-1}y + \cdots\cdot + a_1 xy^{n-1} + a_0 y^n$ in $\mathbb{Z}[x,y]$, where the a_i's are to be determined.*

2. *Use the Hessian determinant of $f(x,y)$ to construct $h(x,y)$ as follows:*

$$h(x,y) = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix}.$$

3. Construct $g(x,y)$ using the Jacobian determinant of $f(x,y)$ and $h(x,y)$:

$$g(x,y) = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{vmatrix}.$$

4. Choose $a_0, a_1, ..., a_n$ such that $[g(x,y)]^p + [h(x,y)]^m = C[f(x,y)]^q$ is satisfied.

The process to find a set of polynomials that solve the Diophantine equation of interest is demonstrated in the next example.

Example 2.3 *Algorithm 2.2 is applied to determine a set of relatively prime polynomial solutions over $\mathbb{Q}$ that satisfy*

$$X^2 + Y^3 = 6912Z^2.$$

1. *Let $f(x,y) = a_3x^3 + a_2x^2y + a_1xy^2 + a_0y^3$ of a total degree of 3.*

2. *Construct $h(x,y)$ using the Hessian determinant of $f(x,y)$:*

$$\begin{aligned} h(x,y) = &\begin{vmatrix} 6a_3x + 2a_2y & 2a_2x + 2a_1y \\ 2a_2x + 2a_1y & 2a_1x + 6a_0y \end{vmatrix} \\ =&12a_0a_2y^2 - 4a_1^2y^2 - 4a_1a_2xy + 36a_0a_3xy + 12a_1a_3x^2 - 4a_2^2x^2 \\ =&(12a_1a_3 - 4a_2^2)x^2 + (36a_0a_3 - 4a_1a_2)xy + (12a_0a_2 - 4a_1^2)y^2. \end{aligned}$$

3. *Construct $g(x,y)$ by using the Jacobian of $f(x,y)$ and $h(x,y)$:*

$$\begin{aligned} &g(x,y) = \\ &\begin{vmatrix} 3a_3x^2 + 2a_2xy + a_1y^2 & a_2x^2 + 2a_1xy + 3a_0y^2 \\ (-4a_2a_1 + 36a_3a_0)y + (24a_3a_1 - 8a_2^2)x & (-8a_1^2 + 24a_2a_0)y + (-4a_2a_1 + 36a_3a_0)x \end{vmatrix} \\ =&(3a_3x^2 + 2a_2xy + a_1y^2)[(-8a_1^2 + 24a_2a_0)y + (-4a_2a_1 + 36a_3a_0)] \\ &- (a_2x^2 + 2a_1xy + 3a_0y^2)[(-4a_2a_1 + 36a_3a_0)y + (24a_3a_1 - 8a_2^2)x] \\ =&(-36a_3a_1a_2 + 108a_3^2a_0 + 8a_2^3)x^3 \\ &+ (108a_3a_0a_2 - 72a_3a_1^2 + 12a_2^2a_1)yx^2 \\ &+ (72a_2^2a_0 - 12a_2a_1^2 - 108a_1a_0a_3)y^2x \\ &+ (36a_1a_0a_2 - 8a_1^3 - 108a_0^2a_3)y^3. \end{aligned}$$

4. *Let $a_3 = 1, a_2 = -1, a_1 = 1, a_0 = -1$. The Diophantine equation has infinitely many solutions depending on the values of a_i's. After examining different values of a_0, a_1, a_2, a_3, the values chosen above lead to polynomials that have useful properties in investigating the integer values of the polynomials, which are discussed in Remark 2.4.*

With $a_3 = 1$, $a_2 = -1$, $a_1 = 1$, $a_0 = -1$, one set of polynomial solutions that satisfy $X^2 + Y^3 = 6912Z^2$ is:

$$\begin{aligned} g(x,y) &= -80x^3 + 48x^2y + 48xy^2 - 80y^3 \\ h(x,y) &= 8x^2 - 32xy + 8y^2 \\ f(x,y) &= x^3 - x^2y + xy^2 - y^3, \end{aligned} \tag{2}$$

where $(X, Y, Z) = (g, h, f)$, which are polynomials in $\mathbb{Z}[x, y]$. These polynomials $f(x, y)$, $g(x, y)$ and $h(x, y)$ are relatively prime in $\mathbb{Q}[x, y]$. However, they are not necessarily relatively prime as integers for a fixed pair of integers x, y. My goal is to investigate the greatest common divisors of the integer values of these polynomials $f(x, y)$, $g(x, y)$ and $h(x, y)$ in $\mathbb{Z}$.

Remark 2.4 *Some of the useful properties of the polynomials are shown below.*

$$\begin{aligned} &g(y,x) = -80x^3 + 48x^2y + 48xy^2 - 80y^3 = -16(x+y)(x^2 - 5xy + y^2) \\ &h(y,x) = 8x^2 - 32xy + 8y^2 = 8(x^2 - 4xy + y^2) \\ &f(y,x) = -x^3 + x^2y - xy^2 + y^3 = (x-y)(x^2+y^2). \end{aligned}$$

Note that $g(x, y) = g(y, x)$, $h(x, y) = h(y, x)$ and $f(x, y) = -f(y, x)$. Then, for fixed integer values of x and y,

$$GCD(f(x,y), g(x,y), h(x,y)) = GCD(f(y,x), g(y,x), h(y,x)),$$

since the negative sign does not affect the greatest common divisors of integers.

3 The Greatest Common Divisor: $D(x, y)$

Since the focus of the paper is on the greatest common divisors of the integer values of f, g, h, the next definition introduces notation for the GCDs of the integer values of $f(x, y)$, $g(x, y)$ and $h(x, y)$.

Definition 3.1 *(The Greatest Common Divisor)*

1. *For any integers a, b, $GCD(a, b)$ is the greatest common divisor of a and b.*
2. *For $x, y \in \mathbb{Z}$, let $D(x, y)$ be the greatest common divisor of $f(x, y), g(x, y)$ and $h(x, y)$ in $\mathbb{Z}$.*

Example 3.2 *For $x = 4$ and $y = 2$,*

$$D(4,2) = GCD(f(4,2), g(4,2), h(4,2)) = GCD(40, -3454, -96) = 8.$$

Similarly, by Remark 2.4 $D(2, 4) = 8$.

A special divisibility notation is introduced in the next definition, which can be found in [2].

Definition 3.3 *Let p be prime and $n \in \mathbb{Z}^+$. We say p^a exactly divides n, if p^a divides n (denoted by $p^a \mid n$), but p^{a+1} does not, denoted as $p^a \| n$.*

Example 3.4 *The integer 8 exactly divides 40; that is, $8 \| 40$, since 8 divides 40 but 16 does not divide 40.*

All parts of the following lemma will be used extensively throughout the paper, which are well-known elementary number theory results.

Lemma 3.5 *Let a, b, c be integers.*

1. *If p is a prime number and $p \mid ab$, then $p \mid a$ or $p \mid b$.*
2. *If $c \mid a$ and $c \mid b$, then $c \mid sa + tb$ for all $s, t \in \mathbb{Z}$.*
3. *If $d = GCD(a, b)$, then $d = au + bv$ for some integers u and v.*
4. *If $ab \mid c$ and $GCD(a, c) = 1$, then $b \mid c$.*
5. *$\forall\ n \in \mathbb{Z}^+$, n can be written as $n = p_1^{a_1} p_2^{a_2} ... p_r^{a_r}$, where $a_i \geq 0$ and $p_1, ..., p_r$ are distinct primes.*

In particular, the interest of this paper is to determine the possible prime divisors of $D(x, y)$ for fixed values of x, y as integers.

4 Research Goals

Since the goal of the paper is to investigate the greatest common divisors of the integer values of $f(x, y)$, $g(x, y)$ and $h(x, y)$ in $\mathbb{Z}$, the focus is on the specifics of these values. Below are some questions I will answer in this paper regarding the greatest common divisor of $f(x, y)$, $g(x, y)$, $h(x, y)$ in $\mathbb{Z}$.

Recall $D(x, y) = GCD(f(x, y), g(x, y), h(x, y))$ as integers. Let $x, y \in \mathbb{Z}$.

1. For what values of x and y, does $D(x, y) = 1$?
2. What is $D(x, y)$ when x and y are identical, i.e. if $x = y$?
3. What are the possible prime divisors for $D(x, y)$?
4. What is the distribution of the prime divisors of $D(x, y)$?
5. Can the solution triple (f, g, h) be used to find solutions for other similar Diophantine equations?

5 Construction of Multiples of $D(x, y)$

It is well known that the greatest common divisor of a finite set of integers can be written as a linear combination of these integers. And, all prime divisors of the GCD will divide any such combination. There are different methods to obtain appropriate combinations. In this paper, we apply Gröbner basis techniques and the Euclidean Algorithm to construct multiples of $D(x, y)$. These multiples will provide information on the possible prime divisors of $D(x, y)$.

Gröbner bases have been used to find greatest common divisors and solve systems of equations because any set of polynomials can be transformed into a useful set that form a Gröbner basis. Since my goal is to investigate the greatest common divisors of the polynomials $f(x, y)$, $g(x, y)$, $h(x, y)$ in $\mathbb{Z}$, I use a set that forms a Gröbner basis of the ideal generated by f, g, h in $\mathbb{Q}[x, y]$ to investigate the GCD. To transform a set of polynomials into a Gröbner basis, one must choose a term order.

Let $\mathbb{F}$ be a field and $R = \mathbb{F}[x_1, \ldots, x_n]$. A term order (or monomial order), denoted as $>_\sigma$, on the set of monomials $\{\, \mathbf{x}^\alpha \mid \alpha \in \mathbb{Z}^n_{\geq 0} \,\}$ of R, is a total and well ordering such that $\mathbf{x}^\alpha >_\sigma \mathbf{x}^\beta$ implies $\mathbf{x}^{\alpha+\gamma} >_\sigma \mathbf{x}^{\beta+\gamma}$ for all $\alpha, \beta, \gamma \in \mathbb{Z}^n_{\geq 0}$ [5,10].

To transform the polynomials of interest into a Gröbner basis, the term ordering used is the following:

Definition 5.1 ***Lexicographic Term Order*** *Let $\alpha = (\alpha_1, ..., \alpha_n)$ and $\beta = (\beta_1, ..., \beta_n)$ be in $\mathbb{Z}^n_{\geq 0}$. We say that $\alpha >_{lex} \beta$ if, in the vector difference $\alpha - \beta \in \mathbb{Z}^n_{\geq 0}$, the left-most non-zero entry is positive. We write $\mathbf{x}^\alpha >_{lex} \mathbf{x}^\beta$; that is, $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} >_{lex} x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$, if $\alpha >_{lex} \beta$ [5,10].*

Example 5.2 *Let $>_{lex}$ be the lexicographic term order defined as above, where $x_1 >_{lex} x_2 >_{lex} x_3$.*

1. $x_1^5 x_2^2 x_3^3 >_{lex} x_1 x_2^5 x_3^5$, *since* $\alpha = (5, 2, 3)$, $\beta = (1, 5, 5)$ *and* $\alpha - \beta = (4, -3, -2)$.

2. $x_1 x_2^2 x_3^3 >_{lex} x_1 x_2 x_3$, *since* $\alpha = (1, 2, 3)$, $\beta = (1, 1, 1)$ *and* $\alpha - \beta = (0, 1, 2)$.

Definition 5.3 ***Gröbner Basis*** *Let I be an ideal of R and $>_\sigma$ be a term order on the monomials of R. Let $G = \{g_1, \cdots, g_s\}$ be a generating set of I. We say G is a Gröbner basis of I if the ideal generated by all of the leading terms of elements in I is also generated by the leading terms of $g_1, \cdots, g_s$.*

If a set of polynomials can be transformed into a Gröbner basis, then every element in the Gröbner basis can be written as a combination of these polynomials. Consider f, g, h as before:

$$\begin{aligned} g(x, y) &= -80x^3 + 48x^2y + 48xy^2 - 80y^3 \\ h(x, y) &= 8x^2 - 32xy + 8y^2 \\ f(x, y) &= x^3 - x^2y + xy^2 - y^3. \end{aligned}$$

Let I be an ideal generated by f, g, h; that is, $I =< g, h, f >$. Then, the Gröbner basis, G, of I with lexicographic term order, $x >_{lex} y$, is

$$G = \{y^3, xy^2, (x-y)^2 - 2xy\},$$

which was computed in *Maple*.

Then, every element in G can be written as a combination of the polynomials $f(x,y)$, $g(x,y)$, $h(x,y)$ as follows, where the computation is done in *Maple*:

$$\begin{aligned} y^3 &= (-\frac{1}{128})g(x,y) + (-\frac{1}{32}y)h(x,y) + (-\frac{5}{8})f(x,y) \\ xy^2 &= (-\frac{1}{384})g(x,y) + (-\frac{1}{96}x - \frac{1}{24}y)h(x,y) + (-\frac{1}{8})f(x,y) \\ (x-y)^2 - 2xy &= 0 \cdot g(x,y) + \frac{1}{8}h(x,y) + 0 \cdot f(x,y). \end{aligned}$$

When the denominators are eliminated, so that all polynomials are in $\mathbb{Z}[x,y]$, the following combinations are obtained in the matrix below:

$$\begin{bmatrix} \mathbf{-1} & \mathbf{-4y} & \mathbf{-80} \\ -1 & -4x-16y & -48 \\ 0 & 48 & 0 \end{bmatrix} \begin{bmatrix} g(x,y) \\ h(x,y) \\ f(x,y) \end{bmatrix} = \begin{bmatrix} \mathbf{2^7 \cdot y^3} \\ 2^7 \cdot 3 \cdot xy^2 \\ (x-y)^2 - 2xy \end{bmatrix},$$

where the bolded entries will be investigated in Theorem 5.4.

Then, for x and y in $\mathbb{Z}$, since $D(x,y)$ divides each of $f(x,y)$, $g(x,y)$, $h(x,y)$, it will also divide all the entries in the matrix on the right side of the equation; that is, $D(x,y) \mid 128y^3$, $D(x,y) \mid 384xy^2$, and $D(x,y) \mid (x-y)^2 - 2xy$.

In general, for any combination of the polynomials $f(x,y)$, $g(x,y)$, $h(x,y)$, such as

$$f(x,y)s(x,y) + g(x,y)t(x,y) + h(x,y)u(x,y) = w(x,y),$$

where $s, t, u, w \in \mathbb{Z}[x,y]$, for integer values of x and y, then $D(x,y) \mid w(x,y)$. In other words, $w(x,y)$ is a multiple of $D(x,y)$ in $\mathbb{Z}$. Any combination of the polynomials $f(x,y)$, $g(x,y)$, $h(x,y)$ will be multiples of $D(x,y)$, which will lead to the possible prime divisors of $D(x,y)$.

Several combinations of $f(x,y)$, $g(x,y)$, $h(x,y)$ are obtained by using the Euclidean Algorithm, computed by *Maple*, in order to determine more multiples of $D(x,y)$. These combinations are represented in the matrix equation below:

$$\begin{bmatrix} -x^2 - xy - 2y^2 & 0 & -80x^2 - 112xy - 96y^2 \\ \mathbf{0} & \mathbf{-3x^2 + 2xy - 3y^2} & \mathbf{24x - 88y} \\ 5x - 19y & 50x^2 - 20xy - 46y^2 & 0 \end{bmatrix} \begin{bmatrix} g(x,y) \\ h(x,y) \\ f(x,y) \end{bmatrix} = \begin{bmatrix} 2^6 \cdot 3^2 \cdot y^5 \\ \mathbf{2^6 \cdot y^4} \\ 2^7 \cdot 3^2 \cdot y^4 \end{bmatrix},$$

where the entries in bold will be investigated in the theorem below. All the combinations obtained by the Gröbner basis or the Euclidean Algorithm yield to the following theorem.

Theorem 5.4 *For any $x, y \in \mathbb{Z}$,*

$$D(x,y) \mid 64 \cdot GCD(x^3, y^3).$$

Proof. Consider the following combinations, which were in bold in the previous matrix equations.

$$0 \cdot g(x,y) + (-3x^2 + 2xy - 3y^2)f(x,y) + (24x - 88y)h(x,y) = 64y^4$$
$$(-1)g(x,y) + (-4y)h(x,y) + (-80)f(x,y) = 128y^3.$$

Since $D(x,y) \mid 64y^4$ and $D(x,y) \mid 128y^3$, then $D(x,y) \mid 64y^3$. By Remark 2.4, $D(x,y) \mid 64x^3$. Therefore,

$$D(x,y) \mid 64 \cdot GCD(x^3, y^3).$$

Then, by the theorem above, the only prime divisors of $D(x,y)$ are 2 or those prime divisors that divide both x and y.

6 Special Cases

The first case I investigate is $y = 1$, where I observe the different values of the greatest common divisors of the three polynomials as integers. For different values of x, there is a pattern for $D(x,1)$, where the only possible prime divisor is 2. Another interesting fact about $D(x,1)$ is that the values have a period of 8. Also, when x is even, the three polynomials are relatively prime, which means $D(x,1) = 1$. (See Table 1 in the Appendix for the distribution of $D(x,1)$.) These observations lead to the following theorem.

Theorem 6.1 *For $y = 1$ and $x \in \mathbb{Z}$,*

$$D(x,1) = \begin{cases} 1 & \text{if } x \equiv 0,2,4,6 \pmod 8 \\ 2^2 & \text{if } x \equiv 3 \text{ or } 7 \pmod 8 \\ 2^3 & \text{if } x \equiv 5 \pmod 8 \\ 2^4 & \text{if } x \equiv 1 \pmod 8. \end{cases}$$

Proof. Consider the two cases where x is even or odd.

1. If x is even, $x \equiv 0,2,4,6 \pmod 8$, then $f(x,1) \equiv 1 \pmod 2$. Therefore, $2 \nmid f(x,2)$ and $D(x,1) = 1$.

2. For $x = 8k + r$, where $r = 1,3,5,7$,

$$f(8k+r) = (r - 1 + 8k)(64k^2 + 16kr + 1 + r^2). \tag{3}$$

For $r = 7$, $4 \| f(8k+7,1)$. Also, 4 divides both $g(8k+7,1)$ and $h(8k+7,1)$. Therefore, $D(x,1) = 4$.

For $r = 3$, $4 \| f(8k+3,1)$. Furthermore, $4 \mid g(8k+3,1)$ and $4 \mid h(8k+3,1)$. Then, $D(x,1) = 4$.

For $r = 5$, $8 \| f(8k+5,1)$. In addition, 8 divides both $g(8k+5,1)$ and $h(8k+5,1)$. Thus, $D(x,1) = 8$.

Lastly, for $r = 1$, $16 \| h(8k+1,1)$. And, 16 divides $g(8k+1,1)$ and $h(8k+1,1)$. Therefore, $D(x,1) = 16$.

If $y = 2$, then the greatest common divisor of the three polynomials as integers exhibit the same pattern as the case where $y = 1$. The period of $D(x,2)$ is 8 and the only prime divisor of $D(x,2)$ is 2. However, in this case, the polynomials are relatively prime when x is odd. (See Table 2 in the Appendix for the details on the distribution of $D(x,2)$.) The following theorem reveals all possible values of $D(x,2)$.

Theorem 6.2 *For $y = 2$ and $x \in \mathbb{Z}$,*

$$D(x,2) = \begin{cases} 1 & \textit{if } x \equiv 1,3,5,7 \pmod 8 \\ 2^3 & \textit{if } x \equiv 0 \textit{ or } 4 \pmod 8 \\ 2^5 & \textit{if } x \equiv 6 \pmod 8 \\ 2^6 & \textit{if } x \equiv 2 \pmod 8. \end{cases}$$

Proof. There are two cases to consider.

1. If x is odd, $x \equiv 1,3,5,7 \pmod 8$, then $f(x,2) \equiv x^3 \equiv 1 \pmod 2$. Therefore, $2 \nmid f(x,2)$ and $D(x,2) = 1$.

2. If x is even, then $x = 8k + r$ for some integer k and $r = 0,2,4,6$. Then,

$$f(8k+r,2) = (r - 2 + 8k)(64k^2 + 16kr + r^2 + 4) \tag{4}$$

For $r = 6$, $32 \| f(8k+6,2)$ and 32 divides both $g(8k+6,2)$ and $h(8k+6,2)$. Therefore, $D(x,2) = 32$.

For $r = 4$, $8 \| f(8k+4,2)$. Since $8 \mid g(8k+4,2)$ and $8 \mid h(8k+4,2)$, $D(x,2) = 8$.

For $r = 0$, $8 \| f(8k,2)$. Then, $D(x,2) = 8$, since 8 divides both $g(8k,2)$ and $h(8k,2)$. Therefore, $D(x,2) = 8$.

Lastly, for $r = 2$, $64 \| h(8k+2,2)$. Also, $64 \mid f(8k+2,2)$ and $64 \mid h(8k+2,2)$. Therefore, $D(x,2) = 64$.

In the next section, the distribution of $D(x,y)$ is determined for any integer values of x and y.

7 General Cases and Main Results for $D(x, y)$

The special cases that I have investigated and other observations that were made about the GCD of the integer values of the polynomials lead to the following theorem that generalize to results regarding $D(x, y)$ for any x and y in $\mathbb{Z}$.

Theorem 7.1 *Let x, y, a be in $\mathbb{Z}$ and recall polynomials f, g, h:*

$$\begin{aligned} g(x,y) &= -80x^3 + 48x^2y + 48xy^2 - 80y^3 \\ h(x,y) &= 8x^2 - 32xy + 8y^2 \\ f(x,y) &= x^3 - x^2y + xy^2 - y^3, \end{aligned}$$

where $D(x, y) = GCD(f(x, y), g(x, y), h(x, y))$ for x, y in $\mathbb{Z}$.

1. *If $GCD(x, y) = 1$ and $x + y$ is odd, then $D(x, y) = 1$.*
2. *For $x = y$, $D(x, y) = h(x, y)$.*
3. *If $a = GCD(x, y)$, then $a^2 \mid D(x, y)$.*
4. *If $a = GCD(x, y)$ with $x = am$, $y = an$, and $m+n \equiv 1 (\mod 2)$, $D(x, y) = a^2 d$, where $d = GCD(a, h(m, n))$.*

Proof. Let x and y be any integers in $\mathbb{Z}$ and $f(x, y)$, $g(x, y)$ and $h(x, y)$ defined as in the previous sections.

1. Factor $f(x, y)$ as $f(x, y) = (x - y)(x^2 + y^2)$. If $x + y$ is odd, then $x - y$ and $x^2 + y^2$ are odd. Therefore, $f(x, y) \equiv 1(\mod 2)$. Then, $2 \nmid D(x, y)$.

2. If $x = y$, then

$$f(x, y) = 0, \quad g(x, y) = -64x^3 \quad \text{and} \quad h(x, y) = 16x^2.$$

   Thus,

$$D(x, y) = 16x^2 = h(x, y), \quad \text{for} \quad x = y.$$

3. If $GCD(x, y) = a$, then $x = ma$ and $y = na$ for some integers m, n, which are relatively prime. Then,

$$\begin{aligned} f(x, y) &= f(ma, na) = a^3(m - n)(m^2 + n^2) \\ g(x, y) &= g(ma, na) = -16a^3(n + m)(5n^2 - 8mn + 5m^2) \\ h(x, y) &= h(ma, na) = 8a^2(n^2 - 4mn + m^2). \end{aligned}$$

   Therefore, a^2 divides all of $f(x, y)$, $g(x, y)$ and $h(x, y)$ for all $x, y \in \mathbb{Z}$. Thus, $a^2 \mid D(x, y)$.

4. For $GCD(a, h(m,n)) = d$, $d \mid a$ and $d \mid h(m,n)$. Thus, $d \mid af(m,n)$ and $d \mid ag(m,n)$. Since $d \mid GCD(af(m,n), ag(m,n), h(m,n))$ and $a^2 \mid D(x,y)$,

$$a^2 d \mid D(x,y).$$

Now, let $a = q_1 d$ and $h = q_2 d$, where $q_1, q_2 \in \mathbb{Z}$ and $GCD(q_1, q_2) = 1$. Then, d divides each of $q_1 df(m,n)$, $q_1 dg(m,n)$, $q_2 d$. Thus,

$$D(x,y) = a^2 d \cdot GCD(q_1 f(m,n), q_1 g(m,n), q_2).$$

Assume there exists $d_1 > 1$ such that

$$d_1 \mid GCD(q_1 f(m,n), q_1 g(m,n), q_2).$$

Since $d_1 \mid q_2$ and $GCD(q_1, q_2) = 1$, $GCD(d_1, q_1) = 1$. Then, d_1 divides both $f(m,n)$ and $g(m,n)$. Furthermore, $d_1 \mid q_2 d = h(m,n)$. Therefore,

$$d_1 = GCD(f(m,n), g(m,n), h(m,n)) = D(m,n) = 1,$$

which is a contradiction, since $d_1 > 1$. Then,

$$D(x,y) = a^2 d \cdot GCD(af(m,n), ag(m,n), h(m,n) = a^2 d \cdot 1 = a^2 d.$$

In the next section, I investigate all possible prime divisors of $D(x,y)$ and determine the lowest and highest power of each prime divisor that divides $D(x,y)$.

8 Behavior of Prime Divisors of $D(x,y)$

Any positive integer can be written as a product of its prime divisors. Then, for fixed values of x and y, the greatest common divisor of integer values of $f(x,y)$, $g(x,y)$, $h(x,y)$ can also be written as a product of its prime divisors. Each of the prime divisors of $D(x,y)$ can be investigated individually in order to obtain integer bounds on the powers of these prime divisors of $D(x,y)$.

Theorem 8.1 *For fixed integer values of x and y, let*

$$D(x,y) = 2^{e_0} q_1^{e_1} \cdots q_s^{e_s},$$

where the q_i's are distinct odd primes and $e_i \geq 0$. Then, the following cases give the possible integer values of e_0.

1. *Case I: If $x + y$ is odd, $2 \nmid D(x,y)$. Then, $e_0 = 0$.*

2. *Case II: Assume x, y are both even such that $2^{r_1} \| x$, $2^{r_2} \| y$. Without loss of generality, let $r_1 \leq r_2$.*

   - *For $0 < r_1 < r_2$, $e_0 = min\{3r_1, 2r_1 + 3\}$.*

- *For* $0 < r_1 = r_2$, $e_0 \geq min\{3r_1 + 2, 2r_1 + 4\}$.

3. *Case III: If* x, y *are both odd, such that* $2^{r_1} \| (x-1)$, $2^{r_2} \| (y-1)$ *with* $r_1 \leq r_2$, *then*

   - *For* $0 < r_1 < r_2$, $e_0 = min\{r_1 + 1, 4\}$.
   - *For* $0 < r_1 = r_2$, $e_0 \geq min\{r_1 + 2, 4\}$.

Proof.

1. If $x + y$ is odd, then either one of x or y is even and the other one is odd. Let x be odd and y be even with

$$x = 2k + 1 \quad \text{and} \quad y = 2l,$$

for k, l in $\mathbb{Z}$. Then,

$$f(x,y) = f(2k+1, 2l) = (2k + 1 - 2l)(4k^2 + 2k + 1 + 2l).$$

Since, $f(x, y) \equiv 1 \pmod 2$, $2 \nmid D(x, y)$. Then $D(x, y)$ does not have 2 as a prime divisor. Therefore, $e_0 = 0$.

Similarly, if x is even and y is odd, by Remark 2.4, $f(x, y) \equiv 1 \pmod 2$. Therefore, $e_0 = 0$.

2. The second case is where both x and y are even, write both x and y so that

$$x = 2^{r_1} k \quad \text{and} \quad y = 2^{r_2} l,$$

where $k, l \in \mathbb{Z}$ and $2 \nmid k, l$. Then

$$g(2^{r_1}k, 2^{r_2}l) = -16(2^{r_1}k + 2^{r_2}l)(5k^2 2^{2r_1} - 8kl 2^{r_1+r_2} + 5l^2 2^{2r_2}) \tag{5}$$
$$f(2^{r_1}k, 2^{r_2}l) = (2^{r_1}k - 2^{r_2}l)(2^{2r_1}k^2 + 2^{2r_2}l^2) \tag{6}$$
$$h(2^{r_1}k, 2^{r_2}l) = 8((2^{r_1}k - 2^{r_2}l)^2 - 2^{r_1+r_2+1}kl). \tag{7}$$

Since $D(x, y) = D(y, x)$, it is sufficient to consider only the case where $r_1 \leq r_2$. For this case, the polynomials (5), (6), (7) lead to the following:

$$2^{3r_1+4} \| \ g(x,y)$$
$$2^{3r_1} \| \ f(x,y)$$
$$2^{2r_1+3} \| \ h(x,y).$$

Then,

$$e_0 = min\{3r_1, 2r_1 + 3\}.$$

For $r_1 = r_2$, using (5), (6), (7), the following holds:

$$\begin{aligned} 2^{3r_1+6} &\mid g(x,y) \\ 2^{3r_1+2} &\mid f(x,y) \\ 2^{2r_1+4} &\mid h(x,y). \end{aligned}$$

Unlike the other case, instead of an exact value, there is a lower bound for e_0 such that $e_0 \geq min\{3r_1 + 2, 2r_1 + 4\}$.

3. For the last case where both x and y are odd, x and y can be represented as follows:

$$x = 2^{r_1}k + 1 \quad \text{and} \quad y = 2^{r_2}l + 1,$$

where k, l in $\mathbb{Z}$. Then

$$g(2^{r_1}k + 1, 2^{r_2}l + 1) = -64(2^{r_1-1}k + 2^{r_2-1}l + 1)(2^{r_1-1}m_1 + 1) \tag{8}$$
$$f(2^{r_1}k + 1, 2^{r_2}l + 1) = 2^{r_1+1}(k - 2^{r_2-r_1}l)(2^{r_1}m_2 + 1) \tag{9}$$
$$h(2^{r_1}k + 1, 2^{r_2}l + 1) = -16(2^{r_1}m_3 - 1), \tag{10}$$

where $m_1, m_2, m_3 \in \mathbb{Z}$.

For $r_1 < r_2$, $e_0 = min\{r_1 + 1, 4\}$ since

$$\begin{aligned} 2^{6} &\mid g(x,y) \\ 2^{r_1+1} &\| f(x,y) \\ 2^{4} &\| h(x,y). \end{aligned}$$

For $r_1 = r_2$,

$$\begin{aligned} 2^{6} &\| g(x,y) \\ 2^{r_1+2} &\mid f(x,y) \\ 2^{4} &\| h(x,y). \end{aligned}$$

Then, similar to the other case, there is a lower bound for e_0, such that $e_0 \geq min\{r_1 + 2, 4\}$.

Results obtained in Theorem 8.1 can be restated using $64y^3$ and $64x^3$ as the maximum integer bounds on the powers of prime divisors by Remark 2.4. For the case where x and y are both even, $64x^3 = 2^{6+3r_1}k^3$ and for the case where both x and y are odd, $64x^3 = 2^6(2^{r_1}k + 1)$.

Corollary 8.2 *(Minimum and Maximum Values of e_0)*

1. *Case I: If $x + y$ is odd, $e_0 = 0$.*

2. *Case II: If x, y are both even, then*

$$\text{for } r_1 < r_2, \quad \begin{cases} \text{If } r_1 \leq 2, & e_0 = 3r_1 \\ \text{If } r_1 > 2, & e_0 = 2r_1 + 3, \end{cases}$$

$$\text{for } r_1 = r_2, \quad \begin{cases} \text{If } r_1 = 1, & 5 \leq e_0 \leq 9 \\ \text{If } r_1 > 1, & 2r_1 + 4 \leq e_0 \leq 3r_1 + 6. \end{cases}$$

3. *Case III: If x, y are both odd, then*

$$\text{for } r_1 < r_2, \quad \begin{cases} \text{If } r_1 \leq 2, & e_0 = r_1 + 1 \\ \text{If } r_1 > 2, & e_0 = 4, \end{cases}$$

$$\text{for } r_1 = r_2, \quad \begin{cases} \text{If } r_1 = 1, & 3 \leq e_0 \leq 6. \\ \text{If } r_1 > 1, & r_1 + 2 \leq e_0 \leq 6. \end{cases}$$

The following section gives the distribution of the powers of the odd prime divisors of $D(x, y)$.

9 Odd Prime Divisors of $D(x, y)$

Recall that $D(x, y)$ can be written as a product of its prime divisors as in the previous section:

$$D(x, y) = 2^{e_0} q_1^{e_1} \cdots q_s^{e_s}.$$

Theorem 9.1 *Let p be an odd prime divisor of $D(x, y)$, i.e. $p \in \{q_1, ..., q_s\}$, and $p^e \| D(x, y)$ with $e > 0$. Also, let $p^{r_1} \| x$ and $p^{r_2} \| y$, with $0 < r_1 \leq r_2$. Then,*

$$2r_1 \leq e \leq 3r_1.$$

Proof. Let $x = p^{r_1}k$ and $y = p^{r_2}l$, where $k, l \in \mathbb{Z}$ and $p \nmid k, l$. We then substitute these values of x and y into f, g, h and obtain the following factorizations:

$$\begin{aligned} f(x, y) &= f(p^{r_1}k, p^{r_2}l) = (p^{r_1}k - p^{r_2}l)(p^{2r_1}k^2 + p^{2r_2}l^2) \\ g(x, y) &= g(p^{r_1}k, p^{r_2}l) = -16(p^{r_1}k + p^{r_2}l)(5p^{2r_1}k^2 - 8p^{r_1+r_2}kl + 5p^{2r_2}l^2) \\ h(x, y) &= h(p^{r_1}k, p^{r_2}l) = 8(p^{2r_1}k^2 - 4p^{r_1+r_2}kl + p^{2r_2}l^2). \end{aligned}$$

Then for all $0 < r_1 \leq r_2$,

$$p^{3r_1} \mid f(x, y), \quad p^{3r_1} \mid g(x, y), \quad \text{and} \quad p^{2r_1} \mid h(x, y),$$

which implies $p^{2r_1} \mid D(x, y)$. Since $D(x, y) \mid 64x^3$ and $D(x, y) \mid 64y^3$, we get $3r_1$ as the upper bound on e. Then,

$$2r_1 \leq e \leq 3r_1.$$

Example 9.2 *Let $x = 2^5 \cdot 3^2$ and $y = 2^7 \cdot 3^4$. Then, $p = 3$, $r_1 = 2$ and $r_2 = 4$. Based on the theorem,*

$$2 \cdot 2 \leq e \leq 2 \cdot 3,$$

Then,

$$4 \leq e \leq 6,$$

which means $e = 4, 5, 6$. Then, $3^4 \| D(2^5 \cdot 3^2, 2^7 \cdot 3^4)$, $3^5 \| D(2^5 \cdot 3^2, 2^7 \cdot 3^4)$ or $3^6 \| D(2^5 \cdot 3^2, 2^7 \cdot 3^4)$. On the other hand,

$$f(2^5 \cdot 3^2, 2^7 \cdot 3^4) = -2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 1297$$
$$g(2^5 \cdot 3^2, 2^7 \cdot 3^4) = -2^{19} \cdot 3^6 \cdot 37 \cdot 6197$$
$$h(2^5 \cdot 3^2, 2^7 \cdot 3^4) = 2^{13} \cdot 3^4 \cdot 1153.$$

Then, $D(2^5 \cdot 3^2, 2^7 \cdot 3^4) = 2^{13} \cdot 3^4$, where $e = 4$. Then, $3^4 \| D(2^5 \cdot 3^2, 2^7 \cdot 3^4)$, which confirms our results.

10 New Diophantine Equations Derived from the Original Equation and Resulting Polynomial Solutions

Recall that polynomials $f(x,y), g(x,y), h(x,y)$ below satisfy the Diophantine equation

$$X^2 + Y^3 = 6912Z^2,$$

where

$$\begin{aligned} g(x,y) &= -80x^3 + 48x^2y + 48xy^2 - 80y^3 \\ h(x,y) &= 8x^2 - 32xy + 8y^2 \\ f(x,y) &= x^3 - x^2y + xy^2 - y^3. \end{aligned}$$

For the cases where $D(x,y) \neq 1$, new polynomials that are relatively prime in $\mathbb{Z}[x,y]$ can be obtained. For example, when $y = 1$ and $x = 8k+1$, by Theorem 6.1, $D(8k+1,1) = 16$. Thus,

$$Z_1 = \frac{f(8k+1,1)}{16} = 32k^3 + 8k^2 + k$$
$$X_1 = \frac{g(8k+1,1)}{16} = -2560k^3 - 768k^2 - 48k - 4$$
$$Y_1 = \frac{h(8k+1,1)}{16} = 32k^2 - 8k - 1,$$

which satisfy the following Diophantine equation:

$${X_1}^2 + 16{Y_1}^3 = 6912{Z_1}^2.$$

Given specific values of x and y, the next example shows new Diophantine equation and integer polynomial solutions derived from the original polynomials of interest and the Diophantine equation $X^2 + Y^3 = 6912Z^2$.

Example 10.1 *For $x = 8k + 1$ and $y = 1$, let $k = 1$. Thus, $x = 9$ and $D(9, 1) = 16$. The following are the integer values of the new polynomials:*

$$\begin{aligned} X_1 = g_1(9,1) &= \frac{g(9,1)}{16} = \frac{-54080}{16} = -3380 \\ Y_1 = h_1(9,1) &= \frac{h(9,1)}{16} = \frac{368}{16} = 23 \\ Z_1 = f_1(9,1) &= \frac{f(9,1)}{16} = \frac{656}{16} = 41, \end{aligned}$$

which are relatively prime in $\mathbb{Z}$ and are solutions to the new equation: ${X_1}^2 + 16{Y_1}^3 = 6912{Z_1}^2$.

The solutions to the original Diophantine equations result in solutions to new Diophantine equations derived from the original equation. For example, the triple $(g(9,1), h(9,1), f(9,1)) = (-54080, 368, 656)$ is a solution to the original Diophantine equation $X^2 + Y^3 = 6912Z^2$. When the triple $(-54080, 368, 656)$ is divided by their GCD; which is 16, the new triple is obtained: $(-3380, 23, 41)$, whose values are now relatively prime and a solution to the new Diophantine equation: ${X_1}^2 + 16{Y_1}^3 = 6912{Z_1}^2$. This is true for all the new equations shown below and their corresponding solutions.

1. For $x = 8k + 3$ we can write f, g, h in the following form:

$$\begin{aligned} Z_2 &= \frac{f(8k+3,1)}{4} = 128k^3 + 128k^2 + 44k + 5 \\ X_2 &= \frac{g(8k+3,1)}{4} = -10240k^3 - 10752k^2 - 3648k - 416 \\ Y_2 &= \frac{h(8k+3,1)}{4} = 128k^2 + 32k - 4, \end{aligned}$$

   which satisfy the following Diophantine equation:

$$ {X_2}^2 + 4{Y_2}^3 = 6912{Z_2}^2. $$

   Also, the case where $x = 8k + 7$ leads to the same Diophantine equation, where the polynomials are evaluated at $x = 8k + 7$ and $y = 1$, since by Theorem 6.1, $D(8k + 1, 3) = D(8k + 7, 1) = 4$.

2. The Diophantine equation

$$ {X_3}^2 + 8{Y_3}^3 = 6912{Z_3}^2 $$

   is obtained for $x = 8k + 5$, with the following polynomials:

$$\begin{aligned} Z_3 &= \frac{f(8k+5,1)}{8} = 64k^3 + 112k^2 + 66k + 13 \\ X_3 &= \frac{g(8k+5,1)}{8} = -5120k^3 - 9216k^2 - 5472k - 1080 \\ Y_3 &= \frac{h(8k+5,1)}{8} = 64k^2 + 48k + 6, \end{aligned}$$

since $D(8k+5,1) = 8$ by Theorem 6.1.

3. For $x = 8k+6$, with $D(8k+6,2) = 32$, we write the three polynomials in the following form:

$$Z_4 = \frac{f(8k+6,2)}{32} = 16k^3 + 32k^2 + 22k + 5$$
$$X_4 = \frac{g(8k+6,2)}{32} = -1280k^3 - 2688k^2 - 1824k - 416$$
$$Y_4 = \frac{h(8k+6,2)}{32} = 16k^2 + 8k - 2,$$

which satisfy the following equation:

$$X_4{}^2 + 32Y_4{}^3 = 6912Z_4{}^2.$$

4. For $x = 8k$ and $x = 8k+4$, with $D(8k,2) = D(8k+4,2) = 8$, the Diophantine equation becomes:

$$X_5{}^2 + 8Y_5{}^3 = 6912Z_5{}^2,$$

with the following polynomials where $x = 8k$:

$$Z_5 = \frac{f(8k,2)}{8} = 64k^3 - 16k^2 + 4k - 1$$
$$X_5 = \frac{g(8k,2)}{8} = -5120k^3 + 768k^2 + 192k - 80$$
$$Y_5 = \frac{h(8k,2)}{8} = 64k^2 - 64k + 4.$$

Same equation is derived when $x = 8k+4$:

$$Z_5 = \frac{f(8k+6,2)}{32} = 16k^3 + 32k^2 + 22k + 5$$
$$X_5 = \frac{g(8k+6,2)}{32} = -1280k^3 - 2688k^2 - 1824k - 416$$
$$Y_5 = \frac{h(8k+6,2)}{32} = 16k^2 + 8k - 2.$$

5. For $x = 8k+2$, the following polynomials are obtained:

$$Z_6 = \frac{f(8k+2,2)}{64} = 8k^3 + 4k^2 + k$$
$$X_6 = \frac{g(8k+2,2)}{64} = -640k^3 - 384k^2 - 48k - 8$$
$$Y_6 = \frac{h(8k+2,2)}{64} = 8k^2 - 4k - 1,$$

satisfying the Diophantine equation:

$$X_6{}^2 + 64Y_6{}^3 = 6912Z_6{}^2,$$

since $D(8k+2,2) = 64$ by Theorem 6.2.

11 Future Work

I would like to develop an explicit formula for $D(x, y)$ for all x and y in $\mathbb{Z}$. For any given values of x and y, is it possible to get the exact value of $D(x, y)$ by examining the values of x and y?

I would like to examine $D(x, y)$ when the triple $f(x, y)$, $g(x, y)$, $h(x, y)$ satisfy specific conditions.

If there is no explicit formula for $D(x, y)$, are there any other techniques that are applicable to improve the bounds on the powers of the prime divisors of $D(x, y)$?

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A Tables

Table 1: GCD for $y = 1$: $D(x, 1)$

x	$D(x,1)$	x	$D(x,1)$	x	$D(x,1)$	x	$D(x,1)$	x	$D(x,1)$
1	16	36	1	71	4	106	1	141	8
2	1	37	8	72	1	107	4	142	1
3	4	38	1	73	16	108	1	143	4
4	1	39	4	74	1	109	8	144	1
5	8	40	1	75	4	110	1	145	16
6	1	41	16	76	1	111	4	146	1
7	4	42	1	77	8	112	1	147	4
8	1	43	4	78	1	113	16	148	1
9	16	44	1	79	4	114	1	149	8
10	1	45	8	80	1	115	4	150	1
11	4	46	1	81	16	116	1	151	4
12	1	47	4	82	1	117	8	152	1
13	8	48	1	83	4	118	1	153	16
14	1	49	16	84	1	119	4	154	1
15	4	50	1	85	8	120	1	155	4
16	1	51	4	86	1	121	16	156	1
17	16	52	1	87	4	122	1	157	8
18	1	53	8	88	1	123	4	158	1
19	4	54	1	89	16	124	1	159	4
20	1	55	4	90	1	125	8	160	1
21	8	56	1	91	4	126	1	161	16
22	1	57	16	92	1	127	4	162	1
23	4	58	1	93	8	128	1	163	4
24	1	59	4	94	1	129	16	164	1
25	16	60	1	95	4	130	1	165	8
26	1	61	8	96	1	131	4	166	1
27	4	62	1	97	16	132	1	167	4
28	1	63	4	98	1	133	8	168	1
29	8	64	1	99	4	134	1	169	16
30	1	65	16	100	1	135	4	170	1
31	4	66	1	101	8	136	1	171	4
32	1	67	4	102	1	137	16	172	1
33	16	68	1	103	4	138	1	173	8
34	1	69	8	104	1	139	4	174	1
35	4	70	1	105	16	140	1	175	4

Table 2: GCD for $y = 2$: $D(x, 2)$

x	$D(x,2)$	x	$D(x,2)$	x	$D(x,2)$	x	$D(x,2)$	x	$D(x,2)$
2	64	37	1	72	8	107	1	142	32
3	1	38	32	73	1	108	8	143	1
4	8	39	1	74	64	109	1	144	8
5	1	40	8	75	1	110	32	145	1
6	32	41	1	76	8	111	1	146	64
7	1	42	64	77	1	112	8	147	1
8	8	43	1	78	32	113	1	148	8
9	1	44	8	79	1	114	64	149	1
10	64	45	1	80	8	115	1	150	32
11	1	46	32	81	1	116	8	151	1
12	8	47	1	82	64	117	1	152	8
13	1	48	8	83	1	118	32	153	1
14	32	49	1	84	8	119	1	154	64
15	1	50	64	85	1	120	8	155	1
16	8	51	1	86	32	121	1	156	8
17	1	52	8	87	1	122	64	157	1
18	64	53	1	88	8	123	1	158	32
19	1	54	32	89	1	124	8	159	1
20	8	55	1	90	64	125	1	160	8
21	1	56	8	91	1	126	32	161	1
22	32	57	1	92	8	127	1	162	64
23	1	58	64	93	1	128	8	163	1
24	8	59	1	94	32	129	1	164	8
25	1	60	8	95	1	130	64	165	1
26	64	61	1	96	8	131	1	166	32
27	1	62	32	97	1	132	8	167	1
28	8	63	1	98	64	133	1	168	8
29	1	64	8	99	1	134	32	169	1
30	32	65	1	100	8	135	1	170	64
31	1	66	64	101	1	136	8	171	1
32	8	67	1	102	32	137	1	172	8
33	1	68	8	103	1	138	64	173	1
34	64	69	1	104	8	139	1	174	32
35	1	70	32	105	1	140	8	175	1