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Independent Dominating Sets in Unicyclic Graphs

Rachel Lopez

Abstract

Wilf found the maximum number of independent dominating sets of a tree using algebraic methods, while Sagan gave an elementary proof. In this thesis, we maximize the number of independent dominating sets of unicyclic graphs, giving a new proof of a result of Jou and Chang. In our proof, we are able to reduce the problem to finding independent dominating sets of single-legged caterpillar graphs. We also study the number of single-legged caterpillar graphs, both oriented and unoriented, which are related to the Fibonacci Sequence. Finally, this thesis also examines the domination ratio in unicyclic graphs. The domination ratio is the quotient of the minimum size of an independent dominating set and the minimum size of a dominating set. Using a generalization of a technique of Furuya et al., we find a new bound on this ratio for unicyclic graphs.

MONTCLAIR STATE UNIVERSITY

Independent Dominating Sets in Unicyclic Graphs

by

Rachel Lopez

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Chapter 1

Introduction

This chapter will introduce the necessary terminology to identify an independent dominating set, as well as define unicyclic graphs. We will also discuss previous results on maximizing independent dominating sets and the domination ratio.

1.1 Terminology

First and foremost, we can begin by defining some graph theory basics relevant to this research.

Definition 1.1.1. A *graph* G has a vertex set V and an edge set E such that E is a set of 2-element subsets of V. Elements in V are known as *vertices*, and elements in E are known as *edges*.

Definition 1.1.2. Suppose u and v are two distinct vertices in a vertex set of a graph G. If $\{u, v\}$ is in the edge set of G, then u and v are *adjacent*. One can say that adjacent vertices are *neighbors*.

Definition 1.1.3. The number of neighbors of a vertex v has is known as the *degree* of v, denoted as $d(v)$. A vertex of degree 1 is referred to as a *leaf*.

Definition 1.1.4. The *open neighborhood* of a vertex v, denoted as $N(v)$ is the set of vertices which are adjacent to v. Similarly, the *closed neighborhood* of a vertex v, denoted as $N[v]$ is the set of vertices in which are adjacent to v, including v in the set (or $N[v] = N(v) \cup \{v\}$).

Now we can define independent sets and dominating sets for a graph G.

Definition 1.1.5. In a graph G, a subset of the vertices is *independent* if it contains no edges. Figure 1.1 shows an independent set in a graph.

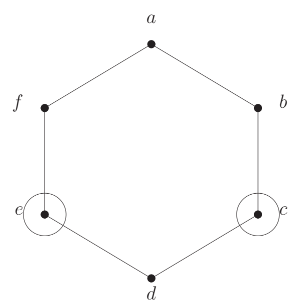

Figure 1.1: $\{a, c\}$ is an independent set since a and c do not share a common edge.

Definition 1.1.6. A subset U of the vertex set V is *dominating* if every vertex in V but not in U has at least one neighbor in U. Figure 1.2 shows a dominating set in a graph.

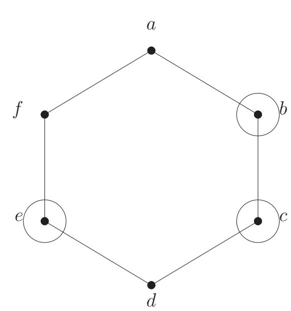

Figure 1.2: $U = \{b, c, e\}$ is a dominating set since a, d, and f have at least one neighbor in U.

A set that is both independent and dominating is also a maximal independent set. This is because if we were to add another vertex to the set, it would be adjacent to at least one vertex already in the set, making it no longer independent. Figure 1.3 shows an independent dominating set in a graph.

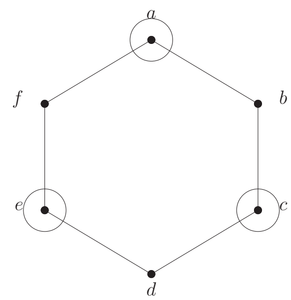

Figure 1.3: $\{a, c, e\}$ is an independent dominating set since each vertex in the set does not share a common edge, and all vertices not in the set have two neighbors in the set.

We let $\partial_i(G)$ be the number of independent dominating sets of G.

Since we are focusing on a smaller set of graphs, unicyclic graphs, we must also define what they are using ideas of other sets of graphs.

Definition 1.1.7. A graph G is a *path* on n distinct vertices $x_1, x_2, \cdots, x_n$, denoted as P_n, if edge set $E(P_n) = \{\{x_i, x_{i+1}\} | i = 1, 2, \cdots, n-1\}$. A graph G is a *cycle*, denoted as C_n if edge set $E(C_n) = E(P_n) \cup \{x_1, x_n\}$ for $n \geq 3$.

Definition 1.1.8. A graph G is *connected* if there exists a path from vertex u to vertex v, for every u and v in the vertex set of G.

Definition 1.1.9. A graph G is *acyclic* if there is no cycle in G.

Definition 1.1.10. A graph G is a *tree* if G is connected and acyclic.

It turns out that one can prove that every tree has $n-1$ edges. In order to do that, we need to state and prove some properties of trees.

Lemma 1.1.1. *Every tree T on n vertices such that $n \geq 2$ has at least two leaves.*

Proof. Suppose P is a maximal path in T. Then because P is maximal, the endpoints of the path must be adjacent to vertices already in the path. But if we add another edge to P, we end up getting a cycle, making T no longer a tree. Therefore, T must have at least two leaves. $\square$

We can also say that removing a leaf from a tree produces a tree containing $n-1$ vertices.

Lemma 1.1.2. *If a tree T contains n vertices, then removing a leaf produces a tree on $n-1$ vertices.*

Proof. Suppose T has n vertices, and vertex v is a leaf in T. Then deleting v only removes one edge, making its neighbor an endpoint. Since T is acyclic and connected, removing that edge does not add a cycle or disconnect the graph. Thus, we have a new tree on $n-1$ vertices. $\square$

Now we have enough information to show that a tree has $n-1$ edges.

Theorem 1.1.3. *For any tree G on n vertices, there always exists $n-1$ edges.*

Proof. Suppose G is a tree. We can prove this using induction for n.

When $n = 1$, there contains no edges since the vertex is isolated. So suppose $n \geq 2$ such that for any tree G on k vertices, there always exists $k-1$ edges. Also let v be a leaf in G. Then by Lemma 1.1.1 and Lemma 1.1.2, removing v would give us a tree on $n-1$ vertices on $(n-1)-1 = n-2$ vertices. Therefore, every tree with n vertices must have $n-1$ edges. $\square$

Definition 1.1.11. A connected graph G on three or more vertices is *unicyclic* if it contains exactly one cycle.

Unicyclic graphs, graphs with exactly one cycle, can be obtained by adding a single edge in a tree, making the graph unicyclic with n edges.

Alternatively, we can "build" a unicyclic graph by starting with a cycle, say C_k. Then perhaps we can add trees to any vertex of C_k. (C_k,T_1,T_2,...,T_k) where the vertices of C_k are labeled $v_1, ..., v_k$ and T_i is a tree off of v_i.

1.2 Previous Results

1.2.1 Maximizing the Number of Independent Dominating Sets

Results on maximizing the number of independent dominating sets have been proven on different sets of graphs. Wilf [12] first was able to maximize the number of independent dominating sets of trees. Theorem 1.2.1 shows this result.

Theorem 1.2.1 (Wilf [12]). *If T is a tree on n vertices, then*

$$\partial_i(T) \leq \begin{cases} 2^{\frac{n}{2}-1}+1 & \textit{if } n \textit{ is even} \\ 2^{\frac{n-1}{2}} & \textit{if } n \textit{ is odd.} \end{cases}$$

Sagan [10] reproved Theorem 1.2.1. In his paper, he mentions extremal graphs, which are defined in Definition 1.2.1.

Definition 1.2.1. A graph G is *extremal* on n vertices if it contains the maximum number of independent dominating sets among all graphs of a given type on n vertices.

Since Sagan [10] reproved Theorem 1.2.1 for T, he was able to find extremal trees on n vertices such that no other trees on n vertices have a greater number of independent dominating sets. The extremal graphs he found consist of a path of length ℓ, where $\ell = 0$ or $\ell = 1, 3$ when n is odd or even respectively, such that any number of paths of length 2 were attached to the endpoints. He referred to these extremal graphs as batons.

In fact, Sagan [10] reproved the result of Wilf [12] using elementary methods to obtain extremal graphs on trees containing n vertices, as Wilf proved the upper bound of the number of independent dominating sets of trees using algebraic methods.

Griggs et al. [4], on the other hand, used the ideas of Sagan [10] and Wilf [12] to find the maximal independent set for connected graphs, a larger set of graphs, along with obtaining the corresponding extremal graphs. Their result is the following.

Theorem 1.2.2 (Griggs et al. [4])**.** *If G is a connected graph on n vertices, then*

$$\partial_i(G) \leq \begin{cases} 2 \cdot 3^{\frac{1}{3}n-1} + 2^{\frac{1}{3}n-1} & \text{if } n \equiv 0 \text{ (mod 3)} \\ 3^{\lfloor \frac{1}{3}n \rfloor} + 2^{\lfloor \frac{1}{3}n \rfloor -1} & \text{if } n \equiv 1 \text{ (mod 3)} \\ 4 \cdot 3^{\lfloor \frac{1}{3}n \rfloor -1} + 3 \cdot 2^{\lfloor \frac{1}{3}n \rfloor -2} & \text{if } n \equiv 2 \text{ (mod 3).} \end{cases}$$

Koh et al. [7] were actually able to maximize the number of independent dominating sets of connected unicyclic graphs and were able to find the corresponding extremal graphs for this result.

Jou and Chang [5] were able to maximize the number of independent dominating sets of graphs containing at most one cycle.

Theorem 1.2.3 (Jou and Chang [5])**.** *If G is a graph on $n \geq 3$ vertices that contain at most one cycle, then*

$$\partial_i(G) \leq \begin{cases} 3 \cdot 2^{\frac{n-4}{2}} & \text{if } n \text{ is even} \\ 2^{\frac{n-1}{2}} + 1 & \text{if } n \text{ is odd.} \end{cases}$$

We will be reproving this result using an alternative method by modifying Sagan's

[10] approach.

While proving Theorem 1.2.3 using our alternative method, we will be seeing a relationship between unicyclic graphs and caterpillar graphs. Ortiz and Villanueva [8] were able to maximize the number of independent dominating sets of caterpillar graphs in some sense. Caterpillar graphs consist of a path containing pendant edges hanging off of the path's vertices. In fact, if a given vertex is incident to multiple pendant edges, all but one can be removed as it does not impact the number of independent dominating sets. This was also proven by Ortiz and Villanueva [8], which is shown below.

Lemma 1.2.4 (Ortiz and Villanueva [8])**.** *In a graph G, if any given vertex is incident to multiple pendant edges, then removing all but one pendant edge does not impact the number of independent dominating sets.*

To summarize, this is because the leaves on the pendant edges share the same neighbor, a vertex in the path. We refer to these leaves as *twin vertices*. This idea will be important as we analyze how we will maximize the number of independent dominating sets of unicyclic graphs.

1.2.2 Domination Ratio

In a graph G, Suppose $\gamma_i(G)$ is the independent domination number of G, or the minimum size of an independent dominating set of G. We can also let $\gamma(G)$ be the domination number of G, or the minimum size of a dominating set of G. Let the domination ratio $\frac{\gamma_i(G)}{\gamma(G)} = \rho(G)$.

Rad and Volkmann [9] were able to prove $\rho(G)$ for a graph G.

Theorem 1.2.5 (Rad and Volkmann [9]). *If G is a graph, then*

$$\rho(G) \leq \begin{cases} \frac{\Delta(G)}{2} & \text{if } 3 \leq \Delta(G) \leq 5 \\ \Delta(G) - 3 + \frac{2}{\Delta(G)-1} & \text{if } \Delta(G) \geq 6. \end{cases}$$

Furuya et al. [1] were able to prove a stronger upper bound by maximizing $\rho(G)$ for simple undirected graphs of G using the maximum degree of G, denoted as $\Delta(G)$. Below shows this result.

Theorem 1.2.6 (Furuya et al. [1]). *For any graph G,*

$$\rho(G) \leq \Delta(G) - 2\sqrt{\Delta(G)} + 2.$$

Goddard et al. [2], on the other hand, were able to maximize $\rho(G)$ for connected cubic graphs, graphs where for all $v \in V(G)$, $d(v) = 3$.

Theorem 1.2.7 (Goddard et al. [2]). *If G is a cubic graph, then*

$$\rho(G) \leq \frac{3}{2}.$$

Meanwhile Knor et al. [6] maximized $\rho(G)$ for connected k-regular graphs, graphs where every vertex is of degree k.

Theorem 1.2.8 (Knor et al. [6]). *If G is a k-regular graph such that $k \geq 3$, then*

$$\rho(G) \leq \frac{k}{2}.$$

Wang and Wei [11] were able to do the same for trees using the structure of Furuya et al. [1] proof and the fact that every tree has a leaf.

Theorem 1.2.9 (Wang and Wei [11]). *If G is a tree, then*

$$\rho(G) \leq \frac{\Delta(G)}{2},$$

provided $\Delta(G) \geq 3$.

As researchers have been able to calculate $\rho(G)$ for different graphs of G, we become interested in the upper bound for the domination ratio in unicyclic graphs. We get the following result which will be discussed in further detail in Chapter 4.

Theorem 1.2.10. *If G is a unicyclic graph, then*

$$\rho(G) \leq \frac{2}{3}\Delta(G) - 1.$$

Chapter 2

Maximizing Independent Dominating Sets

2.1 Theorem for Unicyclic Graphs

We were able to adapt a proof technique introduced by Sagan [10] to prove a similar result for unicyclic graphs, reproving the result of Jou and Chang [5]. The theorem we will prove is as follows.

Theorem 2.1.1. *If G is a unicyclic graph on n vertices, then*

$$\partial_i(G) \leq \begin{cases} 3 \cdot 2^{\frac{n-4}{2}} & \textit{if } n \textit{ is even} \\ 2^{\frac{n-1}{2}} + 1 & \textit{if } n \textit{ is odd.} \end{cases}$$

This maximizes the number of independent dominating sets of unicyclic graphs. Those bounds can be obtained from the extremal graphs on n vertices. The extremal graphs in Figures 2.1-2.2 give us the maximum number of independent dominating sets of unicyclic graphs.

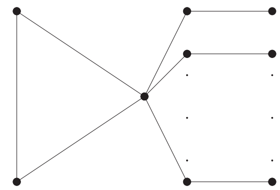

Figure 2.1: Extremal graph when n is odd.

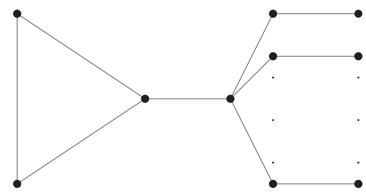

Figure 2.2: Extremal graph when n is even.

When n is odd, the extremal graph is a triangle with legs off one vertex such that all legs have two edges. When n is even, the extremal graph is a triangle with an edge off one vertex such that legs of length two hang off the edge on the vertex not corresponding to the triangle.

2.2 Preliminaries

When constructing unicyclic graphs, there may be situations where there are pendant edges hanging off a vertex in a cycle, which we will call "curly caterpillars". Suppose a vertex v in a cycle has multiple pendant edges. Then when observing any given independent dominating set, either all the pendant vertices are in the set, or they are all not in the set. As Ortiz and Villanueva [8] previously discovered, the set of pendant vertices are independent, but not maximal. This is because they all have the same neighborhood. Thus by Lemma 1.2.4, what we can do to make it maximal is to remove all but one pendant edge to the curly caterpillar. This will not affect the number of independent dominating sets because all of the pendant vertices are either in the independent dominating set, or not in the independent dominating set.

When examining the number of independent dominating sets of any graph, we can partition it based off the number of independent dominating sets containing a vertex v, denoted as $\partial_i(G, v)$ and the number of independent dominating sets not containing v, denoted as $\partial_i(G, \overline{v})$. Below we have this lemma.

Lemma 2.2.1. *If G is a graph with a vertex v,*

$$\partial_i(G) = \partial_i(G, v) + \partial_i(G, \overline{v}).$$

Proof. Suppose G is a graph having a vertex v with ℓ neighbors, such that $x_1, x_2, \cdots, x_\ell \in N(v)$. Let S be an independent dominating set such that $v \in S$. Then $x_1, x_2, \cdots, x_\ell \notin S$ as otherwise the set would not be independent.

Similarly, let S' be an independent dominating set such that $v \notin S'$. Then by Definition 1.1.6, at least one vertex in $N(v)$ must be in the independent dominating set so that v is dominated. Thus there is no set that is both S and S', making them disjoint sets.

And since $\partial_i(G, v)$ will count all the possible S sets, and $\partial_i(G, \overline{v})$ will count all the possible S' sets, both of which are disjoint, we can conclude that the lemma holds. □

2.3 Proof of Theorem 2.1

When proving this theorem, we will split into cases depending on whether or not the graph has a penultimate vertex, defined below.

Definition 2.3.1. A vertex v in a graph G is said to be *penultimate* if $d(v) \geq 2$ and all but at most one of its neighbors is a leaf. If v is a penultimate vertex, then $G - v$ contains $d(v) - 1$ isolated vertices and one component, which we call the *penultimate component corresponding to v*. (Note that if v is the center of a star, we choose one of its neighbors to be the penultimate component.)

When G contains a penultimate vertex, we will use the following generalization of a lemma of Sagan [10] that finds a bound on $\partial_i(G)$.

Lemma 2.3.1. *If G is a graph with penultimate vertex v and corresponding component P, then*

$$\partial_i(G) \leq 2\partial_i(P).$$

Proof. Suppose G is a graph with a penultimate vertex v and corresponding to component P. Let q be the vertex in P and adjacent to v, and $x_1, x_2, \cdots, x_\ell$ such that $\ell \geq 1$ be leaf vertices not in P and adjacent to v. By Lemma 2.2.1, $\partial_i(G) = \partial_i(G, q) + \partial_i(G, \overline{q})$.

We first consider independent dominating sets of G containing q. If S is an independent dominating set of G and $q \in S$, then $v \notin S$, as otherwise q and v would no longer be independent. Since $v \notin S$, then $x_1, \cdots, x_\ell \in S$ as their only neighbor v is not in S. In the penultimate component P, we can obtain all of the independent dominating sets containing q, the number of said sets being $\partial_i(P, q)$, and then add $x_1, \cdots, x_\ell$ to every set of order to obtain the independent dominating sets of G. Thus we can say that $\partial_i(G, q) = \partial_i(P, q)$.

Now we must consider independent dominating sets of G not containing q. If S' is an independent dominating set of G and $q \notin S'$, then either $v \in S'$ or $x_1, \cdots, x_\ell \in S'$. In both cases, we can obtain all of the independent dominating sets of P not containing q, the number of said sets being $\partial_i(P, \overline{q})$, and then either include v in every independent dominating set obtained, or all of $x_1, \cdots, x_\ell$ in every independent dominating set obtained to get the independent dominating sets of G. Since we are ultimately using the independent dominating sets of P not containing q, and then either adding v or $x_1, \cdots, x_\ell$ to every set, we can say that $\partial_i(G, \overline{q}) = 2\partial_i(P, \overline{q})$.

Now that we have defined $\partial_i(G, q)$ and $\partial_i(G, \overline{q})$ in terms P, we can say that

$$\begin{aligned}
\partial_i(G) &= \partial_i(G, q) + \partial_i(G, \overline{q}) \\
&= \partial_i(P, q) + 2\partial_i(P, \overline{q}) \\
&\leq 2\partial_i(P, q) + 2\partial_i(P, \overline{q}) \\
&= 2[\partial_i(P, q) + \partial_i(P, \overline{q})] \\
&= 2\partial_i(P).
\end{aligned}$$

Therefore, $\partial_i(G) \leq 2\partial_i(P)$. □

When maximizing the number of independent dominating sets of a graph G, we must make sure that there exists no graph with a greater number of independent dominating sets. G would then need to be an extremal graph. And since there exists a penultimate vertex, we would need to figure out its maximum degree for the graph to remain extremal, which is shown in the following Lemma.

Lemma 2.3.2. *If G is extremal and v is a penultimate vertex, then $d_G(v) \leq 2$.*

Proof. Suppose G has a penultimate vertex v with a corresponding penultimate component P. We label the leaves neighboring v by $x_1, x_2, \cdots, x_\ell$ such that $\ell \geq 2$. We also label the neighbor of v in P by u. By Lemma 2.2.1, $\partial_i(G) = \partial_i(G, v) + \partial_i(G, \overline{v})$.

We will show that in the case that $\ell \geq 2$ that G cannot be extremal by finding a graph with more independent dominating sets than G. Let G' be the graph formed from G by removing the edges $vx_{\ell-1}$ and vx_ℓ and adding the edges $ux_{\ell-1}$ and $x_{\ell-1}x_\ell$. Note that if $\ell \geq 3$, then v is still penultimate in G' with corresponding component P. However, if $\ell = 2$, then while v is no longer penultimate (since it is a leaf), we have that $x_{\ell-1}$ is penultimate with corresponding component P.

We first consider independent dominating sets containing v. If S is an independent dominating set of G with $v \in S$, then none of the x_i, for $i = 1, 2, \cdots, \ell$, are in S. While S is not an independent dominating set of G' (since x_ℓ and $x_{\ell-1}$ are not dominated), we can get two independent sets by taking $S \cup \{x_\ell\}$ or $S \cup \{x_{\ell-1}\}$. Thus, we've shown that $\partial_i(G', v) = 2\partial_i(G, v)$.

Now, we consider independent dominating sets not containing v. If S' is an independent dominating set of G with $v \notin S'$, then it must be that $x_i \in S'$ for all $i = 1, 2, \cdots, \ell$. We can get an independent dominating set of G' by taking $S' \setminus \{x_{\ell-1}\}$. Thus, there are at least as many independent dominating sets of G' not containing v as there are in G. (We may get more since $S' \setminus \{x_\ell\}$ may also be an independent dominating set of G'.) Thus, $\partial_i(G', \overline{v}) \leq \partial_i(G, \overline{v})$.

Therefore, provided $l \geq 2$, or $d(v) \geq 3$, where $d(v)$ is the degree of v, we have

$$\begin{aligned}
\partial_i(G') &= \partial_i(G', v) + \partial_i(G', \overline{v}) \\
&\geq 2\partial_i(G, v) + \partial_i(G, \overline{v}) \\
&> \partial_i(G, v) + \partial_i(G, \overline{v}) \\
&= \partial_i(G).
\end{aligned}$$

Thus for G to be extremal, $d_G(v) \leq 2$. □

By the definition of a penultimate vertex, and by the previous lemma, all penultimate vertices in an extremal graph must have degree 2.

Now we have to consider situations on how the graph is not extremal and determine it's number of independent dominating sets to ensure Theorem 2.1.1 still holds. The first situation is when the penultimate component is not extremal.

Lemma 2.3.3. *Suppose G is a unicyclic graph where all penultimate vertices have degree 2. If there exists a penultimate vertex whose corresponding component is not* *extremal, then*

$$\partial_i(G) \leq \begin{cases} 3 \cdot 2^{\frac{n-4}{2}} & \textit{if } n \textit{ is even} \\ 2^{\frac{n-1}{2}} + 1 & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. Suppose G is a unicyclic graph where all penultimate vertices have degree 2, and let v be a penultimate vertex whose corresponding component is not extremal. We can prove this result using induction.

We can start by using Lemma 2.2.1 to prove our base cases for $n = 5, 6$, shown in Figure 2.3.

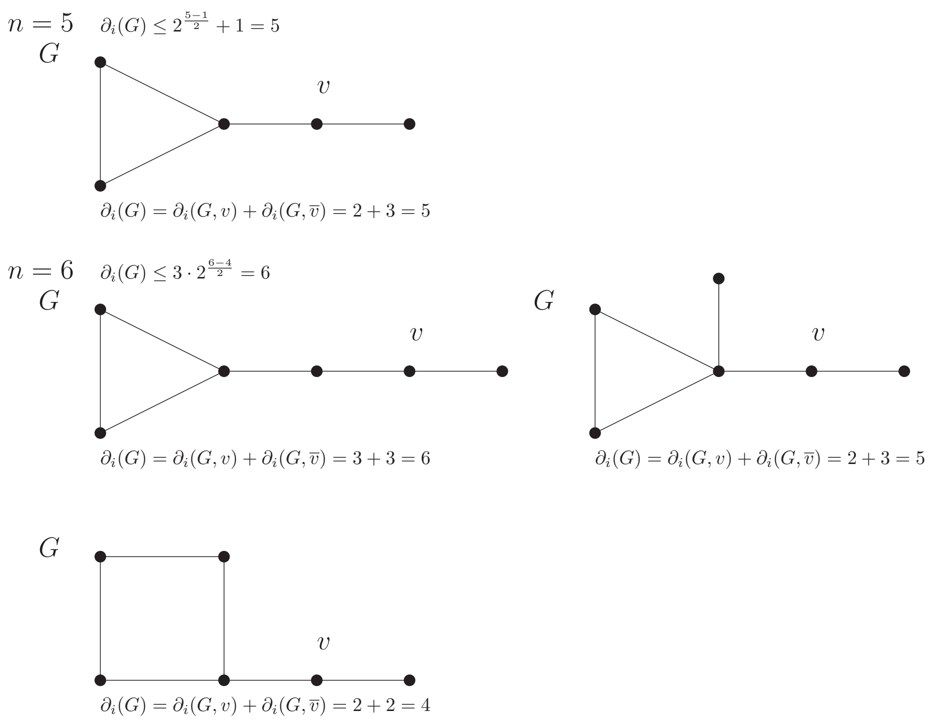

Figure 2.3: Corresponding component of penultimate vertex v is not extremal for $n = 5$ and $n = 6$.

We can see that the inequality holds for $n = 5$ and $n = 6$, where we get equality from the extremal graphs.

Assume P is not extremal on $n - 2$ vertices, since the penultimate vertex and it's neighboring leaf is not in the penultimate component. In this case we can use Lemma 2.3.1.

If n is odd, using induction, we can say that

$$\partial_i(G) \leq 2\partial_i(P) < 2(2^{\frac{(n-2)-1}{2}} + 1) = 2^{\frac{n-1}{2}} + 2.$$

Therefore, we can say that $\partial_i(G) \leq 2^{\frac{n-1}{2}} + 1$.

Similarly if n is even,

$$\partial_i(G) \leq 2\partial_i(P) < 2(3 \cdot 2^{\frac{(n-2)-4}{2}}) = 3 \cdot 2^{\frac{n-4}{2}}.$$

Therefore, we can say that $\partial_i(G) \leq 3 \cdot 2^{\frac{n-4}{2}}$. □

Now we have that in an extremal graph, every penultimate vertex has degree 2 and its corresponding component is extremal on $n-2$ vertices. We have to consider situations where the penultimate component is extremal, but not the graph itself. That way, we can show that there exists a different penultimate component that is not extremal. Then we can say that by Lemma 2.3.3, the maximum number of independent dominating sets still holds from Theorem 2.1.1.

Lemma 2.3.4. *Suppose G is a unicyclic graph where all penultimate vertices have degree 2. If there is a penultimate vertex whose corresponding component is extremal, but the graph is not extremal, then there exists a penultimate vertex whose component is not extremal.*

Proof. If there exists a penultimate vertex attached to a wrong spot, say vertex v, then we must check if G is still extremal. Suppose we start with the extremal graph on $n-2$ vertices such that x is a penultimate vertex. Let a be a different penultimate vertex adjacent to a leaf vertex b, and let d be a vertex such that d is not adjacent to any penultimate vertex.

To start, suppose n is odd. Figure 2.4 shows the extremal graph on $n-2$ vertices, given the defined vertices.

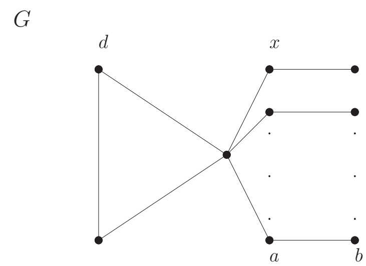

Figure 2.4: Extremal unicyclic graph when n is odd.

For when n is odd, Suppose we look at x and its corresponding penultimate component. We can conclude that the penultimate component of x is not extremal on $n-2$ vertices in three different cases, following Lemma 2.3.3. Thus $\partial_i(G) \leq 2^{\frac{n-1}{2}} + 1$. Below shows the three different cases in which the the penultimate vertex v is attached to the wrong spot.

1. Suppose penultimate vertex v is adjacent b. Figure 2.5 shows the new graph G on n vertices.

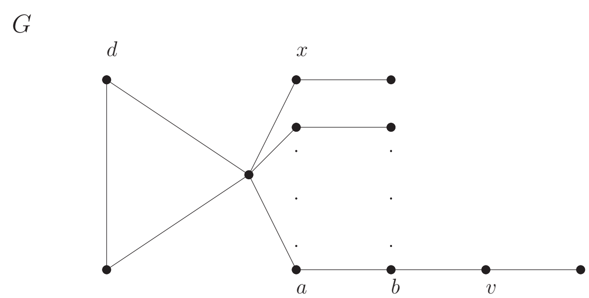

Figure 2.5: Case 1: Penultimate vertex v attached to the wrong spot when n is odd.

2. Suppose penultimate vertex v be adjacent is a. Figure 2.6 shows the new graph G on n vertices.

3. Suppose Penultimate vertex v be adjacent is d. Figure 2.7 shows the new graph G on n vertices.

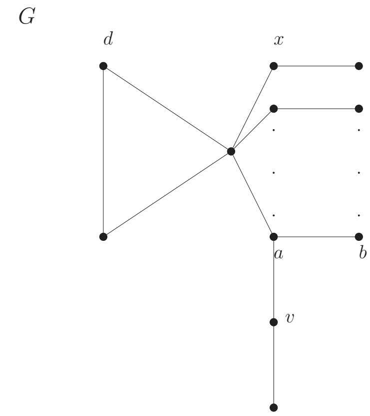

Figure 2.6: Case 2: Penultimate vertex v attached to the wrong spot when n is odd.

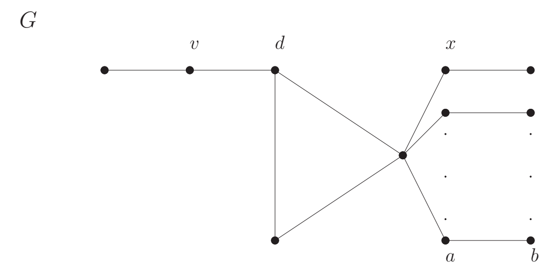

Figure 2.7: Case 3: Penultimate vertex v attached to the wrong spot when n is odd.

For every case, if we looked at the corresponding penultimate component of v, we can conclude that the penultimate component is extremal on $n - 2$ vertices, but G itself is not extremal.

Similarly, suppose n is even. We have an additional vertex f such that f is adjacent to d. Figure 2.8 shows the extremal graph on $n - 2$ vertices, given the defined vertices.

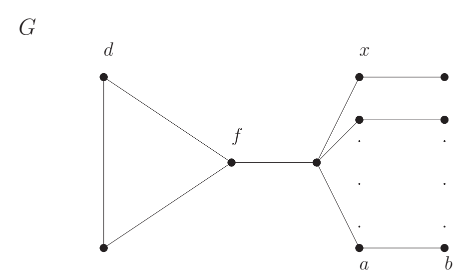

Figure 2.8: Extremal unicyclic graph when n is even.

For when n is even, there are four cases on how a penultimate vertex v is attached

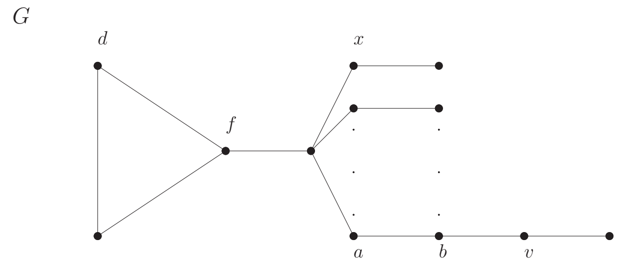

Figure 2.9: Case 1: Penultimate vertex v attached to the wrong spot when n is even.

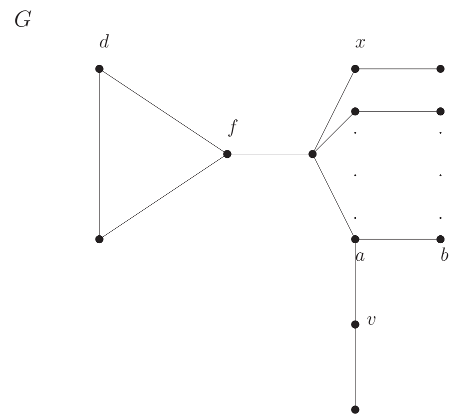

Figure 2.10: Case 2: Penultimate vertex v attached to the wrong spot when n is even.

to the wrong spot. Again, we can conclude that the penultimate component of x is not extremal on $n-2$ vertices in the four different cases below, following Lemma 2.3.3. Thus $\partial_i(G) \leq 3 \cdot 2^{\frac{n-4}{2}}$.

1. Suppose penultimate vertex v be adjacent is b. Figure 2.9 shows the new graph G on n vertices.

2. Suppose penultimate vertex v be adjacent is a. Figure 2.10 shows the new graph G on n vertices.

3. Suppose penultimate vertex v be adjacent is f. Figure 2.11 shows the new graph G on n vertices.

4. Suppose penultimate vertex v be adjacent is d. Figure 2.12 shows the new graph G on n vertices.

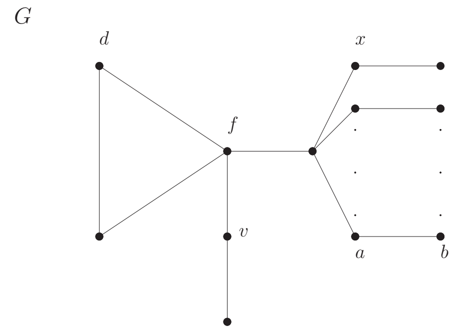

Figure 2.11: Case 3: Penultimate vertex v attached to the wrong spot when n is even.

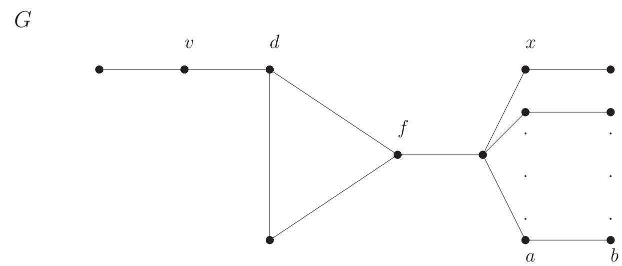

Figure 2.12: Case 4: Penultimate vertex v attached to the wrong spot when n is even.

Similar to when n is odd, looking at the penultimate component of v in every case concludes that the penultimate component is extremal on $n-2$ vertices, but G itself is not extremal.

Thus this lemma holds true, where there exists a penultimate vertex who's component is not extremal. □

Because of Lemma 2.3.3, all the new graphs constructed in Lemma 2.3.4 shows that Theorem 2.1.1 holds true when there exists a penultimate vertex.

We have outlined how we would maximize the number of independent dominating sets of unicyclic graphs that have a penultimate vertex. But what if such a vertex doesn't exist? Then we can assume that it is a "curly caterpillar", or a unicyclic graphs that have pendant edges on the vertices in the cycle.

Throughout the proof, we see that curly caterpillars can be split into 2 "(uncurly) caterpillars" using a recursion, or paths with pendant edges on any given vertex. And from there, we can maximize the number of independent dominating sets from the single-legged caterpillars. Recall that we are only considering single-legged caterpillars because of Lemma 1.2.4.

Theorem 2.3.5. *If D is an (uncurly) caterpillar on n vertices, then $\partial_i(D) \leq \partial_i(P_n)$.*

Proof. Let D be a caterpillar with a spine length, or path, of $k+2$. We can define this caterpillar as a binary code with length k, where 0 represents no leg, or pendant edge, and 1 represents a leg on the the i^{th} vertex on the spine, such that $1 \leq i \leq k$. If $k = 0$, then we have a path of length 2 as our single-legged caterpillar as there exists no vertex for a pendant edge to be incident to. We would denote this single-legged caterpillar as an empty binary code, $\emptyset$, in the case when $k = 0$.

Keep in mind that the first and last vertex on the spine is omitted in the binary code. This is because if either the first or last vertex on the spine has a leg, then the leg(s) can be used to extend the spine, making the new end vertices of the spine not have legs.

By Lemma 2.2.1, $\partial_i(D) = \partial_i(a_1a_2\cdots a_k) = \partial_i(D, a_1) + \partial_i(D, \overline{a_1}) = \partial_i(a_2'\cdots a_k') + \partial_i(a_3''\cdots a_k'')$ where the number of vertices

$$n = \sum_{i=1}^{k} a_i + 2 + k.$$

We can say that

$$a_2' = \begin{cases} 0 & \text{if } a_2 = 1 \\ \emptyset & \text{if } a_2 = 0 \end{cases}$$

and $a'_i = a_i$ for $3 \leq i \leq k$. Similarly,

$$a''_3 = \begin{cases} 0 & \text{if } a_3 = 1 \\ \emptyset & \text{if } a_3 = 0 \end{cases}$$

and $a''_i = a_i$ for $4 \leq i \leq k$.

For the recursion, we would need to know the number of independent dominating sets for all single-legged caterpillars on n vertices with $k = 0, 1, 2$, shown in Figure 2.13.

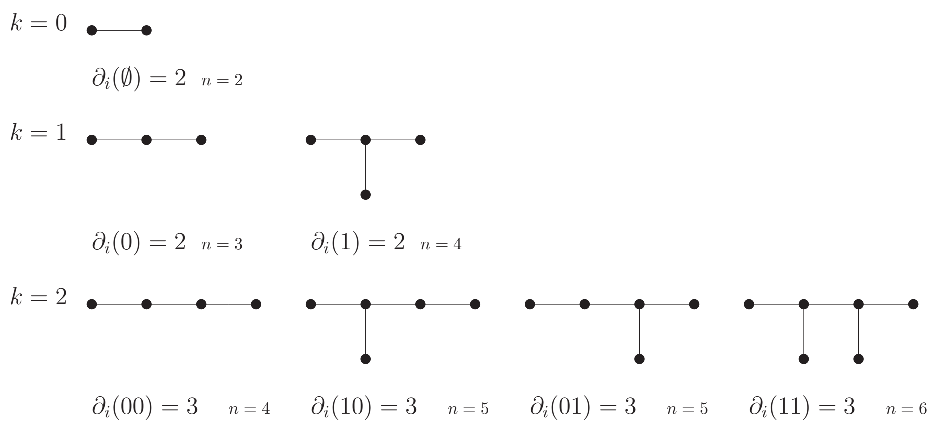

Figure 2.13: $\partial_i(D)$ for all single-legged caterpillars for $k = 0, 1, 2$.

We can already see that the number of independent dominating sets of single-legged caterpillars for $n = 2, 3, 4$ does not exceed the results of the no-legged caterpillars. We can prove the same for $n = 5$ using the recursion for the remaining single-legged caterpillars. We already have $D = 10$ and $D = 01$ for our single-legged caterpillars containing 5 vertices. But what about the path?

Using the recursion,

$$\partial_i(000) = \partial_i(0) + \partial_i(\emptyset) = 2 + 2 = 4.$$

For when $n = 5$, we can also see that the number of independent dominating sets on single-legged caterpillars is maximized by P_5. Now we can prove this lemma using induction.

Thus $\partial_i(a'_2 \cdots a'_k) + \partial_i(a''_3 \cdots a''_k) \leq \partial_i(0 \cdots\cdots 0) + \partial_i(0 \cdots 0)$. The caterpillar $0 \cdots\cdots 0$ is a path on $n-2$ or $n-3$ vertices, depending on whether a_1 does not or does contain a leg, respectively. Similarly, $0 \cdots 0$ is a caterpillar on $n-3$ or $n-5$ vertices, depending if neither a_1 and a_2 have legs, or if both a_1 and a_2 have legs, respectively.

A caterpillar with no legs are paths. And since the more vertices a path has, the greater number of independent dominating sets, we can use induction to say the following: $\partial_i(D) = \partial_i(a_1 a_2 \cdots a_k) \leq \partial_i(P_{n-2}) + \partial_i(P_{n-3}) = \partial_i(P_n)$. Therefore, the number of independent dominating sets on (uncurly) caterpillars on n vertices is maximized by the number of independent dominating sets of path on n vertices. $\square$

Now that we know how to maximize the number of independent dominating sets of curly caterpillars, we can maximize the number of independent dominating sets of curly caterpillars, which we can recall as cycles with pendant edges off of any vertex in the cycle. We will see the split of a curly caterpillar into two single-legged caterpillars in the following lemma.

Lemma 2.3.6. *If C is a curly caterpillar on n vertices, then $\partial_i(C) \leq \partial_i(C_n)$.*

Proof. We can assume that each vertex of C has at most 1 leaf neighboring it due to Lemma 1.2.4.

We can let v be a vertex on the cycle, then by Lemma 2.2.1, $\partial_i(C) = \partial_i(C-v) + \partial_i(C-N[v])$ where $N[v]$ represents the closed neighborhood of v. Both $C-v$ and $C-N[v]$ are (uncurly) caterpillars, or paths with a pendant edge hanging off of any vertex. We can say that $\partial_i(C) = \partial_i(C-v) + \partial_i(C-N[v]) \leq \partial_i(P_{n-1}) + \partial_i(P_{n-3}) = \partial_i(C_n)$. $\square$

Now we have enough information to maximize the number of independent dominating sets in unicyclic graphs.

Proof of Theorem 2.1.1. Assume G is an extremal (with respect to ∂_i) unicyclic graph on n vertices. If G contains a penultimate vertex, we can apply Lemmas 2.3.2, 2.3.3, and 2.3.4 to show that G is of the desired form.

If, on the other hand, G does not contain a penultimate vertex, then we apply Lemma 2.3.6 to see that $\partial_i(G) \leq \partial_i(C_n)$. However, it is well known that $\partial_i(C_n) = P_n$, where P_n is the n^{th} Perrin number. (The Perrin numbers satisfy the recurrence $P(n) = P(n-2) + P(n-3)$ where $P(1) = 0$, $P(2) = 2$, and $P(3) = 3$.). One, however, can check that P_n is less than our bounds for all $n \geq 5$, and so C_n is not extremal. (Asymptotically, $P_n \sim \rho^n$ where ρ is the so-called plastic constant, the root of $x^3 - x - 1$. It is known that $\rho \approx 1.32472$.) $\square$

Chapter 3

Single-Legged Curly Caterpillars

We were able to find a recursion that gives the number of independent dominating sets of a single-legged curly caterpillar which can be partitioned into two (uncurly) single-legged caterpillars. We did this by giving a binary code to each caterpillar and showing a recursion, then proving this result using induction.

3.1 Oriented Version of Single-Legged Caterpillars

Writing the code that gave us the number of independent dominating sets for a single legged caterpillar C allowed us to ask ourselves, how many oriented single-legged caterpillars are there on n vertices, $c(n)$? Keep in mind, $c(n)$ is an oriented count, meaning that every possible single-legged caterpillar is distinct.

Before we talk about how to get $c(n)$, it's important that we discuss binomial coefficients.

A binomial coefficient is denoted as $\binom{n}{k}$ such that we are counting the number of subsets of size k from a set of size n, such that $0 \leq k \leq n$. There is an array of such binomial coefficients known as Pascal's Triangle, shown below for $n = 0, 1, 2, \cdots, 6$.

$$
\begin{array}{ccccccccccccc}
 & & & & & & \binom{0}{0} & & & & & & \\
 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & \\
 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \\
\binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6}
\end{array}
$$

Numerically, Pascal's Triangle is represented as the following for $n = 1, 2, \cdots, 6$.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

Notice how for any value on the n^{th} row, we can add the 2 numbers right above it in row $n-1$ to get that exact value. For example, to get 15 in the 6^{th} row, we would have to add 5 and 10 from the 5^{th} row. Using binomial coefficients, we can say that $\binom{6}{2} = \binom{5}{1} + \binom{5}{2}$.

We can generalize this, known as Pascal's Identity such that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Now that we've learned about binomial coefficients, lets state the theorem for $c(n)$.

Theorem 3.1.1. *For any single-legged caterpillar on n vertices, where $c(n)$ is the* *number of oriented single-legged caterpillars,*

$$\begin{aligned} c(n) &= 1 + \binom{n-3}{1} + \binom{n-4}{2} + \binom{n-5}{3} + \cdots \\ &= \sum_{k=0}^{\infty} \binom{n-2-k}{k}. \end{aligned}$$

Proof. To figure out the number of single-legged caterpillars on n vertices, we have to count the number of caterpillars on n vertices with 1 leg, 2 legs (each leg on separate vertices), etc. We must keep in mind that there will be no leg on the first and last vertex on the spine, or path, otherwise those legs could be extended onto the spine.

If the caterpillar has no legs, then there will be no legs among $n-2$ vertices, again, omitting the first and last vertex on the spine. The number of ways this can occur is $\binom{n-2}{0} = 1$.

If the caterpillar has 1 leg in total, that means we have $n-3$ vertices on the spine, omitting the first and last vertex on the spine as well as the vertex needed for that 1 leg. Thus, the number of ways we can have 1 leg among $n-3$ vertices is $\binom{n-3}{1}$.

Throughout this process, we can note that for each leg we add onto the caterpillar, a vertex on the spine is lost to maintain n vertices. We continue this process until we have no vertices left on the spine to consider. To generalize the number of 1-legged caterpillars on n vertices, we can say the following.

$$\begin{aligned} c(n) &= \binom{n-2}{0} + \binom{n-3}{1} + \binom{n-4}{2} + \binom{n-5}{3} + \cdots + \binom{\lceil \frac{n-2}{2} \rceil}{\lfloor \frac{n-2}{2} \rfloor} \\ &= \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-2-k}{k}. \end{aligned}$$

We can say that $\binom{n}{k} = 0$ if $k > n$. So those binomial for the sake of our calculation, suppose

$$\begin{aligned} c(n) &= 1 + \binom{n-3}{1} + \binom{n-4}{2} + \binom{n-5}{3} + \cdots \\ &= \sum_{k=0}^{\infty} \binom{n-2-k}{k}. \end{aligned}$$

□

As it turns out, there is a relationship between $c(n)$ and the Fibonacci numbers, where $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

The Fibonacci Sequence is a sequence of numbers such that a value, say F_n is the sum of the 2 preceding values. The sequence is the following

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \cdots$$

Lemma 3.1.2 states the relationship between $c(n)$ and the Fibonacci Sequence.

Lemma 3.1.2. *For any single-legged caterpillar on n vertices, where $c(n)$ is the number of single-legged caterpillars,*

$$c(n) = F_{n-1}$$

for $n \geq 2$.

Proof. First we can start with a few base cases. Figures 3.1-3.4 show the cases when $n = 2, 3, 4, 5$.

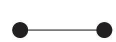

Figure 3.1: $n = 2$: $c(2) = 1$.

With these base cases thus far, we can see that $c(n) = c(n-1) + c(n-2)$.

Figure 3.2: $n = 3$: $c(3) = 1$.

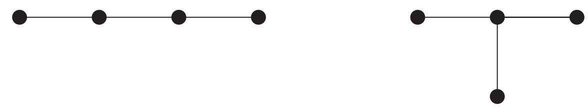

Figure 3.3: $n = 4$: $c(4) = 2 = c(3) + c(2)$.

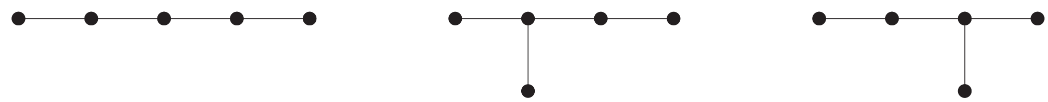

Figure 3.4: $n = 5$: $c(5) = 3 = c(4) + c(3)$.

Using Theorem 3.1.1, we can continue the calculation using summation properties,

$$
\begin{aligned}
c(n) &= 1 + \binom{n-3}{1} + \binom{n-4}{2} + \binom{n-5}{3} + \cdots \\
&= \sum_{k=0}^{\infty} \binom{n-2-k}{k} \\
&= \sum_{k=0}^{n-2} \binom{n-2-k}{k} \\
&= 1 + \sum_{k=1}^{n-2} \binom{n-2-k}{k} \\
&= 1 + \sum_{k=1}^{n-2} \left[\binom{n-3-k}{k} + \binom{n-3-k}{k-1} \right] \\
&= 1 + \sum_{k=1}^{n-2} \binom{n-3-k}{k} + \sum_{k=1}^{n-2} \binom{n-3-k}{k-1} \\
&= \sum_{k=0}^{n-2} \binom{n-3-k}{k} + \sum_{k=1}^{n-2} \binom{n-3-k}{k-1} \\
&= c(n-1) + \sum_{k=1}^{n-2} \binom{n-3-k}{k-1} \\
&= c(n-1) + \sum_{k=1}^{n-2} \binom{n-4-(k-1)}{k-1}.
\end{aligned}
$$

We can use substitution and let $\ell = k-1$, so

$$\sum_{k=1}^{n-2} \binom{n-4-(k-1)}{k-1} = \sum_{\ell=0}^{n-2} \binom{n-4-\ell}{\ell} = c(n-2).$$

Therefore,

$$\begin{aligned} c(n) &= 1 + \binom{n-3}{1} + \binom{n-4}{2} + \binom{n-5}{3} + \cdots \\ &= c(n-1) + \sum_{k=1}^{n-2} \binom{n-4-(k-1)}{k-1} \\ &= c(n-1) + c(n-2). \end{aligned}$$

$\square$

So, $c(n) = F_{n-1}$ where $c(n)$ counts the number of oriented one-legged caterpillars. With this calculation, caterpillars that are isomorphic to each other are counted as distinct caterpillars.

3.2 Unoriented Version of Single-Legged Caterpillars

We found a way to count the number of unoriented caterpillars, $c^*(n)$. After some investigation, we realized that $c^*(n)$ had been discovered in different context and corresponds to the number of domino tilings of a $2 \times (n-2)$ rectangular grid. We found the following.

Theorem 3.2.1. *For any $n \geq 3$, we have*

$$c^*(n) = \begin{cases} \frac{1}{2}\left(F_{n-1} + F_{\frac{n}{2}+1}\right) & \text{if } n \text{ is even} \\ \frac{1}{2}\left(F_{n-1} + F_{\frac{n-1}{2}}\right) & \text{if } n \text{ is odd.} \end{cases}$$

When comparing any given $c^*(n)$ to the domino tilings, the two endpoints of the path that help construct the caterpillars are not observed in the rectangular grid. For the remaining vertices considered in the path, we have to check if each vertex contains a leg. If there is not a leg, a single domino is placed vertically on the grid. If there is a leg, two dominoes are placed horizontally in each row on the grid, where the two dominoes are lined up with each other.

This problem was actually an American Mathematical Monthly Problem in 1961 which was solved by S.W. Golomb [3]. They first observed the number of ways one can arrange dominoes on a $2 \times n$ grid, which turns out directly being the Fibonacci sequence, exactly how we figured out the number of oriented single-legged caterpillar. Then they took into account the left-to-right mirrored layout of the dominoes as well as the left-to-right symmetries of the dominoes such that matching layouts are only accounted for once. In this case they use a $2 \times m$ grid, where m is split into the even and odd case.

Chapter 4

Domination Ratio

4.1 Domination Ratio in Unicyclic Graphs

Recall that the domination ratio $\rho(G)$ in a graph G is the minimum size of an independent dominating set, denoted as $\gamma_i(G)$, to the minimum size of a dominating size, denoted as $\gamma(G)$. We also let $\Delta(G)$ be the maximum degree of G, and $d_G(x)$ be a degree of a vertex $x \in G$.

Recall that Wang and Wei [11] proved the following.

Theorem 4.1.1 (Wang and Wei [11]). *If T is a tree, then*

$$\rho(T) \leq \frac{\Delta(T)}{2},$$

provided $\Delta(G) \geq 3$.

We were able to find an upper bound the domination ratio $\rho(G)$ for unicyclic graphs G, Theorem 4.1.2. For $\Delta(G) \geq 6$, we use a variation on a technique introduced by Furuya et al. [1]. Essentially, we begin with a minimum size dominating set and then use a sort of greedy algorithm to transform it into an independent dominating set. We then generalize an argument of Wang and Wei [11] to get the result to come through.

Theorem 4.1.2. *If G is a unicyclic graph, then*

$$\rho(G) \leq \frac{2}{3}\Delta(G) - 1.$$

The result follows from Rad and Volkmann [9] for $\Delta(G) \leq 5$.

Proof. Note that we are done if $\Delta \leq 5$ by Rad and Volkmann [9] by Theorem 1.2.5.

To start, let G be a graph and D be a minimum dominating set of G. We'll define a sequence using graph G_i and vertex x_i. For $i = 1$, let $G_1 = G[D]$ and x_1 be a vertex of minimum degree in G_1. For $i > 1$, $G_i = V(G_{i-1}) - N_{G_{i-1}}[x_{i-1}]$ provided it is nonempty and x_i be a vertex of minimum degree in G_i. Suppose this recursion stops after k steps, which is when we run out of vertices in D. Then $X = \{x_1, x_2, \cdots, x_k\}$ is an independent dominating set of $G_1 = G[D]$. Also, $\{N_{G_i}[x_i] : 1 \leq i \leq k\}$ partitions D, so

$$\sum_{i=1}^{k}(d_{G_i}(x_i) + 1) = |D| = \gamma(G).$$

Let I be an independent dominating set of $G - \cup_{i=1}^{k} N_G[x_i]$.We note that $X \cup I$ is an independent dominating set of G. So

$$\gamma_i(G) \leq |X \cup I| = k + |I|.$$

Since D is a minimum dominating set of G and $I \subseteq V(G) - D$, we can say that

$$I = \bigcup_{v \in D-x} (N_G(v) \cap I) = \bigcup_{i=1}^{k} \bigcup_{v \in N_{G_i(x_i)}} (N_G(v) \cap I).$$

Also, since $d_{G_i}(x_i) \leq d_{G_i}(v)$ for every $v \in N_{G_i}(x_i)$, we know

$$|N_G(v) \cap I| \leq d_G(v) - d_{G_i}(v) \leq \Delta(G) - d_{G_i}(x_i).$$

Then by using summation properties,

$$\begin{aligned}
|I| &\leq \sum_{i=1}^{k} \sum_{v \in N_{G_i}(x_i)} |N_G(v) \cap I| \\
&\leq \sum_{i=1}^{k} \sum_{v \in N_{G_i}(x_i)} \Delta(G) - \sum_{i=1}^{k} \sum_{v \in N_{G_i}(x_i)} d_{G_i}(x_i) \\
&= (|D| - k)\Delta(G) - \sum_{i=1}^{k} d_{G_i}(x_i)^2 \\
&= \gamma(G)\Delta(G) - k\Delta(G) - \sum_{i=1}^{k} d_{G_i}(x_i)^2.
\end{aligned}$$

Furthermore, from what we have so far, we can say

$$\begin{aligned}
\gamma_i &\leq k + |I| \\
&\leq k + \gamma(G)\Delta(G) - k\Delta(G) - \sum_{i=1}^{k} d_{G_i}(x_i)^2
\end{aligned}$$

Thus, we have

$$\gamma_i(G) \leq \Delta(G)\gamma(G) - \sum_{i=1}^{k} (\Delta(G) - 1 + d_{G_i}(x_i)^2).$$

Dividing through by $\gamma(G)$, we have

$$\rho(G) \leq \Delta(G) - \frac{1}{\gamma(G)} \sum_{i=1}^{k} (\Delta(G) - 1 + d_{G_i}(x_i)^2),$$

and so it suffices to show that

$$\begin{aligned} \frac{1}{\gamma(G)} \sum_{i=1}^{k} (\Delta(G) - 1 + d_{G_i}(x_i)^2) &\geq \left(\frac{\Delta(G)}{3} + 1 \right) \gamma(G) \\ &= \left(\frac{\Delta(G)}{3} + 1 \right) \sum_{i=1}^{k} (d_{G_i}(x_i) + 1). \end{aligned}$$

But this is equivalent to showing that

$$\frac{2\Delta(G)}{3} k - 2k + \sum_{i=1}^{k} d_{G_i}(x_i)^2 - \left(\frac{\Delta(G)}{3} + 1 \right) \sum_{i=1}^{k} d_{G_i}(x_i) \geq 0. \tag{4.1}$$

If it is the case that $d_{G_i}(x_i) = 0$ or 1 for all i, then $d_{G_i}(x_i)^2 = d_{G_i}(x_i)$ and so the left hand side of (4.1) is equivalent to

$$\begin{aligned} \frac{2\Delta(G)}{3} k - 2k - \frac{\Delta(G)}{3} \sum_{i=1}^{k} d_{G_i}(x_i) &\geq \frac{2\Delta(G)}{3} k - 2k - \frac{\Delta(G)}{3} k \\ &= \frac{\Delta(G)}{3} k - 2k \geq 0, \end{aligned}$$

provided that $\Delta(G) \geq 6$.

On the other hand, if it is not the case that all $d_{G_i}(x_i) = 0$ or 1, then there is at most one vertex whose degree is 2. Assume that (renaming the vertices if necessary) $d_{G_k}(x_k) = 2$ and $d_{G_i}(x_i) = 0$ or 1 for $1 \leq i \leq k-1$. (Note that this labeling may not be consistent with the way we chose the x_is, but it makes the argument cleaner to write.) In this case, we have that the left hand side of (4.1) is equivalent to

$$\begin{aligned}\frac{2}{3}\Delta(G)k - 2k + 4 - 2(\frac{\Delta(G)}{3} + 1) - \frac{\Delta(G)}{3}\sum_{i=1}^{k-1} d_{G_i}(x_i)&\\ \geq \frac{2}{3}\Delta(G)k - 2k + 2 - \frac{2}{3}\Delta(G) - \frac{\Delta(G)}{3}(k-1)&\\ = \frac{\Delta(G)}{3}k + 2 - 2k - \frac{\Delta(G)}{3} = \left(\frac{\Delta(G)}{3} - 2\right)(k-1) \geq 0,&\end{aligned}$$

provided that $\Delta(G) \leq 6$. □

4.2 Further Directions

It would be interesting to investigate this transition from an upper bound of $\frac{\Delta(T)}{2}$ for trees, proved by Wang and Wei [11], to $\frac{2}{3}\Delta(G) - 1$ for unicyclic graphs. We suspect that the correct bound may in fact depend on the clique number of the graph. The clique number is defined in Definition 4.2.1.

Definition 4.2.1. In a graph G, a *clique* is a subset of the vertex set V such that all vertices in the subset are adjacent. The *clique number* is the maximum size of a clique in G.

It would even be interesting to try to extend Theorem 4.1.2 to graphs with clique number at most 3. Or even to extend Theorem 4.1.1 to graphs with clique number at most 2. It seems that this might require a different proof technique.

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