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Magic Squares of Squares of Order 5 Modulo a Prime Number

Imani L. Mosquera

Abstract

In this paper, I examine magic squares of squares (MSS) of order 5 over $\mathbb{Z}_p$ where p is a prime number. The first approach to the problem is to find how many distinct elements an MSS may have (called the degree of the MSS). In the next step, I study the relationship between the magic sum and the center entry of any MSS. In order to develop construction methods and configurations for magic squares of squares of order 5 with desired degrees, I study Pythagorean triples and sequences of consecutive quadratic residues modulo p. Properties of these sequences are provided and applied to construct desired magic squares of squares.

This research focuses on magic squares of squares of order 5 in which the center 3×3 square is a magic square of squares of order 3. I claim that the magic sum of such an MSS M is $5c$, where c is the center element of M and the degree of M must be odd when $p > 5$.

The main results of the thesis include several configurations for the construction of MSS of a given degree and the existence of MSSs of all possible odd degrees over $\mathbb{Z}_p$ for infinitely many primes p. Chapter 1 presents an overview of modular arithmetic as well as some important definitions. Chapter 2 gives the results about the magic sum and degrees. In Chapter 3, I investigate special sequences of quadratic residues and describe properties of them. In Chapter 4, by applying special sequences of quadratic residues, several configurations are developed and they are used to construct MSSs of a given degree. The main results of this thesis are provided in Chapter 4 as well.

Keywords: magic squares, matrix theory, modular arithmetic, order five, prime numbers

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Magic Squares of Squares of Order 5 Modulo a Prime Number

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I would like to dedicate this project to my mother because she's my queen, who supported me in every single endeavor and empowered me long after leaving the earth.

I would like to thank my father because *me he portado bien como siempre me decía* and now I'm Imani Mosquera, M.S.

To my family, friends, and church, thank you for your love and support.

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> "I want to thank me for believing in me, I want to thank me for doing all this hard work. I wanna thank me for having no days off. I wanna thank me for never quitting. I wanna thank me for always being a giver and trying to give more than I receive. I want to thank me for trying to do more right than wrong. I want to thank me for just being me at all times."– Calvin "Snoop Dogg" Broadus Jr.

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Introduction

1.1 History and Background

A Chinese legend dating back to around 2800 B.C. tells the tale of Emperor Yu and the Luo River. There was a devastating flood that destroyed crops and killed livestock. The people offered sacrifices to the God of the Luo River to calm his anger and stop the floods. A tortoise emerged from the water with an unusual pattern, named the *Lo Shu*, on its shell: the integers one through nine arranged in a three-by-three grid. Even odder, each row, column, and diagonal in the square added to the same number: fifteen. Fifteen became the magical number of sacrifices required to make the river God happy and eventually became the number of days in most of the 24 cycles in the Chinese solar year.

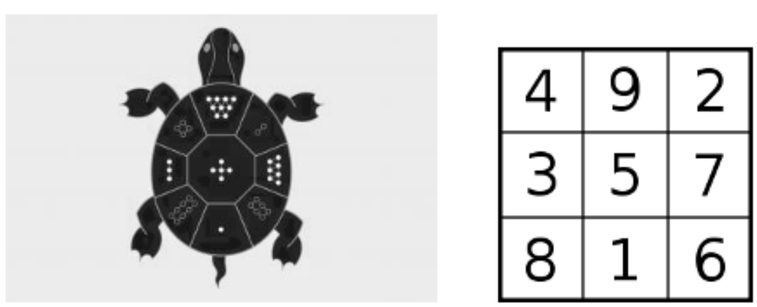

Figure 1.1: Graphical Representation of Lo Shu.
[12]

Magic squares have been around for over 4,000 years and they provide mathematical insights, help explain astrological anomalies, and have been thought to promote longevity and good health when worn. Other magic squares date back to the eleventh or twelfth century in India, Turkey, and Egypt, and were often sewn into clothing or worn as protective charms. Naturally, mathematicians sought larger and larger magic squares, and by the 13th century, a 10×10 magic square had been created.

Oddly enough, the 4×4 magic square rose to fame before its smaller counterpart. As early as 550 B.C., Indian mathematician Varahamihira used the 4×4 magic square to describe a recipe to create a perfume made of 16 distinct ingredients. Magic squares and their various uses were introduced to Europe by Manuel Moschopoulos around 1300 B.C. One of the most famous European 4×4 magic squares was created by German painter and engraver, Albert Durer. In his *Melancholia* (1514), Durer provided the first-documented magic square in European art. In 1770, Leonardo Euler created the first documented magic square of order 4 containing all perfect square entries:

68^2	29^2	41^2	37^2
17^2	31^2	79^2	32^2
59^2	28^2	23^2	61^2
11^2	77^2	8^2	49^2

.

Figure 1.2: Euler's Magic Square of Order 4

Here, each row, column, and diagonal sums to $8,515$. Throughout time, from Subirachs' sculpture at the Sagrada Familia in Spain, to Euler in Switzerland, and founding father Benjamin Franklin's "magical square" doodles, magic squares and the people who tinkered with them are etched in history.

Several mathematicians have developed methods and algorithms to create magic squares.

8	1	6
3	5	7
4	9	2

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

32	29	4	1	24	21
30	31	2	3	22	23
12	9	17	20	28	25
10	11	18	19	26	27
13	16	36	33	5	8
14	15	34	35	6	7

30	39	48	1	10	19	28
38	47	7	9	18	27	29
46	6	8	17	26	35	37
5	14	16	25	34	36	45
13	15	24	33	42	44	4
21	23	32	41	43	3	12
22	31	40	49	2	11	20

64	2	3	61	60	6	7	57
9	55	54	12	13	51	50	16
17	47	46	20	21	43	42	24
40	26	27	37	36	30	31	33
32	34	35	29	28	38	39	25
41	23	22	44	45	19	18	48
49	15	14	52	53	11	10	56
8	58	59	5	4	62	63	1

Figure 1.3: Magic Squares of Various Sizes.
[11]

Maurice Kraitchick's *Siamese method* and John Conway's *Lozenge method* provide paths to creating odd magic squares. In 1984, Martin LaBar raised a question that serves as the motivation for this project:

Question 1. *[4] (Still Open) Can a* 3×3 *magic square be constructed using 9 distinct perfect squares of integers?*

Twelve years later, recreational mathematician Martin Gardner offered a prize of $100 to the first person who could find such a magic square, or prove that it did not exist.

> "So far no one has come forward with a 'square of squares.' If it exists, its numbers would be huge, perhaps beyond the reach of today's fastest supercomputers."– Martin Gardner, *Scientific American*, August 1998 [10].

The question remains open even with the supercomputers of today, but in an earlier work, Hengeveld, Labruna, and Li had a different approach for finding the maximum degree of such a magic square of squares modulo a prime number p [6]. In this project, I investigate whether 5×5 magic squares of squares can be constructed using 25 distinct integers mod a prime number, p. I also want to see how to select elements to fill a 5×5 magic square of squares to reach the maximal degree of 25. More generally, I seek methods to construct magic squares of squares of any possible degree over selected prime numbers.

1.2 Definitions

The following definitions provide a baseline for understanding magic squares and magic squares of squares. Modular arithmetic is also used heavily in this project, thus some relevant information is provided below.

Definition 1. *Given a prime number p and the finite field $\mathbb{Z}_p$ of p elements. The set Q_p of quadratic residues modulo p consists of all the quadratic residues modulo p including 0, that is,*

$$Q_p = \{m \in \mathbb{Z}_p \mid m \equiv a^2 \text{ for some } a \in \mathbb{Z}_p\}.$$

A magic square or magic square of squares of order n can be demonstrated as an $n \times n$ matrix. The entries can be integers or other numbers of interest. We only consider $n \geq 3$.

Definition 2. *Let n be a positive integer. A magic square (MS) of order n over a ring R is an $n \times n$ matrix $M = [a_{ij}]$ with $a_{ij} \in R$ such that all rows, columns, and diagonals add up to the same number $S = S(M)$, which is called the magic sum. If all the entries of M are perfect squares in R, we call M a magic square of squares (MSS) over R.*

In this research, I focus on magic squares of squares of order 5 over the finite field $\mathbb{Z}_p$ for prime numbers p. Specifically, for any given prime p, an magic square of order 5 over $\mathbb{Z}_p$ has the following form:

$$M = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix},$$

where each $a_{ij} \in Q_p$ and

$$S(M) = \sum_{i=1}^{5} a_{ij} = \sum_{j=1}^{5} a_{ij} = \sum_{i=1}^{5} a_{ii} = \sum_{i=1}^{5} a_{i,(5-i)}, \ \forall\ i, j \in \{1, 2, 3, 4, 5\}.$$

Regarding a magic square of squares, one question is "how many distinct entries can it achieve?" It can be measured by its degree.

Definition 3. *Consider a prime number p and let $M = (a_{ij})_{5\times 5}$ be a magic square of squares (MSS) over $\mathbb{Z}_p$. The individual degree of M, denoted as $\deg(M)$, is the number of distinct entries of the matrix M. The global degree for p is defined as*

$$\alpha_p = \max\{\deg(M) \mid M \text{ is an MSS over } \mathbb{Z}_p\}.$$

If $\deg(M) = 1$, that is, $a_{ij} = c$ for some $c \in \mathbb{Z}_p$ and for all $1 \leq i, j \leq 5$, we say M is a trivial MSS.

The focus of this research is on non-trivial magic squares of squares with entries in $\mathbb{Z}_p$.

1.3 Existing Results

In the dissertation by O'Neill [3], the author used an earlier idea to construct larger MSSs from those of smaller order.

Theorem 1. *Let p be any prime and assume u, q, x integers. Then the matrix*

$$B = \begin{bmatrix} 2a-u-q & x-(b+a) & 2q & -x-(b+a) & 2b+u-q \\ b-a+u & a & -(b+a) & b & a-b-u \\ 2u & b-a & 0 & a-b & -2u \\ b-a-u & -b & b+a & -a & a-b+u \\ a-2b-u & b+a-x & -2q & b+a+x & u-2a+q \end{bmatrix}.$$

is an magic square of order 5 with the magic sum $S(B) = 0$.

Remark 1. *1. In the matrix* B, *if all the entries are distinct, then* B *is a magic square of degree 25. If all of the entries are also quadratic residues and are carefully selected, then* B *may be a magic square of squares of degree* 25. *It sets a basis for the development of construction methods.*

2. Note that the inner 3×3 *matrix itself is a magic square of order* 3 *with* 0 *as the magic sum. The matrix* B *gives a configuration for all magic squares of squares of order* 5 *with the magic sum* 0 *and being extended from an MSS of order* 3.

3. If we set $a = b = 0$, *we obtain a trivial MS of order* 3 *in the center with magic sum* 0. *In this case, the maximal degree of* B *is* ≤ 17.

$$B = \begin{bmatrix} -u-q & x & 2q & -x & u-q \\ u & 0 & 0 & 0 & -u \\ 2u & 0 & 0 & 0 & -2u \\ -u & 0 & 0 & 0 & u \\ q-u & -x & -2q & x & u+q \end{bmatrix}.$$

The following theorem is a well-known result about magic squares of order 3. I present a simple proof here.

Theorem 2. *Let* p *be a prime number and* M *be a magic square of order* 3 *over* $\mathbb{Z}_p$. *Then the magic sum of* M *is* $S(M) = 3a_{22}$.

Proof. Let

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

By the definition of a magic square, the following equations are true

$$a_{11} + a_{22} + a_{33} = a_{12} + a_{22} + a_{32} = a_{13} + a_{22} + a_{31} = a_{21} + a_{22} + a_{23} = S(M).$$

By adding these sums and regroup the terms, we obtain

$$(a_{11} + a_{12} + a_{13}) + (a_{21} + a_{22} + a_{23}) + 3a_{22} + (a_{31} + a_{32} + a_{33}) = 4S(M)$$

$$\implies 3S(M) + 3a_{22} = 4S(M), \quad \therefore 3a_{22} = S(M).$$

□

In the paper by Hengeveld, Labruna, and Li [6], it was shown that for any prime number $p > 3$, the degree of a non-trivial magic square of order 3 over $\mathbb{Z}_p$ must be odd. A natural question arises: "Is it the same situation for magic squares of order 5?"

Theorem 3. *Let p be a prime number greater than 3 and M is a non-trivial magic square of order 3 over $\mathbb{Z}_p$. Then* $\deg(M) \in \{3, 5, 7, 9\}$.

When dealing with perfect squares in the field $\mathbb{Z}_p$, Legendre symbols and their properties play important roles in identifying which number is a quadratic residue mod p. Below I give some basics about the Legendre symbol.

Definition 4. *For any odd prime p, the Legendre symbol of an integer a (mod p) is given by*

$$\left(\frac{a}{p}\right) = \begin{cases} 0, & \text{if } p \text{ divides } a; \\ 1, & \text{if } a \text{ is a quadratic residue modulo } p \text{ and } p \nmid a; \\ -1, & \text{if } a \text{ is a quadratic nonresidue modulo } p \text{ and } p \nmid a. \end{cases}$$

Some basic properties of Legendre symbol are well known and show below.

Theorem 4.

Let p be any odd prime. Then

1. $\left(\frac{-1}{p}\right) = \begin{cases} 1, & \text{if } p \equiv 1 \pmod 4, \\ -1, & \text{if } p \equiv 3 \pmod 4. \end{cases}$

2. $\left(\frac{2}{p}\right) = \begin{cases} 1, & \text{if } p \equiv 1,7 \pmod 8, \\ -1, & \text{if } p \equiv 3,5 \pmod 8. \end{cases}$

3. $\left(\frac{3}{p}\right) = (-1)^{(p+1)/6} = \begin{cases} 1, & \text{if } p \equiv 1,11 \pmod{12} \\ -1, & \text{if } p \equiv 5,7 \pmod{12} \end{cases}$

4. $\left(\frac{5}{p}\right) = (-1)^{(2p+2)/5} = \begin{cases} 1, & \text{if } p \equiv 1,4 \pmod 5 \\ -1, & \text{if } p \equiv 2,3 \pmod 5 \end{cases}$

The following theorem guarantees the existence of infinitely many prime numbers in the form of $am + b$, where a, b, m are all integers and a, b are relatively prime. Later, I construct many magic squares of squares of all possible degrees over these types of prime numbers.

Theorem 5. *(Dirichlet's Theorem on Primes in Arithmetic Progressions) [7] If a and b are relatively prime positive integers, then there are infinitely many primes of the form $am + b$ with $m \in \mathbb{Z}$.*

A special interest is given to the prime numbers in the form of $am + 1$, where a, m are positive integers and $a \geq 2$. Obviously, $\gcd(a, 1) = 1$.

1.4 Statement of the Problem and Goals

Consider the prime number p and a 5×5 magic squares of squares over $\mathbb{Z}_p$. I aim to answer the following questions and achieve related goals:

Research Questions and Goals

1. Investigate relationship of the magic sum and the center number in any MSS.

2. For a given prime number p, for what integer d with $1 < d \leq 25$, is d the degree of an MSS of order 5?

3. Identify the prime numbers p such that $\alpha_p = 25$ (full degree).

4. More generally, for an odd integer d with $3 \leq d \leq 25$, identify the prime numbers p such that $\alpha_p = d$.

5. For any given prime number p, what is the value of α_p?

6. Develop methods to construct MSS of any given degree if it exists.

7. Can we give a lower bound and an upper bound for α_p if we cannot find the exact value of α_p?

1.5 Methodology and Preliminary Work

From the above existing results, I attempt to insert a 3×3 MSS with a desired degree. A particular focus is on those with the maximal degree 9. For this purpose, I will insert an MSS of order 3 with degree 9 in the inner square and try to add the most distinct squares for the remaining 16 entries (the "shell"). I also want to find a formula for the relationship between the center entry and the magic sum.

The following example is a magic square of squares over $\mathbb{Z}_{241}$ with a degree of 25. The idea of this construction is to use Pythagorean triples of quadratic residues to build the inner 3×3 magic square.

Example 1. *Let* $p = 241$

$$M = \begin{pmatrix} 1 & 6 & 3 & 229 & 2 \\ 5 & 16 & 200 & 25 & 236 \\ 4 & 9 & 0 & 232 & 237 \\ 233 & 216 & 41 & 225 & 8 \\ 239 & 235 & 238 & 12 & 240 \end{pmatrix} = \begin{pmatrix} 1^2 & 27^2 & 56^2 & 62^2 & 22^2 \\ 103^2 & 4^2 & 21^2 & 5^2 & 85^2 \\ 2^2 & 3^2 & 0^2 & 49^2 & 113^2 \\ 76^2 & 79^2 & 102^2 & 15^2 & 44^2 \\ 38^2 & 41^2 & 31^2 & 112^2 & 64^2 \end{pmatrix}.$$

By Theorem 2, the magic sum of the inner 3×3 submatrix is $3a_{33}$. Naturally, I would expect (or hope) that the magic sum of a magic square of order 5 is also 5 times the center element. It is true in the above example because the magic sum is $0 = 5 \times 0$.

1.6 The Main Results

In this research, I focus on magic squares of squares of order 5 in which the inner 3×3 matrix is a magic square of squares of order 3 modulo a prime number p. Throughout the paper, any magic square of squares of order 5 is referred to one of this type, unless it is especially stated in other types.

In Chapter 2, I give different configurations of magic squares of squares of order 5. It is shown that the sum of any non-trivial magic square of squares M of order 5 over $\mathbb{Z}_p$ is 5 times the center number and $\deg(M)$ must be odd, if $p > 3$.

In Chapter 3, I study special sequences of quadratic residues in order to apply them in the construction of desired magic squares of squares. Two types of sequences are in consideration: Pythagorean triples of quadratic residues (PTQRs) and sequences of consecutive quadratic residues (CQRs) of various lengths. Formulas for the numbers of PTQRs and CQRs for a given prime number p are given. For a given length l, I identify a set of infinite primes over which a sequence of consecutive quadratic residues of length l exists.

In Chapter 4, I prioritize the construction of magic squares of squares using the special sequences found in Chapter 3. The main results of this thesis are given in this chapter and they are summarized below.

1. Regarding the open question by LaBar [4], I show that for infinitely many prime numbers p, there exists a magic square of squares of order 5 with degree 25 over $\mathbb{Z}_p$.

2. For infinitely many primes p, there exists a magic square of squares of order 5 with degree d over $\mathbb{Z}_p$ for any $d \in \{3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25\}$.

3. For certain prime numbers $p > 3$, several configurations in the form of 5×5 matrix function with entries in $\mathbb{Z}_p$, are given by applying special sequences of quadratic residues. These matrices can help to construct desired magic squares of squares of order 5.

Chapter 2

Configurations and Properties of MSS of Order 5

In this chapter, I develop several configurations for building magic squares of squares of order 5 with a desired degree. These configurations are based on the assumption that the 3×3 inner matrix is an MSS of order 3. The main goal is to select appropriate outskirt elements to build the 5×5 magic squares of squares of order 5 with a given degree.

2.1 Configurations of 5 by 5 Magic Squares

Example 1 shows an example of a 5×5 magic squares of squares I constructed with degree 25 whose inner 3×3 submatrix is a magic square of squares of order 3. By Theorem 6, if the center 3×3 matrix is a magic square, then the magic sum of the 5×5 magic square constructed around it is $5a_{33}$.

Configuration 1. *Let p be a prime numbers greater than 3. Over $\mathbb{Z}_p$, the structure of a general configuration for a 5 by 5 magic square with the magic sum $S(M) = 5c$, where c is* *the center element, is given below:*

$$M = \begin{pmatrix} d & j & f & 5c-x & e \\ h & a & 3c-a-b & b & 2c-h \\ g & c-a+b & c & c-b+a & 2c-g \\ 3c-y & 2c-b & a+b-c & 2c-a & y-c \\ 2c-e & 2c-j & 2c-f & x-3c & 2c-d \end{pmatrix},$$

where $j+f+d+e = x$ *and* $d+h+g-e = y$.

Here we see that the general configuration involves 9 variables. A special interest is on the magic squares with the magic sum 0, that is, $c = 0$. When $c = 0$, a reduced configuration is as follows.

Configuration 2. *Let* p *be a prime numbers greater than 3. Over* $\mathbb{Z}_p$*. If* $c = 0$*, then* M *in Configuration 1 becomes*

$$M = \begin{pmatrix} d & j & f & -(j+f+d+e) & e \\ h & a & -a-b & b & -h \\ g & b-a & 0 & a-b & -g \\ e-d-h-g & -b & a+b & -a & d+g+h-e \\ -e & -j & -f & j+f+d+e & -d \end{pmatrix} \quad \text{with } S(M) = 0.$$

Note that M is a magic square and it needs extra conditions to become a magic square of squares. First, -1 must be a quadratic residue mod p. Then all of the variables $a, b, c, d, e, f, g, h, j$ and their combinations involved in M must be quadratic residues. Special considerations are given for the selection of the variables so that all the entries of M are quadratic residues mod p.

Lemma 1. *If* $p \equiv 3 \pmod 4$*, then there is no non-trivial magic square of squares having magic sum equal to* 0.

Proof. Since M is non-trivial, one of the variables must be non-zero, say, $a \neq 0$. By Theorem 4, -1 is not a quadratic residue of p. So a and $-a$ cannot be both quadratic residues mod p. □

2.2 The Magic Sum

It is known that the magic sum of a magic square of order 3 is 3 times the center entry. I claim that a magic square of squares of order 5, with an inner magic square of squares of order 3, has a magic sum equal to 5 times the center entry.

Theorem 6. *Let p be a prime number greater than 3. Given a 5×5 MSS, M, with an inner 3×3 submatrix A, also an MSS, the magic sum of M is $S(M) = 5a_{33}$.*

Proof. Assume $p > 3$ is a prime and M is a 5×5 magic square over $\mathbb{Z}_p$ with inner 3×3 magic square A, shown below.

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix}, \quad A = \begin{pmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{pmatrix}.$$

Since M and A are magic squares and $S(A) = 3a_{33}$,

$$a_{21} + a_{25} = a_{31} + a_{35} = a_{41} + a_{45} = a_{51} + a_{15} = a_{11} + a_{55} = S(M) - 3a_{33}.$$

Combining the above equations yields the result

$$2S(M) = (a_{11} + a_{21} + a_{32} + a_{41} + a_{51}) + (a_{15} + a_{25} + a_{35} + a_{45} + a_{55}) = 5\left(S(M) - 3a_{33}\right),$$

which implies $S(M) = 5a_{33}$. □

2.3 The Degree of an MSS of Order 5

Regarding the degree of a magic square of order 3 modulo a prime number, it is known that the degree must be odd if $p \geq 5$. I claim the same result for the magic squares of order 5 with a magic square of order 3 insert.

Theorem 7. *Let p be a prime greater than or equal to 5. Let $M = [a_{ij}]$ be a 5×5 magic square mod p with a center magic square of order 3. Then the degree of M must be odd.*

Proof. Suppose $M = (a_{ij})_{5\times 5}$ be a magic square mod p and suppose the inner 3×3 matrix is a magic square of order 3. Let $c = a_{33}$. Then M has the following form:

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & 2c - a_{21} \\ a_{31} & a_{32} & a_{33} & a_{34} & 2c - a_{31} \\ a_{41} & a_{42} & a_{43} & a_{44} & 2c - a_{41} \\ 2c - a_{15} & 2c - a_{12} & 2c - a_{13} & 2c - a_{14} & 2c - a_{11} \end{pmatrix}.$$

Considering the outer shell of M and the sets

$$D = \{a_{11}, a_{12}, a_{13}, a_{15}, a_{21}, a_{31}\} \quad \text{and} \quad E = \{a_{12}, a_{13}, a_{14}, a_{21}, a_{31}, a_{41}\}.$$

There are two cases for two elements in the shell to be equal to one another.

Case 1: Two elements in D are identical.

If $u, v \in D$ such that $u = v$, then clearly $2c - u = 2c - v$. Thus, the degree of M is decreased by 2.

Case 2: An element in E is identical to its "opposite" element in the shell.

Pick any $u \in E$ and the opposite element. Without loss of generality, say, $a_{12} \in E$ and $a_{12} = a_{52}$. Since $a_{12} + a_{52} = 2c \implies 2a_{12} = 2c \implies a_{12} = a_{52} = c$. Thus, the degree of M is decreased by 2.

It is already known that the degree of a 3×3 magic square over $\mathbb{Z}_p$ with $p > 3$ must be odd [6]. The maximal degree for M is 25. Each time when the degree is reduced by the above cases (the only possible cases), the degree is decreased by 2. Therefore, $\deg(M)$ must be odd. □

Below are some examples of the possible degrees of a 5×5 MSS for some prime p.

Lemma 2. *For any prime p if there exists a 3×3 magic square of squares of degree d over $\mathbb{Z}_p$, then there exists a 5×5 magic square of squares of degree d.*

Proof. Let A be a 3×3 MS with the magic sum $S(A) = 3c$. We extend it to a 5×5 matrix M by adding c to all the "shell" positions. Then M is an MS of the same degree as that of A. □

In the following example, we see that the outer shell can be filled with elements from the inner submatrix or combined with different addends to admit the same sum. Thus, the 5×5 magic square has the same degree as that of the inner 3×3.

Example 2. *Assume $p \equiv 1 \pmod{120}$ is a prime. Consider two matrices below:*

$$M_1 = \begin{pmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 2 & -1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & 0 & 1 \\ 2 & -1 & 2 & 1 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 4 & -2 & 1 & 2 & 0 \\ -2 & 2 & -1 & 2 & 4 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 0 & 2 \\ 2 & 4 & 1 & 0 & -2 \end{pmatrix}$$

with $\deg(M_1) = 5$ and $\deg(M_2) = 7$. Note that $\deg(M_2)$ is equal to the degree of the inner MS of order 3. But $\deg(M_2) = 7 > 5$, the degree of its inner MS of order 3.

Since we can easily construct an MSS of order 5 with the same degree as that of the inner 3×3 inner magic square by adding 16 cs in the "shell" part, where c is the center element, we turn our focus to those of degree greater that 9. For the construction of these higher degree matrices, additional methods are needed.

Configuration 3. *Let* $p \equiv 1 \pmod{120}$*, then an MSS can be constructed as*

$$M(h,g) = \begin{pmatrix} 8 & 2 & 6 & -25 & 9 \\ h & 1 & -5 & 4 & -h \\ g & 3 & 0 & -3 & -g \\ 1-h-g & -4 & 5 & -1 & h+g-1 \\ -9 & -2 & -6 & 25 & -8 \end{pmatrix},$$

if $h, g, 1-h-g \in Q_p \setminus \{0, \pm 1, \pm 3, \pm 4, \pm 5, \pm 6, \pm 8, \pm 9, \pm 25\}$.

Proof. From Configuration 2, we first choose numbers for the inner 3×3 matrix such that it is a magic square of squares mod p. Additionally, we know that $p \equiv 1$ (mod 3, 4, 5, 8). Then all of the non-variable entries shown above are quadratic residues mod p. If there eixist $h, g, 1-h-g \in Q_p \setminus \{0, \pm 1, \pm 3, \pm 4, \pm 5, \pm 6, \pm 8, \pm 9, \pm 25\}$, then $M(h,g)$ is a magic square of squares. Furthermore, if all the entries of M are distinct, then $M(h,g)$ is an MAA of degree 25.

This configuration gives a way to construct magic squares of squares with degree 25 for any prime number $p \equiv 1 \pmod{120}$. There are infinitely many such p. When p is significantly large, there are more quadratic residues to choose from. Regarding the above configuration $M(h,g)$, the challenge is to find appropriate quadratic residues h and g so that the resulting 5×5 matrix is an MSS over $\mathbb{Z}_p$ of degree 25. In the process, I need to find an 8-tuple of squares to build a desired MSS of degree 25.

Example 3. *Assume* $p \equiv 1 \pmod{120}$ *is a prime. Based on Configuration 3, when* $h = 49$

and $g = 16$*, the following is an MSS of degree* 25 *mod* p*.*

$$M(49,16) = \begin{pmatrix} 8 & 2 & 6 & -25 & 9 \\ 49 & 1 & -5 & 4 & -49 \\ 16 & 3 & 0 & -3 & -16 \\ -64 & -4 & 5 & -1 & 64 \\ -9 & -2 & -6 & 25 & -8 \end{pmatrix}.$$

Note that $h = 49, g = 16 \in Q_p \setminus \{0, \pm1, \pm3, \pm4, \pm5, \pm6, \pm8, \pm9, \pm25\}$.

Proposition 1. *For any prime* $p \equiv 1 \pmod{120}$*, the following are magic squares of squares with degree* 13 *and* 25*, respectively:*

$$M_3 = \begin{pmatrix} 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 5 & -2 & 1 \\ -6 & -1 & 1 & 3 & 8 \\ 10 & 4 & -3 & 2 & -8 \\ 0 & 1 & 1 & 1 & 2 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 8 & 2 & 6 & -25 & 9 \\ 16 & 1 & -5 & 4 & -16 \\ 12 & 3 & 0 & -3 & -12 \\ -27 & -4 & 5 & -1 & 27 \\ -9 & -2 & -6 & 25 & -8 \end{pmatrix}.$$

Proof. Because $120 = 3 \times 5 \times 8$, we know that $p \equiv 1 \pmod{3, 4, 5, 8}$. Thus $\pm1, \pm2, \pm3, \pm5$, their powers, and their products are all quadratic residues mod p. For M_3, the inner 3×3 submatrix is of degree 9 and the outer shell adds 4 more distinct entries. Thus, $\deg(M_3) = 13$. For M_4, all the 25 entries are distinct because the difference between the greatest and the least elements, 27 and -27, is less than the first applicable prime, 241. □

Examples of the remaining odd degrees can be found in the Appendix in Chapter 6.

Chapter 3

Special Sequences of Quadratic Residues

In this chapter, I investigate sequences of a given length of quadratic residues as well as how to build such sequences. These sequences will allow us to build magic squares of squares from the inner 3×3 matrix outward.

3.1 Pythagorean Triples of Quadratic Residues

In Configurations 1 and 2, we see that the inner submatrix involves sums (and differences) of quadratic residues, or squares mod p. In order to obtain a magic square of square from the configurations, all of these sums and differences need to be quadratic residues. They produce triples satisfying the Pythagorean Theorem (called Pythagorean triples.) Thus, we need to investigate Pythagorean triples of quadratic residues and how to construct them. We then use certain Pythagorean triples of quadratic residues to build from the inner 3×3 magic square of squares outward.

Definition 5. *Let p be a prime number. A Pythagorean triple of quadratic residues, denoted as PTQR, is a triple $(a, b, a+b)$ where $a, b, a+b \in Q_p$.*

I am interested in putting together an inner submatrix using Pythagorean triples of quadratic residues then fill in the outer shell. In his 2012 Theory of Numbers lecture, Dr. Abhinav Kumar explained that the number of pairs of consecutive integers such that both are quadratic residues mod p is $\frac{p+2+\left(\frac{-1}{p}\right)}{4}$, but that finding the number of triples of consecutive integers such that all are quadratic residues mod p is "much more complicated."

Theorem 8. *[8] For any prime number p, the number of Pythagorean triples of quadratic residue mod p in the form of $(1, x, x+1)$ is $\left\lfloor \frac{p+3}{4} \right\rfloor$.*

Corollary 1. *For any prime p, the number of PTQRs is given by*

$$\begin{cases} \frac{(p-1)(p+7)}{8}, & \textit{if } p \equiv 1 \pmod 4 \\ \frac{(p-1)(p+5)}{8}, & \textit{if } p \equiv 3 \pmod 4 \end{cases}$$

The following table shows the numbers of Pythagorean triples of quadratic residues for small prime numbers from 5 to 19. Note that the trivial Pythagorean triple $(0, 0, 0)$ is not of interest. We only count the non-trivial PTQRs.

Table 3.1: Table of Number of PTQRs for Primes Up to 19

$p=$	5	7	11	13	17	19
#PTQR	6	9	20	30	48	54

Example 4. *Let $p = 7$. It is known that $Q_7 = \{0, 1, 2, 4\}$. Below are all the non-trivial PTQRs of $p = 7$:*

$$(0,1,1), (0,2,2), (0,4,4), (1,0,1), (1,1,2), (2,0,2), (2,2,4), (4,0,4), \textit{ and } (4,4,8).$$

Thus, the number of PTQRs for $p = 7$ is 9. Two of the triples are $(1, 0, 1)$ and $(1, 1, 2)$ which are from the pairs $(1, 0)$ and $(1, 1)$ respectively.

Example 5. *The set of quadratic residues mod 13 is $Q_{13} = \{0, 1, 3, 4, 9, 10, 12\}$. The first* *five PTQRs mod* 13 *are built from the pairs* $(1,0)$*,* $(1,3)$*,* $(1,9)$*, and* $(1,12)$*.*

$$(0,1,1),(1,0,1),(1,3,4),(1,9,10), \textit{and } (1,12,0).$$

All other PTQRs are permutations of these five, resultant from multiplying these by the remaining non-zero quadratic residues mod 13*. Thus, there are* 30 *PTQRs of* $p = 13$*.*

3.2 Special Consecutive Quadratic Sequences

Now, I focus on consecutive quadratic residues. For any prime p where -1 is a quadratic residue, we have automatically a triple of consecutive quadratic residues centered at 0: $(-1,0,1)$. Furthermore, if both -1 and 2 are quadratic residues mod p, then we obtain consecutive quadratic residues of length 5, centered at 0: $(-2,-1,0,1,2)$. A natural question arises: for a given l, what conditions on the prime p are needed for the existence of the sequence of consecutive quadratic residues of length l centered at 0?

Definition 6. *Let l be a positive integer and p be a prime number. A sequence of quadratic residues mod p of length l, $(a, a+1, \ldots, a+l-1) \in Q_p^l$, is called an l-CQR.*

As mentioned before, we see that if $p \equiv 1 \pmod 8$, then $(-2,-1,0,1,2)$ is an 5-CQR mod p centered at zero. I am interested in finding longer CQRs in order to construct MSSs of higher degrees.

Theorem 9.

1. *For any prime $p \equiv 1 \pmod 8$, there is a 5-CQR $(-2,-1,0,1,2)$ mod p.*

2. *For any prime $p \equiv 1 \pmod{24}$, there is a 9-CQR $(-4,-3,-2,-1,0,1,2,3,4)$ mod p.*

3. *For any prime $p \equiv 1 \pmod{9240}$, there is a 25-CQR $(-12,-11,\cdots,0,\cdots,11,12)$ mod p.*

Proof. For part 1, each of the numbers in $(-2,-1,0,1,2)$ is clearly a quadratic residue for any prime $p \equiv 1 \pmod 8$. For part 2, since $p \equiv 1 \pmod{24}$, each of the numbers in $(-3,-2,-1,0,1,2,3)$ is a quadratic residue mod p. This can be extended to a 9-CQR since ± 4 are also quadratic residues mod p. Finally, for part 3, if $p \equiv 1 \pmod{9240}$, then $-1, 2, 3, 5, 7$, and 11 are quadratic residues. This implies that each of $(-12, -11, \cdots, 0, \cdots, 11, 12)$ is a quadratic residue mod p. □

Example 6. *Let $p = 9241$. By Theorem 9, there is a 25-CQR $(-12,-11,\cdots,0,\cdots,11,12)$ mod p. The following matrix M is an MSS containing the 25 quadratic residues:*

$$M = \begin{pmatrix} 5 & 8 & 6 & -12 & -7 \\ -11 & -1 & -2 & 3 & 11 \\ 9 & 4 & 0 & -4 & -9 \\ -10 & -3 & 2 & 1 & 10 \\ 7 & -8 & -6 & 12 & -5 \end{pmatrix} = \begin{pmatrix} 4513^2 & 3515^2 & 304^2 & 666^2 & 2824^2 \\ 810^2 & 1829^2 & 3220^2 & 849^2 & 2930^2 \\ 3^2 & 2^2 & 0^2 & 3658^2 & 3754^2 \\ 4233^2 & 333^2 & 2863^2 & 1^2 & 1801^2 \\ 623^2 & 2801^2 & 1556^2 & 1698^2 & 2064^2 \end{pmatrix}.$$

Chapter 4

Construction of MSS Using Special Sequences of Quadratic Residues

In this chapter, I use the configurations developed in Chapter 3 to construct several magic squares of squares using special sequences of quadratic residues. I also show that given CQRs of certain lengths, there exist magic squares of squares of all odd degrees from 3 to 25.

4.1 Existence of Full-Degree MSS

Recall that α_p denotes the possible maximum degree of an MSS over $\mathbb{Z}_p$ and $\alpha_p \leq 25$. In this section, I find a set of primes p with $\alpha_p = 25$. For such primes p, an MSS of full degree (degree 25) is constructed by using the Pythagorean triples discussed in Section 3.1.

Theorem 10. *Let p be a prime number satisfying $p \equiv 1 \pmod{120}$. Then $\alpha_p = 25$. In particular, there are at least $\frac{p-1}{2}$ magic squares of squares of full degree. Consequently, there are infinitely many prime numbers p such that $\alpha_p = 25$.*

Proof. For each prime $p \equiv 1 \pmod{120}$, I use a CQR of length 13 centered at 0 to construct an MSS of full degree. The CQR is $(-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6)$. The shell part of the matrix is filled with some numbers from this sequence plus the quadratic residues

± 8 and ± 12. I then fill in the inner 3×3 squares by using certain Pythagorean triples of quadratic residues. In general, if one can find $a, b, c, d \in \mathbb{Z}_p$ such that $a^2 + b^2 = c^2$, $a^2 - b^2 = d^2$, $\pm a^2, \pm b^2, \pm(a^2 + b^2), \pm(a^2 - b^2) \in Q_p \setminus \{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 8, \pm 12\}$, and they are all distinct, then the following matrix $M(a^2, b^2)$ is a magic square of squares of degree 25.

$$M(a^2, b^2) = \begin{pmatrix} 1 & 6 & 3 & -12 & 2 \\ 5 & a^2 & -a^2 - b^2 & b^2 & -5 \\ 4 & b^2 - a^2 & 0 & a^2 - b^2 & -4 \\ -8 & -b^2 & a^2 + b^2 & -a^2 & 8 \\ -2 & -6 & -3 & 12 & -1 \end{pmatrix}.$$

When choosing $a^2 = 36$ and $b^2 = -9$, all of the requirements are satisfied. Thus, we obtain a magic square of squares of full degree:

$$M(36, -9) = \begin{pmatrix} 1 & 6 & 3 & -12 & 2 \\ 5 & 36 & -27 & -9 & -5 \\ 4 & 45 & 0 & -45 & -4 \\ -8 & 9 & 27 & -36 & 8 \\ -2 & -6 & -3 & 12 & -1 \end{pmatrix}.$$

There are exactly $\frac{p-1}{2}$ many non-zero quadratic residues mod p. Each of them multiplying the above matrix $M(36, -9)$ produces a different MSS. Thus, at least $\frac{p-1}{2}$ many MSSs of full degree exist.

Finally, Dirichlet's Theorem 5 guarantees the existence of infinitely many primes $p \equiv 1 \pmod{120}$. □

Example 7. *In particular, when* $p = 241 \equiv 1 \pmod{120}$, *the following is a magic square* *of squares mod p of degree* 25*:*

$$M_1 = \begin{pmatrix} 1 & 6 & 3 & -12 & 2 \\ 5 & 196 & 165 & 121 & -5 \\ 4 & 166 & 0 & 75 & -4 \\ -8 & 120 & 76 & 45 & 8 \\ -2 & -6 & -3 & 12 & -1 \end{pmatrix} = \begin{pmatrix} 1^2 & 26^2 & 55^2 & 62^2 & 22^2 \\ 103^2 & 89^2 & 67^2 & 107^2 & 156^2 \\ 2^2 & 40^2 & 0 & 91^2 & 128^2 \\ 165^2 & 100^2 & 50^2 & 88^2 & 44^2 \\ 38^2 & 200^2 & 41^2 & 31^2 & 112^2 \end{pmatrix}.$$

This proves that when $p = 241$, a magic squares of squares of degree 25 mod p exists.

When filling in the shell part of the 5×5 matrix with all 0s, we obtain an MSS with the same degree as the inner 3×3 MSS. Thus, MSS's of order 5 with degree $3, 5, 7$, and 9 can be constructed easily. We then focus on finding magic squares of squares of each possible (odd) degree, fr is an m 11 up to 25. Each of the following is an MSS over $\mathbb{Z}_{241}$. The construction is based on the similar ideas as shown in Theorem 10.

Example 8. *Consider the field $\mathbb{Z}_{241}$. The following matrices M_2, $M_3, \cdots, M_{10}$ are magic squares of squares mod 241 with degrees* $7, 9, 11, 13, 15, 17, 19, 21,$ *and* 23 *respectively.*

$$M_2 = \begin{pmatrix} 4 & -2 & 1 & 2 & 0 \\ -2 & 2 & -1 & 2 & 4 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 0 & 2 \\ 2 & 4 & 1 & 0 & -2 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 5 & -2 & 1 \\ 1 & -1 & 1 & 3 & 1 \\ 1 & 4 & -3 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\deg(M_2) = 7 \qquad \deg(M_3) = 9.$$

$$M_4 = \begin{pmatrix} 0 & 3 & 3 & 3 & 6 \\ -6 & 0 & 15 & -6 & 12 \\ 18 & -3 & 3 & 9 & -12 \\ 3 & 12 & -9 & 6 & 3 \\ 0 & 3 & 3 & 3 & 6 \end{pmatrix}, \quad M_5 = \begin{pmatrix} 0 & 4 & 4 & 4 & 8 \\ 4 & 0 & 20 & -8 & 4 \\ -24 & -4 & 4 & 12 & 32 \\ 40 & 16 & -12 & 8 & -32 \\ 0 & 4 & 4 & 4 & 2 \end{pmatrix}$$

$$\deg(M_4) = 11 \qquad \deg(M_5) = 13.$$

$$M_6 = \begin{pmatrix} 0 & 2 & 2 & 2 & 4 \\ 24 & 0 & 10 & -4 & -20 \\ -50 & -2 & 2 & 6 & 54 \\ 36 & 8 & -6 & 4 & -32 \\ 0 & 2 & 2 & 2 & 4 \end{pmatrix}, \quad M_7 = \begin{pmatrix} 0 & 3 & 24 & -18 & 6 \\ 36 & 0 & 15 & -6 & -30 \\ -75 & -3 & 3 & 9 & 81 \\ 54 & 12 & -9 & 6 & -48 \\ 0 & 3 & -18 & 24 & 6 \end{pmatrix}$$

$$\deg(M_6) = 15 \qquad \deg(M_7) = 17.$$

$$M_8 = \begin{pmatrix} 0 & -32 & 20 & 32 & 0 \\ 48 & 0 & 20 & -8 & -40 \\ -100 & -4 & 4 & 12 & 108 \\ 72 & 16 & -12 & 8 & -64 \\ 8 & 40 & -12 & -24 & 8 \end{pmatrix}, \quad M_9 = \begin{pmatrix} 12 & -16 & 2 & 8 & 4 \\ 24 & 0 & 10 & -4 & -20 \\ -50 & -2 & 2 & 6 & 54 \\ 36 & 8 & -6 & 2 & -32 \\ -2 & 20 & 2 & -12 & 2 \end{pmatrix}$$

$$\deg(M_8) = 19 \qquad \deg(M_9) = 21.$$

$$M_{10} = \begin{pmatrix} 1 & 9 & 0 & -12 & 2 \\ -3 & 196 & 165 & 121 & 3 \\ -4 & 166 & 0 & 75 & 4 \\ 8 & 120 & 76 & 45 & -8 \\ -2 & -9 & 0 & 12 & -1 \end{pmatrix}$$
$$\deg(M_{10}) = 23.$$

For each of the above matrices M_i, aM_i is a magic square of squares of the same degree mod 241 if a is a non-zero quadratic residue mod 241. The magic sum is $S(aM_i) = aS(M_i)$ then.

We now focus on magic squares of squares with non-zero magic sums.

Theorem 11. *Let $p = 445560m + 1$ be a prime number for some integer m. For any given $c \in Q_p$ with $c \neq 0$, there exists a magic square of squares of full degree with magic sum $5c$. That is, $\exists M$, where M is a magic square of squares such that $S(M) = 5c$.*

Proof. Assume p is prime and $p \equiv 1 \pmod{445560}$. Consider the matrix M as below:

$$M = \begin{pmatrix} 6 & -48 & -10 & 49 & 8 \\ 81 & 0 & 5 & -2 & -79 \\ 18 & -1 & 1 & 3 & -16 \\ -94 & 4 & -3 & 2 & 96 \\ -6 & 50 & 12 & -47 & -4 \end{pmatrix}.$$

The above matrix M is a magic square with magic sum 5 over $\mathbb{Z}_p$. To see if all entries are quadratic residues, it is sufficient to check if $-1, 2, 3, 5, 47, 79 \in Q_p$. Since $4, 8, 5, 47$, and 79 all divide 445560, $-1, 2, 3, 5, 47, 79 \in Q_p$. For example, the number 47 is a quadratic residue mod p because

$$\left(\frac{47}{p}\right) = \left(\frac{p}{47}\right) = \left(\frac{1}{47}\right) = 1 \text{ because } p \equiv 1 \pmod{47}.$$

Thus, all entries of M are quadratic residues mod p. The maximal difference of two entries of M is $96 - (-94) = 190$. Since $p > 190$, all entries of M are distinct. That is, $\deg(M) = 25$.

Finally, if c is a non-zero quadratic residue mod p, then the matrix cM is also an MSS mod p which has the magic sum $5c$. □

We are able to build different magic squares of squares of maximal degree by simply filling the entries with quadratic residues, starting with the inner 3×3 matrix. Below, we see that the inner matrix includes relatively simple quadratic residues, while the outer entries involve larger numbers.

Example 9. *When $p = 241$, the following matrix is an MSS with degree 25:*

$$M_2 = \begin{pmatrix} 6 & -48 & -10 & 49 & 8 \\ 81 & 0 & 5 & -2 & -79 \\ 18 & -1 & 1 & 3 & -16 \\ -94 & 4 & -3 & 2 & 96 \\ -6 & 50 & 12 & -47 & -4 \end{pmatrix} = \begin{pmatrix} 27^2 & 117^2 & 58^2 & 7^2 & 44^2 \\ 9^2 & 0^2 & 103^2 & 38^2 & 43^2 \\ 66^2 & 64^2 & 1^2 & 56^2 & 15^2 \\ 90^2 & 2^2 & 31^2 & 22^2 & 108^2 \\ 61^2 & 110^2 & 112^2 & 26^2 & 113^2 \end{pmatrix}.$$

4.2 Constructions of MSS Using Consecutive Quadratic Residues

For the construction of 3 by 3 MSSs mod p, one method is to use 9 consecutive quadratic residues $x-4, x-3, x-2, x-1, x, x+1, x+2, x+3, x+4$ in the following way:

$$M = \begin{bmatrix} x-1 & x-2 & x+3 \\ x+4 & x & x-4 \\ x-3 & x+2 & x+1 \end{bmatrix}.$$

I apply the same ideas to construct MSSs of order 5. Assume we have 25 consecutive quadratic residues centered at x: $x-12, x-11, \ldots, x-1, x, x+1, \ldots, x+11, x+12$. We define a related matrix function as below:

Definition 7. *Let p is a prime number at least 5 and $\mathbf{r} = (r_1, r_2, \ldots, r_8)$, where $r_i \in \{\pm 5, \pm 6, \pm 7, \pm 8, \pm 9, \pm 10, \pm 11, \pm 12\} \in \mathbb{Z}_p$. Define*

$$M(x, \mathbf{r}) = \begin{pmatrix} x+r_1 & x+r_2 & x+r_3 & x+r_4 & x+r_5 \\ x+r_6 & x-1 & x-2 & x+3 & x-r_6 \\ x+r_7 & x+4 & x & x-4 & x-r_7 \\ x+r_8 & x-3 & x+2 & x+1 & x-r_8 \\ x-r_5 & x-r_2 & x-r_3 & x-r_4 & x-r_1 \end{pmatrix}.$$

It is straightforward to check that the inner 3×3 submatrix at the center of $M(x, \mathbf{r})$ is a magic square of order 3 over $\mathbb{Z}_p$ with the magic sum $3x$. In order for $M(x, \mathbf{r})$ to be a magic square of squares, we need all of the entries to be quadratic residues mod p. That is, $x \pm i \in Q_p$ for all $i = 0, 1, 2, 3, 4$ and $x \pm r_i \in Q_p$ for all $i = 1, 2, \ldots, 8$. For large enough primes p, if we choose r_i appropriately, then all entries in $M(x, \mathbf{r})$ are distinct and are from the consecutive sequence of quadratic residues $x-12, x-11, \ldots, x-1, x, x+1, \ldots, x+11, x+12$. Then $\deg(M(x, \mathbf{r})) = 25$. In addition, $\mathbf{r}$ needs to satisfy the following two conditions:

$$r_1 + r_2 + r_3 + r_4 + r_5 = 0 \quad \text{and} \quad r_1 + r_6 + r_7 + r_8 - r_5 = 0.$$

Lemma 3. *Let p be a prime number. Assume a sequence of consecutive quadratic residues mod p of length 25 exists, say, $(x-12, x-11, \cdots, x-1, x, x+1 \cdots, x+11, x+12)$. If a sequence $\mathbf{r} = (r_1, r_2, \ldots, r_8)$ where $r_i \in \{\pm 5, \pm 6, \pm 7, \pm 8, \pm 9, \pm 10, \pm 11, \pm 12\}$ exists and satisfies $r_1 + r_2 + r_3 + r_4 + r_5 = 0$ and $r_1 + r_6 + r_7 + r_8 - r_5 = 0$, then $M(x, \mathbf{r})$ is a magic square of squares mod p.*

Proof. By definition 7, the conditions $r_1 + r_2 + r_3 + r_4 + r_5 = 0$ and $r_1 + r_6 + r_7 + r_8 - r_5 = 0$

guarantee that all the rows, columns, and diagonals of $M(x, \mathbf{r})$ add to the same sum, $5x$. Thus $M(x, \mathbf{r})$ is a magic square. Since each of the entries is a quadratic residue, $M(x, \mathbf{r})$ is also a magic square of squares. $\square$

From the above analysis, to construct an MSS using a sequence of consecutive quadratic residues of length 25, the main task is to find an appropriate sequence $\mathbf{r}$. We also see that the degree of the inner 3×3 MSS affects that of the 5×5 matrix. If an appropriate sequence $\mathbf{r}$ is selected, it may result in a magic squares of squares of a desired degree.

Example 10. *Assume p is prime and $x \pm i \in Q_p$ all distinct where $i = 1, 2, 3, 4, 5$. When $\mathbf{r} = (0, -5, 5, 0, 0, 0, 0, 0)$, we have an MSS of degree 11:*

$$M(x, \mathbf{r}) = \begin{pmatrix} x & x-5 & x+5 & x & x \\ x & x-1 & x-2 & x+3 & x \\ x & x+4 & x & x-4 & x \\ x & x-3 & x+2 & x+1 & x \\ x & x+5 & x-5 & x & x \end{pmatrix}.$$

Theorem 12. *Let p be any prime number.*

1. *If in $\mathbb{Z}_p$ there is a CQR of length 25, where all entries are distinct in $\mathbb{Z}_p$, then there are MSS's of all possible degrees (odd integers from 3 to 25) over $\mathbb{Z}_p$.*

2. *If $p \equiv 1 \pmod{9240}$, then for each odd integer d from 3 to 25 there exists an MSS of degree d over $\mathbb{Z}_p$ having magic sum 0.*

3. *Let $p \equiv 1 \pmod{120120}$ and $c \in Q_p$. Then for each odd integer d from 3 to 25, there exists an MSS of degree d over $\mathbb{Z}_p$ having magic sum $5c$.*

Proof. It is trivial to produce a magic square of squares of order 5 from the inner magic square of order 3 so that they both have the same degree. If the center element is x, we can just fill in all the "shell" elements by x. The degree of a non-trivial magic square can be

3, 5, 7, or 9. Thus, we focus on magic squares of squares with degrees $11, 13, 15, 17, 19, 21, 23$, and 25. We apply the configuration $M(x, \mathbf{r})$ to construct such matrices. Note that when the 25-CQR in consideration consists of distinct numbers in $\mathbb{Z}_p$, the inner 3×3 matrix is a magic squares of squares of degree 9.

1. Let $(x - 12, \cdots, x, \cdots, x + 12)$ be a CQR of length 25 mod p, where all of entries are distinct in $\mathbb{Z}_p$. We select appropriate sequences $\mathbf{r}$ such that the matrix $M(x, \mathbf{r})$ defined in Definition 7 produces a magic square of squares of degrees $11, 13, 15, 17, 19, 21, 23$, or 25 respectively.

The following sequences $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_8$ produce the magic squares of squares with the degrees $11, 13, 15, 17, 19, 21, 23$, and 25 respectively. One can check easily that $r_1 + r_2 + r_3 + r_4 + r_5 = 0$ and $r_1 + r_6 + r_7 + r_8 - r_5 = 0$.

$$
\begin{aligned}
\mathbf{r}_1 &= (0, -5, 5, 0, 0, 0, 0, 0), & \deg(M(x, \mathbf{r}_1)) &= 11;\\
\mathbf{r}_2 &= (0, -5, 5, 10, 0, 0, 0, 0), & \deg(M(x, \mathbf{r}_2)) &= 13;\\
\mathbf{r}_3 &= (0, 5, 6, -11, 0, 0, 0, 0), & \deg(M(x, \mathbf{r}_3)) &= 15;\\
\mathbf{r}_4 &= (5, -7, 2, 0, 0, 6, -10, -1), & \deg(M(x, \mathbf{r}_4)) &= 17;\\
\mathbf{r}_5 &= (5, -12, 8, 8, -9, 8, -12, -10), & \deg(M(x, \mathbf{r}_5)) &= 19;\\
\mathbf{r}_6 &= (5, 4, 11, -11, -9, -10, -12, 8), & \deg(M(x, \mathbf{r}_6)) &= 21;\\
\mathbf{r}_7 &= (5, 7, 8, -11, -9, -6, -10, 2), & \deg(M(x, \mathbf{r}_7)) &= 23;\\
\mathbf{r}_8 &= (5, 8, 6, -12, -7, -11, -10), & \deg(M(x, \mathbf{r}_8)) &= 25.
\end{aligned}
$$

The resulting matrices $M(x, \mathbf{r}_1)$ and $M(x, \mathbf{r}_8)$ are presented below. The others can be found in the Appendix A.

$$M(x,\mathbf{r}_1) = \begin{pmatrix} x & x-5 & x+5 & x & x \\ x & x-1 & x-2 & x+3 & x \\ x & x+4 & x & x-4 & x \\ x & x-3 & x+2 & x+1 & x \\ x & x+5 & x-5 & x & x \end{pmatrix}, \quad \deg(M(x,\mathbf{r}_1)) = 11.$$

$$M(x,\mathbf{r}_8) = \begin{pmatrix} x+5 & x+8 & x+6 & x-12 & x-7 \\ x-11 & x-1 & x-2 & x+3 & x+11 \\ x+9 & x+4 & x & x-4 & x-9 \\ x-10 & x-3 & x+2 & x+1 & x+10 \\ x+7 & x-8 & x-6 & x+12 & x-5 \end{pmatrix}, \quad \deg(M(x,\mathbf{r}_8)) = 25.$$

2. Assume $p \equiv 1 \pmod{9240}$. Note that $9420 = 8 \cdot 3 \cdot 5 \cdot 7 \cdot 11$. Then $p \equiv 1$ (mod $3, 4, 5, 7, 8, 11$). Thus, -1, 2, 3, 5, 7, 11 are all quadratic residues mod p. Then the sequence $(-12, -11, \cdots, -1, 0, 1, \cdots, 11, 12)$ is a sequence of consecutive quadratic residues mod p of length 25 and all of the 25 numbers are distinct in $\mathbb{Z}_p$. By part 1, there is an MSS of degree d for every odd d from 3 to 25. As the proof in part 1, $M(0,\mathbf{r}_i)$, $i = 1, 2, \ldots, 8$, gives MSSs of degrees 11, 13, 15, 17, 19, 21, 23, and 25 respectively.

For example, the corresponding matrices derived from $\mathbf{r}_1$ and $\mathbf{r}_8$ $(x = 0)$ are shown below, with $\deg(M(0,\mathbf{r}_1)) = 11$ and $\deg(M(0,\mathbf{r}_8)) = 25$. The other matrices can be produced similarly and they are shown in Example 11.

$$M(0,\mathbf{r}_1) = \begin{pmatrix} 0 & -5 & 5 & 0 & 0 \\ 0 & -1 & -2 & 3 & 0 \\ 0 & 4 & 0 & -4 & 0 \\ 0 & -3 & 2 & 1 & 0 \\ 0 & 5 & -5 & 0 & 0 \end{pmatrix}, \quad M(0,\mathbf{r}_8) = \begin{pmatrix} 5 & 8 & 6 & -12 & -7 \\ -11 & -1 & -2 & 3 & 11 \\ 9 & 4 & 0 & -4 & -9 \\ -10 & -3 & 2 & 1 & 10 \\ 7 & -8 & -6 & 12 & -5 \end{pmatrix}.$$

3. For $p \equiv 1 \pmod{120120}$, similar to part 2, $120120 = 8 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 9240 \cdot 13$. This implies that the sequence $(-13, -12, -10, \ldots, -1, 0, 1, 2, \ldots, 12, 13)$ is a CQR consisting of 27 distinct elements. Similarly as in the proof in part 1, for each odd interger d from 3 to 25, we select appropriate $\mathbf{r}$ for $M(x, \mathbf{r})$ to construct an MSS M of degree d with 1 in the center of the matrix. When choosing the same $\mathbf{r}_i$, where $i = 1, 2, \ldots, 8$, as in the proof of Part 2, and placing 1 in the center, the matrix $M(1, \mathbf{r}_i)$ is a magic square of squares of degrees 11, 13, 15, 17, 19, 21, 23, and 25 respectively. And all of them have magic sum 5. For example,

$$M(1, \mathbf{r}_1) = \begin{pmatrix} 1 & -4 & 6 & 1 & 1 \\ 1 & 0 & -1 & 4 & 1 \\ 1 & 5 & 1 & -3 & 1 \\ 1 & -2 & 3 & 2 & 1 \\ 1 & 6 & -4 & 1 & 1 \end{pmatrix}, \quad \deg(M(1, \mathbf{r}_1)) = 11.$$

$$M(1, \mathbf{r}_8) = \begin{pmatrix} 6 & 9 & 7 & -11 & -6 \\ -10 & 0 & -1 & 4 & 12 \\ 10 & 5 & 1 & -3 & -8 \\ -9 & -2 & 3 & 2 & 11 \\ 8 & -7 & -5 & 13 & -4 \end{pmatrix}, \quad \deg(M(1, \mathbf{r}_8)) = 25.$$

We skip presenting the produced matrices $M(1, \mathbf{r}_i)$ for $i = 2, 3, 4, 5, 6, 7$ here. It is straightforward by following the configuration. Note that, for $i = 1, \ldots, 7$, $M(1, \mathbf{r}_i)$ uses the CQR $(-12, -11, \ldots, -1, 0, 1, \ldots, 11, 12)$. However, $M(1, \mathbf{r}_i)$ has to involve the number 13. For each $\mathbf{r}_i = (r_1, r_2, \cdots, r_8)$, $r_1, \ldots, r_8 \in \{\pm 5, \pm 6 \pm 7, \pm 8, \pm 9, \pm 10, \pm 11, \pm 12, 13\} \subseteq Q_p$. It guarantees that $M(1, \mathbf{r}_i)$ is a magic square of squares over $\mathbb{Z}_p$. Obviously, the magic sum $M(1, \mathbf{r}_i)$ is 5. Furthermore, for every $c \in Q_p$, if $c \neq 0$, $cM(1, \mathbf{r}_i)$ is also a magic square of squares with $\deg(cM(1, \mathbf{r}_i)) = \deg(M(1, \mathbf{r}_i))$ and its magic sum is $5c$. If $c = 0$, the matrices provided in part 2 give the MSSs of all odd degrees with magic sum 0, because $p \equiv 1 \pmod{120120} \Longrightarrow p \equiv 1 \pmod{9240}$. Thus MSSs with all possible degrees and all possible sums exist. □

Example 11. *For each of the following matrices, assume* $p \equiv 1 \pmod{9240}$. *Using the configurations from Theorem 12, we are able to construct an MSS over* Q_p *for each odd degree* 3 *to* 25*:*

$$M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -6 & 6 & 0 & 0 \\ 0 & 6 & 0 & -6 & 0 \\ 0 & 0 & -6 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 & 0 \\ 0 & 4 & 0 & -4 & 0 \\ 0 & -2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\deg(M_1) = 3 \qquad \deg(M_2) = 5.$$

$$M_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -3 & 6 & 0 \\ 0 & 9 & 0 & -9 & 0 \\ 0 & -6 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & 3 & 0 \\ 0 & 4 & 0 & -4 & 0 \\ 0 & -3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\deg(M_3) = 7 \qquad \deg(M_4) = 9.$$

$$M_5 = \begin{pmatrix} 0 & -5 & 5 & 0 & 0 \\ 0 & -1 & -2 & 3 & 0 \\ 0 & 4 & 0 & -4 & 0 \\ 0 & -3 & 2 & 1 & 0 \\ 0 & 5 & -5 & 0 & 0 \end{pmatrix}, \quad M_6 = \begin{pmatrix} 0 & -5 & -5 & 10 & 0 \\ 0 & -1 & -2 & 3 & 0 \\ 0 & 4 & 0 & -4 & 0 \\ 0 & -3 & 2 & 1 & 0 \\ 0 & 5 & 5 & -10 & 0 \end{pmatrix}$$

$$\deg(M_5) = 11 \qquad \deg(M_6) = 13.$$

$$M_7 = \begin{pmatrix} 0 & 5 & 6 & -11 & 0 \\ 0 & -1 & -2 & 3 & 0 \\ 0 & 4 & 0 & -4 & 0 \\ 0 & -3 & 2 & 1 & 0 \\ 0 & -5 & -6 & 11 & 0 \end{pmatrix}, \quad M_8 = \begin{pmatrix} 5 & -7 & 2 & 0 & 0 \\ -6 & -1 & -2 & 3 & 6 \\ -10 & 4 & 0 & -4 & 10 \\ 2 & -3 & 2 & 1 & -2 \\ 9 & 7 & -2 & 0 & 0 \end{pmatrix}$$

$$\deg(M_7) = 15 \qquad \deg(M_8) = 17.$$

$$M_9 = \begin{pmatrix} 5 & -12 & 8 & 8 & -9 \\ 8 & -1 & -2 & 3 & -8 \\ -12 & 4 & 0 & -4 & 12 \\ -10 & -3 & 2 & 1 & 10 \\ 9 & 12 & -8 & -8 & -5 \end{pmatrix}, \quad M_{10} = \begin{pmatrix} 5 & 4 & 11 & -11 & -9 \\ -10 & -1 & -2 & 3 & 10 \\ -12 & 4 & 0 & -4 & 12 \\ 8 & -3 & 2 & 1 & -8 \\ 9 & -4 & -11 & 11 & -5 \end{pmatrix}$$

$$\deg(M_9) = 19 \qquad \deg(M_{10}) = 21.$$

$$M_{11} = \begin{pmatrix} 5 & 7 & 8 & -11 & -9 \\ -6 & -1 & -2 & 3 & 10 \\ -10 & 4 & 0 & -4 & 12 \\ 2 & -3 & 2 & 1 & -8 \\ 9 & -4 & -11 & 11 & -5 \end{pmatrix} \quad M_12 = \begin{pmatrix} 5 & 8 & 6 & -12 & -7 \\ -11 & -1 & -2 & 3 & 11 \\ 9 & 4 & 0 & -4 & -9 \\ -10 & -3 & 2 & 1 & 10 \\ 7 & -8 & -6 & 12 & -5 \end{pmatrix}.$$

$$\deg(M_{11}) = 23 \qquad \deg(M_{12}) = 25.$$

Corollary 2. *For infinitely many prime numbers p, for each odd integer d from 3 to 25 (all possible degrees) and for each $c \in Q_p$, there exists an 5×5 MSS of degree d with the magic sum $5c$ (all possible magic sums). In particular, it is true for every prime number $p \equiv 1 \pmod{9240}$.*

Proof. By Dirichlet's Theorem 5, there are infinitely may prime numbers p such that $p \equiv 1$

(mod 120120). Then it is immediate to claim the statement by Theorem 12. ☐

Below we show a magic square of squares by another method.

Example 12. *Here, let* $p = 2041$ *and let* $x = 6 \in Q_p$. *Then,* M *is a magic square of squares of degree 25 over* $\mathbb{Z}_{2041}$*:*

$$M = \begin{pmatrix} 11 & 13 & 14 & -5 & -3 \\ 0 & 5 & 4 & 9 & 12 \\ -4 & 10 & 6 & 2 & 16 \\ 8 & 3 & 8 & 7 & 4 \\ 15 & -1 & -2 & 17 & 1 \end{pmatrix}.$$

One can check that both 13 and 17 are quadratic residues mod 2401 by calculating the related Legendre symbols.

4.3 Other Construction Methods

In this section, I construct another method of constructing MSS of a desired degree. More prime numbers are involved than those in Theorem 12(2).

Theorem 13. *Consider any prime* $p \equiv 1 \pmod{120}$. *Let* d *be an odd integer such that* $3 \leq d \leq 25$. *Then there are at least* $\frac{p-1}{2}$ *many MSSs of degree* d. *Furthermore, Magic squares of squares of odd degrees up to 23 with all the possible non-zero magic sums exist.*

Proof. The proof is constructive. Different selection methods of the entries are applied. See Appendix 6 Part B for MSS's of odd degrees from 3 to 25. The matrices M_1 to M_{10} give MSS's with the magic sum 5. M_{12} is an MSS of degree 25 with magic sum 0. In order to construct an MSS of degree 23, we use the configuration: $M(a)$, shown below.

$$M(a) = \begin{pmatrix} 2^{-1}(3-a) & -8 & a & 8 & 2^{-1}(7-a) \\ 12 & 0 & 5 & -2 & -10 \\ -25 & -1 & 1 & 3 & 27 \\ 18 & 4 & -3 & 2 & -16 \\ 2^{-1}(a-3) & 10 & 2-a & -6 & 2^{-1}(a+1) \end{pmatrix}.$$

Specifically, when $a = 8$, we have the following MSS of degree 23 with the magic sum 5:

$$M(8) = \begin{pmatrix} -2^{-1}(5) & -8 & 8 & 8 & -2^{-1} \\ 12 & 0 & 5 & -2 & -10 \\ -25 & -1 & 1 & 3 & 27 \\ 18 & 4 & -3 & 2 & -16 \\ 2^{-1}(5) & 10 & -6 & -6 & 2^{-1}(9) \end{pmatrix}.$$

There are $\frac{p-1}{2}$ non-zero quadratic residues. For each M_i, $i = 1, 2, \ldots, 10$, aM_i is also a magic square of squares mod p of the same degree if a is a non-zero quadratic residue. Thus, there are at least $\frac{p-1}{2}$ many MSS of degree d for each $d \in \{3, 5, \ldots, 25\}$.

□

Chapter 5

Conclusions and Future Directions

In this research, I investigated magic squares of squares of order 5 modulo a prime number, in which the center 3×3 matrix is a magic square of squares of order 3. The main results are summarized below.

1. For any prime $p > 3$ and for any magic square $M = [a_{ij}]_{5\times 5}$ of squares over $\mathbb{Z}_p$ with a 3×3 center matrix also a magic square of squares, the magic sum $S(M) = 5a_{33}$ and the degree, $\deg(M)$, must be odd.

2. Multiple methods are provided for constructing magic squares of squares of all odd degrees 3 to 25 using special sequences of quadratic residues. The two types of sequences applied are Pythagorean triples of quadratic residues and sequences of consecutive quadratic residues of certain lengths.

3. The existence of magic squares of squares in different categories are shown:

    - Existence of MSSs of full degree for infinitely many prime numbers p, where p is congruent to 1 modulo 120;

    - Existence of MSSs of all odd degrees 3 to 25 with magic sum 0 for infinitely many primes p ($p \equiv 1 \pmod{9240}$);

- Existence of MSSs of all odd degrees 3 to 25 with all possible magic sums for infinitely many primes p ($p \equiv 1 \pmod{120120}$).

While LaBar's question on magic squares of squares of order 3 over the integers remains unanswered, this research shifted the focus to magic squares of squares modulo prime numbers. For infinitely many prime numbers p, the answer to the existence of a magic square of squares over $\mathbb{Z}_p$ with all distinct elements is "YES". Magic squares of squares of order 5 in which all the entries are distinct squares are constructed over $\mathbb{Z}_p$ for infinitely many prime numbers p. The work done in this project brings forward other questions and goals to be addressed in the future. The following questions are of interest for future projects:

1. Modulo a prime number p, can we construct a magic square of squares of order 5 over $\mathbb{Z}_p$ such that the center 3×3 matrix is not a magic square over $\mathbb{Z}_p$?

2. For a magic square $M = (a_{ij})_{5\times 5}$, if the center 3×3 matrix is not a magic square, is it still true that $S(M) = 5a_{33}$?

3. What are the situations for magic squares of squares of order higher than 5?

4. For those prime numebrs p, where $\alpha_p < 25$, what is the exact value of α_p and how to construct the corresponding MSSs with the corresponding degrees?

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Chapter 6

Appendix

6.1 Part A

In general, given a CQR of length 25, we are able to construct a magic square of squares of all odd degrees.

$$\text{Degree 3:} \quad \begin{pmatrix} x & x & x & x & x \\ x & x-6 & x+6 & x & x \\ x & x+6 & x & x-6 & x \\ x & x & x-6 & x+6 & x \\ x & x & x & x & x \end{pmatrix}$$

$$\text{Degree 5:} \quad \begin{pmatrix} x & x & x & x & x \\ x & x-2 & x & x+2 & x \\ x & x+4 & x & x-4 & x \\ x & x-2 & x & x+2 & x \\ x & x & x & x & x \end{pmatrix}$$

Degree 7: $$\begin{pmatrix} x & x & x & x & x \\ x & x-3 & x-3 & x+6 & x \\ x & x+9 & x & x-9 & x \\ x & x-6 & x+3 & x+3 & x \\ x & x & x & x & x \end{pmatrix}$$

Degree 9: $$\begin{pmatrix} x & x & x & x & x \\ x & x-1 & x-2 & x+3 & x \\ x & x+4 & x & x-4 & x \\ x & x-3 & x+2 & x+1 & x \\ x & x & x & x & x \end{pmatrix}$$

Degree 11: $$\begin{pmatrix} x & x-5 & x+5 & x & x \\ x & x-1 & x-2 & x+3 & x \\ x & x+4 & x & x-4 & x \\ x & x-3 & x+2 & x+1 & x \\ x & x+5 & x-5 & x & x \end{pmatrix}$$

Degree 13: $$\begin{pmatrix} x & x-5 & x-5 & x+10 & x \\ x & x-1 & x-2 & x+3 & x \\ x & x+4 & x & x-4 & x \\ x & x-3 & x+2 & x+1 & x \\ x & x+5 & x+5 & x-10 & x \end{pmatrix}$$

$$\text{Degree 15:} \quad \begin{pmatrix} x & x+5 & x+6 & x-11 & x \\ x & x-1 & x-2 & x+3 & x \\ x & x+4 & x & x-4 & x \\ x & x-3 & x+2 & x+1 & x \\ x & x-5 & x-6 & x+11 & x \end{pmatrix}$$

$$\text{Degree 17:} \quad \begin{pmatrix} x+5 & x-7 & x+2 & x & x \\ x+6 & x-1 & x-2 & x+3 & x-6 \\ x-10 & x+4 & x & x-4 & x+10 \\ x-1 & x-3 & x+2 & x+1 & x+1 \\ x & x+7 & x-2 & x & x-5 \end{pmatrix}$$

$$\text{Degree 19:} \quad \begin{pmatrix} x+5 & x-12 & x+8 & x+8 & x-9 \\ x+8 & x-1 & x-2 & x+3 & x-8 \\ x-12 & x+4 & x & x-4 & x+12 \\ x-10 & x-3 & x+2 & x+1 & x+10 \\ x+9 & x+12 & x-8 & x-8 & x-5 \end{pmatrix}$$

$$\text{Degree 21:} \quad \begin{pmatrix} x+5 & x+4 & x+11 & x-11 & x-9 \\ x-10 & x-1 & x-2 & x+3 & x+10 \\ x-12 & x+4 & x & x-4 & x+12 \\ x+8 & x-3 & x+2 & x+1 & x-8 \\ x+9 & x-4 & x-11 & x+11 & x-5 \end{pmatrix}$$

$$\text{Degree 23:} \quad \begin{pmatrix} x+5 & x+7 & x+8 & x-11 & x-9 \\ x-6 & x-1 & x-2 & x+3 & x+10 \\ x-10 & x+4 & x & x-4 & x+12 \\ x+2 & x-3 & x+2 & x+1 & x-8 \\ x+9 & x-4 & x-11 & x+11 & x-5 \end{pmatrix}$$

$$\text{Degree 25:} \quad \begin{pmatrix} x+5 & x+8 & x+6 & x-12 & x-7 \\ x-11 & x-1 & x-2 & x+3 & x+10 \\ x+9 & x+4 & x & x-4 & x+12 \\ x-10 & x-3 & x+2 & x+1 & x-8 \\ x+7 & x-4 & x-11 & x+11 & x-5 \end{pmatrix}.$$

6.2 Part B

Each of the matrices below is a magic square of squares over $\mathbb{Z}_p$ for some $p \equiv 1 \pmod{120}$.

$$M_1 = \begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 2 & 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 & 1 \\ 1 & 1 & 0 & 2 & 1 \\ 1 & 0 & 2 & 1 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 2 & -1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & 0 & 1 \\ 2 & -1 & 2 & 1 & 1 \end{pmatrix},$$

where $\deg(M_3) = 3$, and $\deg(M_4) = 5$.

$$M_3 = \begin{pmatrix} 4 & -2 & 1 & 2 & 0 \\ -2 & 2 & -1 & 2 & 4 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 0 & 2 \\ 2 & 4 & 1 & 0 & -2 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 5 & -2 & 1 \\ 1 & -1 & 1 & 3 & 1 \\ 1 & 4 & -3 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

where $\deg(M_5) = 7$, and $\deg(M_6) = 9$.

$$M_5 = \begin{pmatrix} 0 & 1 & 1 & 1 & 2 \\ -2 & 0 & 5 & -2 & 4 \\ 6 & -1 & 1 & 3 & -4 \\ 1 & 4 & -3 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \end{pmatrix}, \quad M_6 = \begin{pmatrix} 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 5 & -2 & 1 \\ -6 & -1 & 1 & 3 & 8 \\ 10 & 4 & -3 & 2 & -8 \\ 0 & 1 & 1 & 1 & 2 \end{pmatrix},$$

where $\deg(M_7) = 11$, and $\deg(M_8) = 13$.

$$M_7 = \begin{pmatrix} 0 & 1 & 1 & 1 & 2 \\ 12 & 0 & 5 & -2 & -10 \\ -25 & -1 & 1 & 3 & 27 \\ 18 & 4 & -3 & 2 & -16 \\ 0 & 1 & 1 & 1 & 2 \end{pmatrix}, \quad M_8 = \begin{pmatrix} 0 & 1 & 8 & -6 & 2 \\ 12 & 0 & 5 & -2 & -10 \\ -25 & -1 & 1 & 3 & 27 \\ 18 & 4 & -3 & 2 & -16 \\ 0 & 1 & -6 & 8 & 2 \end{pmatrix},$$

where $\deg(M_9) = 15$, and $\deg(M_{10}) = 17$.

$$M_9 = \begin{pmatrix} 0 & -8 & 5 & 8 & 0 \\ 12 & 0 & 5 & -2 & -10 \\ -25 & -1 & 1 & 3 & 27 \\ 18 & 4 & -3 & 2 & -16 \\ 2 & 10 & -3 & -6 & 2 \end{pmatrix}, \quad M_{10} = \begin{pmatrix} 1 & -8 & 1 & 8 & 2 \\ 12 & 0 & 5 & -2 & -10 \\ -25 & -1 & 1 & 3 & 27 \\ 18 & 4 & -3 & 2 & -16 \\ -1 & 10 & 1 & -6 & 1 \end{pmatrix},$$

where $\deg(M_{11}) = 19$, and $\deg(M_{12}) = 21$.

$$M_{11} = \begin{pmatrix} -2^{-1}(5) & -8 & 8 & 8 & -2^{-1} \\ 12 & 0 & 5 & -2 & -10 \\ -25 & -1 & 1 & 3 & 27 \\ 18 & 4 & -3 & 2 & -16 \\ 2^{-1}(5) & 10 & -6 & -6 & 2^{-1}(9) \end{pmatrix}$$

$$M_{12} = \begin{pmatrix} 1 & 6 & 3 & -12 & 2 \\ 5 & 36 & -27 & 9 & -5 \\ 4 & 45 & 0 & -45 & -4 \\ -8 & -9 & 27 & -36 & 8 \\ -2 & 6 & -3 & 12 & -1 \end{pmatrix},$$

where $\deg(M_1) = 23$, and $\deg(M_2) = 25$.

6.3 Part C

Each of the matrices below is a magic square of squares over $\mathbb{Z}_p$ for some $p \equiv 1 \pmod{445560}$. The matrices M_i shown below achieve all odd degrees from 3 to 25 and all with magic sum 5, except for M_{10} which has magic sum 0.

The following are of degree 3 and 5:

$$M_1 = \begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 2 & 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 & 1 \\ 1 & 1 & 0 & 2 & 1 \\ 1 & 0 & 2 & 1 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 2 & -1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & 0 & 1 \\ 2 & -1 & 2 & 1 & 1 \end{pmatrix}$$

The following are of degree 7 and 9:

$$M_3 = \begin{pmatrix} 4 & -2 & 1 & 2 & 0 \\ -2 & 2 & -1 & 2 & 4 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 0 & 2 \\ 2 & 4 & 1 & 0 & -2 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 5 & -2 & 1 \\ 1 & -1 & 1 & 3 & 1 \\ 1 & 4 & -3 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The following are of degree 11 and 13.

$$M_5 = \begin{pmatrix} 0 & 1 & 1 & 1 & 2 \\ -2 & 0 & 5 & -2 & 4 \\ 6 & -1 & 1 & 3 & -4 \\ 1 & 4 & -3 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \end{pmatrix}, \quad M_6 = \begin{pmatrix} 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 5 & -2 & 1 \\ -6 & -1 & 1 & 3 & 8 \\ 10 & 4 & -3 & 2 & -8 \\ 0 & 1 & 1 & 1 & 2 \end{pmatrix}$$

The following are of degree 15 and 17.

$$M_7 = \begin{pmatrix} 0 & 1 & 1 & 1 & 2 \\ 12 & 0 & 5 & -2 & -10 \\ -25 & -1 & 1 & 3 & 27 \\ 18 & 4 & -3 & 2 & -16 \\ 0 & 1 & 1 & 1 & 2 \end{pmatrix}, \quad M_8 = \begin{pmatrix} 0 & 1 & 8 & -6 & 2 \\ 12 & 0 & 5 & -2 & -10 \\ -25 & -1 & 1 & 3 & 27 \\ 18 & 4 & -3 & 2 & -16 \\ 0 & 1 & -6 & 8 & 2 \end{pmatrix}$$

The following are of degree 19 and 21.

$$M_9 = \begin{pmatrix} 0 & -8 & 5 & 8 & 0 \\ 12 & 0 & 5 & -2 & -10 \\ -25 & -1 & 1 & 3 & 27 \\ 18 & 4 & -3 & 2 & -16 \\ 2 & 10 & -3 & -6 & 2 \end{pmatrix}, \quad M_{10} = \begin{pmatrix} 1 & -8 & 1 & 8 & 3 \\ 12 & 0 & 5 & -2 & -10 \\ -25 & -1 & 1 & 3 & 27 \\ 18 & 4 & -3 & 2 & -16 \\ -1 & 10 & 1 & -6 & 1 \end{pmatrix}$$

The following are of degree 23 and 25.

$$M_{11} = \begin{pmatrix} -2^{-1}(5) & -8 & 8 & 8 & -2^{-1} \\ 12 & 0 & 5 & -2 & -10 \\ -25 & -1 & 1 & 3 & 27 \\ 18 & 4 & -3 & 2 & -16 \\ 2^{-1}(5) & 10 & -6 & -6 & 2^{-1}(9) \end{pmatrix}$$

$$M_{12} = \begin{pmatrix} 1 & 6 & 3 & -12 & 2 \\ 5 & 36 & -27 & 9 & -5 \\ 4 & 45 & 0 & -45 & -4 \\ -8 & -9 & 27 & -36 & 8 \\ -2 & 6 & -3 & 12 & -1 \end{pmatrix}.$$