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Proper Connection of Bipartite Circulant Graphs

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Proper Connection of Bipartite Circulant Graphs

by

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Abstract

The study of proper edge-colorings of graphs has been a popular topic in graph theory since the work of Vizing. While the proper edge-colorings of entire graphs was the topic of interest when the subject began decades ago, more recent works have focused on the study of properly colored subgraphs, as opposed to proper colorings of graphs as a whole. The types of properly colored subgraphs that we will be most concerned with are paths. The topic of finding certain types of properly colored paths within larger edge-colored graphs, though seemingly specific in nature, has been a topic of much interest lately. The study began with the work of Chartrand et al., with what are called rainbow paths, i.e., paths in which no two edges are of the same color. From the study of finding properly colored subgraphs within a graph G, Chartrand et al. created a new edge-coloring problem, namely the rainbow connection problem, by adding a connectivity requirement which involves finding rainbow paths between any two vertices of G. Rainbow connection is very well-studied, and we will survey some of the more well-known results. Most of these are concerned with a particular parameter of a graph G, called the rainbow connection number, which is defined as the smallest number of edge colors needed so that between any two vertices of a graph there exists a rainbow path. The main original result of this thesis is concerned with a very natural extension of the rainbow connection number, called the proper connection number. In particular, we will look at the proper connection number of a type of bipartite graph, called a circulant graph. As well as being a result in itself, this yields progress on a conjecture of Borozan et al. We will also provide ideas for future work.

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This thesis is dedicated to my master's advisor, Dr. Jonathan Cutler, for all of the time and knowledge he has given me throughout the past two years, to James J. Alexander, for being the first to open my eyes to the beauty of mathematics, and to my parents, Herminia and Dean, who have always supported and loved me unconditionally.

Melissa Marie Fuentes

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Contents

List of Figures

Chapter 1

The edge coloring problem

The edge coloring problem in graph theory consists of coloring all the edges of a graph G with the minimum number of colors possible so that no two edges incident to the same vertex share a color. The notion of properly edge coloring a graph has become an important problem ever since the work of Vizing, who provided an upper bound for the number of colors needed in order to properly edge color any graph, which we will show later in this section. First, to better understand edge colorings, let us consider an application to a real-world problem.

Suppose we have a computer network consisting of several computers connected to some of the others through communication lines. We can think of the computers as vertices and the lines as edges. Say that each computer could only communicate with at most one other computer in a time unit. We would have to create a time schedule such that every computer communicates to all of its neighbors, but for efficiency, we would want to minimize the total necessary time units. We can think of creating such a schedule as an edge coloring problem by letting each color represent a time unit at which a corresponding communication line is used. Then the total number of colors used is the total number of necessary time units.

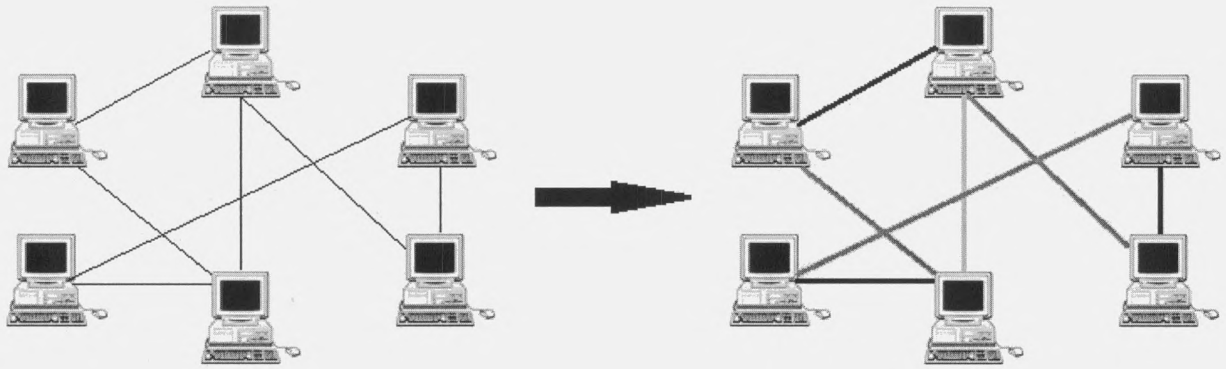

Figure 1.1: A computer network which requires 3 time units.

There are various ways in which we may view an edge coloring. Since we will be working with finite graphs, our edge colorings will use a finite number of colors to color the edges of a graph G. There may be various ways to edge color G using k colors, so we may think of a specific edge coloring as an assignment of colors, or *labels*, to the edges of G. Throughout this thesis we will use numbers as our labellings, although pictorially we will display edge colorings using actual colors.

1 Proper edge colorings and the chromatic index

Consider our edge coloring problem from before. Such an edge coloring which satisfies this problem is called *proper*. For instance, in the example of the computer network we could let the total number of time units be the total number of communication lines, so as to guarantee that each computer would communicate with at most one other computer at a time. This is the equivalence of edge coloring a graph G using $|E(G)|$ colors, thus distinctly coloring each edge, guaranteeing a proper edge coloring. However, these are not always the most time-efficient schedules, nor the minimum number of colors that may be used to properly edge color a graph. We now provide the following formal definitions.

Definition 1.1. A *k-edge coloring* of a graph G is a labelling $c : E(G) \to C$, where the set of distinct labels is $C = \{1, 2, \ldots, k\} = [k]$, and the labels are called *colors.* A k-edge coloring is *proper* if no two edges incident to the same vertex share a common color. The *chromatic index* of G is the least k needed to make a k-edge coloring of G proper, and is denoted $\chi'(G)$.

Informally, we will say that a vertex v "sees x colors" to mean that the edges incident to v are colored using x distinct colors. Since edges incident to a common vertex require different colors in a proper edge coloring of a graph G, then each $v \in V(G)$ must see $d(v)$ colors. However, since $\Delta(G) := \max\{d(v) : v \in V(G)\}$, then $\chi'(G) \geq \Delta(G)$. Theorem 1 below provides a large upper bound for the chromatic index.

Theorem 1.1. *For any graph G, $\Delta(G) \leq \chi'(G) \leq 2\Delta(G) - 1$.*

Proof. In order for an edge coloring of G to be proper, the edges incident to one vertex must all be differently colored, and hence $\Delta(G) \leq \chi'(G)$. We can find an upper bound by considering the adjacency of edges, i.e., edges that are incident to the same vertex. At each of its endpoints, an edge e is adjacent to at most $\Delta(G) - 1$ other edges. Thus by coloring e and the set of its adjacent edges, it suffices to edge color G using $1 + 2(\Delta(G) - 1) = 2\Delta(G) - 1$ colors. Hence, $\chi'(G) \leq 2\Delta(G) - 1$. □

Theorem 1 provides too large of an upper bound. In fact, Vizing and Gupta [1,2] independently proved that at most $\Delta(G) + 1$ colors are necessary to properly color a graph G. We will not provide the proof of this result, although it is far from trivial. The proof assumes that a subgraph $G' = G \setminus e$, where $e \in E(G)$, of a graph G is properly edge colored using $\Delta(G) + 1$ colors, and shows that G can be colored using the same number of colors by cleverly shifting colors between edges.

Theorem 1.2 (Vizing (1964), Gupta (1966)). *If G is a simple graph, then* $\chi'(G) \leq \Delta(G) + 1$.

Given the lower bound $\Delta(G) \leq \chi'(G)$ and the upper bound from Theorem 2, the chromatic index of a graph can only take on one of the two possible values. Thus, we have the following corollary.

Corollary 1.3. *If G is a simple graph, then either $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$.*

2 Chromatic index of different classes of graphs

We conclude Section 1 by providing three results for three classes of graphs. The first result provides the chromatic index of the cycle on n vertices, C_n, the second is on the complete graph on n vertices, K_n, and the third is on the graph whose vertex set can be partitioned into two independent sets A and B, the bipartite graph $G = A \cup B$.

Corollary 1.4. *If n is even, then $\chi'(C_n) = 2$, and if n is odd, then $\chi'(C_n) = 3$.*

Proof. Let $C_n = v_1v_2, v_2v_3, \ldots, v_nv_1$, and denote $c(v_iv_j)$ as the color of an edge v_iv_j. Assume n is even. If we alternately color the edges of C_n using two colors, 1 and 2, then $c(v_1v_2) = 1, c(v_2v_3) = 2, \ldots, c(v_{n-1}v_n) = 1, c(v_nv_1) = 2$. Since $d(v) = 2$ for every $v \in V(C_n)$, then no two edges incident to the same vertex are of the same color, and thus it suffices to use two colors to properly edge color C_n. Hence, $\chi'(C_n) = 2$.

Assume now that n is odd. Suppose we try to color C_n as before, by alternately coloring edges using two colors. Then $c(v_1v_2) = 1, c(v_2v_3) = 2, \ldots, c(v_{n-1}v_n) = 2, c(v_nv_1) = 1$. However, this is not a proper coloring since the edges v_1v_2 and v_nv_1, two edges that are both incident to the vertex v_1, are of the same color. Thus, $\chi'(C_n) > 2$. Since $\Delta(C_n) = 2$, then by Corollary 3, $\chi'(C_n) = 3$. □

Below we display two properly edge colored cycles.

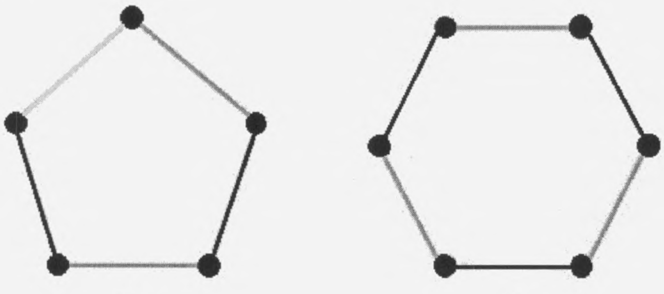

Figure 1.2: Properly edge colored cycles C_5 and C_6.

Corollary 1.5. *If n is even, then $\chi'(K_n) = n - 1$, and if n is odd, $\chi'(K_n) = n$.*

Proof. Since K_n is $(n-1)$-regular, $\chi'(K_n) \geq n - 1$.

Suppose n is even. We show that $\chi'(K_n) \leq n - 1$ by exhibiting a proper $(n-1)$-edge coloring of K_n. Label the n vertices of K_n as $0, 1, \ldots, n-1$. We embed the vertices $1, \ldots, n-1$ in a circle on a plane and place the vertex 0 in the center of the circle. The edges $1(0), 2(n-1), 3(n-2), \ldots, (\frac{n}{2})(\frac{n}{2}+1)$ form a perfect matching of K_n. We can find another $n-2$ distinct perfect matchings by rotating these edges by one vertex to the right. For example, the next perfect matching would be $2(0), 3(1), 4(n-1), \ldots, \frac{n}{2}(\frac{n}{2}+3)$, as shown in Figure 1.3 below. Each of the distinct $n-1$ perfect matchings can be assigned a color, thus properly edge coloring K_n with $n-1$ colors. Hence, $\chi'(K_n) = n - 1$.

Suppose n is odd. By Vizing's Theorem, we have that $\chi'(K_n) \leq n$. However, K_n cannot be properly edge colored with $n-1$ colors. Notice that the size of any matching of K_n can contain no more than $\frac{n-1}{2}$ edges and thus $n-1$ matchings of K_n can contain

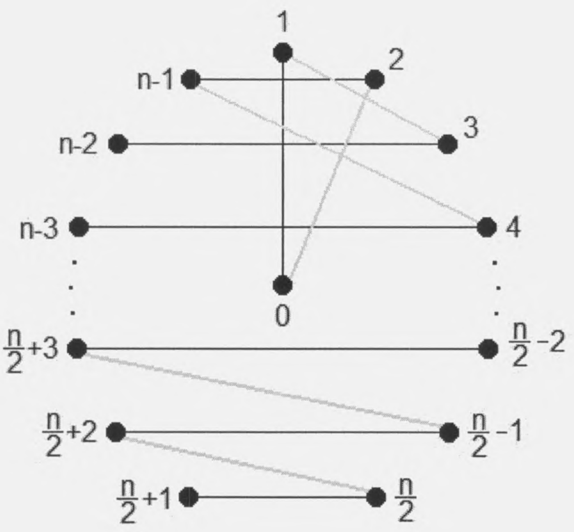

Figure 1.3: Two disjoint matchings of K_n for even n.

no more than $\frac{(n-1)^2}{2}$ edges. But K_n has $\frac{n(n-1)}{2}$ edges. Thus, $\chi'(K_n) > n - 1$, and hence $\chi'(K_n) = n$.

$\square$

Prior to Vizing and Gupta's result, König [3] had proven the exact value of the chromatic index for bipartite graphs.

Theorem 1.6 (König (1916)). *If G is bipartite, then $\chi'(G) = \Delta(G)$.*

Proof. We induct on the number of edges of G. If $|E(G)| = 1$, then the result holds, trivially. Assume that the result holds for all n-vertex subgraphs G' of G such that $1 < |E(G')| < |E(G)|$. Since $\Delta(G) \leq \chi'(G)$, it suffices to show that G has a $\Delta(G)$-edge coloring. Let $\Delta(G) = k$, and let $uv \in E(G)$. Then $G' = G - uv$ has a $\Delta(G')$-edge coloring. Since $\Delta(G') \leq \Delta(G) = k$, then we could properly edge-color G' with k colors. We show that we can use the same k colors to color G.

As $d(u) \leq k$, and the edge uv has yet to be colored, then at least one of the colors incident to u is not colored by one of the k colors. Similarly, at least one color is missing on the incident edges to v. If one of the colors missing at u and v is the same, we use this color to color uv, and we obtain a k-edge coloring of G.

Consider the case when a color i is present at u, but not at v, and a color j is present at v, but not at u. Let S be a maximally connected induced subgraph of G' containing u with edges colored only i or j. We claim that $v \notin V(S)$.

Suppose $v \in V(S)$. Then there exists a u, v-path P in S. Since $uv \in E(G)$, then u and v do not belong to the same partite set, meaning that P is of odd length. Since color i is present at u, the first edge of P is colored i. The edges of P must alternate in edge colors, as S is a subgraph of G', and since P is of odd length, the last edge of P, incident to v, must be colored i. However, this is a contradiction, given that color i is not present at v, proving our claim. Since $v \notin V(S)$, we can interchange the edge colors of S without affecting the colors of the edges incident to v. By doing so, we will have that j is a missing color from u and v, and thus we can color uv with j, making G k-edge colorable. □

Chapter 2

Properly colored subgraphs

A natural question that arises from the study of edge colored graphs is whether under certain conditions, an edge colored graph, not necessarily properly edge colored, is guaranteed to contain a properly edge colored subgraph. The subgraphs of an n-vertex graph G with edge set E that is $|E|$-edge colored are all proper, since these are the most number of colors we can use to edge-color G, and thus every two incident edges to a vertex can never share a common color. However, once fewer colors are used to edge color a graph, the problem of finding at least one properly edge colored subgraph, such as an alternating path or cycle (i.e, an edge colored path or cycle in which no two consecutive edges have the same color), requires careful analysis of the structure of the graph.

1 Alternating cycles

One of the earliest results in the study of finding alternating cycles in an edge colored graph is due to Grossman and Häggkvist [4]. They were able to show that if the edges of a 2-connected graph G can be partitioned into two classes so that each vertex is incident with edges from both classes, then G has an alternating cycle. Notice that whether G contains an alternating cycle is not dependent on any particular edge coloring, especially since an explicit edge coloring is not even provided, but rather on the graph's structure. If the graph satisfies the aforementioned properties, then the partition of edges can be thought of as color classes. There is, however, another possibility: once these two edge classes are assigned colors, say gray and black, G may have a cut vertex v which *separates colors*, i.e., no component of $G - v$ is joined to v by both gray and black edges. Notice that neither a vertex incident with only edges of the same color, nor a vertex that separates colors can be on any alternating cycle.

Fourteen years later, Yeo [5] extended the result by Grossman and Häggkvist to graphs whose edges can be partitioned into k classes such that each vertex is incident with edges from each class. We present these results as one theorem.

Theorem 2.1 (Grossman, Häggkvist (1983), Yeo (1997)). *Let G be a k-edge colored graph, where $k \geq 2$, such that every vertex of G is incident with at least two edges of distinct color. Then either G has a cut vertex separating colors, or G has an alternating cycle.*

We provide two interesting corollaries by Grossman and Häggkvist of Theorem 6. The following proposition is necessary for the proof of Corollary 9. Recall that a *cut-edge* is an edge whose removal increases the number of components of a graph. Thus, if G is a connected graph, the removal of a cut-edge e would result in a disconnected graph, $G \setminus e$.

Proposition 2.2. *An edge is a cut-edge if and only if it is not contained in any cycle.*

Proof. Without loss of generality, assume that G is a connected graph. We show that if an edge e is contained within a cycle, then it cannot be a cut-edge. Suppose

$e = uv$ is a cut-edge contained within a cycle C. Since G is connected, there exists a path between any two vertices of G. Thus the removal of e will not result in the disconnection of G, since u is still connected to v through another u, v-path contained within C other than uv itself. This contradicts that e is a cut-edge.

We show that if e is not contained within a cycle, then it is a cut-edge. Suppose $e = uv$ is not a cut-edge and that it is not contained within any cycle. Then $G \setminus e$ is still connected, and thus there exists a u, v-path within $G \setminus e$. This path together with e forms a cycle in G, which contradicts our original assumption. □

Corollary 2.3. *Let M be a perfect matching in a connected graph G. If no edge of M is a cut edge of G, then G has a cycle whose edges are taken alternately from M and $G \setminus M$.*

Proof. Since no edge of M is a cut-edge, then by Proposition 5, all of the edges of M must be contained within a cycle. Let C be a cycle which contains edges from M. Since M is a matching, then no two edges of M are incident to the same vertex. Thus, the edges of C must alternate in edges from M and edges from $G \setminus M$. □

Corollary 10, below, is based on 2-edge colored graphs G which contain two regular monochromatic subgraphs of distinct color. Monochromatic subgraphs are subgraphs in which each of all edges are assigned the same color.

Corollary 2.4. *Let G be a graph whose edges are colored gray and black so that both of its monochromatic subgraphs are regular and non-trivial. Then G has an alternating cycle.*

We conclude this section with a result by Hilton [6] for 2-edge colored regular, not necessarily connected, bipartite graphs $G = A \cup B$. We may assume that the partite sets of these graphs are both of size $m > 1$, given that they are 2-edge colored.

Notice that by Dirac's theorem [7], the specified degree assumptions of G in Theorem 11 guarantee that G contains a Hamiltonian cycle, i.e., a cycle which visits every vertex of G exactly once.

Theorem 2.5. *Let G be a 2-edge colored regular bipartite graph such that each partite set of G has m vertices and let G' be a black edge colored subgraph of G. If $\delta(G) \geq \lceil \frac{m}{2} + 1 \rceil$ and $\lceil \frac{m}{2} \rceil \leq \delta(G') \leq \delta(G) - 1$, then G has a Hamiltonian alternating cycle.*

Notice that the last theorem is best possible in terms of vertex degree. Consider the graph which consists of two disjoint copies of $K_{\frac{m}{2},\frac{m}{2}}$ with gray colored perfect matchings in both copies and all other edges colored black (Figure 2.1). The vertices of this graph are of one degree less than the minimum degree requirement stated in Theorem 11, and since this graph is disconnected, it cannot contain a Hamiltonian cycle.

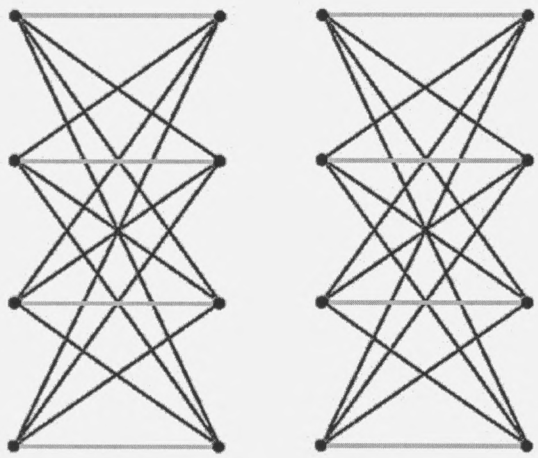

Figure 2.1: Two disjoint copies of $K_{4,4}$ with gray colored perfect matchings and all other edges colored black.

Chapter 3

Rainbow connection of graphs

In the previous section, we surveyed a few results which provided structural properties under which a graph G could be edge-colored such that we could find a particular properly edge colored subgraph. Say we were to add an interesting element, such as a connectivity requirement, to this problem. More specifically, say we wanted to find a properly edge colored path between any two vertices u and v of a graph. In this section, we will survey some results based on finding rainbow paths, a type of properly edge colored path, within a graph. First, let us provide the following definitions.

Definition 3.1. A path P is a *rainbow path* if no two edges of P are colored the same color. A *u, v-path* is a path with endpoints u and v. A graph G is *rainbow-connected* if G contains a rainbow u, v-path for every $u, v \in V(G)$. The minimum k for which there exists a k-edge coloring such that G is rainbow-connected is the *rainbow connection number* and is denoted $rc(G)$.

To better understand the rainbow connection number $rc(G)$, let us begin by looking at some examples. First, let us compute $rc(K_n)$. Since in K_n there is an edge between any two vertices u and v, we can trivially consider uv as a path, and since uv

is a single edge, it is a rainbow path. Thus, $rc(K_n) = 1$. It is worth noting that even though this type of graph can be rainbow-colored with the least number of colors possible, it takes far more colors to properly edge color K_n, as was shown in Section 1.

Let us compute $rc(P_n)$. Since there is exactly one path between any two vertices, we would need every edge to be a different color to ensure a rainbow path can be found between any two vertices. Thus, $rc(P_n) = |E(P_n)| = n - 1$.

1 Maximizing the rainbow connection number

In fact, for any n-vertex tree T_n, i.e., a connected graph without cycles (a path being an example of such a graph), $rc(T_n) = n - 1$. Since $|E(T_n)| = n - 1$, we can equivalently say that the rainbow connection number of a tree is the number of edges it has. Indeed, T_n is the only graph such that $rc(G) = |E|$. Let us prove this.

Proposition 3.1 (Chartrand et al. (2008)). *Let G be a nontrivial connected graph with m edges. Then $rc(G) = m$ if and only if G is a tree.*

Proof. Let G be a tree. Suppose $rc(G) = m - 1$, i.e., that G can be rainbow colored with $m - 1$ colors. Then since $|E(G)| = m$, there are two edges u_1v_1 and u_2v_2 of the same color. Let P be a u_1, v_2-path. Since G is a tree, there exists exactly one u, v-path for every $u, v \in V(G)$. Thus, P must contain the edges u_1v_1 and u_2v_2, in which case, G would have a path which is not rainbow colored.

Assume that G is not a tree and $|E(G)| = m$. Then G contains a cycle $C = v_1v_2, v_2v_3, \ldots, v_kv_1$, where $k \geq 3$. If we color the edges v_1v_2 and v_2v_3 the same color and we color the remaining $m - 2$ edges of G using $m - 2$ distinct colors, then G would be rainbow-connected. Thus, $rc(G) \leq m - 1$. □

Although Proposition 12 is based only on trees, it actually implies something greater. Notice that we can rainbow color a graph G using m distinct colors if $|E(G)| = m$, and thus $rc(G) \leq |E|$ for any graph. Thus, Theorem 10 implies the following corollary.

Corollary 3.2. *The tree on n vertices, T_n, maximizes $rc(G)$.*

2 Minimizing the rainbow connection number

An intuitive lower bound for $rc(G)$ is the diameter of a graph, which is the length of the largest path among all shortest u, v-paths in G, denoted $diam(G)$. We provide a proof for all of the previously mentioned bounds.

Theorem 3.3. *For any graph G with m edges, $diam(G) \leq rc(G) \leq m$.*

Proof. We have $rc(G) \leq m$ as an upper bound from Corollary 13. Suppose that the $diam(G) = d$. Then there exists a u, v-path P such that $e(P) = d$. Suppose we edge color G with fewer than d colors. Then P would not be rainbow colored, and since P is the shortest path between u and v, all other u, v-paths would not be rainbow colored as well. Thus $d \leq rc(G)$. □

Previously, we showed that $rc(K_n) = 1$. The next proposition proves that the complete graph is the only graph with rainbow connection number 1.

Proposition 3.4. *Let G be an n-vertex graph. Then $G = K_n$ if and only if $rc(G) = 1$*

Proof. If $G = K_n$, by our previous explanation, $rc(G) = 1$. We show that if G is not complete, then $rc(G) > 1$. If G is not complete, then G contains two nonadjacent vertices u and v, and thus the shortest u, v-path is of length at least 2, meaning $diam(G) \geq 2$. By Theorem 14, we have that $2 \leq diam(G) \leq rc(G)$. □

Corollary 3.5. *The complete graph on n vertices, K_n, minimizes $rc(G)$.*

Just as Proposition 12 implies Corollary 13, we have that Proposition 15 implies Corollary 16 above. A key observation to make when investigating the lower and upper bounds of the rainbow connection number is that unlike the chromatic index, the rainbow connection number decreases the denser a graph becomes, that is, the more edges a graph has. Naturally, one may want to find a bound for the number of edges in a graph that will guarantee a certain rainbow connection number. Later in this section, we will provide bounds for the number of edges in a graph which guarantee that $rc(G) = 2$.

3 Rainbow connection number of the Petersen graph

In [8], Chartrand et al. provide an insightful example using the Petersen graph to illustrate some of the concepts just discussed. Below we have a 3-edge coloring and a 2-edge coloring of the Petersen Graph, PG. Notice that under the 3-edge coloring, PG is rainbow-connected. Thus, $rc(PG) \leq 3$. Since $diam(PG) = 2$, then by Theorem 14 we have that $2 \leq rc(PG)$. However, under any 2-edge coloring, we will inevitably have two incident edges uv and vw of the same color, since $\delta(PG) = 3$. As the shortest path between any two nonadjacent vertices of PG is of length two, if u and w are nonadjacent, then we will not have any rainbow colored u, w-paths in PG. Hence, $rc(PG) = 3$.

4 A few results based on the minimum degree

Caro et al. [9] investigated the extremal graph-theoretic behavior of the rainbow connection number. They made the observation that there exist graphs G with min-

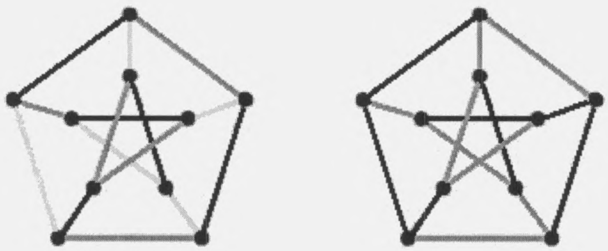

Figure 3.1: A rainbow 3-edge coloring and a 2-edge coloring of the Petersen graph.

imum degree 2 such that $rc(G) = n - 3$, by constructing a graph consisting of two vertex-disjoint triangles (cycles with three vertices), connecting them by a path of length $n - 5$, and using 2 colors to color the triangles and $n - 5$ colors to color the path between them. This observation had motivated them to study the rainbow connection of graphs with minimum degree at least 3. They wondered if for an n-vertex graph G they could find an $\alpha < 1$ such that $rc(G) \leq \alpha n$ when $\delta(G) \geq 3$. In particular, they proved the following.

Theorem 3.6 (Caro et al. (2008)). *If G is a connected graph with n vertices and $\delta(G) \geq 3$, then $rc(G) < 5n/6$.*

Caro et al. conjectured that the factor in the upper bound can be improved from 5/6 to 3/4. However, the upper bound cannot be replaced with a constant smaller than 3/4, since Caro et al. have shown that there exist 3-regular connected graphs G with $rc(G) = (3n - 10)/4$. They have been able to prove the following upper bound for 2-connected graphs.

Proposition 3.7 (Caro et al. (2008)). *If G is a 2-connected graph with n vertices, then $rc(G) \leq 2n/3$.*

In 2009, Schiermeyer [10] proved the conjecture by Caro et al. for 1-connected graphs, thus proving the conjecture for all connected graphs with minimum degree 3, by Proposition 16.

Theorem 3.8 (Schiermeyer (2009)). *If G is a connected graph with n vertices and $\delta(G) \geq 3$ then $rc(G) \leq \frac{3n-1}{4}$.*

As previously discussed, one may look to find bounds for the size of the edge set $E(G)$ of a graph G which guarantee a certain rainbow connection number. Kemnitz and Schiermeyer proved the following for n-vertex connected graphs.

Theorem 3.9. *Let G be a connected n-vertex graph with m edges. If $\binom{n-1}{2} + 1 \leq m \leq \binom{n}{2} - 1$, then $rc(G) = 2$.*

The upper bound in Theorem 20 follows from Proposition 15, since the complete graph has $\binom{n}{2}$ edges, the most possible edges that an n-vertex simple graph G could have.

Chapter 4

Rainbow ℓ-connection of graphs

By definition of rainbow connectivity, we know that if a graph G is rainbow-connected, then between any two vertices u and v of G there exists a u,v-path. Given that there exist $\ell \geq 1$ internally pairwise vertex disjoint paths between any two vertices of G, one may wonder whether there exists an edge coloring under which G contains ℓinternally pairwise vertex disjoint rainbow paths. By Menger's theorem [11], we have that the maximum number of pairwise vertex disjoint u,v-paths for any $u,v \in V(G)$ is equal to the minimum number of vertices whose removal disconnects u from v. Thus, we will define the *ℓ-connectivity* of a graph as follows.

Definition 4.1. Two paths which are internally pairwise vertex disjoint are called *independent* paths. A graph is *ℓ-connected* if between any two vertices u and v of G there exist ℓ independent u,v-paths.

The following definitions are extensions from the definitions of rainbow-connected graphs and the rainbow connection number.

Definition 4.2. An ℓ-connected graph G is *ℓ-rainbow connected* if between any two vertices u and v of G there exist ℓ independent rainbow u,v-paths. The *ℓ-rainbow* *connection number* is the minimum k for which there exists a k-edge-coloring such that G is ℓ-rainbow connected, and is denoted $rc_\ell(G)$.

Notice that the rainbow connection number, $rc(G)$, is equivalent to the 1-rainbow connection number, $rc_1(G)$. An $|E(G)|$-edge coloring of an ℓ-connected graph G trivially contains ℓ independent rainbow paths between any two vertices of G. Thus, we see that $rc_j(G)$ is defined for every integer j such that $1 \leq j \leq \ell$. Furthermore, $rc_i(G) \leq rc_j(G)$ for $1 \leq i \leq j \leq \ell$, since a graph may require the same or more edge colors to find j independent rainbow u,v-paths than to find i independent rainbow u,v-paths.

1 Rainbow j-connectivity of K_n

The complete graph K_n, for $n \geq 2$, has connectivity $n-1$, since for any two vertices u and v, we can consider the edge uv and the $n-2$ paths $P = u,w,v$ of length 2, for distinct $w \in V(G) \setminus \{u,v\}$, as $n-1$ independent u,v-paths. It follows from our previous explanation that $rc_j(K_n)$ is defined for all j such that $1 \leq j \leq \ell$. The following proposition is due to Chartrand et al. [12], who analyzed the ℓ-rainbow connection number of complete graphs and complete bipartite graphs. Recall that by Corollary 5, we have that $\chi'(K_n) = n-1$ when n is even and $\chi'(K_n) = n$ when n is odd. Equivalently, we can say that $\chi'(K_n) = 2\lceil n/2 \rceil - 1$ for any n.

Proposition 4.1 (Chartrand et al. (2009)). *For $n \geq 2$, $rc_{n-1}(K_n) = \chi'(K_n) = 2\lceil n/2 \rceil - 1$.*

Proof. Since $rc_{n-1}(K_2) = 1 = 2\lceil 2/2 \rceil - 1$, we may assume that $n \geq 3$. Suppose we have a proper $\chi'(K_n)$-edge coloring of K_n. Consider any two vertices u and v of K_n and let $x_1, \ldots, x_{n-2}$ be the remaining vertices. Since K_n is properly edge colored, the colors of the edges ux_i and x_iv are different for $1 \leq i \leq n-2$. Thus the path u, v along with the paths ux_iv are $n-1$ independent rainbow u, v-paths. Thus $rc_{n-1}(K_n) \leq \chi'(K_n)$.

We show that $rc_{n-1}(K_n) \geq \chi'(K_n)$. Assume not, and that $rc_{n-1}(K_n) < \chi'(K_n)$. Since $\chi'(K_n) > j$, there are two edges uw and wv incident to the same vertex that have been assigned the same color. Then, since the path uwv is not a rainbow path, then K_n must contain at most $n-2$ independent u, v-paths, contradicting our original assumption. Thus $rc_{n-1}(K_n) \geq \chi'(K_n)$, and so $rc_{n-1}(K_n) = \chi'(K_n)$. □

Given Propositions 15 and 21, we need only be concerned with $rc_j(K_n)$, where $2 \leq j \leq n-2$. Observe that when $n \geq 4$ and $j = 2$ or when $n \geq 5$ and $j = 3$, we can edge color K_n with 2 or more colors in such a way that we can find 2 or 3 rainbow paths between any two vertices of K_n, and thus, $rc_j(K_n) \geq 2$. However, we can create 2-edge coloring of K_n to show that $rc_j(K_n) \leq 2$. Since K_n has all possible edges between n vertices, it contains a Hamiltonian cycle. If we were to 2-edge color K_n with colors 1 and 2 by assigning 1 to each edge of a Hamiltonian cycle and 2 to all other edges of K_n, then we would be able to find three rainbow colored paths of length two between any two vertices (see Figure 4.1 below). Thus, $rc_j(K_n) \leq 2$ for $j = 2, 3$, and we have the Proposition 22 below.

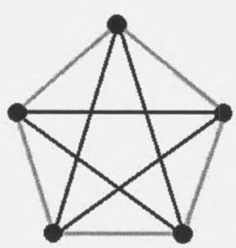

Figure 4.1: A 2-edge coloring of K_5 which shows $rc_j(K_n) \leq 2$ for $j = 2, 3$.

Proposition 4.2 (Chartrand et al. (2009)). *For $n \geq 4$, $rc_2(K_n) = 2$ and for $n \geq 5$, $rc_3(K_n) = 2$.*

Chartrand et al. also proved that for each $k \geq 2$ there exists large enough n such that exactly two colors are necessary in order to k-rainbow connect K_n. Specifically, they showed that there exists an integer dependent on k, $f(k)$, such that $rc_k(K_n) = 2$ for every $n \geq f(k)$. The proof of this result shows that $f(k) \leq (k+1)^2$. However, $(k+1)^2$ is not a sharp upper bound, but thus far, there is no argument that will provide a sharp upper bound.

Chapter 5

Proper connection of graphs

Thus far, we have considered proper edge colorings of graphs, conditions under which a graph contains a properly edge colored subgraph, and the ℓ-rainbow connectivity of an ℓ-connected graph. The definition of the ℓ-proper connection number is a natural extension of the ℓ-rainbow connection number. Instead of finding rainbow colored paths between any two vertices of a graph G, we will now be concerned with finding properly colored paths in general between any two vertices. Recall that a path can be properly edge colored with at least two colors, but that we require $n-1$ colors to rainbow color a path on n vertices. For this reason, the conditions assumed to produce upper and lower bounds on the ℓ-proper connection number are much weaker than those necessary to bound the ℓ-rainbow connection number. We introduce the following definitions.

Definition 5.1. If any two vertices of G are connected by ℓ independent proper paths, then we say G is *ℓ-proper connected.* The *ℓ-proper connection number* of an ℓ-connected graph G is the smallest number of colors needed in order to make G ℓ-proper connected, and is denoted by $pc_\ell(G)$.

Since the n-vertex path P_n is 1-connected, we can edge color P_n with 2 colors by alternating in colors every other edge in order to find one properly edge colored path between any two vertices, and hence $pc_1(P_n) = 2$. As another example, consider the n-vertex cycle, C_n, which is 2-connected. We can also 2-edge color C_n by alternating colors every other edge, and in order to find two proper paths between any two vertices, we can simply traverse the edges of the cycle in either direction. Thus, $pc_2(C_n) = 2$.

Given that the main result of this thesis, Theorem 6.3, is based on a particular class of 2ℓ-connected bipartite graphs, for the remainder of this section, we will only be concerned with the proper connection of 2ℓ-connected bipartite graphs.

1 2ℓ-connected bipartite graphs

In 2011 Borozan, Fujita, Gerek, Magnant, Manoussakis, Montero, and Tuza [13] made the following conjecture on the proper connection of 2ℓ-connected bipartite graphs, dependent on the parameter ℓ.

Conjecvture 1 (Borozan et al. (2011)). *If G is a 2ℓ-connected bipartite graph with $\ell \geq 1$, then $pc_\ell(G) = 2$.*

Throughout our overview of the rainbow connection of graphs we have assumed that the ℓ-rainbow connection number, $rc_\ell(G)$, is based solely on ℓ-vertex connected graphs. Notice that vertex connectivity and edge connectivity are not interchangeable in Conjecture 1 because we can create a 2ℓ-edge connected bipartite graph which is 1-vertex connected, as shown below in Figure 7. Notice that in the graph G below, the removal of the vertex v results in the disconnection of the graph, but that it requires the removal of at least 2 edges in order to disconnect G.

Borozan et al. proved that Conjecture 1 is best possible in the sense of connectivity by showing that if a bipartite graph G has connectivity less than 2ℓ for $\ell \geq 1$, then

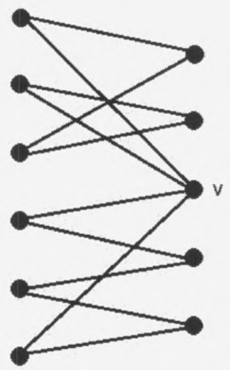

Figure 5.1: A 2-edge connected graph which is 1-vertex connected.

we would require more than 2 colors to properly connect G. The following theorem is based on $(2\ell - 1)$-connected complete bipartite graphs. We denote $c(u, v)$ as the color of an edge uv.

Theorem 5.1 (Borozan et al. (2011)). *Let* $a = 2k - 1$*,* $k \geq 1$*, and* $b > 2^a$*. Then* $pc_k(K_{a,b}) > 2$*.*

Proof. Let $A \cup B = K_{a,b}$, where $|A| = a$ and $|B| = b$. Since $K_{a,b}$ is complete, clearly it is $(2k-1)$-connected since A is the minimal vertex set whose removal would result in the disconnection of $K_{a,b}$. We show that $pc_k(K_{a,b}) > 2$.

Suppose $pc_k(K_{a,b}) = 2$, i.e., that there exists a k-proper coloring of $K_{a,b}$ using exactly two colors, say 0 and 1. For each $v_i \in B$, where $1 \leq i \leq b$, there exists a sequence of colors $c_1c_2 \dots c_a$ corresponding to the set of edges incident to x_i (one could think of this as a sequence of zeros and ones). Each x_i has 2^a ways of coloring its a incident edges. Then since $|B| > 2^a$, there must be two vertices $v_i, v_j \in B$ that share the same incident edge color sequence. As $pc_k(K_{a,b}) = 2$, by assumption, then there exist k internally disjoint proper paths between v_i and v_j. We will use this to arrive at a contradiction. Notice that not all of the paths between v_i and v_j will have at least two internal vertices in A, for this would mean that $|A| \geq 2k$, which is not true. Thus, one of the v_i, v_j-paths must have only one internal vertex

$u \in A$. Then since the edge color sequences of v_i and v_j are the same, we have that $c(v_i, u) = c(v_j, u)$, and therefore there exists an improperly colored path between v_i and v_j, a contradiction. $\square$

Theorem 5.2. *If $m \geq n \geq 2\ell$ for $\ell \geq 1$, then $pc_\ell(K_{n,m}) = 2$.*

Proof. Let $A \cup B$ be the bipartition of $K_{n,m}$. Since $|A|, |B| \geq 2\ell$, we can split each partite set into two subsets $A_1, A_2 \subseteq A$ and $B_1, B_2 \subseteq B$ such that $|A_i|, |B_i| \geq \ell$. We color all edges between the vertices of A_1 and B_1 and all edges between the vertices of A_2 and B_2 gray, and we color all other edges black, as shown Figure 8 below.

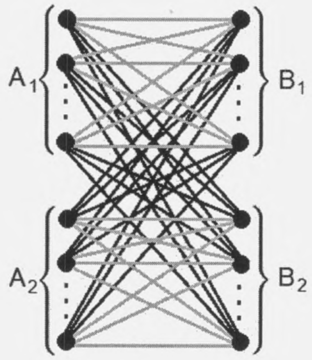

Figure 5.2: A 2-edge coloring of $K_{n,m}$.

We show that this coloring produces ℓ properly colored u, v-paths for all u and v in $K_{n,m}$. We have three cases, depending on which partite sets u and v are in. Without loss of generality, it suffices to consider when $u, v \in A_1$, when $u \in A_1$ and $v \in A_2$, or when $u \in A_1$ and $v \in B_1$, given that there exist ℓ paths of length 1 between the vertices of A_1 and B_2 and the vertices of A_2 and B_1, and hence they are proper.

Let $u, v \in A_1$. Since $|A_i|, |B_i| \geq \ell$, we can form ℓ proper paths of the form $ub_1a_2b_2v$, where $b_1 \in B_1$, $a_2 \in A_2$, and $b_2 \in B_2$. When $u \in A_1$ and $v \in A_2$, we can form ℓ proper paths ubv, where $b \in B$. Finally, if $u \in A_1$ and $v \in B_1$, the path u, v along with the paths ub_2a_2v, where $b_2 \in B_2$ and $a_2 \in A_2$, will form at least ℓ proper paths. This concludes the proof. $\square$

Chapter 6

Original results

We have seen that Conjecture 1 holds for 2ℓ-connected complete bipartite graphs $K_{n,m}$, as shown by Borozan et al. One natural class of graphs that one may look to extend this result to is a class consisting of graphs obtained by the removal of matchings from $K_{n,m}$. In particular, through the removal of perfect matchings from the symmetric complete bipartite graph with partite sets of equal size, $K_{r,r}$. However, the removal of matchings can result in a disconnected graph. For example, the removal of two matching from $K_{6,6}$ can result in two $K_{3,3}$ graphs, as shown below.

Although an unguided attempt at this type of extension can fail in this way, it is through a careful version of this technique that we obtain a class of graphs over which we will prove our extension. One type of graph obtained from a non-disconnecting removal $K_{r,r} \setminus M$, where M is a particular matching, is a graph called a *circulant graph*, and it is on this graph in particular that we will provide an extension. A circulant graph is defined as follows.

Definition 6.1. Given a set $S \subseteq \left\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\right\}$ of integers, the *circulant graph* $C_{n,S}$ is defined as the graph with vertex set $\mathbb{Z}_n$, and edge set $\{uv : |u - v|_n \in S\}$, where $|x|_n := \min\{|x|, n - |x|\}$. The metric $|\cdot|_n$ is referred to as the *circular distance* modulo n.

For visualization purposes, we think of $C_{n,S}$ as being embedded circularly on a plane, with its vertices in consecutive order, as in the graphs of Figure 9. In this configuration, we can think of the elements of $\mathbb{Z}_n$ as being laid down this way, and the edge set of our circulant graph as generated by S, with the placement of edges between vertices relying on whether the circular distance between two vertices falls in the set S. Since the circular distance cannot be of length greater than $\lfloor \frac{n}{2} \rfloor$, we always consider the circular distance to be the smaller of the two possible distances between vertices along the cycle placed this way. It is straightforward to check that circulant graphs are all $2|S|$-regular, by the circular, symmetric nature of this construction. In particular, for each $v \in \mathbb{Z}_n$, we can see that

$$d_{C_{n,S}}(v) = |\{x \in \mathbb{Z}_n : |x - v|_n \in S\}| = 2|S|.$$

We provide the following examples to give a better understanding of the structure, and this described construction, of a circulant graph.

Here we have the circulants $C_{9,\{1,2\}}$ and $C_{9,\{3,4\}}$.

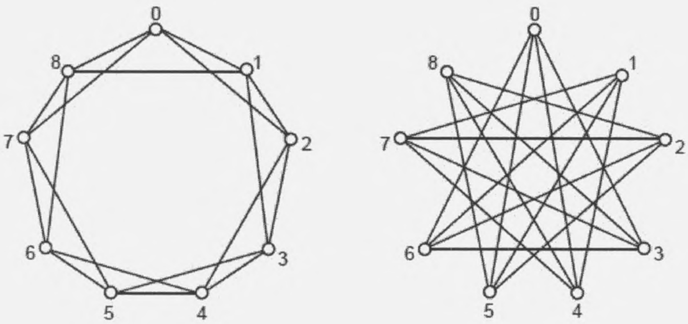

Figure 6.1: The circulant graphs $C_{9,\{1,2\}}$ and $C_{9,\{3,4\}}$

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Notice that when $S = \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$, the circulant $C_{n,S}$ is the complete graph

K_n. Also note that $C_{n,\{1\}}$ is the cycle C_n, and that in general, $C_{n,\{s\}} \cong C_n$ whenever s and n are relatively prime, i.e., the $\gcd(s, n) = 1$.

1 The bipartite circulant graph $C_{4k,\ell}$

The types of circulant graphs that we will be most concerned with are bipartite circulant graphs. Since we have proven Conjecture 1 is true for this particular type of graph, we provide the following formal definition.

Definition 6.2. The *bipartite circulant with parameters k and* ℓ, for $k, \ell \geq 1$, is the circulant graph $C_{4k,\mathcal{O}(\ell)}$, where $\mathcal{O}(\ell) := \{1, 3, 5, \ldots, 2\ell - 1\}$. For simplicity, we will instead denote the bipartite circulant with parameters k and ℓ as $C_{4k,\ell}$.

Since the circular distance between all vertices in $C_{4k,\ell}$ is odd, the generating set $\mathcal{O}(\ell)$ forms a natural bipartition $A \cup B = C_{4k,\ell}$, where $A = \{0, 2, 4, \ldots, 4k - 2\}$ and $B = \{1, 3, 5, \ldots, 4k - 1\}$. Figure 6.2 below shows $C_{16,3}$.

In general, when $\ell = 2k - 1$, we have that $C_{4k,\ell} = K_{2k,2k}$. Notice that in Figure 5 we can see that $C_{16,\{1,3,5,7\}} = K_{8,8}$.

Since our intention is to prove that exactly two colors are necessary to properly k-connect $C_{4k,\ell}$, we must create a 2-edge coloring which will allow us to find k properly colored paths between any two vertices in $C_{4k,\ell}$. Given the circular structure of this graph, it would seem natural to create an edge coloring that will allow us to conveniently find proper u, v-paths, regardless of whether we traverse the internal vertices of these paths consecutively in the clockwise or counterclockwise direction. We have created a symmetric 2-edge coloring using colors labelled 0 and 1 for $C_{4k,\ell}$ in which the colors are evenly distributed amongst all edges. More specifically, if u is an even vertex, then the edges uv in which u precedes v and in which u succeeds

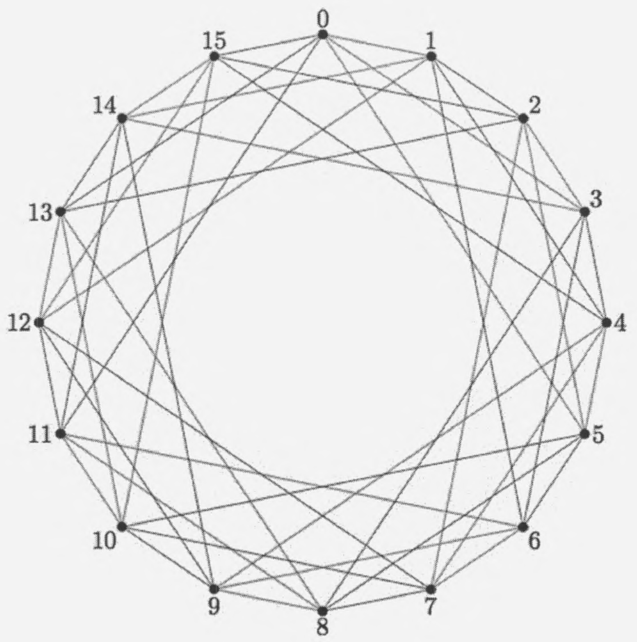


Figure 6.2: The bipartite circulant graph $C_{16,3}$.

v along the circumference of $C_{4k,\ell}$ are colored 0 and 1, respectively. Similarly, if u is odd, then the edges uv in which in which u precedes v and in which u succeeds v along the circumference of $C_{4k,\ell}$ are colored 1 and 0, respectively. Visually, we can see that if u is even, then edges that stem from u in the clockwise direction are colored 0, and edges that stem from u in the counterclockwise direction are colored 1. For odd u, the colors are reversed. Let us formally introduce our edge coloring of $C_{4k,\ell}$.

Definition 6.3. For all $u \in A$ and $v \in B$, we define the coloring $c : E(C_{4k,\ell}) \to \{0, 1\}$ as

$$c(uv) = \begin{cases} 0, & \text{if } u - v < 0, \\ 1, & \text{if } u - v > 0. \end{cases}$$

Below we display $C_{16,3}$ under the coloring c.

We present the following lemma to demonstrate the advantage that our edge coloring c of $C_{4k,\ell}$ provides in finding properly colored paths. For the remainder of

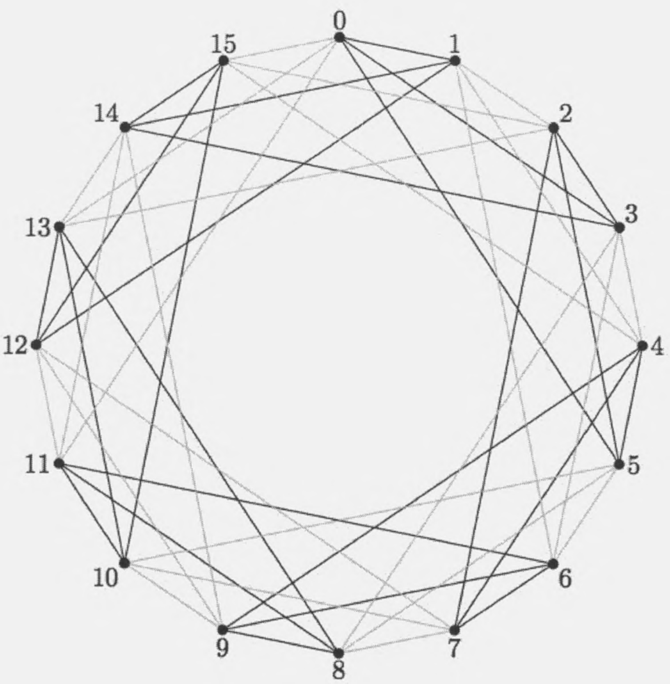


Figure 6.3: The graph $C_{16,3}$ under the coloring c.

this section we will refer to paths in $C_{4k,\ell}$ as clockwise, or counterclockwise, paths, if in the circular embedding of $C_{4k,\ell}$, the vertices of a path proceed successively in a clockwise or counterclockwise direction.

Lemma 6.1. *Every clockwise and counterclockwise u, v-path in $C_{4k,\ell}$ is proper under the coloring $c : E(C_{4k,\ell}) \to \{0, 1\}$.*

Proof. Let $P = uw_1w_2 \ldots w_nv$ be a clockwise u, v-path in $C_{4k,\ell}$ under the coloring c. Recall that $C_{4k,\ell} = A \cup B$, where A and B consist of all the even and odd vertices, respectively. Since no two even and no two odd vertices can be adjacent, then the vertices in P must alternate in parity. Since P is a clockwise path, then $u < w_1 < w_2 < \ldots < w_n < v$. If u is even, then $c(uw_1) = 0$, $c(w_1w_2) = 1$, $c(w_2w_3) = 0$, etc., and if u is odd, then $c(uw_1) = 1$, $c(w_1w_2) = 0$, $c(w_2w_3) = 1$, etc. Thus all clockwise paths in $C_{4k,\ell}$ alternate in edge color, meaning they are proper. The proof is analogous for counterclockwise paths. □

Recall that Conjecture 1 is applicable only to 2ℓ-connected bipartite graphs. The following proposition shows that $C_{4k,\ell}$ satisfies this connectivity requirement. The proof is very similar to the proof by Harary [14], which shows the connectivity for Harary graphs, another class of graphs. In most cases, Harary graphs are circulant graphs or have a circulant as an edge subgraph. Recall that circulant graphs in general are $2|S|$-regular, and thus $C_{4k,\ell}$ is 2ℓ-regular.

Proposition 6.2. *The graph $C_{4k,\ell}$ is 2ℓ-connected.*

Proof. Since $C_{4k,\ell}$ is bipartite and $\delta(C_{4k,\ell}) = 2\ell$, it suffices to show that $\kappa(C_{4k,\ell}) \geq 2\ell$. Let $D \subseteq V(C_{4k,\ell})$ such that $|D| < 2\ell$. We show that $C_{4k,\ell} \setminus D$ is connected.

Let $u, v \in V(C_{4k,\ell}) \setminus D$. Let X be the set of all vertices between u and v in the clockwise direction, and let Y be the set of all vertices between u and v in the counterclockwise direction. Since $|D| < 2\ell$, by the pigeonhole principle there must be fewer than ℓ vertices of D in either X or Y. By construction, each vertex in $C_{4k,\ell}$ is adjacent to ℓ vertices in either direction. Thus, if we remove fewer than ℓ vertices adjacent to a vertex x, we cannot disconnect all u, v-paths containing x because we can always find a u, v-path in $C_{4k,\ell} \setminus D$ via the set X or Y with fewer than ℓ vertices of D. □

Now that we have shown that $C_{4k,\ell}$ is 2ℓ-connected, we are able to present the main result of this thesis, Theorem 6.3, which is the proof of Conjecture 1 for $C_{4k,\ell}$. Recall that Conjecture 1(Section 5) states that if G is a 2ℓ-connected bipartite graph, where $\ell \geq 1$, then exactly two colors are necessary to ℓ-proper connect G, meaning that under some 2-edge coloring of G, we can find ℓ proper paths between any two vertices. In Theorem 6.3, we show that under the 2-edge coloring c, $C_{4k,\ell}$ is ℓ-proper connected, i.e., $pc_\ell(C_{4k,\ell}) = 2$. In the proof, we provide a way of constructing ℓ independent paths between any two vertices by choosing paths that have as few internal vertices as possible. By construction, $C_{4k,\ell}$ has largest circular distance $2\ell - 1$. Our u,v-paths are either clockwise or counterclockwise, which by Lemma 6.1 are proper, and we choose internal vertices which have circular distance $2\ell - 1$ between one another. Figure 6.4 below demonstrates how we find ℓ independent $0,6$-paths in the bipartite circulant $C_{20,4}$. In Figure 6.5, we display all $0,8$-paths and all $0,9$-paths in $C_{20,4}$.

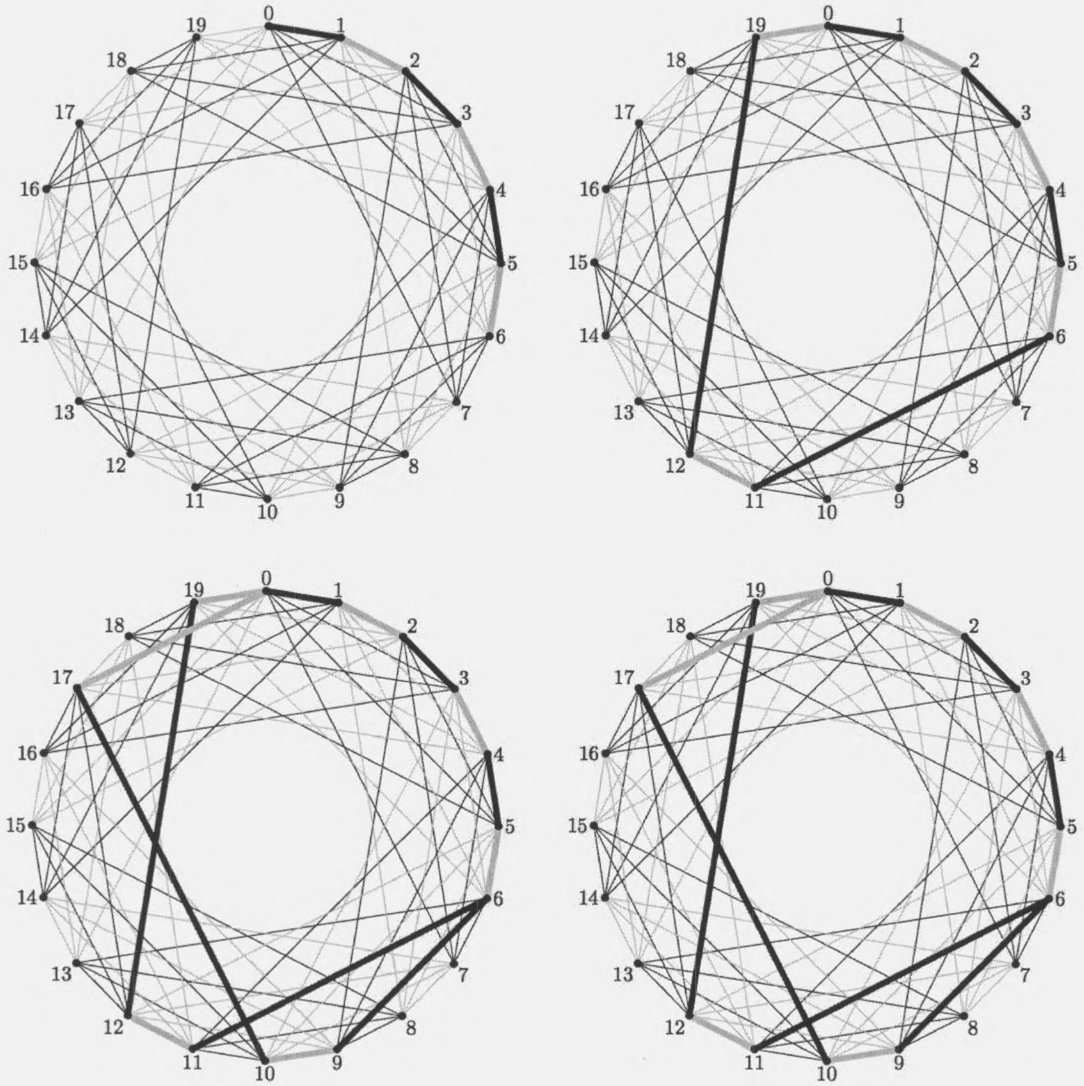
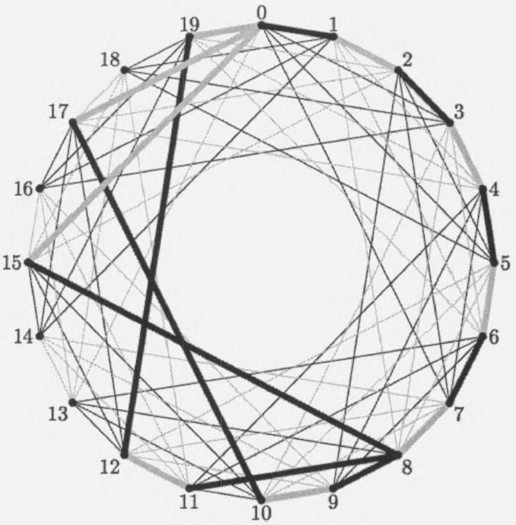


Figure 6.4: Four independent $0,6$-paths in $C_{20,4}$.



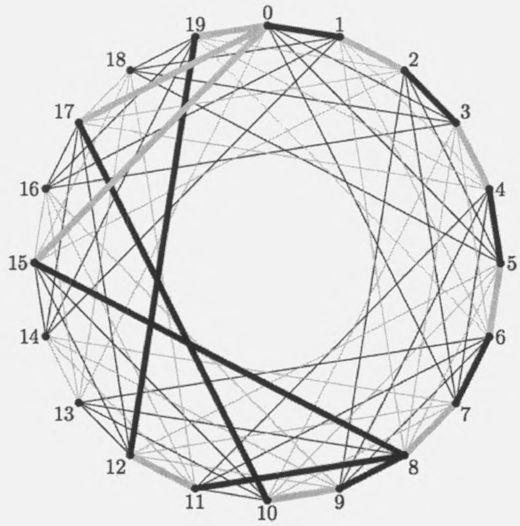


Figure 6.5: All four independent 0, 8-paths (left) and 0, 9-paths (right) in $C_{20,4}$.

Theorem 6.3 (Fuentes(2012)). *For all* $\ell \geq 1$, $pc_\ell(C_{4k,\ell}) = 2$.

Proof. By proposition 26, we have that $C_{4k,\ell}$ is 2ℓ-connected. Let $C_{4k,\ell}$ be edge colored under the coloring c. We show that under c, $C_{4k,\ell}$ is ℓ-proper connected, thus proving $pc_\ell(C_{4k,\ell}) = 2$. It suffices to show that we can find ℓ proper internally pairwise vertex disjoint $0, v$-paths for each $v \in \mathbb{Z}_{4k}$, since we can add the circular distance $|0-u|_n = u$, for any $u \in V(C_{4k,\ell})$, to the vertices of each $0, v$-path to find the set of all internally vertex disjoint proper $u, (v+u)$-paths in $C_{4k,\ell}$. Also, notice that because $C_{4k,\ell}$ is symmetric, we can create a reflection function which maps each vertex of $C_{4k,\ell}$ to the vertex directly across from it over the line of symmetry which passes through the vertex 0. This reflection function $\phi : V(C_{4k,\ell}) \to V(C_{4k,\ell})$ is defined by

$$\phi(v) = \begin{cases} 0, & \text{if } v = 0 \\ 4k - v, & \text{if } v \neq 0. \end{cases}$$

Thus if $P = v_1 v_2 \ldots v_i$ is a clockwise path, which we know is proper by Lemma 18, then ϕ will allow us to create another proper path, $\phi(v_1)\phi(v_2)\ldots\phi(v_i)$, which is counterclockwise across the other side of $C_{4k,\ell}$. Likewise, if P is a counterclockwise path, then $\phi(P)$ is a clockwise proper path. Hence, it suffices to show that there exist ℓ proper disjoint $0, v$-paths for every $v \in \{1, 3, 5, \ldots, 2k\}$, instead of for all $v \in \mathbb{Z}_{4k}$.

We show that there exist ℓ proper $0, v$-paths, for $v \in \{1, 3, \ldots, 2k\}$. If $\ell = 1$ then we only need to find one path from 0 to v, and we can simply choose the clockwise path $0123\ldots v$. If $\ell = 2$, then we can choose the clockwise path $0123\ldots v$ and the counterclockwise path $0(4k-1)(4k-2)(4k-3)\ldots v$. If $\ell \geq 3$, then we will always choose our first path to be $012\ldots v$, and we choose the remaining $\ell - 1$ paths in the counterclockwise direction. Since our goal is to find independent paths, we would like to choose paths with the least number of vertices in order to avoid any two internal vertices from coinciding. By construction, the largest circular distance between adjacent vertices in $C_{4k,\ell}$ is $2\ell - 1$. Thus, if we choose the second vertex of each path to be $(4k - i)$, the third vertex to be $(4k - i - (2\ell - 1))$, the fourth vertex to be $(4k - i - 2(2\ell - 1))$, etc., for distinct $i \in \{1, 3, \ldots, 2\ell - 3\}$, then our counterclockwise paths $0, v$-paths P_i will begin as follows

$$0(4k-i)(4k-i-(2\ell-1))(4k-i-2(2\ell-1))(4k-i-3(2\ell-1))(4k-i-4(2\ell-1))\ldots$$

Notice that for distinct $i, j \in \{1, 3, \ldots, 2\ell - 3\}$, the paths P_i and P_j are thus far internally vertex disjoint, since $4k - i - x(2\ell - 1) \equiv (4k - i) \mod (2\ell - 1)$, for $x \in \mathbb{Z}$ such that $x \geq 0$, and thus $4k - i - x(2\ell - 1) \neq 4k - j - x(2\ell - 1)$. We are careful to say "thus far" since it still remains to show that the path endings, that is, the last few internal vertices of these paths, are disjoint.

For i odd and $1 \leq i \leq 2\ell - 3$, let n_i be the greatest positive integer such that $4k - i - n(2\ell - 1) > v + (2\ell - 1)$. Then $v + (2\ell - 1) \geq 4k - i - (n_i + 1)(2\ell - 1) > v$; otherwise, this would contradict the definition of n_i. The path endings of the paths P_i have two possibilities, depending on whether v is adjacent to $4k - i - (n_i + 1)(2\ell - 1)$.

If v is adjacent to $4k - i - (n_i + 1)(2\ell - 1)$, we label $4k - i - (n_i + 1)(2\ell - 1) = t_i$. Then P_i is of the form

$$0(4k-i)(4k-i-(2\ell-1))(4k-i-2(2\ell-1))\dots(4k-i-n_i(2\ell-1))t_iv. \quad (6.1)$$

If v is not adjacent to $4k - i - (n_i + 1)(2\ell - 1)$, then these vertices are of the same parity, and thus $4k - i - (n_i + 1)(2\ell - 1) - 1$ must be adjacent to v. Then we label $4k - i - (n_i + 1)(2\ell - 1) = t_i + 1$ and $4k - i - (n_i + 1)(2\ell - 1) - 1 = t_i$. Then P_i is the form

$$0(4k-i)(4k-i-(2\ell-1))(4k-i-2(2\ell-1))\dots(4k-i-n_i(2\ell-1))(t_i+1)t_iv. \quad (6.2)$$

By construction of the paths P_i where i is odd and $1 \leq i < 2\ell - 3$, notice that for each vertex u visited on P_i, the vertex $u + 2$ is on P_{i+1}. However, if $i = 2\ell - 3$, so we are considering $P_{2\ell-3}$, if u is a vertex on $P_{2\ell-3}$, then, since the parity of vertices changes with every step, we have that $u+3$ is on P_1. Thus, the gaps (meaning vertices not on any of the P_is) between paths consist of either one or two vertices. So, we consider three cases, depending on the gap near $v + 2\ell - 1$, i.e., if there is an i such that P_i contains $v+2\ell-1$, or such that P_i contains $v+2\ell-2$, or such that P_i contains $v + 2\ell - 3$.

Case 1: There exists an i such that P_i contains $v + 2\ell - 1$.

Since v is adjacent to $v + (2\ell - 1)$, the path P_i will be of the form

$$0(4k-i)(4k-i-(2\ell-1))\dots(4k-i-n_i(2\ell-1))(v+(2\ell-1))v.$$

By our definition of t_i, we have that $t_i = v + 2\ell - 1$. Let $n = n_i$. The paths P_j which follow for $i < j \leq 2\ell - 3$ will be of the same form as P_i, since $n + 1$ edges of circular distance $2\ell - 1$ are needed to reach the vertex t_j. However, if $1 \leq j < i$, then P_j will require an additional step, $4k - j - (n+2)(2\ell - 1)$ in order to reach t_j. But $4k - j - (n+2)(2\ell - 1)$ is of distinct parity than v, and thus, we add the vertex $4k - j - (n+2)(2\ell - 3) - 1$ to P_j in order to reach v. In other words, for $1 \leq j < i$, we have $t_j + 1 = 4k - j - (n+2)(2\ell - 3)$ and $t_j = 4k - j - (n+2)(2\ell - 3) - 1$. Our $\ell - 1$ counterclockwise $0, v$-paths are of the following form.

$$0(4k-i)(4k-i-(2\ell-1))\dots(4k-i-n(2\ell-1))t_i v$$
$$0(4k-(i+2))\dots(4k-(i+2)-n(2\ell-1))t_{i+2} v$$
$$\vdots$$
$$0(4k-(2\ell-3))\dots(4k-(2\ell-3)-n(2\ell-1))t_{2\ell-3} v$$
$$0(4k-1)\dots(4k-1-(n+1)(2\ell-1))(t_1+1)t_1 v$$
$$\vdots$$
$$0(4k-(i-2))\dots(4k-(i-2)-(n+1)(2\ell-1))(t_{i-2}+1)t_{i-2} v$$

Notice that if $i = 1$, then all of the $0, v$-paths that follow will be of the same form as P_1. Since $t_j \equiv (4k - j) \mod (2\ell - 1)$ for $i < j \leq 2\ell - 3$ and $t_j + 1 \equiv (4k - j) \mod (2\ell - 1)$ for $1 \leq j < i$, then $t_i \neq t_j$ for all $i \neq j$. Hence, no two paths P_i and P_j share the same path endings, meaning they are independent.

Case 2: There exists an i such that P_i contains $v + 2\ell - 2$.

The vertex $v + (2\ell - 2)$ is not adjacent to v, but since the vertex $v + (2\ell - 2) - 1$ is adjacent to v, the path P_i will be of the form

$$0(4k-i)(4k-i-(2\ell-1))\dots(4k-i-n_i(2\ell-1))(v+(2\ell-2))(v+(2\ell-2)-1)v.$$

In this case, we have $t_i + 1 = v + (2\ell - 2)$ and $t_i = v + (2\ell - 3)$. Again, let $n = n_i$. The paths P_j which follow, for i odd and $i < j \leq 2\ell - 3$ will be of the same form as P_i, since $n + 1$ edges of circular distance $2\ell - 1$ are needed to reach t_j. However, for

$1 \leq j < i$, P_j will require an additional step, $4k - j - (n+2)(2\ell - 1)$, in order to reach t_j. Since $4k - j - (n+2)(2\ell - 1)$ is of distinct parity than v, it is adjacent to v. For $1 \leq j < i$, we label $4k - j - (n+2)(2\ell - 1) = t_j$. Our $\ell - 1$ counterclockwise $0, v$-paths are of the following form.

$$0(4k - i)(4k - i - (2\ell - 1)) \dots (4k - i - n(2\ell - 1))(t_i + 1)t_i v$$
$$0(4k - (i+2)) \dots (4k - (i+2) - n(2\ell - 1))(t_{i+2} + 1)t_{i+2} v$$
$$\vdots$$
$$0(4k - (2\ell - 3)) \dots (4k - (2\ell - 3) - n(2\ell - 1))(t_{2\ell-3} + 1)t_{2\ell-3} v$$
$$0(4k - 1) \dots (4k - 1 - (n+1)(2\ell - 1))t_1 v$$
$$\vdots$$
$$0(4k - (i-2)) \dots (4k - (i-2) - (n+1)(2\ell - 1))t_{i-2} v$$

Notice again that if $i = 1$, then all of the $0, v$-paths that follow will be of the same form as P_1. By the same argument made in Case 1, $t_i \neq t_j$ for all $i \neq j$, and hence, no two paths P_i and P_j share the same path endings, meaning they are independent.

Case 3: There exists an i such that $t_i = v + (2\ell - 3)$.

Since $v + (2\ell - 3)$ is adjacent to v, our counterclockwise independent $0, v$-paths are of the same form as in Case 1.

We have shown that when $\ell \geq 3$, we can find ℓ properly colored paths. This completes the proof, and thus, $pc_\ell(C_{4k,\ell}) = 2$. □

2 Future directions

Our result can be easily extended to 2ℓ-connected bipartite circulant graphs with an odd number of vertices, $C_{4k-1,\ell}$, although we have yet to formally prove this. Naturally, any bipartite graph G which contains $C_{4k,\ell}$ as a subgraph has ℓ-proper connection number 2. However, such graphs only consist of a small class of bipartite graphs. Proving the conjecture by Borozan et al. becomes much more difficult when considering graphs with a less predictable placement of edges.

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