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The Independence Polynomial of a Graph at −1

Kyle Robbins

Abstract

The independence polynomial of a graph G is given by $I(G;x) = \sum_{k=0}^{\alpha(G)} i_k(G)x^k$ where $\alpha(G)$ is the independence number of G and $i_k(G)$ is the number of independent sets of size k in G. A $\tilde{3}$-cycle of a graph is a cycle with length divisible by 3. Cao and Ren proved for any integer $k \geq 0$ that $|I(G;x)| \leq 2^k$ for a graph G such that all cycles are pairwise disjoint and k is the number of $\tilde{3}$-cycles in G. In this paper, we prove a density result related to this result of Cao and Ren. We show that for every integer $k \geq 0$ and integer q with $|q| \leq 2^k$ there is a graph G such that all cycles are pairwise disjoint, $I(G;x) = q$ and k is the number of $\tilde{3}$-cycles in G.

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The Independence Polynomial of Graphs at -1

by

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O the depth of the riches both of the wisdom and knowledge of God! how unsearchable are his judgments, and his ways past finding out! For who hath known the mind of the Lord? or who hath been his counsellor? Or who hath first given to him, and it shall be recompensed unto him again? For of him, and through him, and to him, are all things: to whom be glory for ever. Amen.
(Romans 11:33-36)

That their hearts might be comforted, being knit together in love, and unto all riches of the full assurance of understanding, to the acknowledgement of the mystery of God, and of the Father, and of Christ; In whom are hid all the treasures of wisdom and knowledge.
(Colossians 2:2-3)

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Chapter 1

Introduction

We begin with a few preliminary definitions. We define a *graph* G to be a pair with a set of vertices $V(G)$ and a set of edges $E(G)$. If two vertices $v, u \in V(G)$ form an edge in G we write $uv \in E(G)$ and say that u and v are *adjacent vertices*. An *independent set* I of G is a subset $I \subseteq V(G)$ such that for every $u, v \in I$ it follows that $uv \notin E(G)$. Therefore, an independent set only contains non-adjacent vertices. The independence number, denoted by $\alpha(G)$, is the size of the largest independent set of a graph G. The *independence polynomial* of a graph G is defined as

$$I(G; x) = \sum_{k=0}^{\alpha(G)} i_k(G)x^k,$$

where $i_k(G)$ is the number of independent sets of size k in G and $\alpha(G)$ is the independence number of G.

Consider an integer n and distinct elements $v_1, \ldots, v_n$. A *path* with n vertices is a graph with a vertex set $V(P_n) = \{v_1, \ldots, v_n\}$ and edge set $E(P_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\}$ such that. Notice, in particular, that $v_n v_1 \notin E(P_n)$. The final vertex and the initial vertex are not adjacent. Figure 1.1 contains an illustration of the path

P_6. A graph G is said to be *connected* if for any vertices $v, u \in V(G)$ there is a path connecting them.

A cycle is defined very similarly to a path except the first and last vertex meet. Let $n \geq 3$ be an integer and $v_1, \ldots, v_n$ distinct elements. A *cycle* with n vertices, denoted by C_n, is a graph with vertex set $V(C_n) = \{v_1, \ldots, v_n\}$ and edge set $E(C_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_n v_1\}$. Figure 1.1 gives an illustration of the cycle C_5. A graph G is *acyclic* if it contains no cycles. We let the decycling number of G, denoted $\phi(G)$, be the minimum size of a set $S \subseteq V(G)$ such that $G - S$ is acyclic.

Figure 1.1: Illustration of P_6 and C_5

What follows is a brief overview of result involving the independence polynomial of a graph at -1. The independence polynomial was first defined by Gutman [6] . Since then, the independence polynomial has been an object of interest in research. The independence polynomial is useful in determining enumerative information about a graph. For example, for a graph G, if $x = 1$, then

$$I(G; 1) = i_0 + i_1 + \cdots + i_{\alpha(G)}.$$

This gives the number of independent sets in the graph G. Or, when $x = -1$,

$$I(G; -1) = i_0 - i_1 + \cdots + (-1)^{\alpha(G)} \cdot i_{\alpha(G)}.$$

This gives the difference of the number of independence sets of even and odd sizes.

Because of its usefulness, one major direction of inquiry has been the computation of the independence polynomial of a graph. It has been shown that finding $\alpha(G)$ is an NP-complete problem [5]. As a result, computation of the independence polynomial in general is difficult to determine explicitly.

The independence polynomials of some special graphs have been determined explicitly. Let P_n be the path with n vertices and C_n be the cycle with n vertices. Hopkins and Staton [7] proved the following which determines explicitly the independence polynomials for a path P_n.

Theorem 1.0.1 (Hopkins, Staton, [7])**.** *For a path P_n with n vertices where $n \geq 0$ the independence polynomial for any $x \in \mathbb{C}$ is given by,*

$$I(P_n; x) = \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} \binom{n-k+1}{k} \cdot x^k.$$

Hopkins and Staton also proved a result that determines the independence polynomial for a cycle C_n.

Theorem 1.0.2 (Hopkins, Staton, [7])**.** *For a cycle C_n with n vertices where $n \geq 3$, the independence polynomial for $x \in \mathbb{C}$ is given by*

$$I(C_n; x) = 1 + \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{n}{k} \binom{n-k-1}{k-1} \cdot x^k.$$

Due to the difficulty of explicit computation of the independence polynomial; attention has been spent on finding bounds for the absolute value of the independence polynomial. Engström proved a major result in this direction using techniques from topological combinatorics. Specifically, he used tools from what is called Discrete Morse Theory first developed by Robin Forman [4]. Forman's ideas and techniques originate from ideas in algebraic topology. Engström [3] was able to prove the following upper bound on the independence polynomial of G evaluated at -1.

Theorem 1.0.3 (Engström 2009)**.** *If G is any graph, then*

$$|I(G;-1)| \leq 2^{\phi(G)}.$$

This same result was later proved using graph theoretic techniques by Levit and Mandrescu [8]. Levit and Mandrescu [8] also conjectured the following theorem which was proved by Cutler and Kahl [2].

Theorem 1.0.4 (Cutler, Kahl 2016)**.** *For every positive integer k and each integer q such that $|q| \leq 2^k$, there is a graph G with $\phi(G) = k$ and $I(G;-1) = q$.*

Levit and Mandrescu [9] were able to find a different upper bound for $I(G;-1)$. Let the *cyclomatic number* of G, denoted $\beta(G)$, be defined by $\beta(G) = |E(G)| - |V(G)| + p$, where p is the number of connected components of G. We note that p is, of course, dependent on G. Levit and Mandrescu proved the following:

Theorem 1.0.5 (Levit, Mandrescu 2013)**.** *If G is any graph, then*

$$|I(G;-1)| \leq 2^{\beta(G)}.$$

If G is a graph a *maximum independent set* is an independent set of maximum size. Notice that $\alpha(G)$ is the cardinality of a maximum independent set in G. A graph G is said to be *well covered* if all its maximal independent sets are of the same cardinality [9]. Levit and Mandrescu also proved the following result:

Theorem 1.0.6 (Levit, Mandescu 2013)**.** *If G is a unicyclic well-covered graph and $G \neq C_3$, then $I(G;-1) \in \{-1, 0, 1\}$.*

This result was then expanded on and generalized by Cao and Ren [1]. It turns out that the condition that Levit and Mandrescu's condition that $G \neq C_3$ hints at a method for determining a sharper bound on $|I(G;-1)|$ later expanded on by Cao and Ren [1].

Definition 1.0.1. We call a cycle in a graph G a $\tilde{3}$-*cycle* if its length is divisible by 3 and a *non*-$\tilde{3}$-*cycle* otherwise.

Cao and Ren [1] proved the following result.

Theorem 1.0.7 (Cao, Ren 2020)**.** *If G contains a non-$\tilde{3}$-cycle, then*

$$|I(G;-1)| \leq 2^{\beta(G)} - \beta(G).$$

There are infinitely many graphs G with $\beta(G) = \phi(G)$ which contain non-$\tilde{3}$-cycles. We will give examples after developing some useful tools for doing so. One idea is to construct graphs with disjoint cycles which we do in Example 5. Because there are infinitely many graphs G with $\beta(G) = \phi(G)$, this result gives a closer bound on $I(G;-1)$ than Engström's result for these graphs.

If G is a graph, two cyclesin G are *vertex disjoint* if they do not share any common vertices. Cao and Ren [1] were also able to prove the following.

Theorem 1.0.8 (Cao, Ren 2020)**.** *If all cycles of G are vertex disjoint, then $|I(G;-1)| \leq 2^k$, where k is the number of $\tilde{3}$-cycles of G.*

In this paper we prove the following density result by adapting the results of Cutler and Kahl [2].

Theorem 1.0.9. *For every integer $k \geq 0$ and integer q with $|q| \leq 2^k$, there is a graph G such that all cycles of G are pairwise disjoint, $I(G;-1) = q$ and the number of $\tilde{3}$-cycles of G is k.*

Chapter 2

Graph Preliminaries

2.1 Explicit Independence Polynomials for Paths and Cycles

We begin by proving the results observed by Hopkins and Staton [7]. The proofs of which give us some interesting ideas and an important foundation for later results.

Recall that Hopkins and Staton determined the independence polynomial for a path P_n explicitly.

Theorem 1.0.1 (Hopkins, Staton, [7])**.** *For a path P_n with n vertices where $n \geq 0$ the independence polynomial for any $x \in \mathbb{C}$ is given by,*

$$I(P_n; x) = \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} \binom{n-k+1}{k} \cdot x^k.$$

Hopkins and Staton also determined the indpendence polynomial for a cycle C_n explicitly.

Theorem 1.0.2 (Hopkins, Staton, [7])**.** *For a cycle C_n with n vertices where $n \geq 3$,* *the independence polynomial for $x \in \mathbb{C}$ is given by*

$$I(C_n; x) = 1 + \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{n}{k} \binom{n-k-1}{k-1} \cdot x^k.$$

These can be proved by way of two lemmas which count the number of independent sets in both paths and cycles. Recall that for a graph G we let $i_k(G)$ be the number of independent sets of size k in the graph G.

Proposition 2.1.1. *Let n, k be integers with $n \geq k \geq 0$, then*

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}.$$

Proof. Let n, k be integers with $n \geq k \geq 0$, then

$$\begin{aligned}
\binom{n}{k} + \binom{n}{k-1} &= \frac{n!}{(n-k)!k!} + \frac{n!}{(n-k+1)!(k-1)!} \\
&= \frac{(n-k+1) \cdot n! + k \cdot n!}{(n-k+1)!k!} \\
&= \frac{(n-k+1+k) \cdot n!}{(n-k+1)!k!} \\
&= \frac{(n+1) \cdot n!}{(n-k+1)!k!} \\
&= \frac{(n+1)!}{n-k+1)!k!} \\
&= \binom{n+1}{k}.
\end{aligned}$$

□

Proposition 2.1.2. *Let $n \geq 1$ be an integer. The maximum size of an independent set of a path P_n with n vertices is $\left\lfloor \frac{n+1}{2} \right\rfloor$. In particular, $\alpha(P_n) = \lfloor (n+1)/2 \rfloor$.*

Proof. Consider a path P_n. If n is even, the biggest independent set can be found by adding the first vertex, call it 1, to our independent set and adding every other vertex. There are $n/2$ such vertices to choose from. Notice that for n even we have $n = 2k$ for some $k \in \mathbb{N}$. In this case

$$\begin{aligned}\left\lfloor \frac{n+1}{2} \right\rfloor &= \left\lfloor \frac{2k+1}{2} \right\rfloor \\ &= \left\lfloor k + \frac{1}{2} \right\rfloor \\ &= k \\ &= \frac{n}{2}.\end{aligned}$$

If n is odd, by similar reasoning we find that there are $(n+1)/2$ vertices we can choose for our maximal independent set. Notice that for n odd we have,

$$\left\lfloor \frac{n+1}{2} \right\rfloor = \frac{n+1}{2}.$$

Thus the maximum size of an independent set for a path P_n with n vertices is given by

$$\left\lfloor \frac{n+1}{2} \right\rfloor.$$

$\square$

Lemma 2.1.3 (Hopkins and Staton, [7]). *Let $n \geq 1$ and P_n be a path with n vertices, then*

(i) $i_0(P_n) = 1$,

(ii) $i_1(P_n) = n$,

(iii) $i_k(P_{n+1}) = i_k(P_n) + i_{k-1}(P_{n-1})$ *for* $0 \leq k \leq \left\lfloor \frac{n+2}{2} \right\rfloor$, *and*

(iv) $i_k(P_n) = \binom{n-k+1}{k}$ *for* $0 \leq k \leq \left\lfloor \frac{n+1}{2} \right\rfloor$.

Remark. Notice that (iv) is true for $0 \leq k \leq \lfloor (n+1)/2 \rfloor$. This is the case because k represents all the possible sizes of the independent sets for P_n. The smallest independent set is of size 0 whereas, for a path P_n, the largest is $0 \leq k \leq \lfloor (n+1)/2 \rfloor$ by Proposition 2.1.2.

To see why (iii) is true for $0 \leq k \leq \lfloor (n+2)/2 \rfloor$ notice that we can apply Proposition 2.1.2 for P_{n+1}. This path has $n+1$ vertices instead of just n.

Proof. Let n be an integer with $n \geq 1$. To prove (i), the number of independent sets of size 0 for any graph G is 1. (ii) The number of independent sets of size 1 for any graph G is the size of its vertex set. This gives $i_1(P_n) = n$ since there are n vertices in P_n.

(iii) Consider a path P_{n+1} with $n+1$ vertices. The idea is to count the number of independent sets of size k which contain the first vertex of the path, call it v, and the number of independent sets of size k which do not contain n. These two cases partition the number of independent sets in the graph. Counting the number of independent sets of size k which contain v is equivalent to counting the number of independent sets of size $k-1$ in a path P_{n-1} with two less vertices since we already have chosen one element of our independent set and have two less vertices to choose from. This gives $i_{k-1}(P_n)$ independent sets of size $k-1$ which contain v. On the other hand, counting the number of independent sets of size k which do not contain v is equivalent to counting the independent sets of size k in a path P_n with one less vertex. This gives $i_k(P_n)$. Putting these two results together gives $i_k(P_{n+1}) = i_k(P_n) + i_{k-1}(P_{n-1})$.

To see (iv) we do induction on n. For the base case $n = 1$ we have

$$i_0(P_1) = 1$$

$$i_1(P_1) = 1$$

and for $k = 0$,

$$\binom{n-k+1}{k} = \binom{1-0+1}{0} = \binom{2}{0} = 1 = i_0(P_1).$$

For $k = 1$,

$$\binom{n-k+1}{k} = \binom{1-1+1}{1} = \binom{1}{1} = 1 = i_1(P_1)$$

The base case follows as a result. Now suppose the result is true for $1 \leq j \leq i$ for integers i and j. Then,

$$\begin{aligned} i_k(P_{i+1}) &= i_k(P_i) + i_{k-1}(P_{i-1}) && \text{by (iii)} \\ &= \binom{i-k+1}{k} + \binom{(i-1)-(k-1)+1}{k-1} && \text{by assumption} \\ &= \binom{i-k+1}{k} + \binom{i-k+1}{k-1} \\ &= \binom{i-k+2}{k} && \text{by Proposition 2.1.1.} \end{aligned}$$

The result follows by induction. $\square$

The following result will be used to prove the result for cycles.

Proposition 2.1.4. *For integers n and k with $n \geq k \geq 1$,*

$$\binom{n}{k} + \binom{n-1}{k-1} = \frac{n+k}{k}\binom{n-1}{k-1}.$$

Proof.

$$\begin{aligned}\binom{n}{k}+\binom{n-1}{k-1} &= \frac{n!}{(n-k)!k!}+\frac{(n-1)!}{(n-k)!(k-1)!}\\ &= \frac{n(n-1)!}{k(n-k)!(k-1)!}+\frac{(n-1)!}{(n-k)!(k-1)!}\\ &= \frac{(n-1)!}{(k-1)!(n-k)!}\left[\frac{n}{k}+1\right]\\ &= \frac{n+k}{k}\cdot\frac{(n-1)!}{(n-k)!(k-1)!}\\ &= \frac{n+k}{k}\binom{n-1}{k-1}.\end{aligned}$$

□

Lemma 2.1.5 (Hopkins and Staton, [7]). *Let $n \geq 3$ and C_n be a cycle with n vertices, then*

(i) $i_0(C_n) = 1$.

(ii) $i_1(C_n) = n$.

(iii) $i_k(C_n) = i_k(P_{n-1}) + i_{k-1}(P_{n-3})$ *for* $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$.

(iv) $i_k(C_n) = \frac{n}{k}\binom{n-k+1}{k-1}$ *for* $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Remark. Note that in this case the range of values for k in (iii) is $1 \leq k \leq \lfloor n/2 \rfloor$. A cycle C_n with n vertices can be viewed like a path with $n-1$ vertices. Applying Proposition 2.1.2 gives the maximum independent set of size $\lfloor n/2 \rfloor$. The smallest number for k must be 1 since division by 0 causes a problem in (iv).

Proof. The proof of (i) and (ii) follow by the same reasoning as above for paths in Lemma 2.1.3.

To prove (iii) we can use similar reasoning as in the proof of Lemma 2.1.3(iii). Consider a cycle C_{n+1} with $n+1$ vertices. Label a vertex v in the cycle. The independent sets of size k partition into those which contain v and those which do not. The number of independent sets which contain v are given by $i_{k-1}(P_{n-3})$ since we get a path with $n-3$ vertices by removing v and its two neighbors. The number of independent sets of size k which do not contain v is given by $i_k(P_{n-1})$ since removing v gives a path of size $n-1$. These two results together give $i_k(C_n) = i_k(P_{n-1}) + i_{k-1}(P_{n-3})$.

We prove (iv) using Lemma 2.1.3 (iv). We see that

$$\begin{aligned} i_k(C_n) &= i_k(P_{n-1}) + i_{k-1}(P_{n-3}) && \text{by (iii)} \\ &= \binom{n-k}{k} + \binom{n-k-1}{k-1} \\ &= \frac{n}{k}\binom{n-k-1}{k-1} && \text{by Proposition 2.1.4.} \end{aligned}$$

□

We are now in a position to prove the results for the independence polynomials of paths and cycles. First let us consider the result for paths.

Theorem 1.0.1 (Hopkins, Staton, [7]). *For a path P_n with n vertices where $n \geq 0$ the independence polynomial for any $x \in \mathbb{C}$ is given by,*

$$I(P_n; x) = \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} \binom{n+1-k}{k} \cdot x^k.$$

Proof. Let $x \in \mathbb{C}$ and n and integer with $n \geq 0$. For a path P_n we saw in Lemma 2.1.3 that the number of independent sets of size k is

$$i_k(P_n) = \binom{n-k+1}{k}$$

for $0 \leq k \leq \lfloor (n+1)/2 \rfloor$. By definition of the independence polynomial and our use of the notation $i_k(P_n)$ we see that

$$I(P_n;x) = \sum_{k=0}^{\alpha(P_n)} i_k(P_n) \cdot x^k = \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} \binom{n-k+1}{k} \cdot x^k$$

$\square$

For cycles we have the following.

Theorem 1.0.2 (Hopkins, Staton [7]). *For a cycle C_n with n vertices where $n \geq 3$, the independence polynomial for $x \in \mathbb{C}$ is given by*

$$I(C_n;x) = 1 + \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{n}{k} \binom{n-k-1}{k-1} \cdot x^k.$$

Proof. Let $x \in \mathbb{C}$ and n be an integer with $n \geq 3$. By Lemma 2.1.5 we saw that the number of independent sets of size k in a cycle C_n is given by

$$i_k(C_n) = \frac{n}{k} \binom{n-k-1}{k-1}$$

for $1 \leq k \leq \lfloor n/2 \rfloor$. Then the independence polynomial is given by

$$\begin{aligned} I(C_n;x) &= \sum_{k=0}^{\alpha(C_n)} i_k(C_n) \cdot x^k \\ &= i_0(C_n) + \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} i_k(C_n) \cdot x^k \\ &= 1 + \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \binom{n-k-1}{k} \cdot x^k. \end{aligned}$$

$\square$

These results now give a relatively straightforward way of computing the independence polynomials for paths and cycles. Let us consider some examples.

Example 1. Let us compute the independence polynomials for paths with 1 through 6 vertices. We will see a glimpse at an interesting pattern when these are evaluated at -1.

Using our main results along with the identities in Lemma 2.1.3 one can show

$$\begin{aligned}
I(P_1; x) &= 1 + x \\
I(P_2; x) &= 1 + 2x \\
I(P_3; x) &= 1 + 3x + x^2 \\
I(P_4; x) &= 1 + 4x + 3x^2 \\
I(P_5; x) &= 1 + 5x + 6x^2 + x^3 \\
I(P_6; x) &= 1 + 6x + 10x^2 + 4x^3.
\end{aligned}$$

We will consider values $I(G; -1)$ for graphs G quite regularly ahead. Consider the polynomials above evaluated at -1. It can easily be shown that we get the following

$$\begin{aligned}
I(P_1; -1) &= 1 - 1 = 0 \\
I(P_2; -1) &= 1 - 2 = -1 \\
I(P_3; -1) &= 1 - 3 + 1 = -1 \\
I(P_4; -1) &= 1 - 4 + 3 = 0 \\
I(P_5; -1) &= 1 - 5 + 6 - 1 = 1 \\
I(P_6; -1) &= 1 - 6 + 10 - 4 = 1.
\end{aligned}$$

One interesting observation here is that if we partition the vertices of the path into multiples of 3 hints at the more general pattern with $I(P_{3n-2}; -1) = 0$ and $I(P_{3n-1}; -1) = I(P_{3n}; -1) = (-1)^n$. This is in fact true for all $n \geq 1$ and is due to Levit and Mandrescu [9]. After developing some simple tools we give a proof of this fact in what follows.

Example 2. Lets find independence polynomials in a similar fashion but for cycles. Using our results in Lemma 2.1.5 we can find the following independence polynomials. Notice in this case we must have values of $n \geq 3$. So, we begin with C_3.

$$\begin{aligned}
I(C_3; x) &= 1 + 3x \\
I(C_4; x) &= 1 + 4x + 2x^2 \\
I(C_5; x) &= 1 + 5x + 5x^2 \\
I(C_6; x) &= 1 + 6x + 9x^2 + 2x^3 \\
I(C_7; x) &= 1 + 7x + 14x^2 + 7x^3 \\
I(C_8; x) &= 1 + 8x + 20x^2 + 16x^3 + 2x^4.
\end{aligned}$$

Substituting -1 gives,

$$\begin{aligned}
I(C_3; -1) &= 1 - 3 = -2 \\
I(C_4; -1) &= 1 - 4 + 2 = -1 \\
I(C_5; -1) &= 1 - 5 + 5 = 1 \\
I(C_6; -1) &= 1 - 6 + 9 - 2 = 2 \\
I(C_7; -1) &= 1 - 7 + 14 - 7 = 1 \\
I(C_8; -1) &= 1 - 8 + 20 - 16 + 2 = -1.
\end{aligned}$$

Likewise there is a more general pattern. One can check that these results are special cases of the identities

$$I(C_{3n}; -1) = 2 \cdot (-1)^n, \quad I(C_{3n+1}; -1) = (-1)^n, \quad I(C_{3n+2}; -1) = (-1)^{n+1}.$$

These results result were likewise proved by Levit and Mandrescu [9]. We prove these results in general in the next section.

2.2 More General Tools

The purpose of this section is to lay down some ground work for our purposes. Doing so will demonstrate some of the thinking and techniques we will use later.

We first consider a generalization of the thinking involved in the recursive formulas we found for paths and cycles in Lemma 2.1.3 and Lemma 2.1.5. The idea was to partition, for example, the path into independent sets which contain the first vertex and those which do not. It turns out that this idea is very useful and can be generalized easily to arbitrary graphs. This result, in turn will be used as a major tool for proving a recursive identity for independence polynomials. We state the result and give its proof.

For a graph G let $V(G)$ be the vertex set and $E(G)$ be the edge set. For two adjacent vertices $u, v \in V(G)$ we say $uv \in E(G)$. Let $v \in V(G)$. Let $N(v) = \{x \in V(G) : xv \in E(G)\}$ be the set of all neighbors of a vertex $v \in V(G)$. $N(v)$ is called the *open neighborhood* of v. Let $N[v] = \{v\} \cup N(v)$ be the set of neighbors of v and v itself; $N[v]$ is called the *closed neighborhood* of v.

Also, for a graph G, a vertex $v \in V(G)$, and some subset $S \subseteq V(G)$ of the vertex set we let $G - S$ be the subgraph with vertex set S and edges defined only for these vertices. So, for a vertex $v \in V(G)$ of a graph G we can consider the subgraph with v removed. Instead of denoting this graph by $G - \{v\}$ we will just write simply write $G - v$.

Proposition 2.2.1. *For an integer $k \geq 0$ and a graph G with $v \in V(G)$,*

$$i_k(G) = i_k(G - v) + i_{k-1}(G - N[v]).$$

Proof. Consider a graph G, an integer $k \geq 0$ and let $v \in V(G)$. To count the number of independent sets in G we can partition them into sets which contain v and those which do not contain v. The number of independent sets which do not contain v is given by $i_k(G-v)$. The number of independent sets which contain v is $i_{k-1}(G-N[v])$. The reason for this is that having already chosen v we must count the number of independent sets of size $k-1$ that can be made and we must exclude v along with its neighbors. Putting these two results together gives

$$i_k(G) = i_k(G-v) + i_{k-1}(G-N[v]).$$

□

Remark. Notice that the proof is very similar for both Lemma 2.1.3 (iii) and Lemma 2.1.5 (iii). Both results follow immediately from observing, for instance, that for a path P_{n+1} with $n \geq 1$ and the first vertex, call it v,

$$P_{n+1} - v = P_n \quad \text{and} \quad P_{n+1} - N[v] = P_{n-1}$$

and for a cycle C_m with $m \geq 3$ and a vertex u,

$$C_m - u = P_{m-1} \quad \text{and} \quad C_m - N[u] = P_{m-3}.$$

We can use Proposition 2.2.1 to prove the following very useful result. We will see this identity more than a few times. Recall that for a graph G we defined the independence polynomial of G to be

$$I(G;x) = \sum_{k=0}^{\alpha(G)} i_k(G)x^k.$$

For simplicity we drop the independence number in the upper limit of the sum, namely

$\alpha(G)$. Note that the independence polynomial is defined for complex values of x. In what follows we assume that x can be a complex number.

Proposition 2.2.2. *For any graph G and vertex $v \in V(G)$,*

$$I(G;x) = I(G-v;x) + xI\left(G - N[v];x\right)$$

Proof. Let G be a graph with $v \in V(G)$. Then,

$$\begin{aligned}
I(G;x) &= \sum_{k=0} i_k(G) \cdot x^k \\
&= \sum_{k=0} \Big[i_k(G-v) + i_{k-1}(G - N[v]) \Big] \cdot x^k && \text{by Proposition 2.2.1} \\
&= \sum_{k=0} i_k(G-v) \cdot x^k + \sum_{k=0} i_{k-1}(G - N[v]) \cdot x^k \\
&= \sum_{k=0} i_k(G-v) \cdot x^k + x \sum_{k=0} i_{k-1}(G - N[v]) \cdot x^{k-1} \\
&= I(G-v;x) + xI(G - N[v];x).
\end{aligned}$$

□

As useful corollary to this fact is the following.

Corollary 2.2.3. *Let G be a graph and $v \in V(G)$, then*

$$I(G;-1) = I(G-v;-1) - I(G-N[v];-1).$$

Proof. Let $x = -1$ in Proposition 2.2.2 and the result follows immediately. □

With these results we can formalize and prove the results which we alluded to in Examples 1 and 2 concerning the independence polynomials of paths and cycles evaluated at -1. We noticed that certain identities seem to hold when partitioning the vertices in multiples of 3. The result was first proved by Levit and Mandrescu [9].

Theorem 2.2.4 (Levit and Mandrescu, [9]). *For $n \geq 1$,*

(i) $I(P_{3n-2};-1) = 0$, $I(P_{3n-1};-1) = I(P_{3n};-1) = (-1)^n$

(ii) $I(C_{3n};-1) = 2 \cdot (-1)^n$, $I(C_{3n+1};-1) = (-1)^n$, $I(C_{3n+2};-1) = (-1)^{n+1}$.

Proof. (i) We prove the result by induction on n. Let $n = 1$. We saw in Example 1 that

$$I(P_1;x) = 1 + x$$
$$I(P_2;x) = 1 + 2x$$
$$I(P_3;x) = 1 + 3x + x^2$$

so that,

$$I(P_1;-1) = 0$$
$$I(P_2;-1) = 1 - 2 = -1 = (-1)^1$$
$$I(P_3;-1) = 1 - 3 + 1 = -1 = (-1)^1.$$

So, the result is true for $n = 1$. Now suppose that the result holds for $1 \leq i \leq k$

where i and k are integers. Then, for a vertex v in each respective graph,

$$\begin{aligned}
I(P_{3(k+1)-2};-1) &= I(P_{3k+1};-1) \\
&= I(P_{3k+1}-v;-1) - I(P_{3k+1}-N[v];-1) && \text{by Corollary 2.2.3} \\
&= I(P_{3k};-1) - I(P_{3k+1};-1) \\
&= (-1)^k - (-1)^k && \text{by assumption} \\
&= 0 \\
I(P_{3(k+1)-1};-1) &= I(P_{3k+2};-1) \\
&= I(P_{3k+2}-v;-1) - I(P_{3k+2}-N[v];-1) \\
&= I(P_{3k+1};-1) - I(P_{3k};-1) \\
&= 0 - (-1)^k && \text{by assumption and last case} \\
&= (-1)^{k+1} \\
I(P_{3(k+1)};-1) &= I(P_{3k+3};-1) \\
&= I(P_{3k+3}-v;-1) - I(P_{3k+3}-N[v];-1) \\
&= I(P_{3k+2};-1) - I(P_{3k+1};-1) \\
&= (-1)^{k+1} - 0 && \text{(by cases above)} \\
&= (-1)^{k+1}.
\end{aligned}$$

The result follows by induction.

To show (ii) we apply (i). Let $n \geq 1$, then for are vertex v in each respective graph,

$$
\begin{aligned}
I(C_{3n};-1) &= I(C_{3n}-v;-1) - I(C_{3n}-N[v];-1) \\
&= I(P_{3n-1};-1) - I(P_{3(n-1)};-1) \\
&= (-1)^n - (-1)^{n-1} \\
&= (-1)^n + (-1)^n \\
&= 2 \cdot (-1)^n, \\
I(C_{3n+1};-1) &= I(C_{3n+1}-v;-1) - I(C_{3n+1}-N[v];-1) \\
&= I(P_{3n};-1) - I(P_{3n-2};-1) \\
&= (-1)^n - 0 \\
&= (-1)^n, \\
I(C_{3n+2};-1) &= I(C_{3n+2}-v;-1) - I(C_{3n+2}-N[v];-1) \\
&= I(P_{3n+1};-1) - I(P_{3n-1};-1) \\
&= I(P_{3(n+1)-2};-1) - I(P_{3n-1};-1) \\
&= 0 - (-1)^n \\
&= (-1)^{n+1}.
\end{aligned}
$$

□

We now prove a useful proposition concerning the product of the independence polynomials of disjoint graphs. First we consider a result counting the number of independent sets in the union of two disjoint graphs.

Proposition 2.2.5. *Let G_1 and G_2 be disjoint graphs, then for an integer $k \geq 0$,*

$$i_k(G_1 \cup G_2) = \sum_{\ell=0}^{k} i_\ell(G_1) \cdot i_{k-\ell}(G_2).$$

Proof. The idea behind the proof of this statement is to count the number of independent sets of size k by partitioning on the number of independent sets we choose first from, say G_1. For $0 \leq \ell \leq k$, the number of ways we can choose an independent set of size ℓ in G_1 is given by $i_\ell(G_1)$. As there are $i_{k-\ell}(G_2)$ ways to choose the remaining $k-\ell$ independent sets we see there that there are $i_\ell(G_1) \cdot i_{k-\ell}(G_2)$ independent sets of size k by choosing ℓ independent sets first from G_1. We sum over all possible values of ℓ and find,

$$i_k(G_1 \cup G_2) = \sum_{\ell=0}^{k} i_\ell(G_1) \cdot i_{k-\ell}(G_2).$$

□

We now prove a result that relates the independence polynomial of the union of two disjoint graphs to the independence polynomials of the individual graphs. We see that it turns out to be their product. Recall that for two sums,

$$\sum_{i=0}^{n} a_i x^i$$

and

$$\sum_{j=0}^{m} b_j x^j,$$

their *Cauchy product* is given by

$$\left(\sum_{i=0}^{n} a_i x^i\right) \cdot \left(\sum_{j=0}^{m} b_j x^j\right) = \sum_{k=0}^{m+n} c_k x^k,$$

where

$$c_k = \sum_{\ell=0}^{k} a_\ell b_{k-\ell}.$$

Proposition 2.2.6. *Let G_1 and G_2 be disjoint graphs and let $G = G_1 \cup G_2$, then*

$$I(G;x) = I(G_1;x) \cdot I(G_2;x)$$

Proof. For two disjoint graphs G_1 and G_2 with $G = G_1 \cup G_2$,

$$\begin{aligned}
I(G;x) &= \sum_k i_k(G) \cdot x^k \\
&= \sum_k i_k(G_1 \cup G_2) \cdot x^k \\
&= \sum_k \left(\sum_{\ell=0}^{k} i_\ell(G_1) i_{k-\ell}(G_2) \right) \cdot x^k && \text{by Proposition 2.2.5} \\
&= \left(\sum_k i_k(G_1) \cdot x^k \right) \cdot \left(\sum_k i_k(G_2) \cdot x^k \right) \\
&= I(G_1;x) \cdot I(G_2;x) && \text{the Cauchy product.}
\end{aligned}$$

$\square$

Chapter 3

Previous Results of Cutler and Kahl

We seek to prove the following density result by adapting the results of Cutler and Kahl [2].

Theorem 1.0.9. *For every positive integer k and integer q with $|q| \leq 2^k$, there is a graph G such that all cycles of G are pairwise disjoint, $I(G;-1) = q$ and the number of $\tilde{3}$-cycles of G is k.*

Consequently, we give a detailed look at the techniques used by Cutler and Kahl in the proof of the following density result first conjectured by Levit and Mandrescu.

Theorem 1.0.4 (Cutler, Kahl 2016)**.** *For every positive integer k and each integer q such that $|q| \leq 2^k$, there is a graph G with $\phi(G)$ and $I(G;-1) = q$.*

We will see that these techniques can be modified to prove our density result related to the work of Cao and Ren.

3.1 (k,q)-graphs, Brackets, Extensions, and Pasting

We begin with definitions and results that serve as a foundation for Cutler and Kahl's proof of the following theorem conjectured by Levit and Mandrescu [8].

Theorem 1.0.4 (Cutler, Kahl 2016)**.** *For every positive integer k and each integer q such that $|q| \leq 2^k$, there is a graph G with $\phi(G)$ and $I(G;-1) = q$.*

We will define a special type of graph called a (k,q)-graph along with some tools for constructing them. We begin by defining the simple notion of a graph with a labeled vertex. We will call such a graph a *rooted graph.* This will enable us to define a pasting operation that will allow us two join two graphs together at a particular vertex.

Definition 3.1.1 (Rooted Graph)**.** Let G be a graph and $v \in V(G)$. The *rooted graph* of G at v, denoted by G_v, is defined to be simply the graph G with the vertex v labeled.

Suppose G is a graph and consider a vertex $v \in V(G)$. By Corollary 2.2.3, we have

$$I(G;-1) = I(G-v;-1) - I(G-N[v];-1).$$

This suggests the following definition.

Definition 3.1.2 (Bracket)**.** Let G be a graph and $v \in V(G)$ be a vertex of G. If we let

$$I(G-v;-1) = a$$

$$I(G-N[v];-1) = b$$

We have,

$$I(G;-1) = a - b.$$

We define the *bracket* of G by $I(G;-1) = a - b = \langle a, b\rangle$.

Definition 3.1.3 (Wedge/Pasting Operation)**.** If G_v and H_w are rooted graphs the graph $G_v \wedge H_w$ is defined to be the graph obtained by pasting G_v and H_w together by identifying v with w. For short when we identify vertices in this manner we will write $v = w$. We call $G_v \wedge H_w$ the *wedge* of G_v and H_w.

Example 3. Let's consider an example of the pasting operation. Set rooted graphs G_v and H_w as shown in Figure 3.1. The pasting operation joins G_v and H_w together

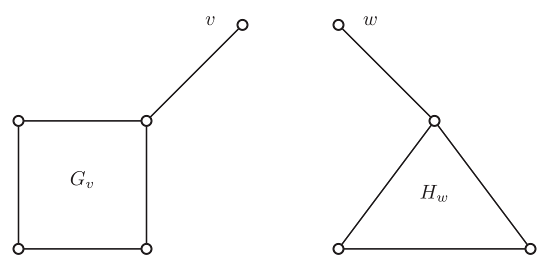

Figure 3.1: Graphs G_v and H_w.

at v and w and declares $v = w$. This is shown in Figure 3.2.

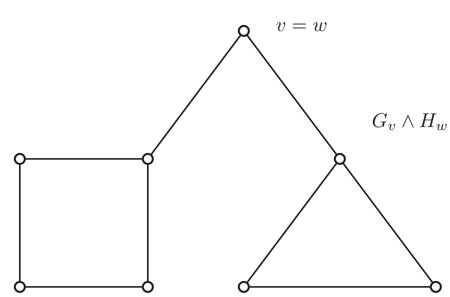

Figure 3.2: Graph $G_v \wedge H_w$.

Lemma 3.1.1 (Pasting Lemma)**.** *Let G_v and H_w be rooted graphs which are disjoint. Let the brackets of G_v and H_w have brackets $I(G_v;-1) = \langle a, b\rangle$ and $I(H_w;-1) = \langle c, d\rangle$, then*

$$I(G_v \wedge H_w;-1) = ac - bd = \langle ac, bd\rangle.$$

Proof. By definition of the bracket we have

$$I(G_v - v; -1) = a$$
$$I(G_v - N[v]; -1) = b$$

and

$$I(H_w - w; -1) = c$$
$$I(H_w - N[w]; -1) = d.$$

Then, because G_v and H_w are disjoint and pasted together at v and w it follows that,

$$\begin{aligned}
I(G_v \wedge H_w; -1) &= I(G_v \wedge H_w - v; -1) - I(G_v \wedge H_w - N[v]; -1) \\
&= I(G_v - v; -1)I(H_v - v; -1) - I(G_v - N[v]; -1)I(H_v - N[v]; -1) \\
&\qquad \text{by Proposition 2.2.6} \\
&= I(G_v - v; -1)I(H_w - w; -1) - I(G_v - N[v]; -1)I(H_w - N[w]; -1) \\
&\qquad \text{since } v = w \\
&= ac - bd \\
&= \langle ac, bd \rangle.
\end{aligned}$$

□

Definition 3.1.4 (ℓ-extension)**.** Let G_v be a rooted graph and $\ell \geq 0$ an integer. An *ℓ-extension*, denoted by G_v^ℓ is defined to be the graph obtained by pasting a path of length ℓ to G_v at v at an endpoint of the path and reassigning v with the other endpoint of the path. When used, G_v^0 will be another way of denoting G_v.

Example 4 (ℓ-extensions of C_3)**.** For illustration, consider $G_v = C_3$. Figure 3.3 shows the graphs G_v^0, G_v^1, and G_v^2.

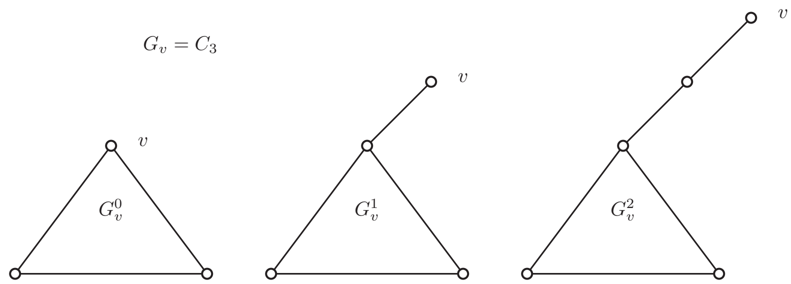

Figure 3.3: G_v^ℓ for $\ell = 0, 1, 2$.

Example 5. We mentioned earlier that $\beta(G) = \phi(G)$ for infinitely many graphs G with non-$\tilde{3}$-cycles. We can use the pasting operation to construct an example which joins disjoint cycles. For each integer $n \geq 1$ we can let

$$G = \underbrace{C_4^1 \wedge \cdots \wedge C_4^1}_{n \text{ times.}}$$

For each integer $n \geq 1$ we see that there are n disjoint cycles. Because there are n disjoint cycles, it follows that $\beta(G) = n$. Also, to make G acyclic we need to delete one vertex from each of the n cycles. Therefore, $\phi(G) = n$. Hence it follows that $\phi(G) = \beta(G)$. As there are infinitely many such integers n, this gives us infinitely many graphs G with $\beta(G) = \phi(G)$. Because each cycle is C_4, each cycle is a non-$\tilde{3}$-cycle.

Lemma 3.1.2 (Extension Lemma)**.** *Let G_v be a rooted graph with bracket $I(G_v; -1) = a - b = \langle a, b \rangle$, then*

$$I(G_v^1; -1) = -b = \langle a - b, a \rangle$$

$$I(G_v^2; -1) = -a = \langle -b, a - b \rangle$$

$$I(G_v^3; -1) = b - a = \langle -a, -b \rangle = -\langle a, b \rangle = -I(G_v; -1).$$

Proof. Suppose $I(G_v, -1) = a - b = \langle a, b \rangle$, then

$$\begin{aligned}
I(G_v^1; -1) &= I(G_v^1 - v; -1) - I(G_v^1 - N[v]; -1) \\
&= I(G_v; -1) - I(G_v - v; -1) \\
&= (a - b) - a \\
&= -b \\
&= \langle a - b, a \rangle \\
I(G_v^2; -1) &= I(G_v^2 - v; -1) - I(G_v^2 - N[v]; -1) \\
&= I(G_v^1; -1) - I(G_v; -1) \\
&= -b - (a - b) \\
&= \langle -b, a - b \rangle \\
I(G_v^3; -1) &= I(G_v^3 - v; -1) - I(G_v^3 - N[v]; -1) \\
&= I(G_v^2; -1) - I(G_v^1; -1) \\
&= -a - (-b) \\
&= \langle -a, -b \rangle
\end{aligned}$$

and notice that

$$\begin{aligned}
-a - (-b) &= -(a - b) \\
&= -\langle a, b \rangle \\
&= -I(G_v; -1).
\end{aligned}$$

So, $I(G_v^3; -1) = b - a = \langle -a, -b \rangle = -\langle a, b \rangle = -I(G_v; -1)$. $\square$

Remark. A simpler way to see the proof of Lemma 3.1.2 is to recognize that for a rooted graph G_v it follows that $(G_v^\ell)^1 = G_v^{\ell+1}$ for any integer $\ell \geq 0$. Thus, for instance, $G_v^2 = (G_v^1)^1$. So if $I(G_v; -1) = \langle a, b \rangle$ it follows that $I(G_v^1; -1) = \langle a - b, a \rangle$.

We can re-apply the first statement to G_v^1 to get

$$\begin{aligned} I(G_v^2;-1) &= I((G_v^1)^1;-1) \\ &= \langle (a-b)-a, a-b\rangle \\ &= \langle -b, a-b\rangle. \end{aligned}$$

We will use this technique in practice whenever we seek to find $G_v^{\ell+1}$ given that we already know G_v^{ℓ}. We illustrate the technique by finding the brackets for C_6 and C_4. We will also see in both examples that the bracket values are cyclic in nature and eventually repeat in multiples of 6 for the value ℓ.

Example 6 (Brackets of C_6)**.** Suppose $v \in V(C_6)$, then

$$\begin{aligned} I(C_6;-1) &= I(C_6-v;-1) - I(C_6-N[v];-1) \\ &= I(P_5;-1) - I(P_3;-1) \\ &= (-1)^2 - (-1) && \text{by Theorem 2.2.4} \\ &= 1-(-1) \\ &= \langle 1, -1\rangle. \end{aligned}$$

Using the Extension Lemma gives,

$$\begin{aligned}
I(C_6^1; -1) &= \langle 1 - (-1), 1\rangle \\
&= \langle 2, 1\rangle \\
I(C_6^2; -1) &= \langle 2 - 1, 2\rangle \\
&= \langle 1, 2\rangle \\
I(C_6^3; -1) &= \langle 1 - 2, 1\rangle \\
&= \langle -1, 1\rangle \\
I(C_6^4; -1) &= \langle -1 - 1, -1\rangle \\
&= \langle -2, -1\rangle \\
I(C_6^5; -1) &= \langle -2 - (-1), -2\rangle \\
&= \langle -1, -2\rangle \\
I(C_6^6; -1) &= \langle -1 - (-2), -1\rangle \\
&= \langle 1, -1\rangle \\
&= I(C_6; -1).
\end{aligned}$$

Notice that $I(C_6; -1) = \langle 1, -1\rangle = I(C_6^6; -1)$. This demonstrates the cyclic nature of the set of brackets for extensions of a graph. This observation turns out to be true in general which we will prove. We summarize these results of the bracket of C_6 in Table 3.1. We will also use these results in later proofs.

Example 7 (Brackets of C_4)**.** We find the brackets of all the extensions of C_4. Sup-

ℓ	$I(C_6^\ell;-1)$	$I(C_4^\ell;-1)$	$I(P_5^\ell;-1)$
0	$\langle 1,-1\rangle$	$\langle -1,0\rangle$	$\langle 0,-1\rangle$
1	$\langle 2,1\rangle$	$\langle -1,-1\rangle$	$\langle 1,0\rangle$
2	$\langle 1,2\rangle$	$\langle 0,-1\rangle$	$\langle 1,1\rangle$
3	$\langle -1,1\rangle$	$\langle 1,0\rangle$	$\langle 0,1\rangle$
4	$\langle -2,-1\rangle$	$\langle 1,1\rangle$	$\langle -1,0\rangle$
5	$\langle -1,-2\rangle$	$\langle 0,1\rangle$	$\langle -1,-1\rangle$
6	$\langle 1,-1\rangle$	$\langle -1,0\rangle$	$\langle 0,-1\rangle$

Table 3.1: Brackets of C_6^ℓ, C_4^ℓ and P_5^ℓ.

pose $v \in V(C_4)$, then

$$\begin{aligned}
I(C_4;-1) &= I(C_4 - v;-1) - I(C_4 - N[v];-1) \\
&= I(P_3;-1) - I(P_1;-1) \\
&= -1 - 0 && \text{by Theorem 2.2.4} \\
&= \langle -1,0\rangle.
\end{aligned}$$

Applying the Extension Lemma gives,

$$\begin{aligned}
I(C_4^1;-1) &= \langle -1,-1\rangle \\
I(C_4^2;-1) &= \langle 0,-1\rangle \\
I(C_4^3;-1) &= \langle 1,0\rangle \\
I(C_4^4;-1) &= \langle 1,1\rangle \\
I(C_4^5;-1) &= \langle 0,1\rangle \\
I(C_4^6;-1) &= \langle -1,0\rangle = I(C_4;-1).
\end{aligned}$$

Notice again that $I(C_4^6;-1) = I(C_4;-1)$. We have summarized these results in Table 3.1 as well.

Example 8. We consider the brackets of the extensions of the path P_5. Let $v \in V(P_5)$

be an endpoint of the path, then

$$\begin{aligned}
I(P_5;-1) &= I(P_5 - v;-1) - I(P_5 - N[v];-1) \\
&= I(P_4;-1) - I(P_3;-1) \\
&= 0 - (-1) && \text{by Theorem 2.2.4} \\
&= \langle 0,-1\rangle.
\end{aligned}$$

Using the Extension Lemma we find,

$$\begin{aligned}
I(P_5^1;-1) &= \langle 1,0\rangle \\
I(P_5^2;-1) &= \langle 1,1\rangle \\
I(P_5^3;-1) &= \langle 0,1\rangle \\
I(P_5^4;-1) &= \langle -1,0\rangle \\
I(P_5^5;-1) &= \langle -1,-1\rangle \\
I(P_5^6;-1) &= \langle 0,-1\rangle.
\end{aligned}$$

Notice once more that the bracket of the original graph is equal to the bracket with an extension of size 6; namely, $I(P_5;-1) = I(P_5^6;-1) = \langle 0,-1\rangle$. We now prove this observation in general.

Proposition 3.1.3. *Let G_v be a rooted graph with $I(G_v;-1) = a - b = \langle a,b\rangle$, then*

$$I(G_v;-1) = I(G_v^6;-1).$$

Proof. If G_v is a rooted graph with $I(G_v;-1) = a - b = \langle a,b\rangle$, then by the Extension

Lemma

$$
\begin{aligned}
I(G_v^3;-1) &= \langle -a, -b\rangle \\
I(G_v^4;-1) &= \langle -a-(-b), -a\rangle = \langle b-a, -a\rangle \\
I(G_v^5;-1) &= \langle b-a-(-a), b-a\rangle = \langle b, b-a\rangle \\
I(G_v^6;-1) &= \langle b-(b-a), b\rangle = \langle a, b\rangle \\
&= I(G_v;-1).
\end{aligned}
$$

□

Remark. This proposition shows that the brackets for the extensions of a graph are cyclic in nature. So, for example we can say for $\ell \geq 0$ it follows that $I(G_v^{\ell};-1) = I(G_v^{\ell \bmod 6};-1)$. This tells us that the bracket for any extension of length $\ell \geq 6$ will already be found on the list of brackets for $\ell < 6$.

3.2 Results on (k, q)-graphs

Definition 3.2.1 ((k,q)-graph)**.** Let k be a positive integer and q be an integer such that $|q| \leq 2^k$. A graph G such that $\phi(G) = k$ and $I(G;-1) = q$ is called a (k,q)-*graph.*

Lemma 3.2.1. *Let G and H be disjoint connected (k_1, q_1) and (k_2, q_2)-graphs, respectively. Set $k_1 + k_2 = k$ and $q_1 \cdot q_2 = q$, then there exists a connected (k,q)-graph F such that $\phi(F) = k_1 + k_2 = k$ and $I(F;-1) = q_1 q_2 = q = I(G \cup H;-1)$.*

Proof. Let $G = G_v$ and $H = H_w$ and set

$$
\begin{aligned}
I(G_v;-1) &= \langle a, b\rangle = q_1 \\
I(H_w;-1) &= \langle c, d\rangle = q_2.
\end{aligned}
$$

It follows by the Extension Lemma that

$$\begin{aligned} I(G_v^1; -1) &= \langle a-b, a\rangle \\ &= \langle q_1, a\rangle \\ I(G_v^2; -1) &= \langle -b, q_1\rangle \end{aligned}$$

and

$$\begin{aligned} I(H_w; -1) &= \langle c-d, c\rangle \\ &= \langle q_2, c\rangle \\ I(H_w; -1) &= \langle -d, q_2\rangle. \end{aligned}$$

The Pasting Lemma gives

$$I(G_v^2 \wedge H_w^2; -1) = \langle bd, q_1q_2\rangle.$$

So applying the Extension Lemma again gives

$$I((G_v^2 \wedge H_w^2)^1; -1) = \langle bd - q_1q_2, bd\rangle = -q_1q_2.$$

Finally, one more application of the Extension Lemma gives

$$\begin{aligned} I((G_v^2 \wedge H_w^2)^4; -1) &= I\left(\left((G_v^2 \wedge H_w^2)^1\right)^3; -1\right) \\ &= -I((G_v^2 \wedge H_w^2)^1; -1) = q_1q_2 = q. \end{aligned}$$

Thus we can set $F = (G_v^2 \wedge H_w^2)^4$. It follows that $I(F; -1) = q_1q_2 = q$. As G and H are disjoint, note that $I(H \cup G; -1) = I(G; -1) \cdot I(H; -1) = q_1q_2$. So, $I((G_v^2 \wedge H_w^2)^4; -1) =$

$I(G \cup H; -1) = q_1 q_2 = q$ by Proposition 2.2.6. Further, because the extension and pasting operations do not add any cycles we have $\phi(F) = k_1 + k_2 = k$. Therefore F is a (k, q)-graph with the desired properties. $\square$

Corollary 3.2.2. *If G is a (k, q)-graph, there exists*

(i) a connected $(k + 1, 2q)$-graph, and

(ii) a connected $(k, -q)$-graph.

Proof. Let G be a (k, q)-graph.

(i) Apply Lemma 3.2.1 with $H = C_6$, then because $I(C_6; -1) = 2$ by Theorem 2.2.4 and $\phi(C_6) = 1$ it follows that there's a $(k + 1, 2q)$-graph F.

(ii) Apply Lemma 3.2.1 with $H = P_5^3$, then because $I(P_5^3; -1) = -1$ by Example 8 and $\phi(P_5) = 0$ it follows that there's a $(k, -q)$-graph. $\square$

Lemma 3.2.3. *Let $k \geq 1$ be an integer. For every odd integer $q \in [0, 2^k]$ there is a (k, q)-graph G_v such that either $I(G_v; -1) = \langle 2^k, 2^k - q\rangle$ or $I(G_v; -1) = \langle -2^k + q; -2^k\rangle$.*

Proof. We use induction on k. If $k = 1$, then $\phi(C_6^1) = 1$ and $I(C_6^1; -1) = \langle 2, 1\rangle = 1$ which we saw in Example 6. Therefore we have a graph G_v with $q = 1$ in the form $I(G_v; -1) = \langle 2^1, 2^1 - q\rangle$. The result is true for $k = 1$.

Now, suppose the result is true for $k - 1$. Let $q \in [0, 2^k]$ be an odd integer. We consider two cases; either $q \in [2^{k-1}, 2^k]$ or $q \in [0, 2^k]$.

Case 1. If $q \in [2^{k-1}, 2^k]$ is an odd integer, then there's an odd integer $r \in [0, 2^{k-1}]$ with $q = 2^k - r$. By assumption, there's a $(k - 1, 2^{k-1} - r)$-graph G_v such that either

$$\begin{aligned} I(G_v; -1) &= \langle 2^{k-1}, 2^{k-1} - (2^{k-1} - r)\rangle \\ &= \langle 2^{k-1}, r\rangle \end{aligned}$$

or

$$\begin{aligned} I(G_v;-1) &= \langle -2^{k-1} + (2^{k-1} - r), -2^{k-1} \rangle \\ &= \langle -r, -2^{k-1} \rangle. \end{aligned}$$

If $I(G_v;-1) = \langle 2^{k-1}, r \rangle$, because $I(C_6^1;-1) = \langle 2, 1 \rangle$ by Example 6 and $\phi(C_6^1) = 1$, it follows by the Pasting Lemma that

$$\begin{aligned} I(G_v \wedge C_6^1;-1) &= \langle 2^{k-1} \cdot 2, r \cdot 1 \rangle \\ &= \langle 2^k, r \rangle \\ &= \langle 2^k, 2^k - q \rangle. \end{aligned}$$

Also, $\phi(G_v \wedge C_6^1) = k - 1 + 1 = k$. Therefore, $G_v \wedge C_6^1$ is a (k, q)-graph with $I(G_v \wedge C_6^1;-1) = \langle 2^k, 2^k - q \rangle$.

If $I(G_v;-1) = \langle -r, -r^{k-1} \rangle$, then because $I(C_6^2;-1) = \langle 1, 2 \rangle$ by Example 6 and $\phi(C_6^2) = 1$, it follows again by the Pasting Lemma that

$$\begin{aligned} I(G_v \wedge C_6^2;-1) &= \langle -r \cdot 1, -2^{k-1} \cdot 2 \rangle \\ &= \langle -r, -2^k \rangle \\ &= \langle -2^k + q, -2^k \rangle. \end{aligned}$$

Further, $\phi(G_v \wedge C_6^2) = k - 1 + 1 = k$. Therefore $G_v \wedge C_6^2$ is a (k, q)-graph with $I(G_v \wedge C_6^1;-1) = \langle -2^k + q, -2^k \rangle$. The result in this case follows then by induction for odd integers $q \in [2^{k-1}, 2^k]$.

Case 2. If $q \in [0, 2^{k-1}]$ is an odd integer, then $q = 2^k - r$ for an odd integer $r \in [2^{k-1}, 2^k]$. By Case 1, there's a (k, q)-graph G_v such that either $I(G_v;-1) = \langle 2^k, 2^k - r \rangle$ or $I(G_v;-1) = \langle -2^k + r, -2^k \rangle$.

If $I(G_v; -1) = \langle 2^k, 2^k - r \rangle = \langle 2^k, q \rangle$, then notice from the Extension Lemma that

$$I(G_v^3; -1) = \langle -2^k, -q \rangle$$
$$I(G_v^4; -1) = \langle -2^k + q, -2^k \rangle.$$

Therefore, G_v^4 is a (k, q)-graph with $I(G_v^2; -1) = \langle -2^k + q, -2^k \rangle$.

On the other hand, if $I(G_v; -1) = \langle -2^k + r, -2^k \rangle = \langle -q, -2^k \rangle$, then by the Extension Lemma,

$$I(G_v^1; -1) = \langle 2^k - q, -q \rangle$$
$$I(G_v^2; -1) = \langle 2^k, 2^k - q \rangle.$$

So, it follows that G_v^2 is a (k, q)-graph with $I(G_v^2; -1) = \langle 2^k, 2^k - q \rangle$. The result in this case for values odd $q \in [0, 2^{k-1}]$ follows by induction.

□

3.3 Density Result of Cutler and Kahl

Having laid the ground work the proof of the conjecture proposed by Levit and Mandrescu follows easily by using an induction argument.

Theorem 1.0.4 (Cutler, Kahl 2016)**.** *Given a positive integer k and an integer q with $|q| \leq 2^k$, there is a connected graph G with $\phi(G) = k$ and $I(G; -1) = q$.*

Proof. The prove uses induction on k. Given any positive integer k notice that we can choose any graph H with $\phi(H) = k$. For example, we can let

$$H = \underbrace{C_3 \wedge \cdots \wedge C_3}_{k \text{ times}}.$$

Because $\phi(C_3) = 1$ and the pasting operation joins k such graphs in a way that H has disjoint cycles, it follows that $\phi(H) = k$. Then, because $I(P_1; -1) = 0$ by Example 1, it follows by Proposition 2.2.6 that $I(H \cup P_1; -1) = I(H; -1) \cdot I(P_1; -1) = 0$. So, for any positive integer k we have a connected graph $G = H \cup P_1$ with $\phi(G) = k$ and $I(G; -1) = 0$. The case when $q = 0$ is handled.

Suppose $k = 1$, then because for integers $\ell \geq 0$ it follows that $\phi(C_6^\ell) = 1$ and

$$I(C_6; -1) = 2$$
$$I(C_6^1; -1) = 1.$$

Therefore, we have $(1, 1)$ and $(1, 2)$-graphs. By Corollary 3.2.2 there are also $(1, -1)$ and $(1, -2)$-graphs. This proves the result for $k = 1$.

Now, suppose the result is true for $k-1$. By Corollary 3.2.2(ii) we need only show the result for positive values of $q \leq 2^k$. So, if q is positive, by assumption, there exist $(k-1, q)$-graphs for $q \leq 2^{k-1}$. By Corollary 3.2.2 there exist $(k, 2q)$-graphs. Which proves the result for even integers $q \in [0, 2^k]$. For the odd integers $q \in [0, 2^k]$ we can apply Lemma 3.2.3. This proves the result. □

Chapter 4

Our work

We now seek to prove the following result. The idea will be to modify and use the series of definitions and lemmas that lead to Theorem 3.3. Recall that we defined a $\tilde{3}$-cycle to be a cycle in a graph G where the length of the cycle is a multiple of 3. Also, recall that when the cycles in a graph G are vertex disjoint, the cyclomatic number $\beta(G)$ counts the number of cycles in the graph.

Theorem 1.0.9. *Let $k \geq 0$. For every positive integer k and integer q with $|q| \leq 2^k$, there is a graph G such that all cycles of G are vertex disjoint, $I(G; -1) = q$ and the number of $\tilde{3}$-cycles of G is k.*

We begin with a few definitions.

Definition 4.0.1. For a graph G we will denote the number of $\tilde{3}$-cycles by $\zeta(G)$.

Definition 4.0.2 (((k, q)-$\tilde{3}$-graphs)**.** Let $k \geq 0$. Define a (k, q)-$\tilde{3}$-graph G to be a graph with $\zeta(G) = k$ and $I(G; -1) = q$.

Example 9. We give a few examples of different type of (k, q)-$\tilde{3}$-graphs in Figures 4.1, 4.2 and 4.3.

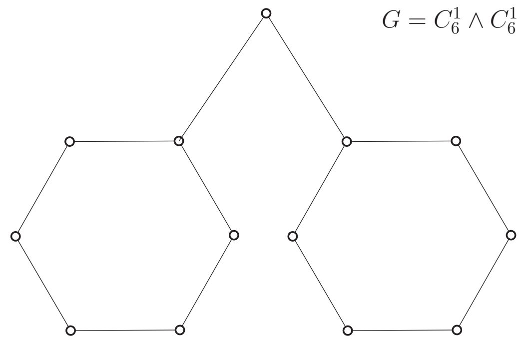

Figure 4.1: Example of a $(2,3)$-$\tilde{3}$-graph.

Note that $\zeta(G) = 2$ and $I(G;-1) = \langle 4, 1\rangle = 3$ since $I(C_6^1;-1) = \langle 2, 1\rangle$.

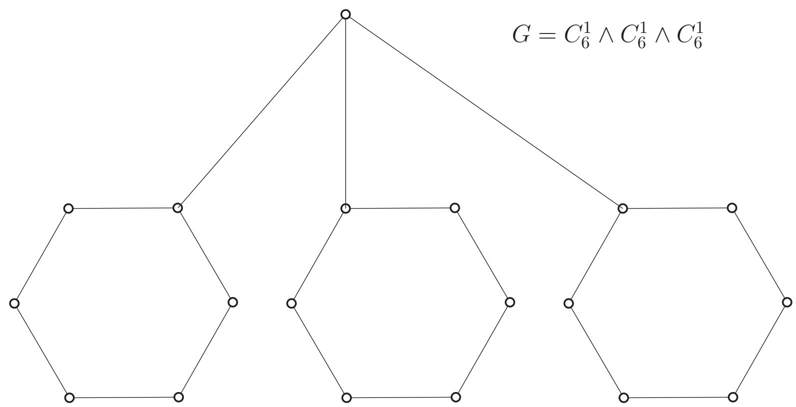

Figure 4.2: Example of a $(3,7)$-$\tilde{3}$-graph.

Note that $\zeta(G) = 3$ and $I(G;-1) = \langle 8, 1\rangle = 7$ since $I(C_6^1;-1) = \langle 2, 1\rangle$.

We restate the major lemmas of Cutler and Kahl in terms of (k,q)-$\tilde{3}$-graphs. Note that for rooted graphs G_v and H_w the pasting operation doesn't create any new cycles in the resulting graph $G_v \wedge H_w$; so that $\beta(G_v \wedge H_w) = \beta(G_v) + \beta(H_w)$. However, we will need to be careful when dealing with disjoint graphs and the pasting operation. It is possible to have the resulting graph after pasting not have vertex disjoint cycles at the joined vertex. For example, if we set $G_v = C_6$ and $H_w = C_6$ where v and w are any two vertices in C_6, the resulting graph $G_v \wedge H_w$ will not have vertex disjoint cycles. This problem can be avoided however by taking extensions before pasting. For example, if $G_v = C_6$ and $H_w = C_6$, as above, notice that $G_v^1 \wedge H_w^1$ has vertex disjoint cycles. If we want the resulting graph to have the same bracket then we can appeal to the cyclic nature of extensions. For instance, $I(G_v \wedge H_w;-1) = I(G_v^6 \wedge G_w^6;-1)$.

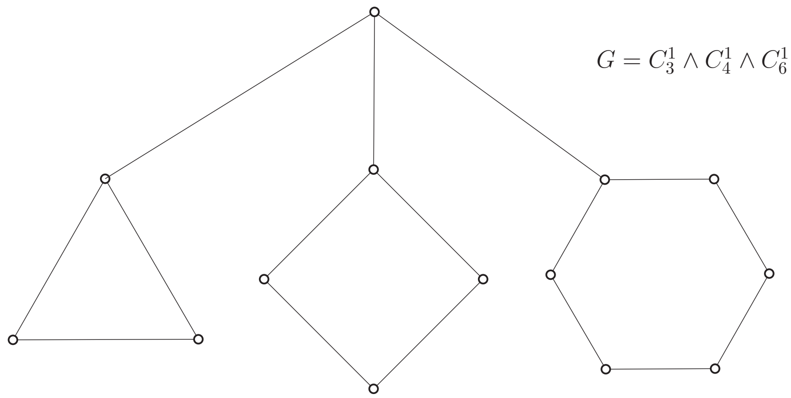

Figure 4.3: Example of a $(2,3)$-$\tilde{3}$-graph.

Note that $\zeta(G) = 2$ and $I(G;-1) = \langle 4,1\rangle = 3$ since $I(C_3^1;-1) = \langle -2,-1\rangle$, $I(C_4^1;-1) = \langle -1,-1\rangle$ and $I(C_6^1;-1) = \langle 2,1\rangle$.

The former doesn't have vertex disjoint cycles whereas the latter does. The following result is analogous to Lemma 3.2.1.

Lemma 4.0.1. *Let G and H be disjoint connected (k_1,q_1) and (k_2,q_2)-$\tilde{3}$-graphs, respectively, which have vertex disjoint cycles. Let $k_1+k_2 = k$ and $q_1q_2 = q$, then there is a connected (k,q)-$\tilde{3}$-graph F with vertex disjoint cycles such that $\zeta(F) = k$ and $I(F;-1) = q$.*

Proof. If $G = G_v$ and $H = H_v$ are (k_1,q_1) and (k_2,q_2)-$\tilde{3}$-graphs we can set $F = (G_v^2 \wedge H_w^2)^4$ as in the proof of Lemma 3.2.1. By the nature of the pasting operation and since extensions add no new $\tilde{3}$-cycles we have $\zeta(F) = \zeta(G_v)+\zeta(H_w) = k_1+k_2 = k$. Further, F has vertex disjoint cycles since G_v and H_w have vertex disjoint cycles and the extensions prevent cycles from overlapping on the pasting vertex. Then, as in Lemma 3.2.1 we have $I(F;-1) = q_1q_2 = q$. □

Next we prove a Corollary similar to Corollary 3.2.2 which allows us to find $(k+1,2q)$-$\tilde{3}$-graphs and $(k,-q)$-$\tilde{3}$-graphs given a (k,q)-$\tilde{3}$-graph. This will allow us to handle even and negative values in the proof of our result.

Corollary 4.0.2. *Let G be a (k,q)-$\tilde{3}$-graph. Then,* *(i) there is a connected $(k+1, 2q)$-$\tilde{3}$-graph with vertex disjoint cycles.*

(ii) there is a connected $(k, -q)$-$\tilde{3}$-graph with vertex disjoint cycles.

Proof. Let G be a (k, q)-$\tilde{3}$-graph.

(i) Apply Lemma 4.0.1 with $H = C_6$. (ii) Apply Lemma 4.0.1 with P_5^5. $\square$

The following is an updated form of Lemma 3.2.3 for (k, q)-$\tilde{3}$-graphs while also emphasizing the fact that we want graphs that have vertex disjoint cycles.

Lemma 4.0.3. *Let $k \geq 1$. For each odd integer $q \in [0, 2^k]$ there is a connected (k, q)-$\tilde{3}$-graph G_v with vertex disjoint cycles such that either $I(G_v; -1) = \langle 2^k, 2^k - q\rangle$ or $I(G_v; -1) = \langle -2^k + q, -2^k\rangle$.*

Proof. The proof follows verbatim to the proof of Lemma 3.2.3. The reason is that we used C_6^1 and C_6^2 which ensure the result has vertex disjoint cycles. Further, we note that $\zeta(G_v \wedge H_w^\ell) = \zeta(G_v) + \zeta(H_w)$ for any rooted graphs G_v and H_w. $\square$

We can now prove our final result.

Theorem 4.0.4. *For every integer $k \geq 0$ and integer q with $|q| \leq 2^k$, there is a graph G such that all cycles of G are vertex disjoint, $I(G; -1) = q$ and the number of $\tilde{3}$-cycles of G is k.*

Proof. The result follows by induction on k. Let $k = 0$, then we need (k, q)-$\tilde{3}$-graphs with $q \in \{-1, 0, 1\}$. We have

$$I(P_5; -1) = 1$$

$$I(P_1; -1) = 0$$

$$I(P_5^3; -1) = -1.$$

Notice that each of the above graphs have no $\tilde{3}$-cycles. Therefore the result follows for $k = 0$. In general, for any $k \geq 0$ for a graph G with $\zeta(G) = k$ take $H = P_1$ in Lemma 4.0.1 and $I(F; -1) = I(G; -1) \cdot I(H; -1) = 0$.

Suppose the result is true for k, then we need only prove there are (k, q)-$\tilde{3}$-graphs for positive $q \leq 2^{k+1}$ by Corollary 4.0.2. It follows that there are (k, q)-$\tilde{3}$-graphs for even integers $q \leq 2^{k+1}$ by Corollary 4.0.2. Finally, for odd values of $q \leq 2^{k+1}$ we apply Lemma 4.0.3. The result follows by induction. $\square$

Chapter 5

Further Directions

We discuss some further questions for investigation. As mentioned, a major result of Cao and Ren [1] is the following result which gives a closer bound than Engsröm's result in the case when $\phi(G) = \beta(G)$.

Theorem 1.0.7 (Cao, Ren 2021)**.** *If G contains a non-$\tilde{3}$-cycle, then*

$$|I(G; -1)| \leq 2^{\beta(G)} - \beta(G).$$

This theorem lends itself to asking whether we can prove a density result in this case.

Question 1. For every positive integer k and each integer q such that $|q| \leq 2^k - k$, is there a graph G which contains a non-$\tilde{3}$-cycle with $\beta(G) = k$ and $I(G; -1) = q$ such that $|q| \leq 2^k - k$? Or is there some further condition needed on k, q or G?

This seems to hold true for the cases where $k = 1, 2$. When $k = 0$ we would need graphs G with $q = I(G; -1) \in \{-1, 0, 1\}$ and $\beta(G) = 0$. We can find the necessary graphs for each value of q as follows.

$$\begin{aligned} I(P_5;-1) = 1, &\quad \beta(P_5) = 0, \\ I(P_1;-1) = 0, &\quad \beta(P_1) = 0, \\ I(P_5^3;-1) = -1, &\quad \beta(P_5^3;-1) = 0. \end{aligned}$$

However, each of the above graphs contain no cycles and therefore do not contain a non-$\tilde{3}$-cycle.

For $k = 1$, we need graphs G with $I(G;-1) \in \{-1, 0, 1\}$ and $\beta(G) = 1$ which contain non-$\tilde{3}$-cycles. Notice that

$$\begin{aligned} I(C_4^3;-1) &= \langle 1, 0\rangle = 1 && \text{by Example 7,} \\ I(C_4^4;-1) &= \langle 1, 1\rangle = 0, \\ I((C_4^3)^3;-1) &= I(C_4;-1) \\ &= -I(C_4^3;-1) = -1. \end{aligned}$$

Each of the above graphs satisfy $\beta(G) = 1$ for each G and each of these graphs contain non-$\tilde{3}$-cycles since the length of C_4 is 4 and $3 \nmid 4$. So, the question holds for $k = 1$.

For $k = 2$, we would need graphs G with non-$\tilde{3}$-cycles with $I(G;-1) \in \{-2, -1, 0, 1, 2\}$ and $\beta(G) = 2$. Notice that applying the Pasting Lemma and Extension Lemma to the brackets of C_6 and C_4 in Examples 7 and 6 gives

$$\begin{aligned}
I((C_6^2 \wedge C_4^2)^3; -1) &= 2,\\
I((C_4^5 \wedge C_4^5)^3; -1) &= 1,\\
I((C_4^3 \wedge C_4^5; -1) &= 0,\\
I(C_4^5 \wedge C_4^5; -1) &= I(((C_4^5 \wedge C_4^5)^3)^3; -1)\\
&= -I(C_4^5 \wedge C_4^5)^3; -1)\\
&= -1,\\
I(C_6^2 \wedge C_4^2; -1) &= I(((C_6^2 \wedge C_4^2)^3)^3; -1)\\
&= -I(C_6^2 \wedge C_4^2)^3; -1)\\
&= -2.
\end{aligned}$$

Each of the above graphs satisfy $\beta(G) = 2$ for each G and contain C_4 which is a non-$\tilde{3}$-cycle.

There seems to be a problem applying extensions and pasting operations when $k = 3$. We would need a graph G with non-$\tilde{3}$-cycles such that $\beta(G) = 3$ and $I(G; -1) = 5$, for example. Notice that the general form of the bracket for $\tilde{3}$-cycles and non-$\tilde{3}$-cycles by Theorem 2.2.4 is given by

$$\begin{aligned}
I(C_{3n}; -1) &= \langle (-1)^n, (-1)^{n-1} \rangle,\\
I(C_{3n+1}; -1) &= \langle (-1)^n, 0 \rangle,\\
I(C_{3n+2}; -1) &= \langle 0, (-1)^n \rangle.
\end{aligned}$$

Recall the bracket values of C_6 and C_4 shown in Table 3.1. These were the following.

Observe that the brackets of C_6 and C_4 encompass all the possible bracket values

ℓ	$I(C_6^\ell;-1)$	$I(C_4^\ell;-1)$
0	$\langle 1,-1\rangle$	$\langle -1,0\rangle$
1	$\langle 2,1\rangle$	$\langle -1,-1\rangle$
2	$\langle 1,2\rangle$	$\langle 0,-1\rangle$
3	$\langle -1,1\rangle$	$\langle 1,0\rangle$
4	$\langle -2,-1\rangle$	$\langle 1,1\rangle$
5	$\langle -1,-2\rangle$	$\langle 0,1\rangle$
6	$\langle 1,-1\rangle$	$\langle -1,0\rangle$

Table 5.1: Brackets of C_6^ℓ, and C_4.

for arbitrary cycles C_n; $\tilde{3}$-cycles will have a similar form as C_6^ℓ whereas any non-$\tilde{3}$-cycle will have a similar form to C_4^ℓ by consideration of the general form of the bracket above.

A problem for the case where $k = 3$ and $q = 5$ seems to occur by taking these ideas into consideration. Optimally, we would seek for a graph G with bracket of the form $\langle 4, -1\rangle$ with $\beta(G) = 3$. We could do this easily by taking $G = C_6^1 \wedge C_6^1 \wedge C_6$. The Pasting Lemma gives $I(G; -1) = \langle 4, -1\rangle$. However, G in this case contains only $\tilde{3}$-cycles. We cannot simply wedge another non-$\tilde{3}$-cycle on G – for instance C_4^4 would be the perfect candidate – the problem then would be that $\beta(G \wedge C_4^4) = 4 \neq 3$.

We could seek for a way to introduce a mixed sign into the bracket $I(C_6^1; -1) = \langle 2, 1\rangle$ with a non-$\tilde{3}$-cycle, say H with bracket $I(H; -1) = \langle 1, -1\rangle$. This would give $I(C_6^1 \wedge H; -1) = \langle 2, -1\rangle$. Then we could take $G = C_6^1 \wedge H \wedge C_6^1$. The bracket here would be $I(G; -1) = \langle 4, -1\rangle$. However the problem here is the fact that it doesn't seem like there are non-$\tilde{3}$-cycles with $I(H; -1) = \langle 1, -1\rangle$ using our construction. We saw that non-$\tilde{3}$-cycles all seem to only have the form of C_4 where there is no bracket $I(C_4^\ell; -1) = \langle 1, -1\rangle$.

It seems that our construction fails to be able to confirm Question 1. The issue may be due to pasting graphs together with disjoint cycles.

Cao and Ren [1] also proved the following result.

Theorem 5.0.1 (Cao, Ren 2021)**.** *Let G be a graph with non-$\tilde{3}$-cycles. If G contains no vertices of degree* 1*, then* $|I(G;-1)| \leq 2^{\beta(G)-1}$.

This theorem suggests the following question for investigation.

Question 2. For every positive integer k and q, are there graphs G with non-$\tilde{3}$-cycles containing no vertices of degree 1 with $I(G;-1) = q$ and $\beta(G) = 1$ with $|q| \leq 2^{k-1}$?

Our techniques certainly seem to fail due to the condition that G contain no vertices of degree 1. This condition forces us to avoid using graphs with extensions. For example, in the proof of Lemmas 3.2.1 and 4.0.1 we set $F = (G_v^2 \wedge H_w)^4$. However, the condition that F contain no vertices of degree 1 causes a problem for choosing F; the reason being that the extension operation produces vertices of degree 1 unless the extension is eventually pasted to another graph.

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