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Investigating Elementary School Students' Reasoning about Dynamic Angles

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Investigating Elementary School Students' Reasoning about Dynamic Angles

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Abstract

Angle measurement is a significant topic in almost all areas of mathematics learning and also in many disciplines outside mathematics education, such as engineering and architecture. According to the literature, there are three common conceptions of angles – as union of rays, rotations, and wedges. Researchers argued that students must consider these three angle concepts together to construct a meaningful understanding of angles. However, the curriculum standards for mathematics often present these angle conceptions separately to students, probably resulting in a fragmented understanding of the angle concept. In addition to this problem, the research literature documents multiple alternative conceptions that students exhibit when they engage with static representations of angles, which is the prevalent way of the current teaching and learning of the concept. Consequently, this dissertation study aimed to explore how students may reason about angles when they engage in tasks that present angles dynamically and bridge the three conceptions. Specifically, this dissertation examined (a) the forms of reasoning that students exhibit as they engaged in dynamic digital tasks that bridged the three conceptions of angles, (b) the characteristics of the design (tasks, tools, and questioning) that supported particular forms of students' reasoning, and (c) how the design evolved to support students' reasoning for angles.

Prior research on dynamic measurement and quantitative reasoning guided the design of tasks in GeoGebra to prompt the students to examine how angles are generated and change dynamically. A design experiment methodology was followed to engineer particular forms of reasoning about angles in these dynamic situations and explore how the specific design supports these forms of reasoning. Video-recorded data were collected from four third-grade students working on the tasks individually. Two phases of data analysis were conducted – ongoing analysis and two levels of retrospective analysis. The ongoing analysis as each design experiment unfolded showed how students' prior knowledge and in-the-moment reasoning about angles influenced the modification of the design. The first level of retrospective analysis conducted at the end of each design experiment illustrated four categories of student reasoning, namely reasoning about the three angle conceptions, constructing multiplicative comparisons between angles, reasoning about an angle as a discrete or continuous quantity, and measuring angles using multiplicative reasoning. The second level of retrospective analysis at the end of all design experiments cross-compared all students' reasoning and demonstrated the specific characteristics of the design that supported students' reasoning about angles in those four categories. These findings can be foundational for supporting students' conceptual understanding of angles in both research and practice.

Keywords: angles, design experiment, dynamic measurement, quantitative reasoning

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why they exhibited limited and recurring responses. Following that advice saved me from many frustrations when I worked with the students in this dissertation. Indeed, the dynamics of my questioning influenced students' responses to illustrate both broad and specific elements of their thinking. In the future, I will be better at my way of questioning and offer opportunities for students to reorganize their thinking.

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and a driver to me. You may have only read one chapter of this dissertation, but I am grateful that you have been part of my writing progress since I applied to the doctoral program. We celebrated every small success and probably went out for dinner about a hundred times in anticipation of my graduation. I look forward to creating our Filipino-Peruvian recipes together.

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Dedication

I heartfully dedicate this dissertation to my family: Estella, Rudy Sr., Rudel, Ellen, Eauffemme, Rudy Jr., Robert, Peter, and Elijah.

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Chapter 1: Introduction

In this introductory chapter, I explain why a conceptual understanding of angles is important. To do that, I discuss how angles are presented in mathematics education and beyond and what kind of understanding students need to have about angles. Through this exploration, I outline what is missing from the current research and practice. I also present the type of contribution my dissertation aims to make by providing opportunities for students to experience angles dynamically and merge the three common conceptions of angles. Finally, I conclude the chapter by presenting the contents of the subsequent chapters.

1.1. Rationale: Angles in Mathematics Education and Beyond

I argue the importance of learning angles and discuss the rationale for this study. First, I explain how learning angles is prevalent in all aspects and levels of mathematics education. Next, I present some aspects of real-world situations where angles are experienced outside the mathematics classroom. Then, I discuss some challenges in the current teaching of angles, aiming to provide an argument of why the current study is needed.

1.1.1. Angles in Mathematics Education: The three categories

> "The concept of angle causes much ambiguity in higher level mathematical studies. It is basic in the sense that no mathematical curriculum can do without it, but it appears to be far from basic in other sense" (Barabash, 2017, p. 31).

A conceptual understanding of angles is significant in almost all areas of mathematics learning, such as in geometry and trigonometry. For instance, angle measures are used in geometry to prove relationships between other geometric objects or statements. In trigonometry, for example, angles in a triangle are used in finding the lengths of the triangle sides or other angles. These examples show that the angle concept is a significant topic in the mathematics curriculum, yet it is too complex to understand its multifaceted nature. Throughout history, an angle has been described in different ways. This is because an angle is a complex concept in which both students and mathematicians have struggled to define (Keiser, 2004; Sinclair & Bruce, 2015).

Early mathematicians defined an angle based on one or all of the three Aristotelian categories – a quantity, a quality, or a relation (Keiser, 2004). According to Proclus (410–485/1970), an angle can be described as a *quantity* that defines its measurement, a *quality* that identifies its attributes, and a *relation* of lines, rays, or planes as its boundaries. Similarly, Freudenthal (1973) distinguished angles in three ways – as a static pair of sides (*quality*), as an enclosed planar or spatial area (*quantity*), and as the process of change of direction (*relation*). While Proclus (1970) and Freudenthal (1973) considered all three categories to define angles, many scholars continued to define angles in terms of one or two of these categories. For instance, in Euclid's *Elements*, an angle is defined as "the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line" (Heath, 1956, p. 153). This definition emphasizes angles as a *relation* and a *quality* between a pair of lines based on the inclination of one line to another. However, according to Euclid's definition, a straight line formed by two colinear rays is not even considered an angle. Therefore, this definition may lead students to form alternative conceptions, which are conceptions that differ from what is expected of them to learn (Mevarech & Kramarsky, 1997).

One way to contrast Euclid's definition is to examine angles in terms of the amount of openness (*quantity*) between two sides. Using this approach, in the Devichi and Munier (2013) study, students who initially argued that there is no angle in 180° reorganized their reasoning by conceiving that one side of the angle opens (increasing) while the other side closes (decreasing) and that the two sides can open as a “flat angle.” Alternative conceptions about angles such as described above take place when the three Aristotelian categories (quantity, quality, relation) of angles are not considered altogether. This view is based on the work of Freudenthal (1973), who suggested that students need to consider multiple angle concepts in order for them to create a meaningful understanding of angles. Consequently, I explored how the three Aristotelian categories are presented in the current mathematics curriculum standards of learning angles.

1.1.2. Learning Angles According to Mathematics Curriculum Standards

Angles are taught in almost every grade level of education according to the Common Core State Standards for Mathematics (CCSSM) (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA & CCSS], 2010), and this shows the significance of the angle concept. Based on the Geometry standards of the CCSSM, kindergarten students begin their learning of angles by using their informal notion of angles as corners or vertices to analyze and compare shapes (CCSS.M.K.G.B.4). This notion of angles often leads students to define angles as corners of a shape (Clements & Battista, 1989). Such a definition of an angle illustrates a *quality* according to the Aristotelian categories because it refers to the attributes of connected sides that form a corner. In second grade, students are expected to recognize and draw shapes using the number of angles (CCSS.M.2.G.A.1) rather than merely counting corners as expected in kindergarten mathematics. This learning standard illustrates a *relation* of the number of angles to define a shape.

In fourth grade, mathematics standards categorize an angle as a *quantity* by pertaining to an angle measurement. At this grade level, students are formally introduced to language such as acute, right, or obtuse to classify different groups of angle sizes measured in degrees (CCSS.M.4.G.A.1). Measuring angles using protractors is also introduced to students

(CCSS.M.4.MD.C.5). In terms of angle measurement, the CCSSM states that an angle is measured with reference to a circle where a "one-degree angle" means 1/360 of a circle (CCSS.M.4.MD.C.5.A). These standards show the angle as a *quantity* that can be measured. In sum, from kindergarten until fourth grade, the three Aristotelian categories are presented to students separately.

Starting in fifth grade, the standards for angles begin to combine two Aristotelian categories together. To elaborate, students are expected to classify geometric figures according to the number and size of their angles (CCSS.M.5.G.B.3). For instance, all rectangles have four right angles, and squares are categorized as rectangles because squares have four right angles. In this standard, an angle is treated as a *relation* between the number of angles in a shape and the name of the shape. It is also a *quantity* because the equality of angles is also considered to categorize a shape.

Similarly, in seventh grade, angles are accounted for both as a *quantity* considering its measure and a *relation* between the angle sides. For example, students are expected to construct triangles from three measures of angles (CCSS.M.7.G.A.2). This standard uses angle measures illustrating an angle as a *quantity*. It also requires knowledge of the *relation* between angle sides and the angle size to construct triangles. At this grade level, students also use derived facts about supplementary, complementary, vertical, and adjacent angles in word-problems involving unknown angle measures (CCSS.M.7.G.B.5). In this standard, the *relation* between the positions of the angle sides or the quantity in terms of angle measures are often used to judge how angles are related to each other. The same categories of angles (a *quantity* and a *relation*) are also illustrated in the teaching of angles in eighth grade. Students learn to verify the congruence of angle measures as a *quantity* by experimenting with the *relations* between the angle size and positions of sides through rotations, reflections, and translations (CCSS.M.8.G.A.1.B).

Freudenthal (1973) argued that students would never learn to carefully distinguish different angle concepts and grasp the meaning of angles if they do not consider all the angle concepts. The standards above show a disintegrated definition of an angle in terms of the three Aristotelian categories (*quality, quantity,* and *relation).* These standards involving the learning of angles in K-8 are applied at the high school level to learn trigonometry, similarity and congruence, proof, and constructions. For example, the fourth-grade learning standard where students learn about right angles (CCSS.M.4.G.A.1) is utilized to learn about angles in circles in high school (CCSS.M.HSG.C.A.2). One way to support students to broaden their conception of angles is to present angles using all three categories, which is more comprehensive and could offer them better preparation for angles in higher level mathematics (Keiser, 2004).

Following the suggestions of Freudenthal (1973) and Keiser (2004), in this dissertation I started with the conjecture that the three Aristotelian categories – a *quantity*, a *quality*, and a *relation*, are necessary to show this multiple nature of angles. In my review of related literature in the next chapter, I discuss in detail the three common conceptions of angles that students often exhibit, which correspond to the three Aristotelian categories. First, students may conceive an angle as a union of rays, which illustrates a *quality* category because this conception shows the qualitative components of angles of two pairs of sides with a common point. Second, students may conceive an angle as a rotation of angle sides, which illustrates a *relation* between the starting position and the terminal position of an angle side. Finally, students may conceive an angle as a wedge, which illustrates a *quantity* because wedges show the amount of openness between two angle sides and are often used to quantify angles. In the next chapter, I argue that all three angle conceptions need to be considered by students.

1.1.3. Angles Beyond Mathematics Education: The Dynamic Component

An understanding of angles is also essential for learning concepts in other disciplines, such as engineering and architecture. For example, in engineering, angles are fundamental in surveying to determine the relative positions of points on the earth's surface. In architecture, angles are often used to create balance and symmetry in designing structures such as windows and doors. Devichi and Munier (2013) documented how the understanding of the angle concept is significant in solving real-life problems that are motivating and meaningful for students.

Most angles in real-life situations are dynamic in nature, and this makes the study of angles more useful and engaging. Individuals engage with angles in practical activities such as playing sports, describing locations, and driving. For instance, children who ride scooters and bicycles estimate the amount of turn they make with their handlebars without formal knowledge of angle measures. In playing basketball, shooting a ball from behind the free-throw line requires a smaller angle between the position of the elbow and the face. In these experiences, dynamic angles can be easily quantified yet do not require computations of numerical values. Instead, angles are given meaning and usefulness to day-to-day activities. Extending the formal learning of angles to situations in applied fields can create more opportunities for students to construct a meaningful conception of an angle.

Although in real-life situations, angles are utilized dynamically, angles continue to be presented statically in mathematics classrooms. A vast amount of research has shown that the current state of teaching angles causes students to have difficulties in understanding the concept and leads them to construct problematic alternative conceptions that they often carry at higher levels of schooling (Lehrer et al., 1998). There is limited research on elementary students' reasoning about angles presented in dynamic situations (e.g., Clements & Battista, 1990; Smith et al., 2014). Even if dynamic rotations are introduced in later years of schooling, the primary focus of students is on measuring the static output of rotations. Additionally, the disconnection from real-life applications is another example of problematic approaches to teaching angles. Students are not encouraged to seek connections between the learning of angles in the classroom and the use of angles in their everyday lives.

To elaborate, Smith and Thompson (2007) argued that students are unable to find meaning and purpose in mathematics because teachers expect them to work with mathematics that is too abstract for them to understand. Students do not think of what they measure when measuring an angle (Thompson, 2013). Often, this results in a vicious cycle of not understanding angles because students do not know which part of an object being measured is an angle. Instead, we need to provide opportunities for students to create angles, manipulate them dynamically, and reason about relationships between the quantities involved in generating angles. One way to offer these opportunities is to illustrate dynamic angles through a digital technology platform. Research has shown that dynamic geometry environments offer the affordance of dynamic manipulation of mathematical objects and immediate feedback with precise measurements (Browning et al., 2007; Smith et al., 2014).

In this dissertation, I engaged students with instructional tasks, tools, and questioning to explore angles and examine how students would reason about angles in dynamic situations. To achieve that, I utilized the power of digital technology to illustrate angles dynamically and bridge multiple angle conceptions – an angle as a union of rays (*quality*), as a rotation (*relation*), and as a wedge (*quantity*). Specifically, the goal was for students to explore the dynamic generation and change of angles through rotations, identify the quantities involved, and construct relationships between those quantities. I also aimed to examine how the design supports students' reasoning.

1.2. Preview of Subsequent Chapters

This dissertation is organized into seven chapters. In Chapter 2, I discuss my literature review that provides insights about the three common conceptions of angles: an angle as a union of rays (*quality*), as a rotation (*relation*), and as a wedge (*quantity*). For each form of angle conception, I discuss the alternative conceptions that students exhibit when engaging with static angles. Knowing these conceptions and alternative conceptions significantly informed the initial designs of my tasks, tools, and questioning. In Chapter 3, I discuss the theoretical framing of my study, which includes radical constructivism, dynamic measurement, and the quantitative reasoning approach. Chapter 4 describes the design experiment and its characteristics as my research methodology. I also detail my design and conjectures, presenting the initial design of tasks, tools, and questioning. Then, I describe my data collection methods and the framework I use to analyze my data. In Chapters 5 and 6, I present my findings. Specifically, Chapter 5 includes the results from my ongoing and first level-retrospective analyses of each of the four design experiments. Chapter 6 is devoted to the continuation of the findings from the second level of retrospective analysis. Finally, in Chapter 7, I present my conclusions, which discuss the contributions of this dissertation for research and practice. I also outline the limitations of this study and suggest areas for future research.

Chapter 2: Literature Review

A conceptual understanding of angles is foundational for working with other geometric concepts such as polygons, symmetry, transformations, as well as for developing arguments in geometric proofs. Understanding angles and their measurement can also support the understanding of non-geometric concepts, such as trigonometric functions (Moore, 2012). Even though an understanding of angles is essential for conceptualizing many aspects of mathematics, students continue to struggle with this concept. For instance, the results of the Trends in International Mathematics and Science Study (TIMSS) in 2015 show that, in the United States, eighth-grade students' achievement in reasoning about angles is relatively low compared to their performance in other cognitive and content domains (Mullis et al., 2016). Indeed, Lehrer et al. (1998) found that students develop a variety of alternative conceptions about angles that they often continue to have even in later years of schooling.

According to Piaget's (1971) genetic epistemology, children develop their conception of ideas over a period of time. Specifically, in geometry, children's imagination is initially attentive to topological relationships (e.g., proximity, separation, order) while ignoring the Euclidean relationships (e.g., length, size, magnitude) of an object in a space (Piaget & Inhelder, 1956). For instance, 4- to 6-year-old children focusing on topological relationships may copy angles by visually estimating the proximity of sides without attempting to measure them (Piaget et al., 1960). In later years of schooling, the reverse may happen where students consider topological relationships may not be enough, and instead, they focus on the Euclidean relationships. For example, 8- to 10-year-old students are more likely to copy angles by measuring the proximity of the sides or the distance between the endpoints of the sides (Piaget et al., 1960). Piaget and Inhelder (1956) argued that students must focus on both angles as the union of straight lines and as the space between the two sides for them to bridge the topological and Euclidean relationships in developing their meanings about angles.

Students form their meanings about angles as they make sense of their everyday experiences, language, and formal and informal aspects of their schooling. In geometry, when tactile and concrete explorations are lacking, children develop their conceptions as they apply an egocentric perspective that is characterized by a lack of awareness of different views (Piaget & Inhelder, 1956). Children's egocentric perspective commonly leads them to develop alternative conceptions. Alternative conceptions are students' conceptual configurations of ideas that are culturally embedded and vary based on students' language and historical backgrounds (Confrey, 1990). Students respond according to their expectations, predictions, confirmations, or rejections (Confrey, 1990). In other words, students exhibit alternative conceptions as they construct ideas within the context of their perspectives and experiences.

In this chapter, I provide an overview of the three most common conceptions of angles that students appear to develop, namely angles as two sides sharing an endpoint, angles as rotations, and angles as wedges. I describe the research studies that focused on each conception and compare and contrast those conceptions based on the findings from the literature. Through this exploration, I also discuss alternative conceptions that students form. This literature review examines how the conceptions of angles are developed based on students' experiences.

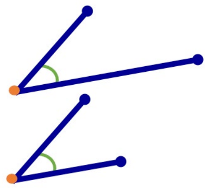
2.1. Angles As Two Sides Sharing a Common Point

When Clements and Battista (1989) asked third-grade students "What is an angle?", students' responses included the term "corners" which, as the researchers interpreted, indicated that a corner must be part of a geometric object and that it involved perpendicularity. Clements and Sarama (2014) also found that as students grow older, they define an angle as the union of two rays. This definition may be due to how angles are introduced in textbooks and standards. For instance, Keiser (2004) examined sixth-grade students' conceptions of angles when they investigated angles of polygons from Lappan et al.'s (1996) Connected Mathematics Project (CMP) textbook unit, Shapes and Designs. According to CMP 3 an angle is where the sides of a geometric shape meet (Lappan et al., 2014). Keiser (2004) found that most students believed that angles must have two sides and that a vertex connects them. In terms of curriculum standards, such as the Common Core State Standards for Mathematics (NGA & CCSS, 2010), fourth-grade students are expected to identify angles in two-dimensional figures (e.g., triangles, quadrilaterals). In other words, students should consider two connecting sides as an angle.

This form of angle conception seems similar to how angles are defined in the ancient history of mathematics. For instance, Euclid defined angle as "the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line" (Heath, 1956, p. 153). Euclid's definition can be problematic in the sense that it does not pay attention to angle as a quantity of space between two sides and it does not include straight angles (i.e., 180°). Students who used textbooks published in the late 20th century continue to reflect angle conceptions similar to Euclid's definition. For example, one student thought of an angle as having two sides and how they are inclined to each other, while another student excluded 180° as an angle because they do not see where the two sides connect (Keiser, 2004). Since the ancient angle definition does not emphasize which feature of an angle must be measured, students often confuse the angle measure as being dependent on side lengths (Keiser, 2004). Instead of understanding that the size of an angle is irrelevant to the length of its sides (Figure 1), students associate longer sides with a bigger angle or shorter sides with a smaller angle. Other studies have reported the same alternative conceptions about angle size and side length relationship (e.g., Clements & Battista, 1989; Devichi & Munier, 2013; Fyhn, 2008; Lehrer et al., 1998; Smith et al., 2014).

Figure 1

Examples of Angles with the Same Measure but Different Side Lengths



Note: Example of two angles with the same measure but different side lengths showing that angle size is irrelevant to the length of its sides.

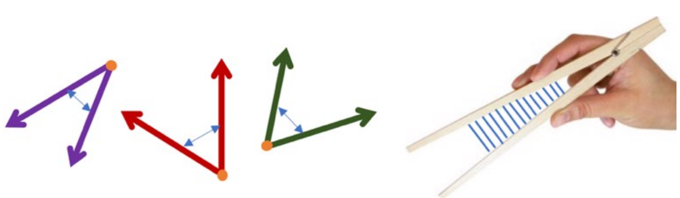
In addition to the above alternative conception, several studies documented that students also demonstrate other conceptions when they conceive angles as union of two rays. For instance, it was reported that students conceive angles as the *linear distance* between the two connected rays (Clements & Battista, 1989; Keiser, 2004; Lehrer et al., 1998; Thompson) (Figure 2a). Elementary students often associate the proximity of the two sides of an angle with the angle size. For instance, Keiser (2004) found that one student referred to an angle as the "width between two lines." But a student who objected to the use of "width between two lines" to refer to angles argued that no linear width could be drawn between the sides of a 270° angle as an example. This student viewed angles as the relationship between two connected sides and that these sides can open to more than 180°. Clements and Battista (1989) also reported that students in their control group associated the size of an angle measure with the length of the line segment between the two sides. For these students, a shorter segment between the two legs corresponds to a smaller angle, while a longer segment that connects the legs of an angle means a bigger angle.

Students who worked with activities involving turns and rotations in the treatment group did not exhibit this alternative conception about the distance between the two segments.

Research shows that this alternative conception is not only prevalent among elementary school students but also among high school students. For example, Hardison (2019) observed that ninth-grade students described the openness of an angle by demonstrating with their two fingers the sweeping of segments through a pair of chopsticks as illustrated in Figure 2b. This demonstration of awareness of an angle is what (Hardison, 2019) refers to as a *segment sweep*. Although students may attempt to show an angle using line segments between the two sides, the linear distance should not be considered to define angle measure (Lehrer et al., 1998). As Clements and Battista (1989) emphasized, this linear distance is not always equidistant from the vertex; therefore, it should not be considered to define the size of an angle. One way to support students moving beyond this conception is to help them notice that it is impossible to draw a linear distance for angles greater than 180°, as exemplified by one student (Keiser, 2004).

Figure 2

Alternative Conceptions about Angles as Linear Distance Between Rays and as Segment Sweep



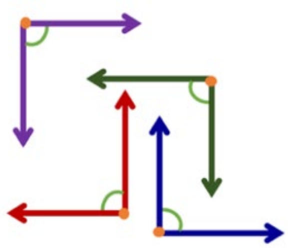
(a) (b)

Note: Angle as (a) a linear distance between two rays and (b) a segment sweep between sides.

Another common alternative conception of angles that research reports is that young students initially develop the idea that all angles are right angles (Devichi & Munier, 2013; Piaget & Inhelder, 1956) (Figure 3). Piaget and Inhelder (1956) noticed that right angles are ubiquitous in children's drawings showing two connected perpendicular segments, even before children develop an understanding of perpendicularity. For these students, an angle has “arms” parallel to the edges of a perpendicular corner, such as a rectangular tabletop. Devichi and Munier (2013) reported that every angle drawn by most third- and fourth-grade students was a right angle. Students’ personal experiences with corners such as tables, doors, or walls as their first examples for angles may influence them to develop this conception. This alternative conception is prevalent among elementary school students and is difficult for some students to develop. Devichi and Munier (2013) argued that children continue to have this alternative conception, which could hinder them from developing other angle concepts. They suggested that students should be confronted with angles of different sizes.

Figure 3

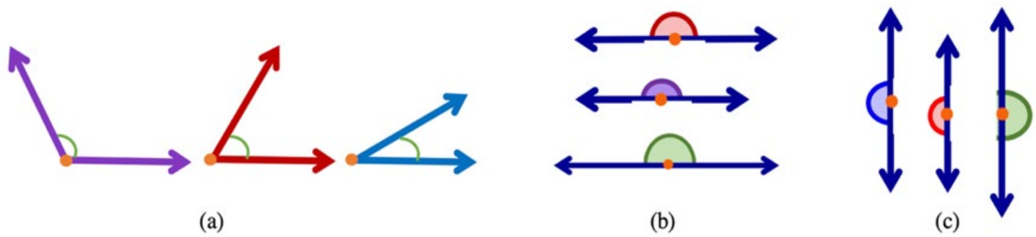
Students Consider Only Right Angles as Angles



Students also consider orientation an essential component in recognizing an angle (Clements & Battista, 1989; Devichi & Munier, 2013; Hardison, 2018). They believe that angles always have a horizontal side as its base, while the other side is non-horizontal (Clements & Battista, 1989; Hardison, 2018; Mitchelmore & White, 2000) (Figure 4a). Children often use languages such as tilted, bent, slanted up, or slope side to describe the openness between the non-horizontal side and the horizontal base (Clements & Battista, 1989; Mitchelmore & White, 2000). Interestingly, some older students continue to experience difficulties in recognizing angles that do not have a horizontal side (Browning et al., 2007).

Figure 4

Students Develop Other Alternative Conceptions of Angles



Note: (a) Some students believe that angles have a horizontal base and another non-horizontal side. When students acknowledge straight angles, they only recognize (b) straight angles oriented horizontally, but they have difficulties with (c) straight angles oriented vertically.

Students also struggle in recognizing an angle formed by two coinciding rays or two rays forming a straight line (Clements & Battista, 1989; Keiser, 2004). For example, when students encounter 0°, 180°, and 360° angles, they could not recognize an angle (Keiser, 2004). Students also think that coterminal angles in a standard position have the same measures (Keiser, 2004). For instance, students may see that the relationship between the initial and terminal rays is the same for the angles with measurements of 90°, 450°, and 270°. Keiser (2004) argued that students struggle because they only experience angles in geometric shapes but not with the turning motion. He suggested that if students are given more opportunities to experience angles in turning objects, they would develop these conceptions. These difficulties are probably due to students' conception of angles as formed by the union of two rays. Students may not recognize angles when the vertex of overlapping rays or rays forming a straight line is not visible to them. For these students, a vertex is an essential component of an angle. Smith et al. (2014) found that when students do recognize straight angles, they often visualize them as horizontal straight angles (Figure 4b). Vertical straight angles (Figure 4c) can be difficult for students who rely on angle examples with horizontal bases.

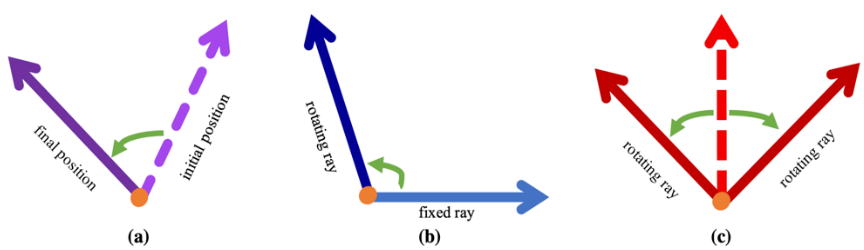
These alternative conceptions arise because students experience them from their personal lives or they have not perceived angles as the openness between the two angle sides. The development of these alternative conceptions is probably because of the use of static representations in exploring angles (Devichi & Munier, 2013; Keiser, 2004; Smith et al., 2014). Static representations are usually found in textbooks that do not involve any movement on the position of angle sides (Mitchelmore & White, 2000). As a result, students create different alternative conceptions and they reason at the visual level of the van Hiele (1984) levels of geometric thinking. Students at the visual level focus on the appearance of a geometric figure (configuration) rather than reasoning about the underlying properties and principles at higher levels of thinking (van Hiele, 1984). Even if students develop their geometric levels of thinking, they often continue to have this alternative conception in higher grades of education (Lehrer et al., 1998). Static representations alone support the conception that angles are formed by the union of two rays and encourage students to develop a variety of alternative conceptions (Smith et al., 2014). Instead, Devichi and Munier (2013) suggested that a combination of static and dynamic representations can help students move beyond their alternative conceptions such as the association of side length and angle size. Research suggests that providing students with dynamic representations of angles, such as conceptualizing angles as rotations may help students overcome these difficulties. In the next section, I discuss how students may develop the concept of angles as rotations.

2.2. Angles as Rotations

The conception of angles as a rotation is primarily characterized by the movement of one or both sides. Recent work in teaching angles as a turn or rotation focused on the transformation of geometric objects. Particularly, Lehrer et al. (2014) use rotation as a transformation of geometric figures which provides a dynamic basis for describing and analyzing geometric relationships. Students' conception of angles as rotations was broadly studied in mathematics education. From the literature, I identified three forms of angles as rotations, namely rotating one ray as a transformation from an initial position to a final position, rotating one ray while the other ray is fixed, and rotating two rays (Figure 5). In case (a), only one ray is rotating, and the angle is the amount of turn from the initial position to the final position. In case (b), there is also one rotating ray while the other ray is fixed, and the angle is the amount of turn made by the rotating ray away from the fixed ray. In case (c), two rays are rotating, and the angle is the openness between the two rays.

Figure 5

The Three Forms of Angles as Rotations



Note: Angles as rotations can be (a) rotating one ray from an initial position to a final position, (b) rotating one ray while the other ray is fixed, and (c) rotating two rays.

Angles as rotations are commonly measured using the non-standard unit of a turn and the standard units of degrees or radians (e.g., Confrey et al., 2012; Keiser, 2004; Moore, 2012). Often, students initially measure rotations in terms of the fraction of a turn, and then relate this to a measure in degrees (e.g., Clements & Sarama, 2014; Keiser, 2004) (Table 1). For example, one full turn is 360°, half of a full turn is equal to 180°, and a quarter of a full turn is 90°. When counting the amount of turn after a full turn, younger students instinctively count further. For instance, Kaur (2020) found that students consider the measure of a turn after a full rotation as bigger than only one full turn.

Table 1

The Equivalent of Turns with Its Degree Measure

Turns	Degree measure
One full turn (1)	360°
$^{3}/_{4}$ turn	270°
Half turn ($^{1}/_{2}$)	180°
Quarter turn ($^{1}/_{4}$)	90°
$^{1}/_{8}$ turn	45°
$^{1}/_{360}{}^{th}$ turn	1°
$^{n}/_{360}{}^{th}$ turn	n°

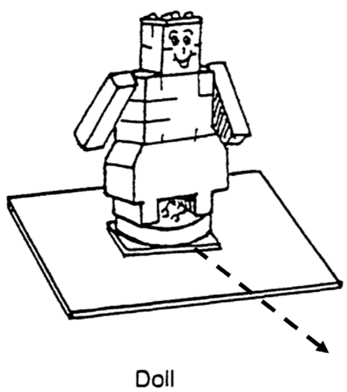
In the next paragraphs, I describe how students may conceptualize angles based on the three forms of angles as rotations mentioned above, and for each one I discuss how students may measure them.

2.2.1. Rotating One Ray

The first type of angle as rotation is illustrated as a transformation of one ray from an initial position to a final position (Figure 5a). Freudenthal (1973) described an angle as a transition from side to side showing continuous change. An angle as a rotation of a ray is often observed through turning or rotating physical objects (Browning & Garza-Kling, 2009; Clements & Sarama, 2014; Mitchelmore & White, 2000). Specifically, students' use of concrete movable objects can offer them a dynamic representation of this type of rotation (Clements, 2000). For example, Mitchelmore (1997) examined 7-years-old students' conception of angles and turns when they used realistic rotation models such as turning a doll. The students were asked to predict the doll's position if they turn the doll model for a half turn, a quarter of a turn, and multiple turns. Although the doll offered a dynamic representation of angles as rotations, Mitchelmore (1997) found that the students could not correctly explain those turns. In particular, students mentally construct a half-line to represent the final position of the doll model (Figure 6). But, a half-line does not visually form an angle.

Figure 6

A Doll Model Representing a Rotation of One Angle Side

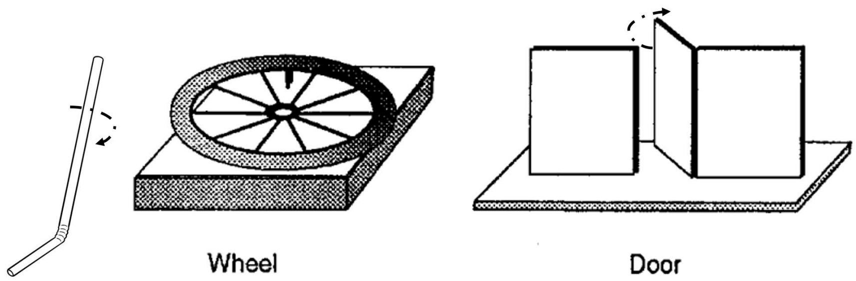


Note: A doll model that students used to mentally construct a half-line to represent the final position after the rotation (Mitchelmore, 1997).

In another study, Mitchelmore and White (2000) presented models of movable situations, such as a wheel or a door, to investigate the development of students' angle concept in grades 2-8. When Mitchelmore and White (2000) asked students to demonstrate how a wheel turns by bending a drinking straw, most students intuitively used only one arm of the straw to represent the rotation (Figure 7a). Although students were able to demonstrate the rotation of one side, they struggled in identifying the center of rotation on the wheel model. Students also found it difficult to determine the initial position of the door model since only one side is being rotated (Figure 7b). The difficulties that students encounter with physical models can be addressed using digital technologies.

Figure 7

Wheel and Door Models Representing Rotation of One Angle Side



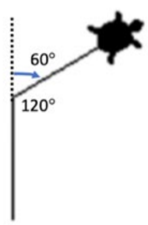
Note: Modeling a rotation of one side using a wheel and a door (Mitchelmore & White, 2000).

Some digital technologies, such as Logo programming (Papert, 1980), were used to investigate students' conception of angles as rotations. In Logo, a turtle icon on a computer screen receives and enacts two commands: the turn and the distance commands. For instance, the command "rt 74" makes a right turn of 74° or "fd 22" makes a forward move of 22 turtle steps. The user can use the visual tracing of a path as a useful feature for understanding geometric shapes and angles as shapes' attributes. Several studies used Logo to explore how students conceptualize angles as dynamic turns (e.g., Browning et al., 2007; Clements et al., 1996; Clements & Burns, 2000). For example, Clements and Battista (1989) found that activities in Logo benefited fourth-grade students in understanding angles as rotations. Browning et al. (2007) also found that Logo activities enabled students to define angle as a turn and measure it based on the amount of turn. They discussed that the Logo environment helped students distinguish between the angle itself and the degree as the angle's standard unit of measure (Browning et al., 2007).

Logo also offers visuals to illustrate an angle as turning a ray and the space between a ray's initial and final positions. It is worth mentioning here that Logo's turtle turns using the exterior angle rather than the interior angles. For instance, when sixth-grade students commanded the turtle to create sides of a polygon, they noticed that the 60° right turn command creates an interior angle of 120° (Browning et al., 2007) (Figure 8). According to Browning et al. (2007), students' experience with Logo potentially helped them discover the relationship between exterior and interior angles and visualize the creation of angles through the turtle's turning motion.

Figure 8

An Illustration of Logo's Turtle Turning 60° Angle Creating a 120° Angle

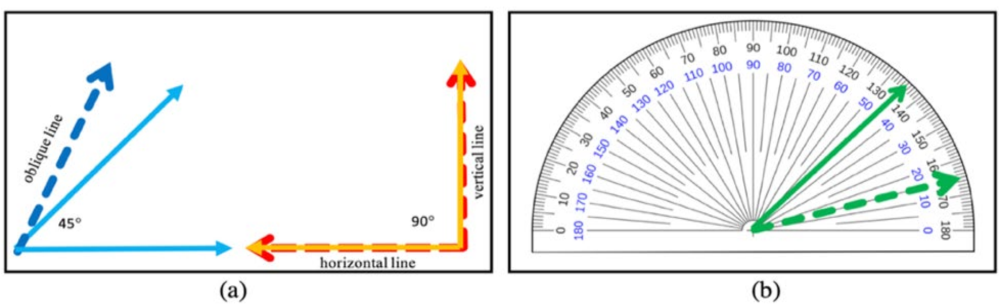


Several studies using Logo also showed evidence that students bridge their understanding of digital turns with their body rotations. In particular, Clements et al. (1996) and Clements and Burns (2000) argued that students used their body movements to verify the direction of turns and approximate the amount of turn. For example, students twisted their bodies to align with the direction of the turtle (Clements et al., 1996). Clements et al. (1996) found that a third-grade student consistently used hand gestures to internalize a strategy for turns that he was struggling with. Another third-grade student would gesture her arm 90° to estimate whether the measure of a turn was equal to, greater than, or less than 90°. Indeed, students used body gestures that they found necessary for representing turns (Clements et al., 1996). Both studies showed that when students used their body movements, they were able to verify their responses and produce correct answers regarding the direction and amount of turns (Clements et al., 1996; Clements & Burns, 2000). It is also interestingly noticeable from these examples of body gestures that students used a single body part (e.g., arm, hand) to relate with the rotation of one ray.

Also, Clements et al. (1996) found that students who worked with the digital features of Logo avoided certain alternative conceptions in measuring turn and angle. An example of an alternative conception that students did not portray was the use of 45° and 90° turns to refer to oblique and horizontal or vertical lines, respectively. For example, an oblique line can be associated with 45° turn while the horizontal and vertical lines are associated with a 90-degree turn (Figure 9a). Students also avoided the incorrect use of a protractor image in which turning an object is based on a protractor in standard position but disregards the initial position of a ray. For instance, if a ray corresponds to 45° on a protractor in standard position regardless of its initial orientation, the turn measures 45° (Figure 9b).

Figure 9

Two Alternative Conceptions That Were Developed When Students Worked with Logo Activities



Note: The two alternative conceptions on (a) creating 45-90 angles, and (b) misusing a protractor were developed when students worked with Logo activities.

As Clements et al. (1996) argued, students with these alternative conceptions make an oblique line by using an input command of "rt 45" but disregard the turtle's initial position. Instead, students used the turtle's perspective, which can be more meaningful to students as they associate their bodies with the rotational view of Logo's turtle. They also found that other common alternative conceptions about angles, such as associating the linear distance with the angle size, were not evident when students were working with Logo.

To examine students' measurement strategies for angles, Clements and Burns (2000) introduced an electronic Logo's protractor tool to guide the students to turn an arrow to direct the turtle's heading. However, students used their estimation on the amount of turn and changed the input in the protractor tool. For instance, when the protractor tool provided the command rt 20 to turn the turtle at 20 to the right, one student adjusted it to rt 22 because he reasoned that it was more accurate than what was shown in the tool. The student's estimation and adjustment on the amount of turn enabled him to work on their task on navigating around a map successfully.

Indeed, students can conceptualize angles as a rotation of one ray when they experience turning physical or digital objects and gesture the turns with their bodies. However, students continue to experience difficulties in visualizing the continuous movement of an angle side as well as in measuring angles when only one side is visible (Mitchelmore & White, 2000). Using these three tools (physical objects, digital objects, and body movements) in representing an angle as a rotation of one side does not seem to offer students constructive opportunities to visualize how angles are created and continuously change. Even though Clements and colleagues (1996; Clements & Burns, 2000) tried to help students quantify angles using the digital protractor in Logo, they found that some students continued to rely on their inaccurate estimates of the amount of turn while some students focused on the number but not on the direction of turn.

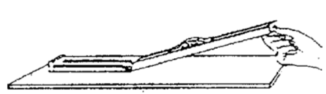
2.2.2. Rotating One Ray While the Other Ray Is Fixed

The second type of an angle as rotation is described as having one visible side fixed (usually a horizontal ray) and the other side rotates to create an angle (Figure 5b) (Clements & Battista, 1989; Mitchelmore & White, 2000). Angle measure in this type is the difference between the position of the rotating ray from its fixed base (Keiser, 2004). For instance, when children perceive angles as rotations, they describe such movement as departures from straight lines (i.e., bending, slanting) (Lehrer et al., 1998; Mitchelmore & White, 2000). One conception that students often develop about angles as rotations is that the rotating side of an angle always goes counterclockwise while the other side is positioned horizontally as in a standard position.

Research found that students may develop their understanding of an angle by turning physical objects which have both angle sides visible (Mitchelmore & White, 2000). An example of this type of rotation can be illustrated using a movable model of a hill with a car on it (Figure 10) (Mitchelmore & White, 1998). As Figure 10 illustrates, the line representing the hill was the movable angle side and the horizontal plane below the hill was the fixed angle side. The slope of the hill represented the size of the angle between the movable and the fixed sides. In each question for the students, the slope of the hill model was set at approximately 30°, then 45°, and then 15°. When students were asked to predict whether the car would travel faster or slower uphill and downhill as the slope changed, they reasoned that the steeper a hill, the more difficult it is for the car to ascend or that the car could not drive up the hill if it was straight up (Mitchelmore & White, 1998). The steepness of the hill is relative to the size of the slope. Although students did not measure the slope, they reasoned about the steepness of the hill based on the openness between the movable and fixed sides. This study illustrated the potential of using angles for studying the change of covarying quantities in the real world.

Figure 10

A Hill Model Representing a Rotation of One Angle Side



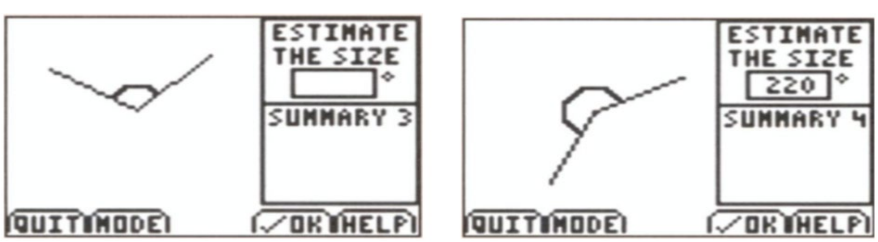
Note: A hill model that represents a rotation of one angle side while the other side is fixed (Mitchelmore & White, 1998).

Aside from turning physical objects, angle as a rotation of one ray while the other ray is fixed can also be demonstrated through body motions. Studies have found that physically experiencing angles through relative positions, body enactment of turns, and hand rotations can help students verify their perception of angles from static configurations (Clements & Burns, 2000; Devichi & Munier, 2013; Fyhn, 2006, 2008; Smith et al., 2014). For instance, seventh-grade students identified angles in climbing activities using relative positions of their body and gestures of their body parts (Fyhn, 2006, 2008). Specifically, (Fyhn, 2008) engaged students in climbing activity, then she asked students to demonstrate, draw, and point out the angles from their climbing experience. One group of students bent their elbows to create acute or obtuse angles between their arms to climb up the wall successfully. The use of body gestures such as bending arms helped students conclude that it is more tiring to climb if your arms are held at a right angle than if they are stretched at an obtuse angle.

Another tool used for understanding this type of angle as rotation is through digital illustrations. Browning et al. (2007) used activities using SmileMath, the angle estimation feature on the T1-73 calculator (Figure 11). The SmileMath tool allows the user to create an angle of a particular measure by pressing the freeze key to stop the moving ray or guess the measure of an angle once the moving ray stops.

Figure 11

The Angle Estimation Feature on SmileMath



Note: Sample screens from the SmileMath application (Browning et al., 2007).

SmileMath engages students with the idea of angles as a rotation of a ray turning away from another ray. The Browning et al. (2007) study found that as students engaged with the tool, they recognized that several angles of different orientations have the same measure. According to Browning et al. (2007), this observation helped the students overcome the limitations of typical angles such that 90° should have at least one horizontal ray. However, angle situations with only one rotating ray lead younger students to respond differently than with situations where both rotating rays are visible (Mitchelmore & White, 2000). The next sub-section describes this difference in conceptions by exploring angles as a movement of both sides.

2.2.3. Rotating Two Rays

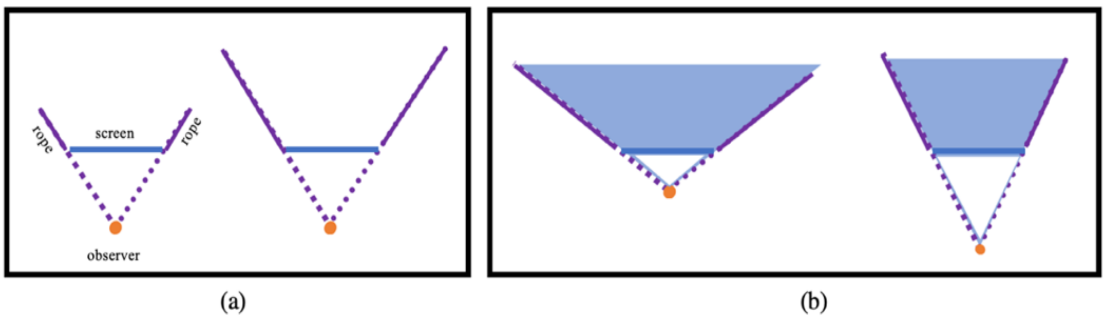
The third type of angle as rotation involves the movement of both angle sides (Figure 5c). The first two types only involve one ray turning and can be measured by the amount of turn. In contrast, angle as rotation of both rays can be measured according to the openness between the two angle sides. One of the benefits of this approach is that moving both sides of an angle does not restrict the other side as the initial position. Instead, both sides of an angle can rotate and can take different orientations. Two physical models of this type of rotation that illustrate the openness of two sides are a pair of scissors and a hand fan. According to Mitchelmore and White (2000), students reasoned that these models have two lines and are considered examples of objects that open but not turn.

The clock hands model is another example of rotation with both sides opening. (Fitz, 2016) used an analog clock to engage middle-school students in identifying and measuring angles between the hands of the clock. In another study, (Pagni, 2005) helped eighth-and-ninth grade students explore the application of angles in identifying the number of degrees formed by the minute hand and the hour hand or the degrees rotated by the hour hand of a clock. In terms of the amount of rotation made by the hour hand, students used proportional reasoning to explain that the hour hand moves a twelfth of 360° of its way around the clock every hour, and that the number of degrees the hour hand moves is a fraction of 30° which is equivalent to the fraction of 60 minutes moved by the minute hand. Although students could quantify the degree measure made by hour hand in multiples of 30, it was not clear from the study if the students were able to measure smaller increments of angles made by both clock hands. Consequently, more research is needed to examine whether students understand the angle measure in terms of discrete smaller increments of degrees or even perceiving angle as a continuous quantity.

Researchers also engaged students in representing the opening of both rays of an angle through physical positions. Devichi and Munier (2013) engaged students in playground activities to physically illustrate the area hidden from an observer behind a screen. The students were presented with two versions of the physical demonstrations (Figure 12) that helped them mitigate the side length-angle size alternative conception discussed in the previous section. Specifically, in the first version, one student facing the center of an obstacle screen served as an observer and as the vertex of the angle as shown in Figure 12a. All the other students lined up behind the screen and were holding cone markers. Each student in the line walked out from behind the screen and away from the observer until the observer could notice them on the left or the right side of the screen. As soon as each student became noticeable from the observer, they put their cone markers on the ground. After the students finished setting the markers, they observed that it formed two oblique straight lines. Then, the students checked the alignment of the boundary markers by connecting the markers with ropes. When the teacher asked the students how far they could go if they could have stood farther apart in the line, some students conjectured that the lines could be extended beyond the length of the ropes. This demonstration was repeated, showing the same observer position with different rope lengths. Since the observer position did not change, the size of the alignment of ropes also stayed the same. This example shows that the angle size is the same regardless of the length of sides; thus, it invalidates the alternative conception that angle size is dependent on side length.

Figure 12

Two Versions of a Dynamic Angle Demonstrated in a Playground



Note: Illustrations of (a) angles with the same observer position but different rope lengths, and (b) angles with different observer positions. The observer is represented by an orange point.

In the second version, the students demonstrated the same procedure in connecting the cones with ropes, but the observer was moved to two different positions (Figure 12b). In this demonstration, students noticed how the area hidden from the observer increased or decreased as the observer moved closer or away from the screen. At the end of this task, the teacher asked the students to represent the shape of the area using body gestures as it corresponds to the observer's successive change in position. The students straightened their arms and moved them as if they were opening and closing the angle's sides when the vertex position moved toward or away from the screen. Devichi and Munier (2013) concluded that students associated the shape of the area with the farther apart opening of both sides of the angle. Although students associated the opening of angle sides with the shape of the area between them, this conception can be problematic. The hidden area between the angle sides does not represent the size of its angle.

In another study, Hardison (2018) investigated students' conceptions of angles as a rotation of both rays using two pairs of chopsticks. One pair of chopsticks was shorter while the other pair was longer. In one of his tasks, he asked the students to use the chopsticks to model two angles with the requirement that the shorter angle model is four times as open as the longer angle model. When students were asked to verify the congruence of the chopsticks' opening, they were not given any protractor, yet they were able to represent the angle size efficiently. To compare the size of the angles represented by the chopsticks, students put one pair of chopsticks on top of the other pair and rhythmically moved both sides of the shorter chopsticks four times wider than the longer chopsticks. In this example of opening both angle sides, the students compared the static and discrete result of openness instead of experiencing the dynamic and visible continuous motion of angle opening.

Similar to the examples of representations for rotating one side, the rotation of both rays was also studied using digital tools. Smith et al. (2014) used a digital environment to examine how embodied activities enhance students' learning of angles. Smith et al. (2014) drew on the context of *embodiment* (Lakoff & Núñez, 2000) that recognizes the close connection between cognition and action, to design a body-based angle task using the Kinect environment for Windows. Kinect is a motion-controlled learning environment that tracks and translates body movements into motions on a digital screen. Smith et al. (2014) used this motion-sensor technology for students to learn angles by positioning their arms and estimating a range of angle measures that were associated with a particular color on the screen. For example, to create a pink screen, students had to raise their arms illustrating an acute angle; to create a light blue screen, they had to create a gesture of an obtuse angle. The design of their tasks aimed for students to recognize and create both static and dynamic representations of angles. For instance, students either used dynamic sweeping motions of their arms to show a range of possible angle measures that would qualify for such color requirement, or a static motion stopped at a particular arm position to show only one angle (Smith et al., 2014).

After the above explorations, Smith et al. (2014) added a digital protractor on the screen to show benchmark degree angles, such as 0°, 45°, 90°, 135°, and 180°. They reported that by using the protractor, a student dynamically opened their arms to associate a range of arm movements from one benchmark angle to the other. The feedback from the screen supported students' arm movements in creating angles. Although associating the opening of both arms with the screen feedback supported students to generate and explore angles dynamically, the design of the tasks did not support students in engaging with quantitative operations on angle measure or developing a conception of an angle as a continuous quantity.

2.2.4. Understanding Angles as Rotations: Pushing Forward

The understanding of angles as rotations has significantly helped students to avoid some common alternative conceptions that they develop when angles are conceptualized as merely a union of two rays. In all forms of angles as rotation, two essential features of an angle are established. First, the generation of an angle by rotating one or both sides of an angle. Second, the generated angle is illustrated as a continuous quantity. In other words, the rotational motion supports students to visualize the continuous generation of angles. Indeed, angles as rotations provide a more dynamic way of showing how angles can be generated and how angles are continuous quantities that can vary.

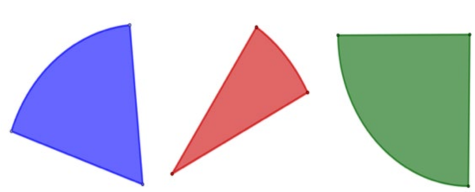
However, multiple and continuous rotations are difficult for students to quantify. Although students were able to conceptualize angles as turning or opening of objects when both angle sides are visible, they mostly focused on the static output of the movable model and not on the dynamic turn (Mitchelmore & White, 1998; Mitchelmore & White, 2000). In other words, students maintain their attention on the static result of a movable model. Also, many students were not able to conceive turning as related to angle size (Mitchelmore & White, 2000). Because of these difficulties, Mitchelmore and White (2000) argued that definitions of an angle as an amount of turn seemed particularly inappropriate in elementary mathematics education and are less helpful to younger students. Instead, they proposed an angle definition as an angular relation between two lines meeting at a common point. This angular relation between two connected lines seems to call for quantifying angles. The next section describes the conception of angle as a wedge that may be the way to support students' quantification of angles.

2.3. Angles as Wedges

Another conception of an angle is what research refers to as a wedge, or the quality of a contained area, or delineated space between the two angle sides (Browning & Garza-Kling, 2009; Browning et al., 2007; Lehrer et al., 1998; Thompson, 2013). As Figure 13 illustrates, a wedge is a static representation of an angle defined by the area created between the two sides. A more dynamic way of looking at a wedge is as the trace of the movement of rotating one side or rotating both sides of an angle. In other words, a wedge can be the traced outcome of a rotation. Indeed, wedges have the potential to support the conception of angles as being a continuous quantity generated from a rotation.

Figure 13

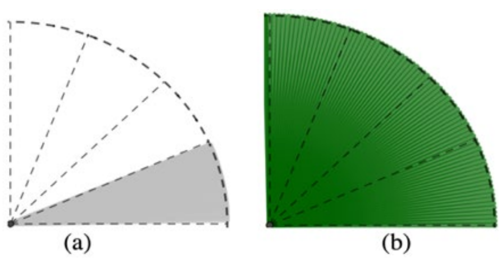
Illustration of Angles as Wedges



Research shows that wedges can be useful for quantifying angles. Browning and colleagues (Browning & Garza-Kling, 2009; Browning et al., 2007) provided the example of middle school students who were given a piece of patty paper to invent a device that would measure angles. They folded the patty paper into wedges of different sizes and reasoned about those wedges as non-standard units of measure. For example, as students folded their paper into small triangular wedges, some described that their wedges are fractions of a corner (i.e., “I have one-fourth of a corner”) (Figure 14a) (Browning et al., 2007). Browning and Garza-Kling (2009) suggested that these experiences can help students develop a sense of a degree as a standard unit of angle measure. For instance, students described an angle in terms of a space filled with small one-degree wedge units (Figure 14b).

Figure 14

Illustrations of Wedges as Non-Standard Units of Measure for Angles



Note: (a) A small wedge as a fraction of a corner and (b) one-degree wedges.

The Measurement standards of the Common Core State Standards for Mathematics (CCSS.M.4.MD.C.5.A) also suggest connecting angle measure to the fraction of a circle. To do so, we need to consider the fraction of the circular arc between the points where the two rays intersect the circle, provided that the center of the circle is at the common endpoint of the rays. The specific standard defines an angle that turns through 1/360 of a circle as a “one-degree angle” that can be used as a measurement unit. In describing this standard, Confrey et al. (2012) provided a table (Table 2) connecting students’ conceptions of a turn, with the fraction of the circle and the degree measure. For instance, ¼ of a turn is ¼ of a circle and it is ¼ of 360°, which is equal to 90°.

Table 2

The Equivalent of Turns as Fractions of a Circle and its Degree Measure

Turns	Fraction of a circle	Degree measure
2 and a quarter turn	2 and a quarter of a circle	810°
1 and a half turn	1 and a half of a circle	540°
One full turn (1)	1 circle	360°
$^{3}/_{4}$ turn	$^{3}/_{4}$ of a circle	270°
Half turn ($^{1}/_{2}$)	$^{1}/_{2}$ of a circle	180°
Quarter turn ($^{1}/_{4}$)	$^{1}/_{4}$ of a circle	90°
$^{1}/_{8}$ turn	$^{1}/_{8}$ of a circle	45°
$^{1}/_{360}{}^{th}$ turn	$^{1}/_{360}$ of a circle	1°
$^{n}/_{360}{}^{th}$ turn	$^{n}/_{360}$ of a circle	n°

Indeed, understanding angles as wedges can help students progress into the understanding of angle measure as it can help them recognize the attribute being measured (Browning & Garza-Kling, 2009) and help them quantify angles. However, defining an angle as a wedge can elicit alternative conceptions among students. In their longitudinal study, Lehrer et al. (1998) found that elementary school students refer to angle size as a relative area or space between its two sides. They also found that this conception of angles continues to prevail even in later years of schooling (Lehrer et al., 1998). This conception is evident in the Browning et al. (2007) study where some middle school students associated the size of an angle with roughly triangular wedges. An angle can contain wedges of different sizes but these wedges do not represent the size of an angle. Moore (2012) also noticed that college students initially refer to the area or space between the angle sides in his attempt to help students make sense of angle measure through arclength measure. As a result, students often misconceive angles as having different measurements when inscribed in circles with different areas (Moore, 2012). This alternative conception probably develops when angles are illustrated only as static wedges with a fixed length of sides.

2.4. Bridging the Three Conceptions of Angle

The exploration of the literature shows that angles are often narrowly defined in mathematics textbooks and classrooms as a union of two rays through a common point (Browning & Garza-Kling, 2009). Because of this, students initially conceive angles as geometric objects with two sides connected by a vertex. This first form of angle conception results in a plethora of alternative conceptions about angles and the relation between the angle sides. This led me to an exploration of angles as rotations and as a wedge. Angles as rotations are usually illustrated through the rotation of a single ray, rotation of one ray while the other ray is fixed, or rotation of both rays. This second common conception of angle is also problematic because it is difficult to quantify smooth and continuous rotations. Angles as wedges, on the other hand, can be used to easily quantify angles in relation to the fraction of the circle that the space or region between the angle sides generates. Additionally, angles as wedge can illustrate the smooth and chunky generation of angles and represent angles as a continuous quantity. Still, researchers have found this third conception of an angle problematic as students reason about the size of an angle based on the area between the rays and not in relation to the arc of the circle.

Freudenthal (1973) emphasized that several concepts of angles must be considered to avoid restricting conventional definitions from the meaningful learning of mathematical ideas. Indeed, using one angle conception in isolation may limit students into memorization, single-word responses, and procedural thinking rather than offering them opportunities for reasoning (Boston & Candela, 2018). Furthermore, there is no proper definition that can describe angles from all areas of personal experiences (Taimina & Henderson, 2005). Since students also recognize angles in one context different from another context, Mitchelmore and White (2000) suggested that they should be presented using various angle models. Considering the above, I would argue that the three conceptions of angles are not enough when taken apart from each other. Instead, combining all three conceptions of angles may help students develop a robust understanding of an angle as being generated through a rotation or turn and being quantified in terms of a fraction of a circle and number of degrees. This might be the kind of activity that would help students to bridge the topological and Euclidean relationships in forming their meanings about angles (Piaget & Inhelder, 1956).

The literature also has shown that the way angles are modeled (static vs. dynamic physical models, body movements, digital tools) influences students' experiences and, therefore, the different conceptions they build about angles. The static models of angles are the output of the rotation of angle sides and lack the rotational motion that dynamic models have. As a result, research documents how static models usually encourage more alternative conceptions that students continue to have even in later years of schooling. Since static models do not help students develop their conception about angles, I looked into the literature on what dynamic representations could offer. Generating and changing angle size using digital technologies can help students resolve limitations and difficulties that students experience with the static representations (e.g., Browning et al., 2007; Clements & Battista, 1989; Clements et al., 1996; Clements & Burns, 2000). Specifically, Browning et al. (2007) proposed the use of digital technology to help students make meaning of dynamic angles beyond paper-and-pencil tasks. The dynamic models of angles offered students opportunities to interact with the rotational motions of angles and are represented through body movements or using digital technologies.

The embodied approaches showed that angles are perceived through body gestures but have limitations when arms can only be rotated to some extent and it is difficult for students to visualize the generation and quantification of angles. These limitations experienced from the physical embodiment of angles can be eliminated using digital tools. Indeed, the exploration of angles in digital environments offered students visual platforms to experience angles as rotations. However, studies about angle conceptions using digital tools were only limited to the generation of angles as rotations and were difficult for quantifying angles.

For this reason, I explored the potential of using a digital environment to engage students in a dynamic exploration of angles that focuses on all three angle conceptions. The theoretical framework that I describe in the next section aims to engage students in exploring angles dynamically by bridging the three conceptions of angles into a unified construct, what I refer to as a dynamic measurement for angle. In particular, I began with the assumption that elementary school students can construct meaning of an angle as a continuous quantity that can be quantified and dynamically changed with careful task design. Specifically, I aimed to explore the following research questions:

1. What forms of reasoning do students exhibit as they engage in dynamic digital tasks that aim to bridge the three conceptions of angles?
2. What characteristics of the design (e.g., characteristics of tasks, tools, and questioning) support the particular forms of students' reasoning for angles?
3. How did the design evolve to support students' reasoning for angles?

Chapter 3: Theoretical Framework

To investigate the research questions that I presented in the previous chapter, I followed the perspective of radical constructivism. Specifically, I explored the theoretical elements of radical constructivism and discussed how this perspective to learning provides a platform to illustrate how students construct their mathematical reasoning about angles and how I could examine those forms of reasoning. Additionally, I discussed how quantitative reasoning could be used as a window for students to construct their meaning of angles and for me to explore students' thinking.

3.1. Looking at Angles from a Radical Constructivism Perspective

Von Glasersfeld (1987, 1995) drew upon his vast collection of experiences to describe constructivism. According to him, radical constructivism posits that knowledge is constructed by students based on their experiences. For von Glasersfeld (1987), learning means drawing conclusions from experiences and activities. In analyzing patterns of student learning, von Glasersfeld (1995) proposed a scheme theory that was built on Piaget's (1976) tripartite conception of reflex. Specifically, this tripartite conception of reflex includes a perceived situation, an activity associated with the situation, and a result of the activity that turned out to be beneficial for the actor (Piaget, 1976). Building on this, von Glasersfeld's (1995) scheme theory specified the three parts of schemes as a) recognition of a certain situation, b) a specific activity associated with that situation, and c) the expectation that the activity produces a certain previously experienced result. As Piaget (1976) promoted the use of clinical interviews to warrant the study on what goes on in students' heads, von Glasersfeld (1995) took a step further to validate the perspective that children's knowledge has an epistemological value though it could fundamentally differ from that of an adult. Specifically, von Glasersfeld (1995) formulated the two principles of radical constructivism:

1. Knowledge is not passively received either through the senses or by way of communication; knowledge is actively built up by the cognizing subject.
2. The function of cognition is adaptive, in the biological sense of the term, tending towards fit or viability; cognition serves the subject's organization of the experiential world, not the discovery of an objective ontological reality (p. 51).

The first principle of radical constructivism highlights an individual's active construction of knowledge drawn from their experiences and reflections on their experiences. It does not follow the transmission-of-knowledge model of learning where an existing knowledge from an external source is viewed to be imparted to students (Cobb et al., 1993). In other words, it asserts that it should be carefully taken as a lens on how children come to understand the world and validate their thinking (Confrey, 2011). The second principle is the radical piece which contends the purpose of cognition where individuals organize their experiential reality to meet their needs. In radical constructivism, an individual's reality is constructed based on their experiences rather than discovered. Confrey (2011) supported that these two principles are inseparable, and one could not coherently and completely accept one without the other.

From the perspective of radical constructivism, mathematics is constructed by individuals and mathematical meanings are developed within the structure of an individual's experiences. The term *meaning* refers to "the space of implications that the current understanding mobilizes – actions or schemes that the current understanding implies, that the current understanding brings to mind" (Thompson et al., 2014, p. 12). In the sense of Piaget (1977/2001), constructing a meaning is synonymous to constructing an understanding by repeatedly engaging in mental actions to assimilate a situation, and reflecting on these experiences, in which meanings are reorganized. In mathematics education, one can study students' construction of mathematical meaning within the space of their experience (Steffe & Kieren, 1994). Steffe and Thompson (2000) refer to mathematical realities that students construct as "students' mathematics." Mathematics is assumed to include more than its definitions and logical relationships, such as forms of representations, evolution of problems, and methods of proof and standards of evidence (Confrey, 1991). According to Confrey (1991), radical constructivism also upholds what students do as reasonable and seeks to describe student activity from the students' view. She supports that to study how students construct their mathematical knowledge, one can use sophisticated mathematical perspectives but recognize that the goal is to uncover students' voices. Student knowledge, as students construct it, exists in their minds and is not readily available, unless students' discourse and actions are made accessible to observers.

To make students' thinking accessible to observers, I refer to Noss and Hoyles' (1996) window metaphor. This window metaphor pertains to how windows mediate what we see and how we notice things through them. Also, the window has a dual nature as it allows students to look through the window and construct their meaning of what they notice; at the same time, this window allows us to see what students think. For instance, (Noss & Hoyles, 1996) refer to the use of computer environments as windows for students in constructing their mathematical meanings and for researchers to study in-depth what the students have constructed. Accordingly, my goal is to engage students in a mathematical activity that may serve as a window for them to construct their mathematical meaning of angles and a window for me, as a researcher, to look into the students' minds. Looking through a window to view the students' minds is what Confrey (1991) suggests as uncovering students' voices and accessing their thinking. Given that the design of the window influences what and how one sees through it, it is also crucial to consider how a window should be designed.

In designing the window for this study, I considered the principle that student knowledge is constructed based on their experience and that experience is dynamic in nature. I also understand that knowledge can be dynamically developed through constructive mathematical activity with specific tasks, tools, and questioning. From this idea of dynamic construction of knowledge, I aimed to document the *reflexive relation* (Cobb et al., 2001) between specific forms of student reasoning about dynamic angles and the design of the tasks, tools, and researcher questioning that constructively shape such reasoning. By studying the reflexive relation, I intended to understand how student reasoning is developed as students interact with the tasks, tools, and questions. Additionally, this type of relation does not prime one element as independent from the other. Instead, the reflexive relation between student reasoning and the design of mathematical activities shows that the two elements co-evolve. Student reasoning about angles emerges within the context of the designed activities, and the design of activities is modified in light of that student reasoning.

To unfold this reflexive relation between student reasoning about angles and the design of mathematical activity, I first considered the possible constructs of an angle. I examined students' reasoning about angles through the interplay of geometry and multiplicative reasoning. The geometric and multiplicative natures of an angle co-exist and complement one another. In particular, the exploration of the literature shows that the conception of an angle is geometric in nature. In order for students to construct their mathematical meaning of angles, it is not enough to only consider their geometric nature. It is also important to consider the multiplicative relationships between quantities involved in angles. To describe the interplay between geometry and multiplicative reasoning in the angle concept, I examined how quantitative reasoning may serve as a platform for bridging these two aspects as a unified construct.

3.2. Looking at Angles from a Quantitative Reasoning Lens

Quantitative reasoning focuses on quantities as measurable attributes of an object in a given situation (Smith & Thompson, 2007; Thompson, 2011). Quantities are described as mental constructions; in other words, quantities are created in mind (Thompson, 2011). Quantitative reasoning refers to the mental process of an individual in conceiving an experiential situation at hand, constructing quantities of the conceived situation, and reasoning about the constructed quantities and their relationships (Smith & Thompson, 2007; Thompson, 2011). Reasoning about quantities involves quantification, a mental process of "conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship with its unit" (Thompson, 2011, p. 37). Although looking through the lens of quantitative reasoning could potentially advance students' mathematical meaning of angles, the literature shows that quantitative reasoning about angles is not commonly encouraged in elementary mathematics education. If this type of mathematical reasoning is cultivated from students' elementary years of education, it can potentially produce flexible and generalizable mathematical forms of reasoning that can be further developed in higher levels of schooling (Smith & Thompson, 2007). Accordingly, through the quantitative reasoning lens, I explored how the geometric and multiplicative components of angles can be unified as a single construct that guided the design of mathematical activity for angles. In the following sub-sections, I first describe possible ways of how students may conceive angle situations and identify quantities involved in those situations. Then, I discuss the probable means of students' reasoning about the relationships of those quantities.

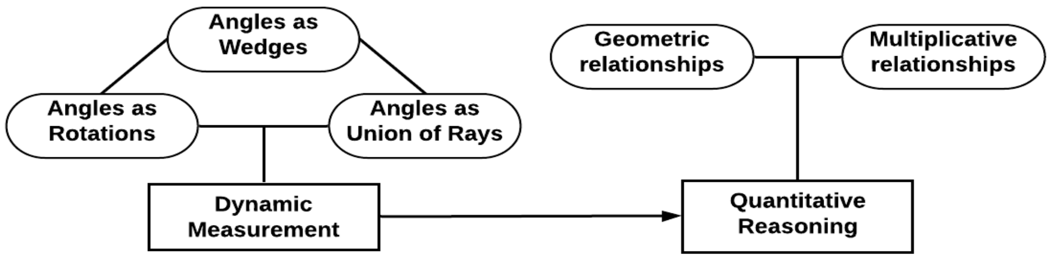
3.2.1. Conceiving angle situation and identifying quantities involved: What is changing?

The exploration of the literature shows that students may conceptualize an angle in three ways, namely as a geometric union of two rays, as a wedge, and as a rotation. The literature also has shown that considering multiple but unified angle conceptions may help avoid the limitations and difficulties encountered from working with individual conceptions (e.g., Boston & Candela, 2018). Considering these suggestions, it is more intuitive to design mathematical tasks involving the three conceptions of angles rather than focusing on each form as isolated constructs.

The *dynamic measurement* (DYME) (Panorkou, 2017, 2021) approach can unify the three angle conceptions and can potentially support how students may use quantitative reasoning to bridge the multiplicative and geometric attributes of angles (Figure 15). The notion of DYME can offer an entry point to examining the interplay between geometric and multiplicative reasoning perspectives of angles. Panorkou (2017, 2021) defined DYME as an approach to geometric measurement focusing on how a space is measured by lower-dimensional objects that generated it. Kobiela and Lehrer (2019) talked about this generation as a physical enactment of a dynamic sweep of one quantity to generate another quantity. The process of generation of an object may potentially support students in bridging the discrete and continuous conception of a quantity (Kobiela & Lehrer, 2019; Panorkou, 2017, 2021). For instance, tracing the continuous change in a rotation of a ray can illustrate the generation of an angle as a continuous quantity. Moreover, experiencing the dynamic sweep can foster thinking about the multiplicative composition of a geometric object, which in turn, can serve as a springboard for students to generate quantities through more sophisticated multiplicative operations (Kobiela & Lehrer, 2019).

Figure 15

Dynamic Measurement Bridges the Three Angle Conceptions and Quantitative Reasoning Bridges Geometric and Multiplicative Relationships



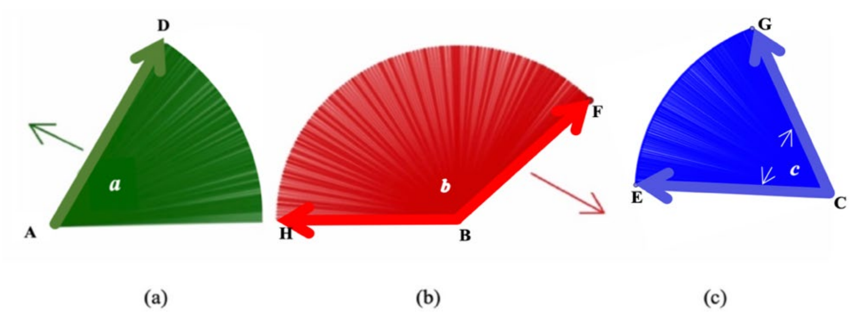
To visualize the generation process, researchers used the tracing feature of either physical tools or dynamic geometry environment (DGE) in their studies. Kobiela and Lehrer (2019) used physical squeegees in tracing areas and found that the tracing of the generation process offered feedback to students to see their errors and modify their interaction with the tool. Meanwhile, Panorkou (2021) who used the tracing feature of GeoGebra, a DGE, found that this feature assisted students to imagine the sweeping of a 2-D surface both as iterated into chunky identical layers and also as the smooth transformation of the 2-D surface into a 3-D shape. These experiences led students to construct both *smooth* and *chunky* images of change.

According to Castillo-Garsow and colleagues (Castillo-Garsow, 2012; Castillo-Garsow et al., 2013), chunky thinking involves imagining change in discrete chunks, regardless of the sizes of chunks. Smooth thinking, on the other hand, involves imagining a change in progress when students may envision the intermediate and infinite amounts of change in a continuous and smooth way (Castillo-Garsow, 2012). These smooth and chunky images of change have implications for students' conceptions of situations involving quantities that change (Castillo-Garsow et al., 2013). For instance, Castillo-Garsow et al. (2013) argued that smooth thinking encourages a smooth conception of change, while chunky thinking encourages chunky conception of change. These two forms of thinking may also imply different conceptions of angles as quantifiable objects. For instance, the literature in the previous chapter has shown that smooth and continuous rotations are difficult to quantify and that chunky images of wedges can be easily used in quantifying angles. Examining angles from a dynamic measurement perspective would ultimately help students construct both smooth and chunky images of change about angles and conceptualize an angle as both a rotation and a wedge.

In applying the DYME approach for angles using the quantitative reasoning lens, students may conceive an angle as generated by the rotation of connected rays and identify the quantities involved in generating the angle. For example, students may enact the rotation motion of rays to generate an angle by a quantity (e.g., amount of rotation). The literature shows three ways that an angle may be conceived via rotation, namely rotating one ray, rotating one ray while the other ray is fixed, or rotating both rays (Figure 16).

Figure 16

Illustrations of Dynamic Generation of Angles via Rotation



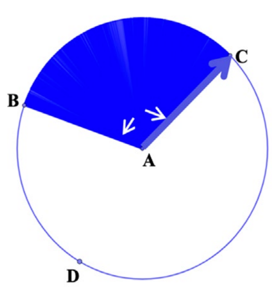
Note: Angles can be dynamically generated through (a) rotation of one ray, (b) rotation of one ray while the other ray is fixed, and (c) rotation of two rays.

To illustrate the generation of angles via rotation of rays, the tracing feature of GeoGebra may help students visualize the rotation of $\overrightarrow{AD}$ to generate the wedge $\angle a$ and they may identify the amount of rotation a (Figure 16a). Similarly, students may conceive that while $\overrightarrow{BH}$ is fixed, the rotation of $\overrightarrow{BF}$ generates $\angle FBH$ and identify the amount of rotation *b* (Figure 16b), or that the rotation of both $\overrightarrow{CG}$ and $\overrightarrow{CE}$ generates $\angle GCE$ by an amount of rotation *c* (Figure 16c).

In addition to situations perceiving angles as a rotation of connected rays, students may conceive angles as wedges being generated via rotation of a radius of a circle. While these quantities are recognized as being generated, students may reason that these quantities also change. This angle situation illustrates a combination of three angle conceptions as suggested in the literature. Students may identify the quantities involved in generating the wedge *BAC* such as the amount of rotation and the openness between the two radii $\overline{AB}$ and $\overline{AC}$ (Figure 17). Students may also recognize other quantities that are changing such as the space inside the wedge *BAC* and the circumference of circle *A*. The literature shows that the wedge exemplifies the idea of an angle as a continuous quantity that may support students' construction of smooth and continuous images of change. A wedge is also a static representation that can be split, iterated, and used to create other angles as composites of an angle. In other words, it may also support students' chunky images of change that would eventually help them with quantification.

Figure 17

The Generation of Angles Conceived as Wedge and the Identification of Quantities Involved



Note: Quantities involved in generating an angle may include the amount of rotation and the openness between the two radii in a circle.

In both angle situations, students may not only conceive an angle as a dynamic rotation or wedge but also as a static openness of space being formed by two rays and a vertex. This dynamic versus static perception of angle from these situations may support students' bridging of the three conceptions of an angle as a rotation, a wedge, and formed by two rays and a vertex. Also, both examples may help students avoid the alternative conception that the length of angle sides influences the angle size, as discussed in the section "Angles As Two Sides Sharing a Common Point"1.1. Instead, students may reason that although the amount of rotation is changing, the length of sides remains the same because they are radii of a circle. Similarly, students may explain that the size of wedge is changing, but the side length is the same.

In conceiving angle situations, students may not only construct the quantities, but they may also conceive that the quantities are changing. Students may characterize these quantities as dynamic rather than having one static value. This characterization of changing quantities, in turn, offers the space for students to construct multiplicative reasoning such that an angle can be generated in chunks or as a quantity that increases or decreases. From the quantitative reasoning perspective, there are possible ways that students may construct multiplicative relationships between the quantities conceived from the generation of dynamic angles.

3.2.2. Constructing relationships about angles: How is it changing?

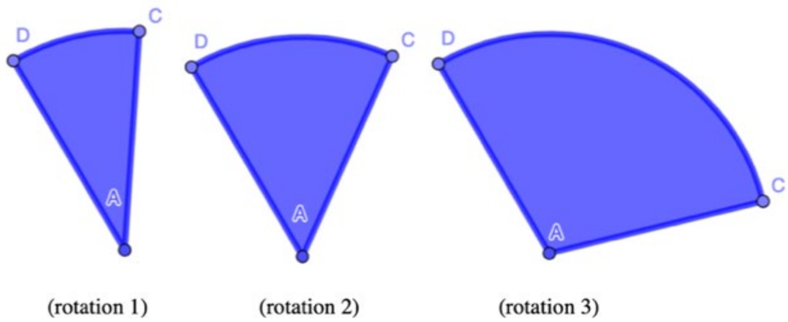
In exploring the DYME for volume, Panorkou (2021) found three forms of reasoning, namely reasoning about the quantities involved in the generation of 2D and 3D space, reasoning about the multiplicative change of those quantities, and coordinating the change in those quantities. Students from the DYME study did not only conceive volume through the generation process and identify the quantities that are changing, but they also have constructed relationships between those quantities. Similarly, students may also construct relationships between the quantities in an angle situation. For instance, in conceiving an angle with rotated rays, students may reason about the angle generation as getting wider, more open, or bigger. When the generation is reversed such that the rays approach each other, the angle measure is decreasing, approaching zero. From this reverse action, students may reason about angles as getting thinner, narrower, less open, or smaller.

Students may coordinate the change in one quantity with the change in another quantity by using covariational reasoning. Covariation is the relationship of change in quantities, describing how one quantity varies in relation to another varying quantity (Confrey & Smith, 1995). Coordinating two quantities changing while focusing on how they change in relation to each other is what research refers to as *covariational reasoning* (Confrey & Smith, 1995; Thompson & Carlson, 2017). Covariational reasoning may be non-numeric or numeric. Smith and Thompson (2007) argued that reasoning about quantities does not necessarily require numerical values. In non-numeric reasoning, students are not required to carry out the numerical measurement. Instead, they only need to coordinate the direction of change and amount of change in quantities.

In the exploration of dynamic angles, students may engage in covariational reasoning as they construct the relationship between the change in quantity involved in the generation of angles with the change in other quantities. For example, students may reason about the amount of rotation as a magnitude and how this rotation relates to the wedge it creates as another magnitude, and not on specific values. As students increase the amount of rotation for $\overline{AC}$, the space inside the wedge *DAC* also increases (Figure 18). As students construct covariational relationships between quantities involved in this example, I conjecture that they may also not exhibit the alternative conception of associating the side length with the change in the size of the angle. A series of rotations may offer students the idea that the length of $\overline{AC}$ did not change as the amount of rotation and the size of the wedge did.

Figure 18

An Illustration of Increasing Rotation and Size of the Wedge



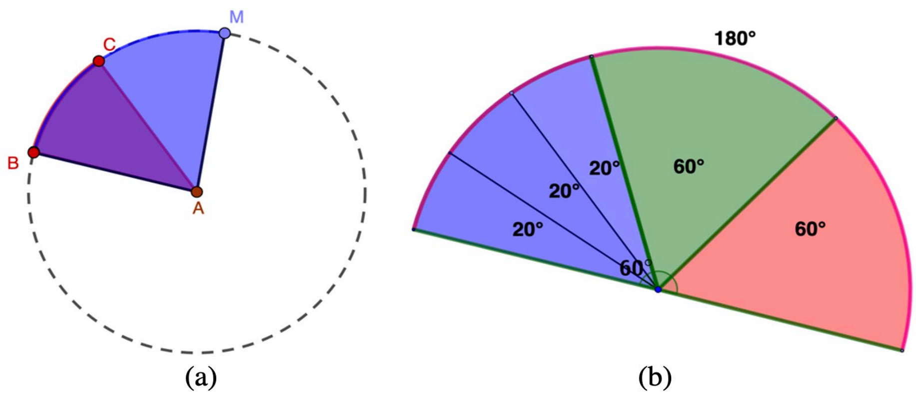
As students mentally coordinate the change in quantities involved in generating angles, they may exhibit chunky or smooth images of change (Castillo-Garsow et al., 2013). Reasoning about the multiplicative change of the quantities illustrates a chunky form of reasoning. This form of reasoning may provide students a space to use the values in constructing multiplicative relationships between quantities involved in generating an angle.

3.3. Comparing Angles

The previous section focuses on the possible ways that students may conceive the generation of a single angle and how they may reason about what is changing and how quantities are changing in that angle. It is also interesting to examine students' conceptions as they compare the sizes of angles (Figure 19). Students may construct multiplicative relationships as they compare two angles. They may conceive the reciprocal relationship of the relative size of two quantities (Thompson & Saldanha, 2003).

Figure 19

Comparing Angles Multiplicatively



Note: Students may compare angles multiplicatively (a) when an angle is $1/n$ the size of another angle or n times bigger than the other, or (b) when an angle is a product of groups of unit angles.

For instance, reasoning that the size of $\angle BAC$ is $1/n$ the size of $\angle BAM$ means that $\angle BAM$ is n times bigger than $\angle BAC$ (Figure 19a). To examine students' multiplicative reasoning about the change in angle measure, I also consider the definition of multiplication as coordinated measurement $N \cdot M = P$ formulated by Izsák and Beckmann (2019). This multiplication is applicable to situations involving a product quantity (P) that is simultaneously measured using two other units called base units and groups, where N is the number of base units that make one group, and M is the number of groups that make the exact product amount P. For instance, one may coordinate three base units of a 20°-angle to make 1 group of a 60°-angle, and then coordinate three groups of this 60°-angle to make a 180°-angle as shown in Figure 19b.

This multiplication of angles in degree units via coordinated measurement is what Hardison (2018) calls *extensive quantification* of angles via units coordination. The units coordination of angles involves conceptualizing the openness of an angle and the number of angle base units needed to cover the openness. Geometrically, this is what Battista (2004) defines as *composite units*. Students may construct composite units to reason multiplicatively about angles. For example, the iteration of a 1°-angle 45 times creates a 45°-angle. This 45°-angle may serve as a composite unit for creating larger angles, such as iterating it three times to create a 135°-angle. The 135°-angle is a composite of three 45°-angles but also a composite of 135 groups of 1°-angles. Students may reason that an angle unit is created through an iteration of smaller angle units. If students engage in multiplicative operations, they may successfully work with different levels of units (Reynolds & Wheatley, 1996; Steffe, 1992). Students' reasoning about units coordination is an example of a form of reasoning that describes the interplay between the multiplicative and geometric perspectives of dynamic angles.

3.4. Concluding remarks

The purpose of this chapter was to set the theoretical basis that could guide the design of mathematical activities that could be used for developing elementary school students' reasoning about angles, and for characterizing the forms of reasoning of angles that students could exhibit as they engage with those mathematical activities. I discussed how the notion of DYME in generating angles could bridge the three angle conceptions making learning about angles more accessible to students. I also discuss how DYME could support student reasoning about angles. Reasoning about angles involves both reasoning geometrically and multiplicatively and this chapter presented how quantitative reasoning could be utilized for describing this interplay. Thompson (2013) argued that quantitative reasoning is important in the development of mathematical meaning because it is established in the conception of situations that are experienced by students. Quantitative reasoning as a lens involves examining the ways that students perceive angle situations, construct quantities that are changing in these situations, and also construct relationships about how these quantities change simultaneously. Since quantitative reasoning is established in the conception of situations that are often experienced by students, Thompson (2013) argued that quantitative reasoning is important in the development of mathematical meaning. In terms of angles, I conjectured that mathematical activities designed with the lenses of DYME and quantitative reasoning in mind could engage students in constructing their mathematical meaning about angles and avoid the difficulties and alternative conceptions found in the literature.

Chapter 4: Methodology

The goal of my dissertation was to develop mathematical tasks and examine how students could reason as they engage with my design. In this chapter, I discuss my research methodology, which is the design experiment (Barab & Squire, 2004; Brown, 1992; Cobb et al., 2003). I describe the characteristics of design experiments and then discuss the three phases of my design experiment, namely, the design and conjectures phase, the data collection phase, and the data analysis phase. In the design and conjecture phase, I discuss the initial task design, and my conjectures on how the design could help achieve the intended goal, which was for students to construct reasoning about angles as discussed in the Theoretical Framework. In the data collection phase, I describe the participants, the research setting, and the methods for collecting the data. Finally, in the data analysis phase, I discuss the framework that I used to analyze the data to seek answers to the following research questions:

1. What forms of reasoning do students exhibit as they engage in dynamic digital tasks that aim to bridge the three conceptions of angles?
2. What characteristics of the design (e.g., characteristics of tasks, tools, and questioning) support the particular forms of students' reasoning for angles?
3. How did the design evolve to support students' reasoning for angles?

4.1. Design Experiment

The primary principle of the design experiment methodology is the design of educational interventions to engineer particular forms of learning and the study of those forms of learning with the goal of supporting them (Cobb et al., 2003). Design experiments also aim to develop learning theories and inform pedagogical practices (Barab & Squire, 2004; Cobb et al., 2003; Schoenfeld, 2006). Additionally, this methodology focuses on the development and evaluation of educational interventions (e.g., Anderson & Shattuck, 2012; Cobb et al., 2003; Plomp, 2013). In my research, I aimed to engineer particular ways of student reasoning about angles and to study students' forms of reasoning as they engage with my design. I conjectured that the design of my tasks, tools, and questioning could significantly influence students' construction of reasoning about angles. By conducting a design experiment, I aimed to evaluate this educational intervention with the goal to develop a learning theory about students' reasoning of angles, which in return could inform pedagogical practices.

The following paragraphs describe some characteristics of the design experiment methodology, namely being systematic, iterative, and authentic, that are the prominent features adopted in my research.

4.1.1. Systematic

Schoenfeld (2006) described the design experiment as a test-bed for innovation and argued that a design experiment is conducted with a goal toward the systematic data generation and examination and theory refinement. In a design experiment, researchers systematically study the relationships and interactions between design interventions and the impact of those interventions in learning that takes place in naturalistic but complex settings (Cobb et al., 2003; Cobb et al., 2001; Schoenfeld, 2006). For instance, Cobb et al. (2001) systematically worked through their data by continually testing and revising their conjectures. This means that the design experiment is a systematic way of constructing designs and testing and refining both the theory about learning and the design. In this dissertation, I anticipated that my design could evolve to support students' quantitative reasoning as discussed in the Theoretical Framework chapter. Furthermore, I studied how my design and student reasoning about angles co-evolve. By following how my design influenced students reasoning, I pointed out the affordances and drawbacks of my design and suggested possible ways to develop it in future iterations.

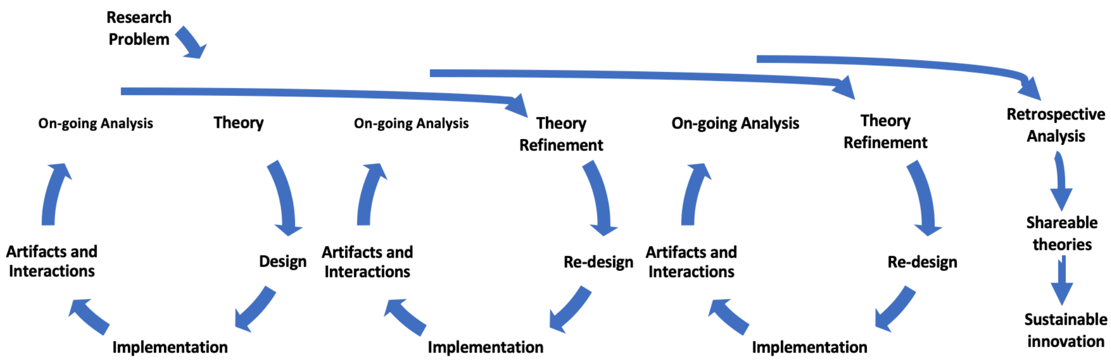
4.1.2. Iterative

One unique quality of a design experiment is its nonlinear cycles of design, implementation, analysis, and redesign (Anderson & Shattuck, 2012; Cobb et al., 2003; Cobb et al., 2001; Design Based Research Collective [DBRC], 2003). These iterations are individual experiments of systematic design modifications to generate an effective intervention (Barab & Squire, 2004). In other words, the design experiment undergoes an ongoing, recursive design process which offers more flexibility in generating a better and desired output than traditional experimental methods (Wang & Hannafin, 2005). According to the DBRC (2003), innovative interventions also involve multiple iterative steps. The first step is theory development for hypothetical solutions. Then, researchers design and re-design interventions based on the initial theory. Next, a thought experiment is implemented in classroom settings according to the hypothetical solution and design. Then, design researchers analyze the artifacts for theory refinement, achieve satisfactory design intervention, and explain how the evolution of design might support students' learning.

Figure 20 illustrates how design researchers first identify the problem and explore existing theories that can potentially address some elements of the problem (Anderson & Shattuck, 2012; Brown, 1992). Theories, which design researchers describe as humble theories, are developed on domain-specific learning processes and used to inform the design of the intervention (Cobb et al., 2003). Humble theories are often presented as instructional activities, associated materials, and the norms and discourse that are existing in a classroom setting (Cobb & Gravemeijer, 2008).

Figure 20

The Stages of the Design Experiment



In this study, my humble theories included engaging students with tasks that bridged the three angle conceptions that could support their reasoning about dynamic angles. I also conjectured that my design could help students avoid exhibiting alternative conceptions about angles that I discussed in the Literature Review chapter. The Design and Conjectures section offers additional details on my design and humble theories (conjectures).

4.1.3. Authentic

Another characteristic of a design experiment is that educational interventions are implemented in authentic settings such as classrooms (Anderson & Shattuck, 2012; Barab & Squire, 2004; Brown, 1992; Design Based Research Collective [DBRC], 2003). Design experiments also examine classroom interactions, an immediate feature of the setting (DBRC, 2003). According to Brown (1992), the realistic interactions in a classroom delineate why interventions work and make them reliable and reproducible designs situated in authentic contexts. This study was conducted during the global pandemic when classroom settings and interactions with students were limited to virtual platforms. Considering this limitation, I employed a series of virtual in-person interviews with students through Zoom (https://zoom.us/). Students in this study experienced meeting with their classes virtually that they were already familiar with the virtual classroom norms such as using a computer, its tools, and sharing their computer screens.

4.2. Design and Conjectures

The purpose of this study was to design dynamic digital tasks that aim to bridge the three conceptions of angles and examine students' possible forms of reasoning as they engage in these tasks. My main conjecture was that by engaging with my design, students could reason about angles generated dynamically. Specifically, I conjectured that they could identify the changing quantities in situations modeling an angle as a union of rays, a rotation, and a wedge and that they may reason about the relationships between those quantities.

Student reasoning about dynamic angles can be best supported using digital technology. As discussed in the literature review, several studies show that digital technologies can offer dynamic explorations of generating angles and changing angle measures. This dynamic exploration could support students to bridge the three angle conceptions and recognize the quantities involved. It could also support them in making conjectures about the relationships between quantities and use the immediate feedback that these technologies provide to verify or refine their conjectures. This kind of exploration could help mitigate difficulties that students encounter from working with static illustrations only (e.g., Browning et al., 2007; Clements & Burns, 2000; Smith et al., 2014). For instance, Smith et al. (2014) reported that students associated their arm sweeping movement with the change in an angle size when they engaged in activities using a digital software which would be difficult for students if using static images.

In addition to using digital technology as a tool for students to explore angles dynamically, I also considered the *dynamic measurement* (DYME) approach (Panorkou, 2017, 2021) to illustrate the *generation* of angles in my tasks design. After exploring the construction of angle tasks in different digital software, I chose the sweeping, tracing, and measurement feedback offered by GeoGebra (https://www.geogebra.org) as the most suitable tools to help students construct their reasoning about angles. The goal of my design was for students to dynamically generate angles and the sweeping tool may help achieve this objective. The tracing tool could help students visualize the angles they generated. This visualization could also aid students to conceive the quantities involved in the generation process and construct relationships between those quantities. The measurement feature of the GeoGebra could help students verify the relationships they constructed.

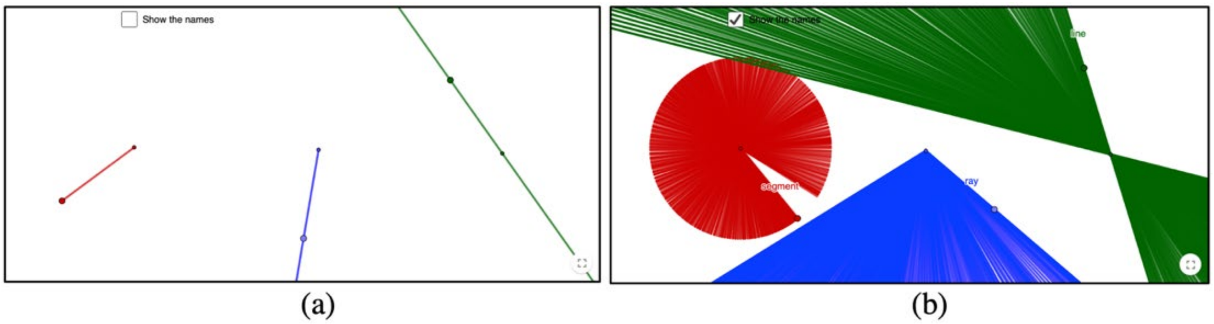
In the following two sub-sections, I used the two forms of quantitative reasoning discussed in my theoretical framework – what is changing and how it is changing – to present the three sets of tasks that I designed (https://www.geogebra.org/m/axvjtjxm). I also discuss my conjectures about student reasoning in each task. The first sub-section describes the Task Set 1 and how students could conceive an angle situation and identify quantities that are changing. The second sub-section presents Tasks Sets 2 and 3 and discuss the possible ways that students could reason about the change of quantities and the relationships between these quantities.

4.2.1. Task Set 1: Conceiving angle situations and identifying quantities (What is changing?)

Task Set 1 consisted of tasks designed to help students explore how they could generate angles by rotating angle sides and identify the quantities involved in this generation. In this section, I present three examples of these tasks. First, in the "Three Pairs of Different Objects" task, I designed a red segment, a blue ray, and a green line that can be rotated around in a full circle (Figure 21). The goal of this first task was for students to explore the program and differentiate between a segment, a ray, and a line. I conjectured that this task could provoke them to recognize that the three geometric objects can be used to generate angles.

Figure 21

Illustrations of the Three Pairs of Different Objects Task

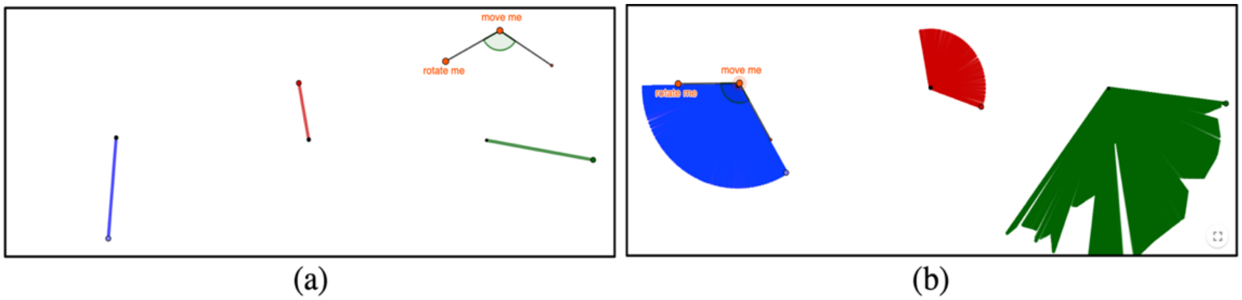


Note. The "Three Pairs of Different Objects" task as shown in (a) and after the generation of three angles via rotation of a segment, a ray, and a line as shown in (b).

In the "Comparing Angles with a Fixed Angle" task, the goal was to help students reason that the side length is not relevant to the size of the angle being generated. This kind of reasoning does not illustrate the alternative conception of associating the side length with the angle size as discussed in the literature review. To achieve this goal, I designed three angles that could be generated with the same openness but have different side lengths (Figure 22). The purpose of this design was to enable students to compare angles with side lengths that do not change and an angle with side length that changes. I first asked students to explore the task. I conjectured that students could create angles with the same or different amount of openness. Then, I asked them questions such as "What is changing in each?" and "What stays the same in each?" to provoke them to identify the quantities that were changing and quantities that were not changing. I also asked them questions such as "How are they the same?" and "How are they different?" to further examine if they could identify other quantities involved in the generation of angles. If students created all angles with the same amount of openness, they could use the fixed angle object (Figure 22a) to verify the equality of their openness. Otherwise, students could generate angles of different measures and reason that these measures make the angles different from each other.

Figure 22

Illustrations of the Comparing Angles with a Fixed Angle Task

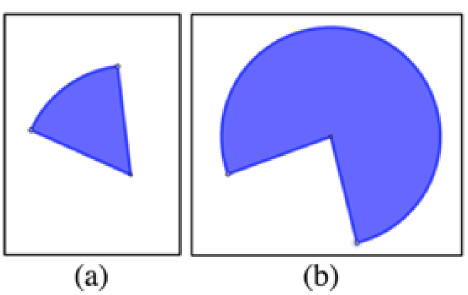


Note. The "Comparing Angles with a Fixed Angle" task as shown in (a) and after the generation of three angles of the same size via rotation of different side lengths as shown in (b).

The "Blue Wedge" task aimed to bring in the conception of angles as wedges (Figure 23). In this task, students could drag one or both angles sides, and identify the openness, the amount of rotation of one side or both sides of the angle, and the space inside as the quantities that change. Students could also recognize the side length as the quantity that did not change.

Figure 23

Illustrations of the Blue Wedge Task



Note. The "Blue Wedge" task as shown in (a) and after the generation of an angle as a wedge via dragging the angle sides as shown in (b).

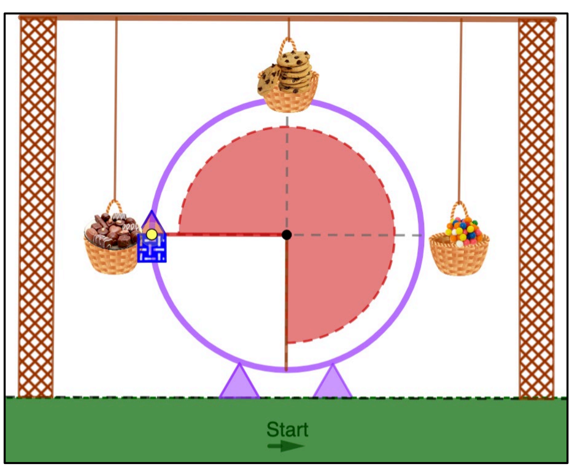
In this task, I asked students "What makes this different from other tasks?" to help them reason about the difference between angles generated by tracing the rotation and angles as wedges.

4.2.2. Task Sets 2-3: Constructing relationships about angles (How is it changing?)

The primary goal of Task Sets 2 and 3 was to examine students' reasoning about the relationships between the quantities. To begin with, the primary goal of Task Set 2 (Multiplicative Comparison) was for students to construct multiplicative relationships when comparing the openness of angles. In the "Ferris Wheel" task, I designed a cart that could go around the Ferris wheel (Figure 24). In each consecutive quarter turns from the starting point, the cart aligned with candies, cookies, and chocolates, respectively. The goal of this task was for students to reason multiplicatively about angles as a quarter of a full turn. For example, I asked students questions "How much of a turn will make the cart from Start to Start?", "How much of a turn will make the cart reach for the cookies?", "How much of a turn will make the cart reach for the candies?", and "How much of a turn will make the cart reach for the chocolates?" This series of questioning could provoke students to associate a complete turn with a whole circle, half a turn with half a circle, a quarter of a turn with a quarter of a circle, and three-quarters of a turn with three-quarters of a circle, respectively (e.g., Confrey et al., 2012).

Figure 24

Illustration of the Ferris Wheel Task

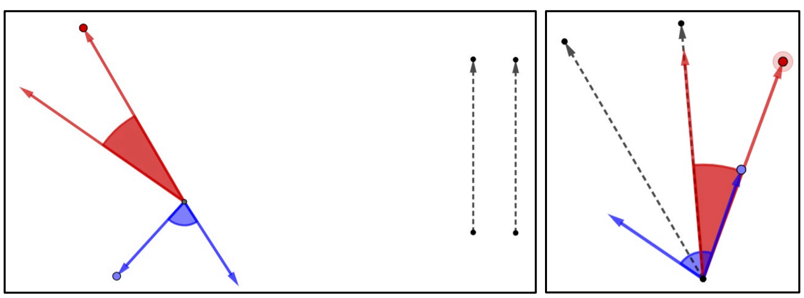


Note. This task shows an angle is generated as 3/4 of a full turn and a wedge of 3/4 of a circle.

In the “Comparing the Openness Between Two Angles” task, the goal was to examine how students may multiplicatively compare two angles of different openness. In this series of tasks, my goal was to provoke students to reason about two angles with one angle being three times (Figure 25), four times (Figure 26), or six times (Figure 27) bigger than the other angle. This progression was intended for students to reason from less sophisticated to more complex multiplicative reasoning about quantities (Thompson & Saldanha, 2003). I asked students questions “Which is more open?” or “How many times bigger is that angle than the other?” To help students identify these relationships, I created some supportive rays on the side that students could use as they iterate a smaller angle within the bigger angle.

Figure 25

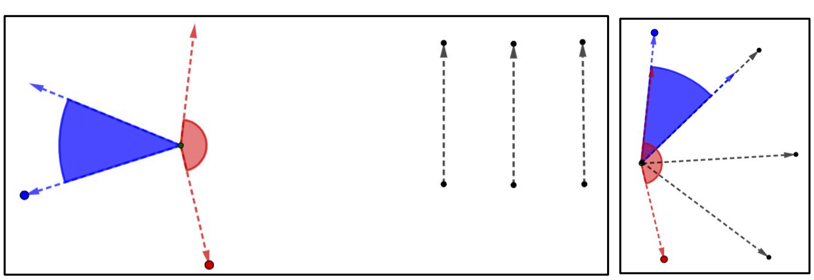
Illustration of Comparing an Angle Three Times Bigger than the Other Angle Task



Note. Comparing two angles where the blue angle is three times bigger than the red angle.

Figure 26

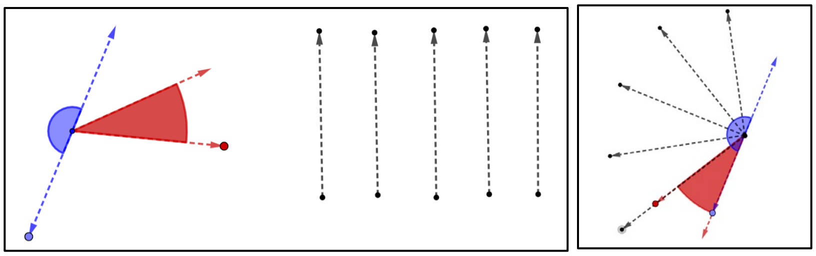
Illustration of Comparing an Angle Four Times Bigger than the Other Angle Task



Note. Comparing two angles where the red angle is four times bigger than the blue angle.

Figure 27

Illustration of Comparing an Angle Six Times Bigger than the Other Angle Task

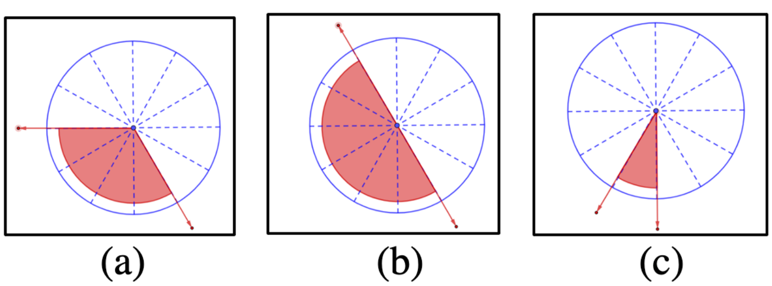


Note. Comparing two angles where the blue angle is six times bigger than the red angle.

The previous three tasks engaged students to compare two angles multiplicatively while the "Growing and Shrinking Angles" task engaged students to multiplicatively change one angle. In this task, I designed an angle and a circle with 12 equal partitions with the goal to prompt students in reasoning about the multiplicative change in an angle (Figure 28). Specifically, I asked them to use the circle to double the angle, triple the angle, make the angle four times bigger, and make the angle half than what it was.

Figure 28

Illustrations of the Growing and Shrinking Angles Task



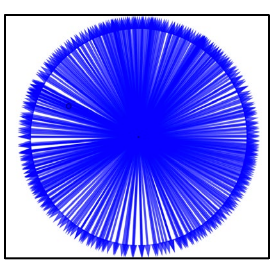
Note. The "Growing and Shrinking" task where an angle is changed multiplicatively such as (a) doubled, (b) tripled, or (c) halved than what it was.

In Task Set 3 (Numeric Angles), I aimed to examine students use of continuous quantitative reasoning (Castillo-Garsow, 2012) about an angle as a quantity. In this reasoning, students could recognize a smooth generation of an angle as a full rotation or imagine smaller chunks of angles within this full rotation. Then, students could repeat the process of imagining a smooth generation of intermediate angles within the smaller chunks. This kind of reasoning could lead students to construct multiplicative relationships between an angle and its intermediate angles. If a circle that represents a full rotation was split into finite numbers of n-degree angles, students could construct what Battista (2004) refers to as composite units to reason multiplicatively about angles.

In the "Many Very Small Angles" task, I designed a blue ray that could be dragged in a full rotation while leaving traces of the rotation. The goal was for students to recognize that the angle they generated was composed of an infinite number of smaller angles (Figure 29). I asked students "What did you create?" to examine their use of smooth or chunky thinking. Then, I asked students "How many of these did you create?" I conjectured that student reasoning could illustrate their thinking about an infinite (or uncountable) number of angles within a full rotation. This response could show the initial step to continuous quantitative reasoning (Castillo-Garsow, 2012). Students could use smooth thinking of change when they envision a change in progress until this change has been completed in the form of a full rotation, and visualize the smaller angles within the full rotation.

Figure 29

Illustration of the Many Very Small Angles Task

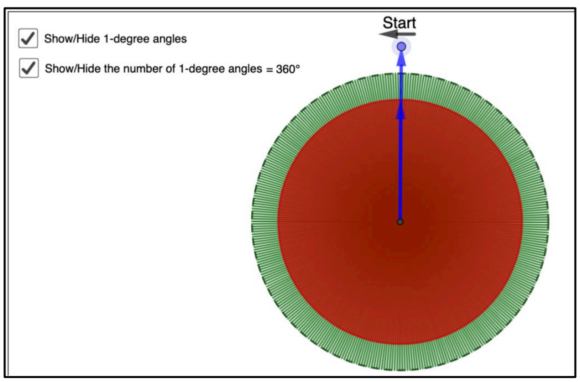


Note. The "Many Very Small Angles" task shows a full rotation with many very small angles.

In “360 Angles” task, I introduced degrees as a unit of measurement of an angle. I designed a circle split into 360 equal parts. Each part was a small angle that measures one-degree. This design aimed to help students discover that a full rotation is 360°. I asked students to drag the ray around the circle to make a full rotation. I also designed two checkboxes that students could use to show or hide the one-degree angles and the number of these one-degree angles that they could cover after dragging the blue ray (Figure 30).

Figure 30

Illustration of 360 Angles Task

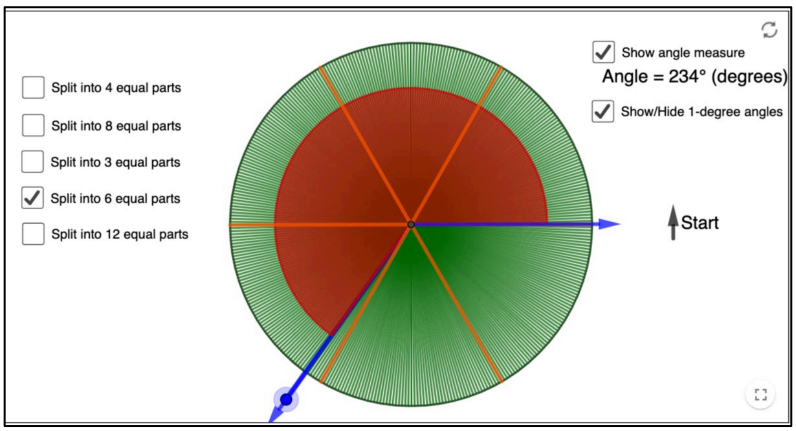


Note. The “360 Angles” task shows a full rotation has 360 one-degree angles.

In the “Splitting a Circle” task, the goal was for students to construct composite units of angles. To do this, I designed a circle that could be split into four equal parts, eight equal parts, three equal parts, six equal parts, and 12 equal parts (Figure 31). There were also two checkboxes that students could use to show the one-degree angles and the “show angle measure” tool to show the measure of the angle they created. I first asked students to recall the number of one-degree angles in a full rotation. Then, I engaged them in constructing their theory about the number of degrees in half a turn and a quarter of a turn. Also, students were asked to determine the number of degrees in each fraction if a full turn was either split into four equal parts, eight equal parts, three equal parts, or six equal parts, respectively. This task, therefore, engaged students in splitting 360° into how many equal parts was required. For instance, a student creates a 234-degree angle with a full rotation split into six equal parts (Figure 31). This student could reason that if a full rotation is composed of six 60-degree angles, then the angle created has four 60-degree angles minus six one-degree angles, or three 60-degree angles and 54 one-degree angles, illustrating a coordinated measurement approach to multiplication as described by Izsák and Beckmann (2019).

Figure 31

Illustration of the Splitting a Circle Task



Note. The "Splitting a Circle" task shows a circle is split into six equal parts and students could create a 234-degree angle.

In this section, I described some conjectures about the ways that students could reason when they engage with tasks that illustrate angles dynamically. Although students could reason about different aspects of the tasks, the questions that I designed guided them to focus on specific quantities and the relationships between the changes in these quantities.

4.3. Data Collection

In this section, I discuss the data collection phase of my study. I first describe the research participants and research setting. Then, I describe the method of gathering data.

4.3.1. Research Participants and Research Setting

The primary goal of my study was to examine how elementary students could reason about dynamic angles. To achieve my goal, I conducted design experiments with elementary school students. According to the Common Core State Standards for Mathematics (CCSSM) (NGA & CCSS, 2010), second-grade students are expected to recognize and draw shapes using the number of angles (CCSS.M.2.G.A.1) while students in fourth grade are formally introduced to language such as acute, right, or obtuse to classify different groups of angle measure (CCSS.M.4.G.A.1). Since second grade students have developed a familiarity about angles as part of geometric figures and students in fourth grade already developed a formal language in naming angles, it is essential to work with third-grade students because they are not yet provided with formal instructions of angles. It is possible that third-grade students have not encountered common or technical terms from textbooks pertaining to angles. Students at lower-grade levels could develop their *own* reasoning about angles without being influenced by the language provided in textbooks. In this dissertation, I worked with four third-grade students. My participants were recruited via personal pleas, emails, and social media posts. I had a diverse group of participants of different intellectual capacities, race, gender, and economic situations.

4.3.2. Audio, Video, and Screen recordings

I conducted individual design experiments with four third-grade students: Jordan, Angelie, Axel, and Alicia. In each design experiment, I employed a virtual interview with each student. The students were in their homes and their guardians were allowed to be present. The students already had experiences in meeting virtually with their classes and were familiar with using a computer, a virtual meeting application, and screen-sharing. Interviews were conducted outside class hours for 45 to 55 minutes each session for three to five sessions until students finished all the tasks. All interviews were audio-video and screen-recorded using the Camtasia Studio software. The audio and video recording were used to capture students' verbal interactions with my questioning and body gestures. The screen recordings were used to capture students' work on the screen. My dissertation advisor, Dr. Nicole Panorkou, joined me during the first few sessions with Jordan and Angelie to aid me in interviewing the students since the two students were interviewed one after the other within the same week.

4.4. Data Analysis

I conducted a series of four design experiments involving one third-grade student for each iteration. Each iteration of the design experiment was a macro-cycle of students' daily micro-cycle interactions with the tasks, tools, and researcher questioning. The unit of analysis in this research was students' qualitatively different ways of reasoning as they engage with my designed tasks, tools, and questioning. This analysis documented the reflexive relation between the ways that students reasoned and the design that supported those forms of reasoning. Table 3 summarizes the theoretical framework that I used to analyze students' quantitative reasoning about angles.

Table 3

Framework for Analyzing Students' Reasoning About Angles

Tasks	Theories on Student Reasoning
Part 1 SET 1 tasks What is changing?	***Bridging the three angle conceptions*** • Given an angle situation, students may conceive angles as a union of two angle sides, as a rotation, and as a wedge. - "I traced the rotation of a ray and I created an angle." - "I created many traces that cannot be counted." - "I created a circle out of a segment."
	Conceiving quantities • Identifying quantities that are changing and not changing.

	- “The openness changes but side lengths do not change.” - “The side length changes, but the openness stays the same.” - “The space inside the two sides changes and the amount of rotation also changes.” • Construct smooth and chunky images of change. - “There is a large number of angles in a full rotation.” - “There are 360 tiny wedges in a full rotation.”
Part 2 SET 2-3 tasks How is it changing?	***Comparing Quantities*** • Constructing relationships between quantities. - “The space between sides in one angle is larger than the other angle.” - “One angle is more open than the other angle.” ***Multiplicative reasoning*** • Reasoning about the multiplicative change in an angle. - “The angle is doubled than what it was before.” • Constructing multiplicative relationships between two angles. - “The blue angle is three times bigger than the red angle.” • Reasoning about angles in relation to a circle. - “The cart turned half a circle.” • Constructing composite units of angles. - “A 90-degree angle is composed of three 30-degree angles, and it can also be composed of 90 one-degree angles.”

4.4.1. Ongoing Analysis

Design experiments are implemented to generate data in the form of artifacts or interactions; then, these data are collected and analyzed (DBRC, 2003). After analyzing the data from the initial design experiment, researchers make refinements to the theory and designs for reimplementation (Anderson & Shattuck, 2012; Brown, 1992). The approach that a design experiment follows in analyzing a wealthy data from multiple iterations of design experiments is similar to Glaser and Strauss’ (1999) constant comparison method (Cobb et al., 2001). In an ongoing analysis, new data are compared with the current conjectured themes which leads to constant refinements of the overarching theories. The theory refinement, design, implementation, analysis, and stages are repeated in as many iterations until satisfactory design principles are achieved. In each micro-cycle of my design experiments, I conducted an ongoing analysis to examine emerging and reproducible patterns on students' reasoning within and across pairs as they engaged with my design. I took notes in every micro-cycle of the experiments to record in-the-moment analysis and possible modifications on the tasks, tools, and questioning for future iterations. Starting from the design experiment with one student, I monitored how specific elements of my design would support particular forms of students' reasoning about angles in the next iteration. Then, I repeated the processes of theorizing and refining the design of the tasks for the succeeding design experiments.

4.4.2. Retrospective analysis

After completing each macro-cycle of design experiment, I conducted two levels of retrospective analysis (Brown, 1992; Plomp, 2013) to study the data set and evaluate the theoretical basis and effectiveness of my design. In retrospect, I analyzed the chronological accounts of student reasoning and how their reasoning was influenced by my design. As for my data, all recordings were transcribed and chronologically analyzed using the framework I summarized in Table 3. First, I looked for student episodes where they bridged the three angle conceptions, namely, a union of ray, wedge, and rotation. I also identified student episodes of reasoning. For example, in Part 1 tasks, I looked for student excerpts that exhibited their qualitatively different ways of describing what they created when they dragged the rays and the quantities they may have recognized as changing. In Part 2 and 3 tasks, I looked for student reasoning about how the quantities they identified were related to other quantities. For instance, in multiplicative reasoning, I looked for student episodes when they compared angle measures as fractions of a circle. In similar multiplicative reasoning tasks, I looked for instances where students used their language for multiplication when comparing two angles or when comparing angle measures with the amount of rotation.

In the second part of the retrospective analysis, I analyzed how my tasks, tools, and questioning may have prompted students in constructing their reasoning. A retrospective analysis informs succeeding or new macrocycles of design experiments in new settings and with different participants (Plomp, 2013). With this principle in mind, I followed the evolution of my design and analyzed how the changes in the design may have supported students' reasoning.

4.5. Concluding Remarks

In this chapter, I discussed the methodology for examining students' quantitative reasoning about angles. I followed the design experiment methodology to develop the initial design of my tasks, tools, and questioning, which I hoped could help students construct their reasoning about dynamic angles. I followed three phases in my design experiment, namely, the design and conjectures phase, the data collection phase, and the data analysis phase. In the design and conjectures phase, I used quantitative reasoning (Thompson, 2011) and Dynamic Measurement (Panorkou, 2021) to design my tasks and conjecture how students could reason about angles. In the next two phases, I collected data during the micro-cycles and macro-cycles of the design experiments and studied students' reasoning that I presented in order to refine my task design.

Chapter 5: Findings – Part 1 (Cases-by-Case Analysis)

In this chapter, I present the case-by-case analysis of four individual design experiments with Jordan, Angelie, Axel, and Alicia. The goal of the design experiments was to examine the forms of reasoning that students exhibited as they engaged in dynamic digital tasks that aimed to bridge the three conceptions of angles: as union of rays, as rotations, and as wedges. I also aimed to study the characteristics of the design (tasks, tools, and questioning) that supported the particular forms of students' reasoning for angles and how the design evolved to support such reasoning. I analyzed my data into three phases.

In the first phase, I conducted ongoing analyses in each design experiment studying the chronological accounts of students' reasoning while engaging in the design of my tasks, tools, and questioning. I discuss the chronological account for each student considering their varied prior knowledge and how their prior knowledge might have influenced their forms of reasoning. After each experiment, I present the forms of reasoning of every student in a summary table outlining how each reasoned in four distinct categories: angle conceptions, multiplicative comparisons, discrete or continuous conception of an angle as a quantity, and numeric multiplicative reasoning about angles.

In the second phase, I conducted the first level of retrospective analysis at the end of each design experiment. I reflected on student's reasoning progression pointing to the tasks and questioning that potentially prompted different forms of reasoning among students. I also reflected on the changes on my design and conjectures in each iteration to follow the evolution of the design that support student reasoning.

The third phase is the second level of retrospective analysis which I discuss in the Findings – Part 2 (Cross-Case Analysis) chapter. In this phase, I cross-compared students' reasoning after all the design experiments were completed and discuss the similarities and differences in their reasoning. I also cross-compared the characteristics of the design (tasks, tools, and questioning) that potentially elicited different forms of reasoning about angles.

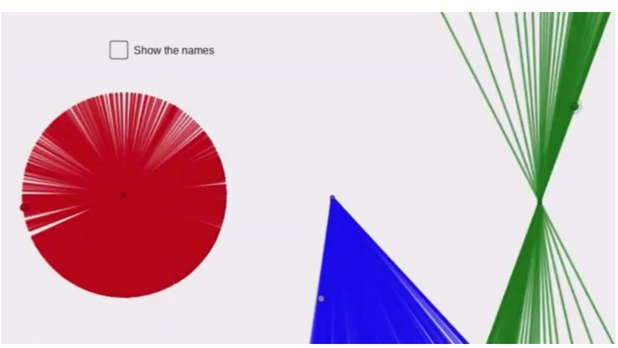
5.1. Case 1: Jordan

Jordan conceived angles as fractions of a circle, then he constructed multiplicative relationships between the sizes of angles by iterating the smaller angle within the bigger angle. I infer that he used this progression of reasoning to connect fractions of a circle with the degree measures of an angle. I describe this progression in detail in the succeeding paragraphs.

In the “Three Pairs of Different Objects” tasks (Figure 32), I asked Jordan to rotate the traceable segments to create wedges. He reasoned about the full rotations as circles and that they were “4/4 covered.” When he created 3/4 of a circle, Jordan described the wedge as “3/4 covered.” For Jordan, “3/4 mean that three parts out of four are covered,” showing that he probably imagined the full rotation as a circle is split into four equal parts. He also reasoned that the 4/4 wedge has more space “covered” than that of a 3/4 wedge.

Figure 32

The Three Pairs of Different Objects Task to Generate Angles



Likewise, in the “Blue Wedge” task in Figure 33, he created a quarter of a wedge and called it a “1/4.” Then, he iteratively rotated the side to make “2/4, 3/4, and 4/4,” respectively. I interpret Jordan's reasoning to show that he could associate the quarter rotations with the quarter sizes of a circle. When he was asked what was changing, Jordan stated, "there is also 90 kind of degrees on the side [creating an imaginary corner sign on the ¼ missing part of the wedge] because it also has the same thing as the other side [pointing at the ¾ blue wedge]." I infer that he used 90° to describe the 1/4 and 3/4 wedges. At this stage, Jordan could only associate a 90-degree angle for 1/4 and 3/4 wedges, probably because the 3/4 wedge has a 1/4 missing piece. Jordan's reasoning showed his preliminary conception of a quarter wedge and connected it with a 90-degree angle.

Figure 33

The Blue Wedge Task

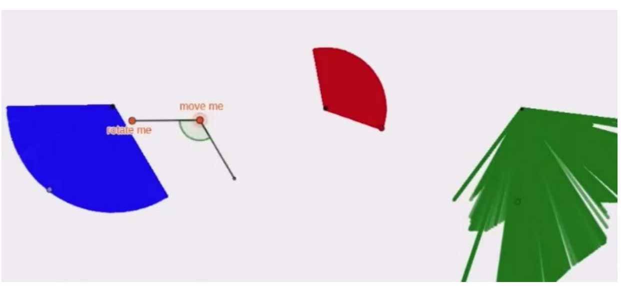
Note. Jordan associated the quarter rotations with the quarter sizes of a circle.

In "Comparing Angles with a Fixed Angle Object" task Jordan was asked to compare the sizes of wedges using an object of two connected segments with a fixed opening, and he called this object an angle as shown in Figure 34. For Jordan, "an angle is like a corner of a square of any, box thing like a square… the shape has to be closed at a corner." He seemed to refer to angles as corners where a shape is enclosed and that it has to be a right angle. For him, whenever he sees a "tiny square" symbol at the corner of a wedge, "it is 90 degrees," but Jordan could not explain what 90 degrees meant. As Jordan compared two angles of the same amount of opening but different side lengths and sizes of wedges, he focused on the space covered by each wedge. Jordan stated, "the blue one [wedge] covers more space and then, the red one covers less space."

His reasoning showed his alternative conception of an angle as the amount of space created by the blue and red wedges instead of looking at the openness. To further examine how he would reason about angles that he could change, Jordan was asked to go back to the "Three Pairs of Different Objects" task (Figure 32). He began to reason about the amount of rotation in creating an angle. To make a wedge smaller, he had to rotate less "because if you move past that [pointing at the rotating side], you make more space, that every time you [rotate], it gets a little bit bigger." The rotation motion seemed to offer him a constructive experience to reason about the size of wedge and amount of space changing dependently on the amount of rotation. I also infer that this task probably continued to support Jordan's alternative conception of angle as the amount of space when comparing angles as wedges. I conjectured that by removing the wedges in the next task could help him compare angles in terms of openness.

Figure 34

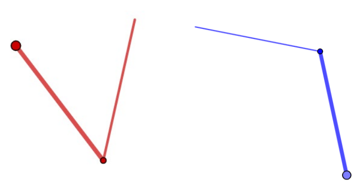
The Comparing Angles with a Fixed Angle Object Task



To examine whether Jordan could connect the amount of space between angle sides with the openness of an angle without a visual reference to the wedge, I designed an additional task, "Comparing Two Non-Wedge Angles", that involved two angles of the same side lengths but different openness as shown in Figure 35. He was asked to compare the openness between the two angles. Jordan reasoned that the blue object was more open because it "has more blank space" while the red object "has less blank space." He further explained that the blue angle was “more open” because there was “more space" between the two sides. When he was asked what he was looking at to compare the angles, he stated, “I’m looking at the line where you move it like this [rotating one side of the blue angle].” Then, when he was asked how he knew the blue object was more open, he replied “because when I look at it, I try to imagine like from there [pointing at one blue segment] to there [pointing at the other blue segment], and I imagine it’s going to stop right there. So now I know this [blue angle] one is more open.” I infer that as his effort to “imagine” the rotation of one side from the other side was showing that he was creating a connection between the amount of rotation and the amount of openness of an angle. From Jordan’s exploration, he seemed to identify the space between the two angle sides and the amount of openness as the changing quantities while changing the amount of rotation. I also interpret this reasoning to show that he connected the amount and the direction of rotation with the openness between the two angle sides. Probably, the progression of the tasks from involving wedges into tasks without wedges prompted Jordan to construct his generalization about the amount of rotation and the openness of angles.

Figure 35

The Comparing Two Non-Wedge Angles Task



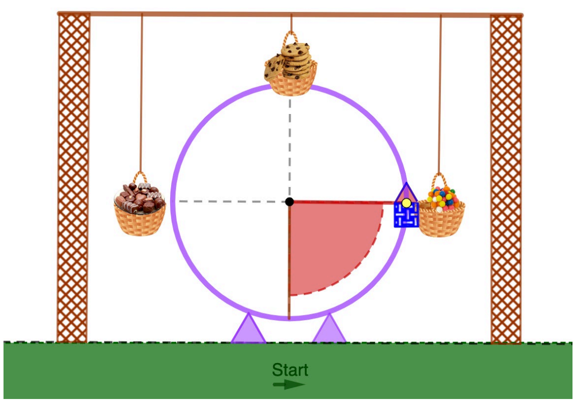
Jordan developed this reasoning further in tasks involving angles with multiplicative and numeric relationships. In the “Ferris Wheel” task in Figure 36, I asked Jordan about the amount of angle that he had to turn the cart to reach for the jellybeans and he stated that, “You have to do ¼.” When I asked him to show how he knew it was ¼, he reasoned about the vertical and horizontal lines that divided the circular Ferris wheel “by making it equal” (four equal parts).

Next, when I asked Jordan the amount of turn he needed to reach for the brownies, he stated that "You have to do ¾." Then, he was asked about the amount of turn to go back to Start, Jordan reasoned about the rotation in four fourths to define the amount of full turn the cart had to make.

> Jordan: [Rotated the cart back to Start.] That is 4/4 because this is 1/4, 2/4, 3/4 and then this is 4/4 [while pointing at each quarter part from the bottom right of the wedge, counterclockwise]. So now, it's also split into four parts, which also helps. So now I know, it's 4/4.

Figure 36

The Ferris Wheel Task



Note. Jordan turned the cart one-fourth around the "Ferris Wheel" task.

Jordan was also able to reason about rotations in fourths when the beginning position of the cart was not from the Start position. For example, he stated, "if we start from here [pointing at the jellybeans position], it will be 1/4 [rotating the cart from the jellybean candies to cookies]." When I asked him how much turn he would have to make from the jellybean candies to the brownies, he reasoned,

> Jordan: [Rotated the cart from cookies to brownies] I think 2/4, because when I counted this line here [pointing at the vertical dotted line], so it helps me split them up into two. And the answer is two because I just said it. And but if we start

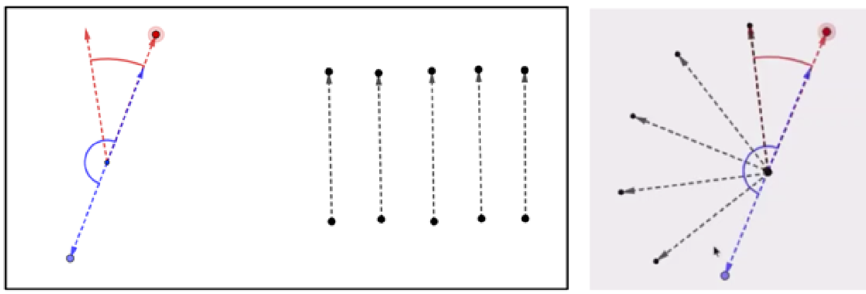
> the thing from here [rotating the side without the cart from jellybeans to Start], it will take 3/4. Because when I look at the model, there's also a line here [pointing at the horizontal dotted line], so that splits the thing into 3/4.

I infer that his visualization of splitting a circle into four parts may have supported his reasoning in the "Ferris Wheel" task. However, he struggled when was asked to rotate 5/4. Jordan reasoned, "that is going to be off this [quarter partitions in Ferris wheel] because they can only fit four." This shows that Jordan has difficulty in visualizing angles bigger than a full turn. It is also important to note that Jordan seemed to intuitively use the amount of rotation with the size of wedges in a circle.

Next, I asked him to work on the "Comparing the Openness between Two Angles" task (Figure 37) to examine how he would reason about one angle more open than the other angle. A set of black arrows was provided for Jordan to use as markers to iterate the smaller angle within the bigger angle. For example, in the task presented in Figure 37a, Jordan was asked to compare the blue and red angles' openness. He stated that "the blue" angle is more open and reasoned, "because I compare every line of, that makes it more, of it more open." Jordan probably mentally imagined the number of times he could iterate the smaller angle within the bigger angle as he experienced in the prior tasks. When I asked him to estimate the number of times the blue angle was more open than the red, he conjectured that the blue angle is "four times more open." Then, I asked Jordan to show how the blue angle is four times more open than the red angle. Jordan rotated the red angle, iterating it within the blue angle showing the iterations using the black arrows as illustrated in Figure 37b. Then, he counted up to six iterations and explained that "every time when you do this, it counts as one space [rotating the red angle to show one iteration within the blue angle]." By counting the iterations of the red angle he was able to observe that the red angle fits six times instead of four times that was his initial conjecture. His reasoning illustrated his decomposition of the bigger angle to compare its openness with the smaller angle and the actual iterative generation of space using the smaller angle to compose the bigger angle.

Figure 37

The Comparing the Openness between Two Angles Task



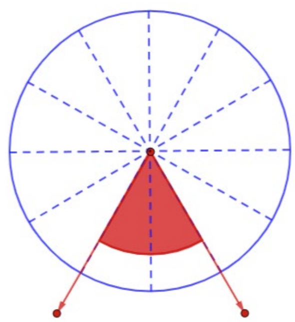
(a) (b)

Note. (a) Task for Jordan to conjecture which openness is bigger, and (b) Jordan iterated the red angle within the blue angle to construct a multiplicative relationship between them.

In the "Growing and Shrinking Angles" task, my goal was to examine how Jordan would reason about a single angle growing and shrinking. The task involved an angle that formed a central angle within a circle split into twelve equal parts. I asked Jordan to double the given 2/12 angle (Figure 38). He explained that he counted the original angle as two while pointing at the two 1/12 wedges, then to double the angle, "it would be two [pointing at the two adjacent wedges on both sides]." For Jordan, doubling is to "add two more" pieces of 1/12. Then, I asked Jordan to triple the original 2/12 angle. He stated, "I'd add two more [rotated by 2/12 wedge more than the 4/12 wedge to create a 6/12 wedge] because when I do this, there will be one double here, because the three big things, count them by the big things." I infer that he was referring to the 2/12 wedge as "the big things" he added to the 4/12 wedge to create a triple of the original angle. Jordan's reasoning illustrated his use of additive iteration of the original angle to construct multiplicative changes. This construction of multiplicative changes through additive iterations of the original angle is similar to his iterative generation of space to compose a bigger angle that he exhibited in the “Ferris Wheel” and the “Comparing the Openness between Two Angles” tasks.

Figure 38

The Growing and Shrinking Angles Task

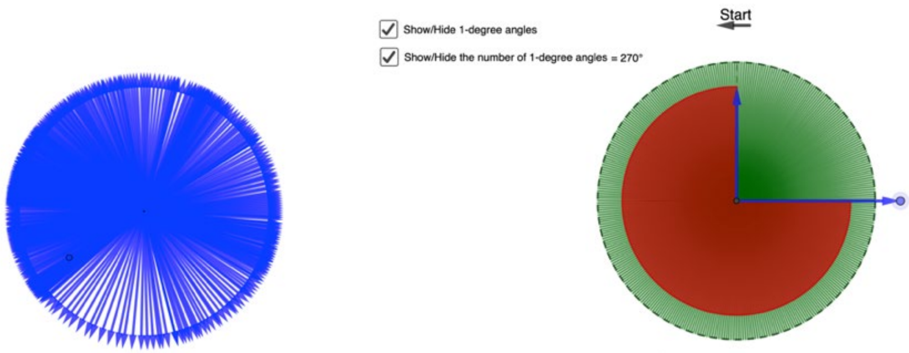


Note. A task for Jordan to reason multiplicatively about an angle shrinking and growing.

To examine how Jordan would reason about the very large number of very thin angles in a full rotation, I asked him to work on the “Many Very Small Angles” task with a traceable ray that can be rotated in a full circle (Figure 39a). Jordan rotated the traced ray in a full circle without being prompted, probably because he was accustomed to rotating rays or segments in the previous tasks. When I asked him what he was making, he stated, “I’m making a big circle,” and “there are spiky things on the outside.” I infer that Jordan was focusing on the rays of the very small angles he was creating. When I asked him how many of those he created, he said, “a lot” and “that would be too long” to count them. I interpret Jordan’s reasoning to show that he perceived a circle as having a very large number of “spiky things” or what we refer to as rays of the angles. His reasoning that there were “a lot” of angles illustrates a more sophisticated understanding of angle as a discrete quantity. The perception of a circle as being composed by many very thin angles may be considered to be the reversible mental action to recognizing that the circle can be split in any number of angles.

Figure 39

The "Many Very Small Angles" and "360 Angles" Tasks



(a) (b)

Note. (a) The "Many Very Small Angles" task where Jordan reasoned about the degree measure in a full rotation, and (b) the "360 Angles" task where he reasoned about the angle measure as a fraction of 360°.

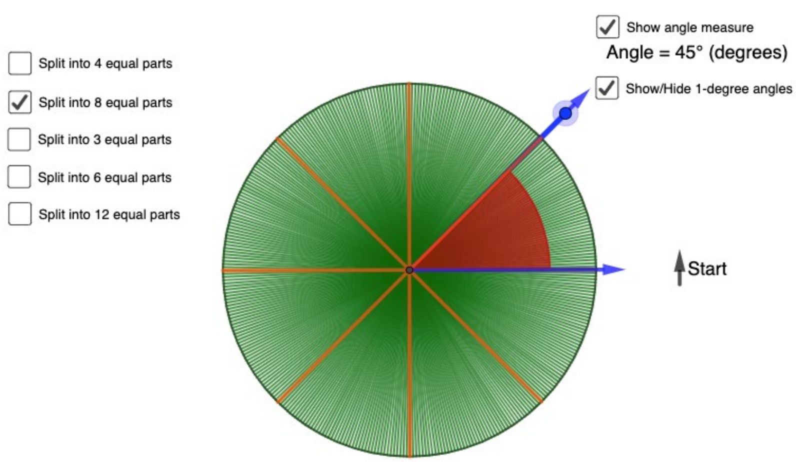
Next, I aimed to examine whether he could use this understanding of a very large number of very thin angles in a circle to connect the idea of the fractions of a circle to the degree measure. To introduce the idea that there are 360 one-degree angles in a full rotation, I designed a "show angle measure" tool that shows the degree measure within the rotation and a "show one-degree angle" that shows all the one-degree angles in a circle ("360 Angle" task, Figure 39b). When Jordan rotated the ray and created a ¼ wedge, half a wedge, a ¾ wedge, and a full wedge, I asked him to reason about the number of degrees of the angle he created. Similar to what he exhibited during the exploration tasks, Jordan associated 1/4 rotation with a 90-degree angle, but he could not explain how he knew it as 90°. However, when I asked Jordan about the number of degrees for two quarters of a circle, he stated that it would be 180 because "you're supposed to add another 90 every 1/4 you move." For the number of degrees in three quarters, Jordan immediately answered, "that would be 270 because I added another 90 to the 180, and that gave me 270. And 4/4 will be 360." Jordan's responses illustrated his use of iterative addition of 90°

to identify the degree measure for each quarter turn. He associated different degree measures with other quarter angles illustrating more complex reasoning than what he exhibited at the exploratory stage of the experiment.

To examine whether Jordan could identify the degree measure for every single partition of splitting a circle and not just 90°, I asked Jordan to split the circle into eight equal parts and reason about the degrees of the angle created in one part ("Splitting a Circle" task, Figure 40). When I asked Jordan how many degrees 1/8 was before he checked the "show angle measure" tool, he estimated, "I think it is 27" degrees. To prompt Jordan to identify the angle measure for an eighth, I asked him how 1/8 and 1/4 are related to each other. Jordan pointed at the 1/8 wedge and reasoned that 1/8 was "only halfway to 1/4." When I probed him again to identify the number of degrees for an eighth, he continued to reason that "it is 27" degrees. When I allowed Jordan to use the "show angle measure" tool to check his answer, he found the 45° answer to be reasonable explaining "that 45 plus 45 equals 90. Since it's 1/8, and if it's half, it has to be 45."

Figure 40

The Splitting a Circle into Eighths Task

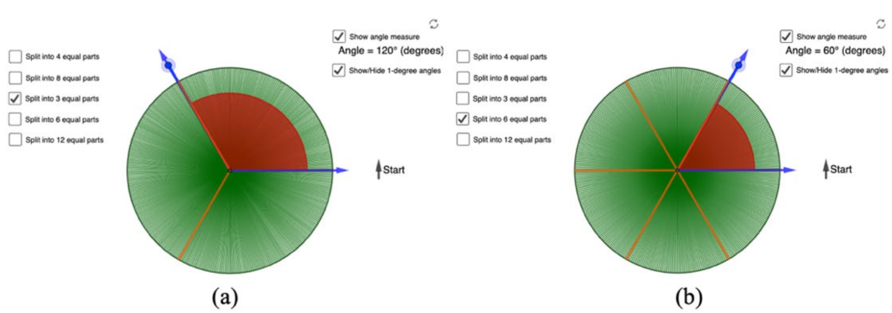


Note. Jordan to split the circle into eight equal parts and reasoned about the degrees in an eighth.

Next, I asked Jordan to split the circle into three equal parts (Figure 41a). Jordan used the "show angle measure" tool to know that 1/3 is 120°. When I challenged him to uncheck the tool and explain how he would get the number of degrees for 2/3 of the circle, he responded "240" and reasoned that "120 plus 120 which would give me 240." Jordan's use of the "show angle measure" tool to find the degrees for an eighth and a third helped him argue about the compositions of 90° as two groups of 45° and 240 as two groups of 120°. When I asked Jordan to split the circle into six equal parts (Figure 41b), he estimated a sixth of a circle as 57° "because it is a little bit higher than eight equal parts [45°]." Using the "show angle measure" tool, Jordan realized that a sixth is 60°, and he reasoned that he can "add 60 every time" to get the degrees for multiples of 1/6. From this exploration, he then recalled that a 1/3 wedge has 120°. So, he created a 2/6 wedge and showed that it is the same as 1/3 with 120°. These forms of reasoning illustrated Jordan's construction of multiplicative relationships between the fractional parts of splitting a full turn. This sequence of tasks also offered a constructive platform for Jordan to reason about equivalency between degrees and fractions of a circle.

Figure 41

The Splitting a Circle into Thirds and Sixths Tasks



Note. Jordan (a) split the circle into three equal parts and reasoned about the degrees of a third, and (b) split the circle into six equal parts and reasoned about the degrees of a sixth.

At the end of the experiment, I asked Jordan what he learned. He stated, "I learned that all circles are supposed to be 360 [degrees]. Angles have something like for example one-fourth. They have to be split in a certain way." Jordan's reasoning shows that he expanded his initial understanding of an angle as a corner of a shape and as fractions of a circle to include conceptions of angle as composed of 360° when generated through a full rotation. Additionally, Jordan's reasoning showed a conception that angles in a full rotation could be split into a number of partitions, similar to what he experienced from the Splitting a Circle task.

5.1.1. Retrospective Analysis on the First Iteration

In Table 4, I present a summary of Jordan's reasoning and the tasks that elicited his different forms of reasoning. In each form of reasoning that he exhibited, I created sub-categories that are important when comparing the levels of sophistication in reasoning among students in all design experiments.

Table 4

Jordan's Forms of Reasoning

Student Reasoning	Task
Angle Conception	
A. Angle as union of rays e.g., "an angle is like a corner of a square of any, box thing like a square… the shape has to be closed at a corner." **B. Angle as wedge** e.g., "The blue one covers more space and then, the red one covers less space." **C. Angle as rotation** e.g., "Because if you move past that [the rotating side], you make more space, that every time you [rotate], it gets a little bit bigger."	A. Comparing Angles with a Fixed Angle Object (Figure 34) B. Comparing Angles with a Fixed Angle Object (Figure 34) C. Three Pairs of Different Objects (Figure 32)
Multiplicative Comparisons	
A. Initial splitting strategy: Associated quarter rotations with the quarter sizes of a circle e.g., Reasoned that "3/4 mean that three parts [of a circle] out of four are covered." **B. Iteration and splitting strategies in reasoning about full turn**	A. Three Pairs of Different Objects (Figure 32) B. Ferriswheel (Figure 36)

e.g., "That is 4/4 because this is 1/4, 2/4, 3/4 and then this is 4/4 [wedge]. So now, it's also split into four parts, which also helps. So now I know, it's 4/4." **C. Decomposition and composition strategies to compare angles** e.g., the blue angle is "four times more open, every time when you do this [rotated one side], it [each iterated rotation] counts as one space." **D. Additive iteration to multiplicative changes an angle** e.g., Doubling is to "add two more" pieces of 1/12.	C. Comparing the Openness between Two Angles (Figure 37) D. Growing and Shrinking Angles (Figure 38)
Discrete/ Continuous Conception of Angle	
Reasoned about a circle as having a very large number of angles e.g., Reasoned about a discrete number of angles that there were "a lot" and "that would be too long" to count them.	Many Very Small Angles (Figure 39a)
Numeric Multiplicative	
A. A 90-degree benchmark to reason about quarters in a circle e.g., Two quarters of a circle "would be 180" because "you're supposed to add another 90 every 1/4 you move." **B. Composition of 90° as two groups of 45°** e.g., 1/8 was "only halfway to 1/4… that 45 plus 45 equals 90. Since it's 1/8, and if it's half, it has to be 45." **C. Composition of 240° as two groups of 120** e.g., 2/3 angle is "240... 120 plus 120 which would give me 240." **D. Multiples of sixths and its relationship with thirds** e.g., 1/6 is 60° and "add 60 every time" to get the degrees for multiples of 1/6 of a turn.	A. 360 Angles (Figure 39b) B. Splitting a Circle in Eighths (Figure 40) C. Splitting a Circle in Thirds (Figure 41a) D. Splitting a Circle in Sixths (Figure 41b)

Jordan's prior knowledge about angles as a corner of a shape was similar to what was found in the literature (e.g., Clements & Battista, 1989). Researchers interpreted student conception of angles as significantly attributed to the understanding of corners of a geometric shape that often progresses into defining angles as union of rays (Clements & Sarama, 2014). As Jordan engaged with tasks involving rotations of a segment that generate wedges, he reasoned about each quarter rotations as quarters of a circle. He needed to connect the output of his rotations with fractions of a circle probably because the wedges he generated potentially prompted him to express his prior knowledge about fractions of a circle. The wedges seemed to offer him a constructive space to imagine equipartitioning a full circle into quarters similar to what Browning et al. (2007) found that students used fractions of wedges to quantify angles. It is also possible that Jordan's prior knowledge about fractions of a circle played a vital role in exhibiting a form of multiplicative numeric reasoning at the beginning of the experiment. In reasoning further about a ¼ wedge, Jordan also brought up his prior knowledge about 90°, although he could not explain what 90° meant. He probably needed to connect what he conceived about the quarter wedges with the degrees. This progression in Jordan's reasoning is similar to what Confrey et al. (2012) supported, that students need to connect their conception of turns with fractions of a circle, then with angle measure in degrees.

Jordan's multiplicative reasoning about angles when he equipartitioned a circle into different number of parts illustrated a combination of decomposition and composition strategies. This reasoning started when he engaged with the "Ferris Wheel" task where Jordan seemed to combine the splitting and iterating strategies when he reasoned about the wedges created by the rotations of the Ferris wheel cart as turning by iteration of fourths into a whole turn while he imagined the whole turn being split into four equal parts. His reasoning about splitting a whole circle resulting into reasoning multiplicatively by fourths is an initial form of what Steffe (1992) referred to as constructing levels of units. It would be interesting to study how his reasoning is similar or different from other students who also worked at the "Ferris Wheel" task.

Jordan exhibited a similar reasoning in the "Comparing the Openness between Two Angles" task. He decomposed a bigger angle into equal smaller angle as his benchmarks and multiplicatively reasoned about the number of times he could compose a smaller angle into the bigger angle. Jordan's reasoning illustrated the reflexive relationship between the two interiorized measurement processes by envisioning the decomposition of an angle and iterating back the smaller angle into the bigger angle resulting into multiplicative comparison between the two angles (Moore, 2012).

Additionally, Jordan reasoned in ways that resonated with what Izsák and Beckmann (2019) described as a coordinated measurement approach to multiplication. In coordinated measurement, a product quantity is simultaneously measured using two other units, the number of units that make one group and the number of groups to make the product quantity. In the "Growing and Shrinking" Angles task, Jordan used the 2/12 wedge as the unit (group) which he iterated to create an angle consisting of two groups of the 2/12 wedge. In the analyses of the other students that follow, I further investigate whether this form of reasoning is also evident with other students at the same task.

Jordan progressed in reasoning about the decomposition and composition of an angle when he worked on the "Many Very Small Angles" task where he reasoned that a full rotation is decomposed of a "lot" of angles and that "it would be too long" to count them. His reasoning illustrated a more sophisticated understanding of an angle as a discrete quantity envisioning smaller chunks of angles within the full rotation. It is possible that the traces left while Jordan was rotating a segment in circle prompted him to reason about angle as a large but discrete quantity. This kind of reasoning reflects a chunky thinking (Castillo-Garsow, 2012). Furthermore, these chunky images of small angles might have potentially prompted Jordan to reason multiplicatively about the decomposition and composition of a full circle.

Similar to what Jordan exhibited at the "Three Pairs of Different Objects" task and the "Ferris Wheel" task, he reasoned about a full circle being split into four equal parts in the "360 Angles" task. This repetition of reasoning about the quarters of a circle is a more sophisticated illustration of the right angle conception represented by a quarter wedge that is prevalent among third-and-fourth grade students (Devichi & Munier, 2013). My questioning then prompted him to connect the quarters of a circle with degrees, where he reasoned that he added 90° "every one-fourth" of the rotation. Another way to interpret why Jordan exhibited this reasoning more frequently throughout the three different tasks was because of his prior knowledge about 90° angle, a right angle. Jordan built on this knowledge to reason further about the two quarters of a circle as 180°, three quarters of a circle as 270°, and four quarters of a circle as 360°. Prior to the experiment, I initially conjectured that students may need to use the "show angle measure" tool to prompt them in talking about the full rotation as containing 360°. In Jordan's case, we turned off the tool when he was reasoning about the degrees in quarter turns. It is more evident that Jordan used his prior knowledge about 90° to decompose the whole rotation into fourths and compose it back using the multiplicative iteration of 90° angle. It would be an interesting comparison on students' use of the "show angle measure" tool to probe their reasoning about the degrees in fractions of a turn.

In contrast to not using the "show angle measure" tool in the "360 Angles" task, Jordan needed the tool to make sense with the non-quarter fractions of a turn in the "Splitting a Circle in Eighths" task and "Splitting a Circle in Sixths" task. He exhibited loose estimations for 1/8 turn as 27° and 1/6 turn as 57°. Jordan probably had difficulty in identifying the degrees for these fractions of turns because he did not recognize their relationship to 90°. I conjectured that he would use his knowledge of 90° to successfully reason about the degrees of 1/8 of a turn. So, I prompted him to talk about the relationship between 1/8 and 1/4. Using the "show angle measure" tool to show the degrees for 1/8 as 45°, Jordan did not only successfully identify its degrees but he also exhibited a more complex form of reasoning about 90° as a composition of two groups of 45° because 1/8 is half of 1/4. He continued to express this composition reasoning with "Splitting a Circle in Thirds" and "Splitting a Circle in Sixths" tasks. Jordan exhibited these interiorized measurement processes of decomposition and composition of degrees to quantify angles (Moore, 2012). His reasoning about fractions of a circle and multiplicative relationships between angles seemed to support his bridging of angles as fractions of a circle and the degrees.

In terms of design, generating angles as wedges is seemingly important for Jordan to reason about the fractional partitions of a full rotation. However, these wedges often hindered Jordan from focusing on the openness of the angle. Throughout the experiment, Jordan's reasoning showed that he significantly associated angles with a fraction of a circle probably because most of the tasks involved wedges or traces that when he generated angles he created circles. Consequently, for the next iteration, I modified the tasks to remove the traces of segments on angles at the exploration stage. The generation of wedges were also removed from the "Growing and Shrinking Angles" tasks for the next iteration. I conjectured that this modification would prompt the students to reason about angles as openness prior to conceiving them as wedges. For the "Many Very Small Angles", "360 Angles", and "Splitting a Circle" tasks, I did not remove the traces tool and the wedges to investigate how other students would reason while working on the tasks with these features. Also, in the design experiments that followed, I restricted students from using the "show angle measure" tool before they created a conjecture about the degree measure of an angle.

Additionally, since Jordan defined angle as a corner of a shape, I decided that it would be interesting to investigate how Angelie, the next student, would reason about the corners of different shapes. To support students in developing an understanding of a corner as an angle that can change, I designed a "Triangle" task with vertices that students can drag to modify the size of an angle and prompted them to reason about how these angles change.

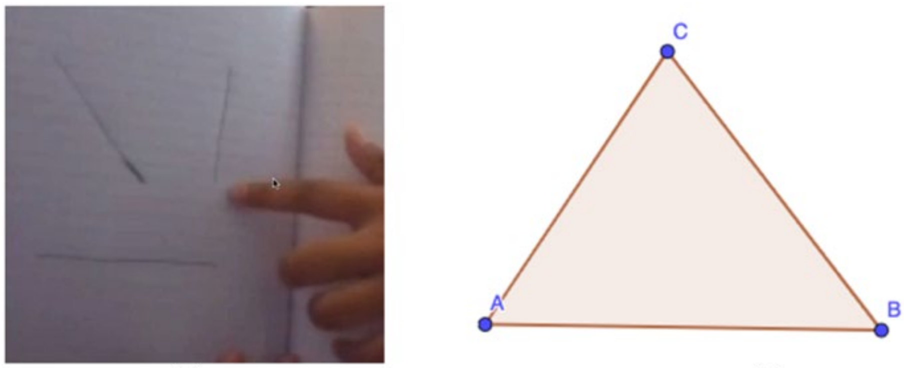
5.2. Case 2: Angelie

Angelie's prior knowledge of angles involved perceiving angles as illustrating orientations of lines inclining or pointing to different directions, which was different from Jordan's conception of angles as corners. After engaging in some exploratory tasks, her reasoning progressed into conceiving angles as the space between the lines and arguing that the side lengths are irrelevant when changing an angle. Similar to Jordan, the series of tasks and questioning also seemed to probe Angelie to reason about the multiplicative relationships between angles as fractions of a circle and then connect them to the respective degree measures. In the following paragraphs, I discuss Angelie's progression of reasoning during the design experiment in detail.

At the onset of the design experiment with Angelie, she was asked what an angle is. She stated that "an angle is like a way of how you see something or how it is drawn or standing… An angle is a way of how something looks. It could be diagonal, it could be vertical, sideways." I infer that Angelie viewed angles as illustrating orientation and offered different ways an object can be seen at an angle. When asked to draw an angle, Angelie drew a diagonal line, a vertical line, and a horizontal line (Figure 42a). From her drawings, I interpret that she was aware of three different orientations of lines, and she associated angles with these orientations. Another possible interpretation of her reasoning is that Angelie probably mentally imagined a horizontal line as the beginning of a rotation of a single ray. In the "Triangle" task (Figure 42b), I asked Angelie to identify the angles. She reasoned that "each line shows a different angle" while pointing at the lines in a triangle. I interpret that her reasoning and illustrations are influenced by her prior knowledge of angles as orientations of lines.

Figure 42

Angelie's Illustration of an Angle and the "Triangle" Task



(a) (b)

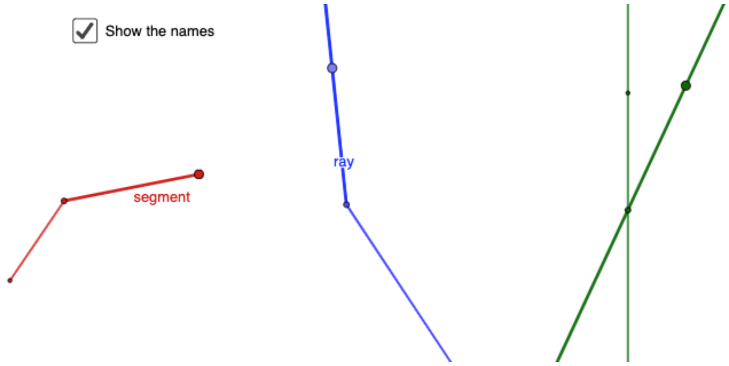
Note. (a) Angelie's drawings of different angles and (b) the "Triangle" task where Angelie identified the lines of a triangle as the angles.

As mentioned in the reflection of Jordan's design experiment, the "Three Pairs of Different Objects" task was modified in the second design experiment by removing the traces on the objects so that they do not form a wedge as shown in Figure 43. In this task, Angelie was asked to drag the red segment and describe what she was creating. Angelie responded, "a different angle line… if I move it around, it becomes a different way of drawing." Similar to her previous reasoning about angles as different orientations, she reasoned about the rotation of the line segment as a "different angle line" and "different way of drawing." When Angelie was asked to make a very big angle, she rotated the red segment further away from the other segment. To prompt Angelie to reason about the changes in an angle, she was asked to identify what she was looking at when making a big or a small angle, as shown in Figure 43. Angelie stated,

> Angelie: With the smaller angle, [rotated the red segment towards the other red segment] there's much smaller space between each line. But the bigger angle [rotated the red segment away from the other segment], there's much more space [pointed at the space between the two segments] between both of the line(s).

Figure 43

The Modified Three Pairs of Different Objects Task

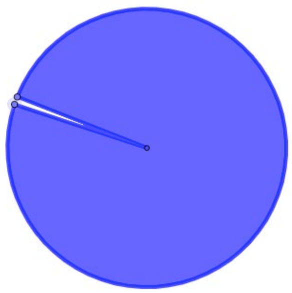


The questioning above seemed to probe Angelie to illustrate reasoning about angles as the space between the two lines. When she was prompted to identify what was changing, she reasoned that "what is changing is definitely the angles, because once you move the line farther away from the other line, the angle becomes larger; putting it closer to the other line will make it much smaller." I infer from Angelie's statement that she conceived an angle as how far or close the sides are from each other. When asked if this reasoning would also work for the blue and the green objects, she stated that "it will do the same thing for all of the lines…it does not matter the length." Angelie's reasoning showed that rotating an object (segment, ray, or line) closer or farther away from the other object would change the angle regardless of the lengths of the sides.

To explore whether Angelie could connect the size of an angle to the fractions of a circle, I asked her to explore the "Blue Wedge" task that she could grow or shrink a wedge to create different fractions of a circle (Figure 44). When asked how this task differed from the previous task, Angelie reasoned that "it is like making a circular space" by referring to the blue wedge shown on the screen. When Angelie was asked about the largest angle she could create, she rotated one side of the wedge and created a figure that was almost a whole circle. In this task, Angelie did not reason about angles in terms of fourths of a circle as Jordan did in the same task (Figure 33).

Figure 44

The "Blue Wedge" Task where Angelie Created a Figure that was Almost a Circle

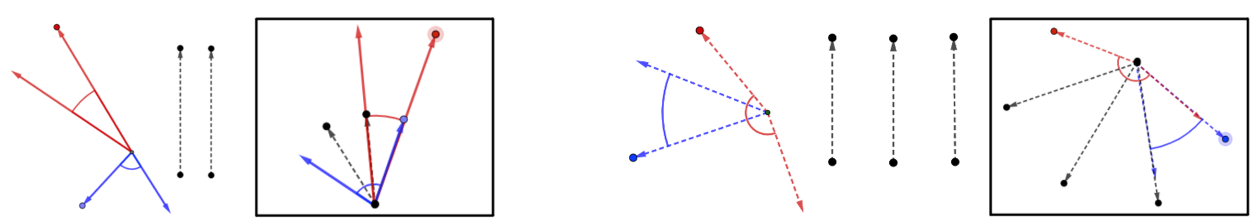


Angelie only began to reason about angles as fractions of a turn in a circle while working on the "Ferris Wheel" task (Figure 36). When she was asked to identify how much she needed to turn the cart from the Start to the jellybeans, she correctly stated, "you need a quarter of it…a quarter of this circle." I infer from her reasoning that she already has prior knowledge about fractions in fourths. However, this reasoning was not evident in the previous tasks until I engaged her with a task illustrating a circle split into four equal parts. When Angelie was probed to identify the angle she needed to turn the cart from the Start to the cookies, she responded, "You need half of the circle to get to the cookies." Next, when she was asked about the amount of turn she needed to get to the chocolate from the Start, Angelie reasoned that "To get to the chocolate, you'll need to use half of the circle and another quarter of it…or three quarters." Lastly, Angelie was asked to talk about the angle when she turns the cart from the Start and then back to the Start. She stated that, "It takes the whole circle or you could say four quarters." Her statements showed that she connected the generation of angles as fractions of a full rotation with the fractions of a circle.

Angelie constructed multiplicative relationships between two angles when she engaged with the "Comparing the Openness between Two Angles" task (Figure 45). When she was asked to compare two angles by estimating the number of times the blue angle was bigger than the red angle, she conjectured that the red angle was "like one-quarter of the space of the blue." Next, Angelie was asked to use the black arrows to mark the number of times she could fit the red angle into the blue angle. She iterated the red angle within the blue angle, such as shown in Figure 45a. When asked to state the relationship between the blue and the red angle, she reasoned that "the blue is the size of three of those red spaces." I followed the same questioning structure to engage Angelie in comparing a different pair of blue and red angles (Figure 45b). Angelie responded in a similar manner stating that "the red angle is bigger than the blue angle…the blue line is much closer to this (other) blue line than the red line(s)," and "the angle of the red lines is four times bigger than the angle for the blue lines." The analysis of Angelie's reasoning showed that she compared the closeness of the sides to reason about the sizes of the angles. She seemed to utilize her previous experience in iterating the smaller angle within the bigger angle to construct a multiplicative relationship between the two angles.

Figure 45

The Comparing the Openness Between Two Angles Task



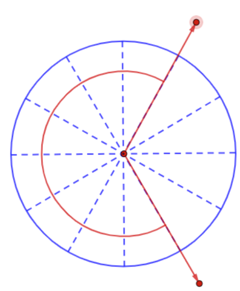
(a) (b)

Note. Tasks for Angelie to compare two angles where (a) the blue angle is three times bigger than the red angle, and (b) the red angle is four times bigger than the blue angle.

In the next task, my goal was to investigate whether she could reason multiplicatively about the change in a single angle. Angelie was asked to double a 2/12 angle in the "Modified Growing and Shrinking Angles" task (Figure 46). This task was modified for students to reason about angles as openness prior to conceiving them as wedges. She rotated the ray to double the angle while explaining, "the red space can hold by itself two other pieces of the circle. So, if I am going to double it [rotated the ray to double the angle], it will hold four pieces of this circle." Angelie recognized that the pieces "are twelfths," and she explained that she "doubled it by putting two more pieces from where it was before." When I asked Angelie to make the angle four times bigger than what it was before, she reasoned, "It would be 8 (8/12)… I have to add 2 (2/12) four times" (Figure 46). I infer that Angelie used 2/12 as a unit which she multiplied by a number of times to make angles double or four times bigger. When I asked her to triple the 2/12 angle, she stated "I would get six piece[s], 6/12." I infer from her reasoning that she multiplied the size of 2/12 angle into two groups when doubling and multiply the same into three groups when tripling the angle.

Figure 46

The "Modified Growing and Shrinking Angles" Task



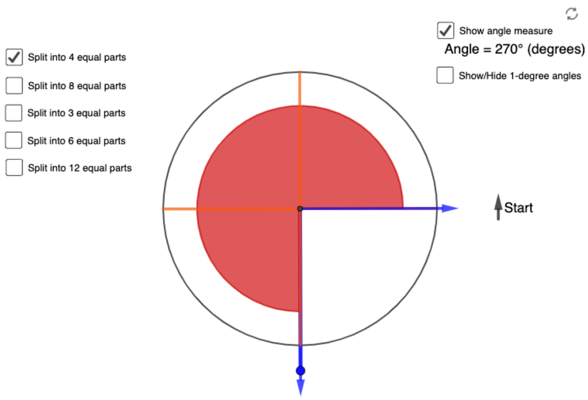
Note. Angelie made the angle four times bigger than what it was before.

To examine whether Angelie could connect the fractions of a circle to degrees, I prompted Angelie to work on the "Many Very Small Angles" task (Figure 39a). When she rotated the ray around, she explained that she created “one circle.” When I asked her to identify the number of angles she could make inside the circle, she reasoned that “it depends on the angle… like if I use one quarter, I would use four of those to cover the whole (circle) thing.” I infer that she imagined the whole circle being split into a number of equal parts. To introduce Angelie to reasoning about angles in terms of degrees, I asked her to use the “show angle measure” tool at the “360 Angles” task (Figure 39b) that shows that one full turn equals 360°. Using this tool, Angelie learned that “a whole circle is 360 degrees.” She struggled to recognize multiple angles smaller than a quarter wedge until she used the “show angle measure” tool. While exploring this tool, Angelie conceived that “the smallest you can make is one-degree, I’ll keep on adding it until I get 360.” This kind of reasoning showed that Angelie could iterate the “one-degree” angle multiple times to create 360° in a full turn.

In the “Splitting a Circle” task, I aimed to examine whether Angelie could connect this new understanding of angles in terms of degrees to her conception of angles as fractions of a circle. First, I asked Angelie to explore the task. During her exploration, she made a full rotation. Then, I asked her to split the circle into four equal parts (Figure 47). Similar to what she exhibited in the “Ferris Wheel” tasks, Angelie identified “four quarters”. When I asked Angelie to make a conjecture about the number of degrees for one quarter, she estimated it as “271” degrees. After using the “show angle measure” tool, she learned that one quarter was 90° and she explained, “that makes sense because 90 times four equals 360.” Then I followed her up on what it meant to have two-quarters of an angle. Angelie reasoned, “that would be 180° because I got it by adding 90° plus 90°.” For three quarters, Angelie thought that she would “add 90 again to 180. It should be 270,” and the four quarters “would be 360° because that would be all the quarters together, which equals 360." Angelie used the multiplicative iteration of 90° to identify the degree measure for each quarter turn.

Figure 47

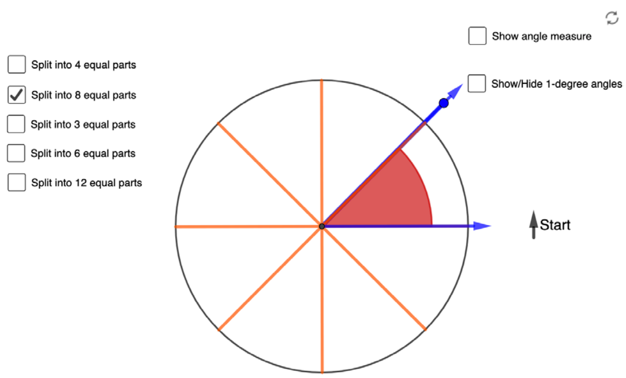
The Splitting a Circle in Fourths Task



Next, when I asked Angelie to split the circle into eight equal parts as shown in Figure 48, Angelie conjectured that "it would be 30 degrees… because I learned that one quarter is 90 degrees, which is 2/8. So, 90 is not an even number; I can't really split it into two equal numbers." Although she only estimated the degree measure for 1/8 angle because she could not split 90 into two equal numbers, she created a connection between 1/8 and 90° by stating that 90 was 2/8, or 1/4 in a simplified form. When I allowed Angelie to use the "show angle measure" tool to verify her conjecture, she exclaimed, "45 plus 45 equals 90." Then, she expanded the connection she made between the eighths and the quarters by stating that "6/8 equals three quarters because 2/8 equal one quarter." Before I asked her to split the circle into three equal parts (Figure 41a), the angle measure in degrees was left open on the screen. So, when I asked Angelie to show me a 1/3 angle, she already knew that 1/3 was 120°. Then, I asked her to explain the number of degrees for 2/3 angle while hiding the degree measure. She responded, "240 because 1/3 equals 120 degrees… so I added 120 plus 120, which equals 240 degrees." Angelie's engagement with the "show angle measure" tool to find the degree measures for an eighth and a third offered her a constructive space to compose 45° and 45° into a 90-degree angle, and 120° and 120° into a 240-degree angle.

Figure 48

The Splitting a Circle in Eighths Task



In splitting the circle into six equal parts (Figure 41b), I asked Angelie to make a conjecture about the number of degrees for 1/6. She then reasoned that "1/6 is like little more than a half of a quarter," but she realized that associating 1/6 with a quarter did not help her. When I allowed Angelie to use the "show angle measure" tool, she learned that "1/6 is 60°, 60 plus 60 equals 120 degrees." Then, I asked her to identify the degree measure for 1/12 angle. She recalled that she knew "1/6 is 60 degrees. And since I can see that 1/12 is half of 1/6, it should be 30 degrees because 30 is half of 60." Her reasoning in this series of tasks shows that although Angelie struggled with dividing 360 into smaller parts, when the degrees of a unit fraction was given (e.g., 1/3 equals to 120°, 1/6 equals to 60°), she was able to treat this as an angle unit that she iterated or split to find bigger and smaller angles respectively.

At the end of the design experiment, I asked Angelie on what she learned about angles. She explained:

> I learn that angles is the space between two lines. Not the way a line is shown. It's the space between two different lines. And if the lines get closer to each other, the angles become smaller, if the lines move farther away from each other the angle will become larger. I also learned that different fractions equal different angles… And I also learned that you can measure angle with degrees, like how one half is 180 degrees, and one quarter is 90 degrees.

Her reasoning shows that she reconstructed her thinking from her initial conception of angle as an orientation to include a conception of angles as the space between angle sides. When she was prompted to reason about the changes in an angle, she started to exhibit the conception of angles as rotations and that as the rotation changes the distance between the lines and changes the size of an angle. Angelie also exhibited reasoning about angles as fractions of a circle and associated this fraction with the degrees to identify angle measure.

5.2.1. Retrospective Analysis on the Second Iteration

Table 5 shows the progression of Angelie's reasoning and the tasks that prompted her to express different forms of reasoning.

Table 5

Angelie's Forms of Reasoning

Student Reasoning	**Task**
Angle Conception	
A. Angle as union of rays e.g., "A different angle line… if I move it around, it becomes a different way of drawing." **B. Angle as rotation** e.g., "What is changing is definitely the angles, because once you move the line farther away from the other line, the angle becomes larger; putting it closer to the other line will make it much smaller." **C. Angle as wedge** e.g., "It is like making a circular space"	A. Modified Three Pairs of Different Objects (Figure 43) B. Modified Three Pairs of Different Objects (Figure 43) C. Blue Wedge (Figure 44)

Multiplicative Comparisons	
A. Initial splitting strategy: Associated quarter rotations with the half or quarter sizes of a circle e.g., "To get to the chocolate, you'll need to use half of the circle and another quarter of it…or three quarters." **B. Iteration and splitting strategies in reasoning about full turn** e.g., "It takes the whole circle or you could say four quarters." **C. Decomposition and composition strategies to compare two angles** e.g., "The blue is the size of three of those red spaces." **D. Additive iteration of the original angle to construct multiplicative changes in an angle** e.g., "Doubled it by putting two more pieces from where it was before."	A. Ferris Wheel (Figure 36) B. Ferris Wheel (Figure 36) C. Comparing the Openness between Two Angles (Figure 45) D. Modified Growing and Shrinking Angles (Figure 46)
Discrete/ Continuous Conception of Angle	
Reasoned about a circle as having a very large number of angles e.g., "It depends on the angle… like if I use one quarter, I would use four of those to cover the whole [circle] thing."	Many Very Small Angles (Figure 39a)
Numeric Multiplicative	
A. Iterations of one-degree compose 360° in a whole e.g., "the smallest you can make is one-degree, I'll keep on adding it until I get 360." **B. Composition of n-quarter of a circle as multiplicative iterations of 90°** e.g., One quarter of a circle is "90 degrees" using the "show angle measure" tool, and she reasoned "that makes sense because 90 times four equals 360." **C. Composition of 90° as two groups of 45°** e.g., Using the "show angle measure" tool to show that an eighth is 45, she reasoned "45 plus 45 equals 90." **D. Composition of 240° as two groups of 120** e.g., "240 because 1/3 equals 120 degrees… so I added 120 plus 120, which equals 240 degrees." **E. Composition of 120° as two groups of 60** e.g., Using the "show angle measure" tool, "1/6 is 60 degrees, 60 plus 60 equals 120 degrees."	A. 360 Angles (Figure 39b) B. Splitting a Circle in Fourths (Figure 47) C. Splitting a Circle in Eighths (Figure 48) D. Splitting a Circle in Thirds (Figure 41a) E. Splitting a Circle in Sixths (Figure 41b)

Angelie exhibited a prior knowledge about angles as orientations. This conception of orientation was found to be an essential component in recognizing an angle in terms of the initial and terminal positions of angle sides (Devichi & Munier, 2013). While students with a conception of angle as orientation have seen to show a reliance on a horizontal side as the base of an angle (Browning et al., 2007), Angelie did not exhibit this form of dependence. Instead, her conception of angle as orientation potentially supported her construction of mental images of the amount of space between the lines she rotated. Her engagement with the tasks involving rotations also led her to reason that the lengths of sides did not affect the changes in the angle size. Her reasoning showed that she did not have the common alternative conception that the angle size depends on the length of its sides (Smith et al., 2014).

When Angelie engaged with the "Ferris Wheel" task, she began to reason multiplicatively about angles as quarters of a circle, illustrating her prior knowledge about fractions in fourths. I infer that she only exhibited this form of reasoning because the task involved a circle split into four equal parts. This design of the task might have supported her to connect her understanding of fractions to angle size.

Then, Angelie associated the quarter rotations with the quarter size of a circle. However, she did not connect the fractions of a circle with the degrees (Confrey et al., 2012) until after she worked at the "Splitting a Circle" tasks, probably because the "show angle measure" tool showed her the degrees of an angle. When prompted to identify the degree measures for an eighth, a third, and a sixth of a circle, Angelie used decomposition and composition of degrees to quantify angles (Moore, 2012). Angelie illustrated her use of the operations of decomposing an angle into groups of smaller identical angles and composing back the original angle by adding the smaller angles. This kind of reasoning also illustrated the reversibility of the decomposition and composition operations. In all cases, Angelie used the "show angle measure" tool to help her because the values are too big for Angelie to compute. It would be interesting to examine how the progression of Jordan and Angelie are similar or different from the other students' reasoning in the "Blue Wedge", "Ferris Wheel", and "Splitting a Circle" tasks.

When working with the "Comparing the Openness between Two Angles" task, Angelie showed a form of reasoning that combined composition and decomposition strategies when she reasoned that one angle is the size of three times of the other angle or that an angle is four times bigger than the other. Angelie constructed an angle unit that she iterated to find the bigger angle or split the bigger angle to find the smaller angle. This shows that comparing the sizes of two angles could prompt students to express multiplicative reasoning via decomposition and composition of angles (Moore, 2012).

Angelie also reasoned adding two more pieces of a unit angle to double the angle in the "Modified Growing and Shrinking" task. This reasoning seemed to resonate what Izsák and Beckmann (2019) described as coordinated measurement approach to multiplication in which the product quantity 4/12 angle is measured by using two groups of the 2/12 angle. I also noted that this task was modified for the second iteration by removing the wedges so that students could conceive angles as openness prior to conceiving them as wedges. However, the similarities between the two students' reasoning imply that removing the wedges did not influence Angelie's reasoning to differ from Jordan's. In the design experiments that follow, I further investigated whether the other two students exhibit a similar reasoning. For the next two iterations, I did not make any significant modifications on the tasks besides fixing minor codes for aesthetic purposes. I conjectured that following the same sets and sequences of tasks and similar questioning with the iteration with Axel could potentially offer a better comparison between students' forms of reasoning.

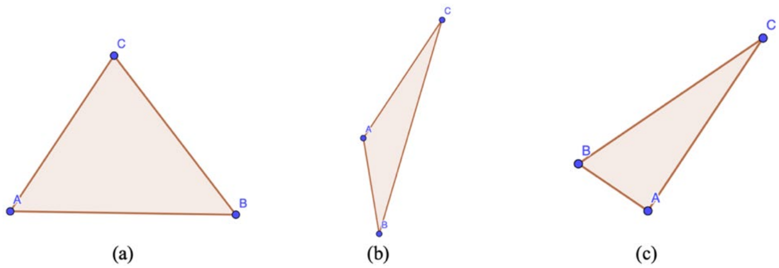
5.3. Case 3: Axel

Axel's prior knowledge of angles included conceptualizing an angle as a composition of two rays and a common point. He was also familiar with some angle concepts used in the mathematics curriculum, such as identifying an "obtuse" angle that is larger than 90° and knowing that a whole circle has 360°. During the experiment, he was able to use this prior knowledge of a circle as consisting of 360° to reason about angles in terms of the fractions of a circle. I describe his progression of reasoning in detail in the following paragraphs.

At the beginning of the design experiment, I asked Axel to define angles and he stated that "an angle is two rays that point in certain directions." Axel exhibited his conception of an angle as a composition of two rays. He also reasoned about angles as pointing in different directions. To examine Axel's prior knowledge about the size of angles, I asked him to work on the "Triangle Task" (Figure 49). When I asked Axel how he could make angle A bigger (Figure 49a), he pulled vertex B downward as shown in Figure 49b and stated that he was "stretching it, like pulling it out." I infer that Axel was referring to changing the angle by pulling one of the vertices away to stretch the opposite side of the included angle A. To further examine his thinking about the size of the angle he created, I asked if he could describe to someone who had not seen the task how angle A was bigger compared to the other angles in the triangle. Axel said that it was "obtuse." Then, when I probed him to explain what he meant by obtuse, Axel stated, "larger than 90 degrees, basically." Axel exhibited a prior knowledge about angle measures relative to 90°. When I asked him to show angle A as a 90-degree angle, he moved the vertices B and C to make segments AC and AB approximately perpendicular to each other (Figure 49c). His illustration of a 90-degree angle also shows that he was familiar with 90° where angle sides are perpendicular to each other.

Figure 49

The Triangle Task

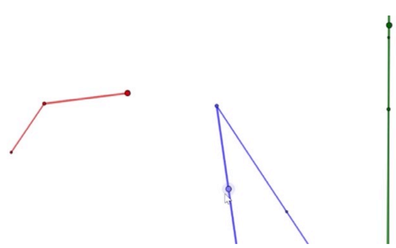


Note. Axel dragged vertex B as shown in (a) to create an obtuse angle (b), and a right angle (c).

When I asked Axel to explore the “Modified Three Pairs of Different Objects” task (Figure 50), he rotated a blue ray and stated, “You can make angles. Same thing with the red [segments].” When I probed Axel to explain how he created angles, he reasoned, “I’m pulling it away from one side.” Similar to his reasoning in the “Triangle” task, I interpret that Axel conceived an angle as a transformation (rotation) by pulling away one side from the other. Axel also showed evidence of his prior knowledge about angles as having 360° by stating that he could create “360” angles with a pair of segments because he learned that in his “class.” As he argued, “when you do tricks on, for example, bicycles, you can also twist yourself around in a circle making 360 degrees.”

Figure 50

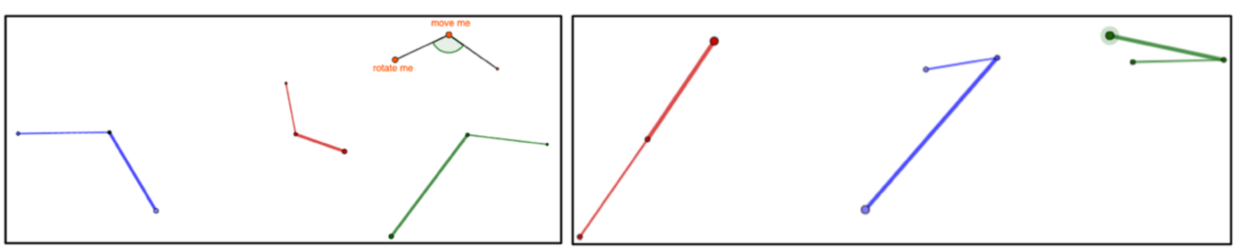
The Modified Three Pairs of Different Objects Task



Next, I asked Axel to work on the “Modified Comparing Angles with a Fixed Angle Object” task (Figure 51a). When I asked him to compare the openness between each angle, Axel estimated the measure of the blue angle, “like a 90-degree angle and maybe about 45 more degrees… actually maybe 15, 105 [degrees in total].” He also argued that the red and the green angles measure 105° and “they’re all the same angle.” I then asked Axel to use the fixed angle tool to verify his claims and he confirmed that the angles were “the same.” Axel’s reasoning shows that he did not associate the angle size with the lengths of the segments. Similarly, in the “Comparing Three Pairs of Segments with Different Lengths” task (Figure 51b), Axel reasoned an angle larger than 180° as 195° “because if this is 180 degrees [creating a straight line using the two red segments], because it gets larger…the farther you point it out here.” He knew that “180 degrees is a straight line” and “that a straight line is half of 360 degrees.” When he explored the rotation of the other pairs of segments, the student described the blue pairs as it “only folds out to at least smaller than 90 degrees” while the green “you can fold it out all the way, the whole 360 degrees.” I interpret his reasoning to show that his prior knowledge included being able to imagine the size of an angle in terms of degrees and being able to reason about angles relative to 90°, 180°, and 360° angles.

Figure 51

The Modified Comparing Angles with a Fixed Angle Object and Comparing Three Pairs of Segments with Different Lengths Tasks



(a) (b)

Note. (a) Axel reasoned that the three angles have 105° each, and (b) angle with the red pair of segments has 180°.

Next, I asked him to rotate a pair of segments and he created a wedge in the "Blue Wedge" task (Figure 52). Then, I asked him to explain how this task was different than the previous tasks. He stated, "this one has the angle measure… it doesn't only have these two lines." I then probed him to talk about the part that he was calling angle measure and he explained, "this would be a part, that measuring, like right here [pointing at the blue wedge], it fills in the space between the two angles [referring to the angle sides], so you know this is in degrees." I infer that Axel was referring to the amount of "space between" referring to the blue wedge as the "angle measure" and that he thought of this measure "in degrees."

Figure 52

The Blue Wedge Task with Axel

Note. Axel reasoned about the wedge as being measured in an angle.

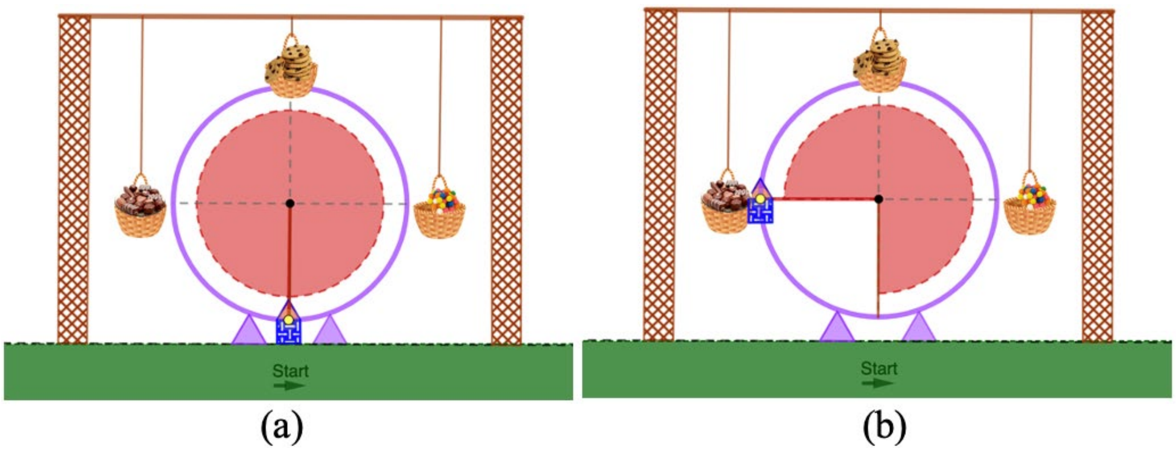
In the "Ferris Wheel" task that followed (Figure 53) Axel stated,

Axel: "I can change the angle. Also, there are marks here that this, I think this one is 90 degrees [turned the cart from Start to the jellybeans to create a quarter wedge], 180 degrees [turned the cart to the cookies to create two-quarters wedge], and 270 degrees [turned the cart to the chocolates to create three-quarters wedge], and 360 [turned the cart back to Start to create a full wedge]."

When I asked him how he knew that turning from Start to the jellybeans was 90°, he reasoned, "it is a quarter of a whole circle, 360 degrees." I then probed him to explain his reasoning further about the 180° and 270° and stated, "It is 1/2 [rotated the cart to create ½ wedge]. And this is 2/4, 3/4 [rotated the cart to create 3/4 wedge] of 360 degrees." When I asked Axel how much of a turn he had to make from Start to return to Start as illustrated in Figure 53a, he stated, "360 degrees," and it is a "full turn." His reasoning in this task shows that Axel was able to associate a fraction of a full rotation to a fraction of a circle and also to fractions of 360°. I infer from this association that he was able to find the degrees for each quarter turn by splitting 360° into four equal parts.

Figure 53

The Ferris Wheel Task Where Axel Reasoned about Quarter Turns



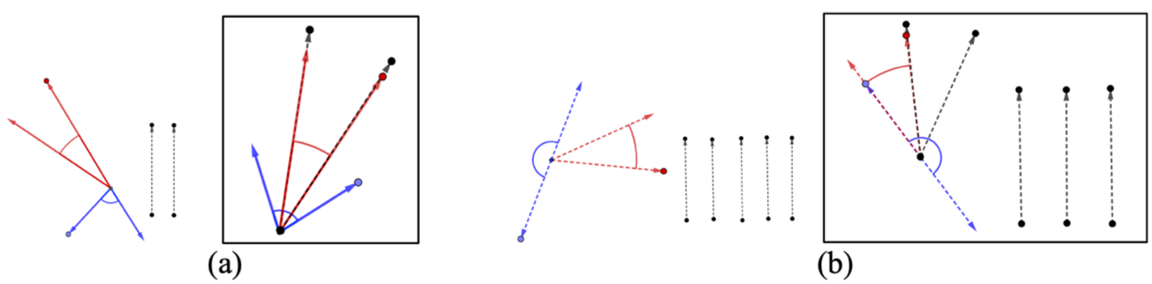
Note. Axel reasoned about (a) each quarter turn and (b) 7/4 turn around the Ferris wheel.

I was intrigued by the sophisticated reasoning that Axel had about fractions that supported his understanding of angles. To prompt his reasoning even further, I asked him where the cart would go if he made five quarters of a turn from the Start. Axel reasoned, "there's a full turn [turned the cart from Start back to Start], and then also 90 degrees [turned the cart to jellybeans], five quarters." Then, I asked him how he knew it was five quarters. Axel explained, "because four quarters as a whole, which means this is 360 degrees and also you added another quarter… you go around another 90 degrees." When I asked Axel to turn the cart by 7/4 of a turn, he stated, "one full turn [turned the cart from the Start back to Start], and then up to here [turned the cart from the Start to chocolates] (Figure 53b).” Then, I asked Axel how he knew that he should stop at the chocolates. He reasoned, “there are four (quarters) for full (turn)… and then four plus three equals seven… one full turn and three quarters.” Axel accurately represented the amount of turn bigger than a full turn. For instance, he explained that 5/4 is equivalent to “a full turn” and “another quarter,” and 7/4 is “one full turn and three quarters.” Axel seemed to have a deep understanding of fractions of a circle and was able to leverage this knowledge to construct 1/4 as an angle unit which he iterated to find angles larger than 360°.

In the “Comparing the Openness between Two Angles” task (Figure 54a), I asked Axel which of the two angles is bigger than the other. He identified the blue angle as “more open” and that “the two rays are farther apart” than the red angle. When I asked Axel how many times bigger was the blue angle than the red angle, he conjectured, “maybe about two times bigger… I don’t know, two or three.” I asked Axel to verify his conjecture. Axel iterated the red angle within the blue angle using the black arrows as his markers, he reasoned that the blue angle was “three times bigger” than the red angle.

Figure 54

The Comparing the Openness between Two Angles Task



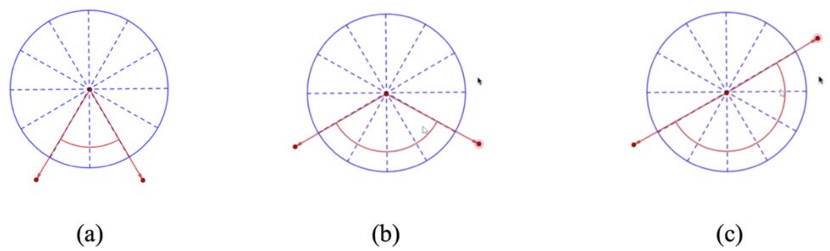
Note. Tasks for Axel to compare two angles where (a) the blue angle is three times bigger than the red angle, and (b) the blue angle is six times bigger than the red angle.

Next, I asked Axel to compare another pair of angles, as shown in Figure 54b. He stated that “the angle on the 180 is bigger” by referring to the blue angle. When I asked Axel to conjecture how many times bigger was the blue angle compared to the red angle, he claimed that it would be “four times bigger,” reasoning about the decomposition of the blue angle. Then, I asked Axel to show how the blue angle was four times more open than the red angle. As he iterated the red angle within the blue angle, Axel counted each iteration and stated, “six times bigger.” Although Axel’s conjecture was incorrect, I interpret his actions as exhibiting the composition of the blue angle using six iterations of the red angle to correctly reason that the blue angle was “six times bigger” than the red angle.

In the “Modified Growing and Shrinking Angles” task (Figure 55a), Axel grouped 2/12 as an angle unit (one group) and multiplied this unit two or three times (groups) to make it two or three times bigger respectively. For instance, when I asked him to double the red angle he reasoned, “you would have 4/12,” and rotated both rays by 1/12 each to create a 4/12 angle (Figure 55b). When I asked him why it was 4/12, he explained, “it was 2/12 before and now, four times, two times two equals four.” Similarly, when I asked him to make the angle three times bigger, as illustrated in Figure 55c, he rotated one ray of the 4/12 angle by another 2/12 to create a 6/12 angle and reasoned, “like this, and then also add another two, that that would be half, 6/12 because three times two is six. So, 6/12.” Axel was also able to name one partition (1/12) as 30 degrees because he already knew that the circle is 360°. Specifically, he reasoned, “So, 36 divided by 12 is 3. So, thirty-six hundred would be thirtieth.” I followed up Axel’s thinking by asking what the “thirtieth” was and he explained, “the angle is a thirtieth like, there’s 12, a twelfth.” I infer that Axel referred to the 30^{th} as 30° for a twelfth of a full turn.

Figure 55

The Modified Growing and Shrinking Angles Task

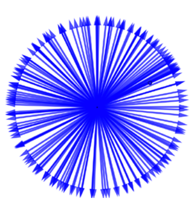


Note. (a) The original 2/12 angle, (b) Axel's illustration of doubling the angle, and (c) Axel's illustration of making the angle three times bigger.

Next, I asked Axel to work on the "Many Very Small Angles" task shown in Figure 56 to examine how he would reason about angle as a quantity. When I asked him to rotate the blue ray and describe what he created, he said, "I don't think I'm creating angles, just lots of rays." Next, when I asked him how many of the rays he created, he stated, "a lot, it's a lot." I interpret from Axel's reasoning that he initially conceived a discrete number of rays. However, it was not clear from his reasoning how he conceived the angles as quantities.

Figure 56

The Many Very Small Angles Task



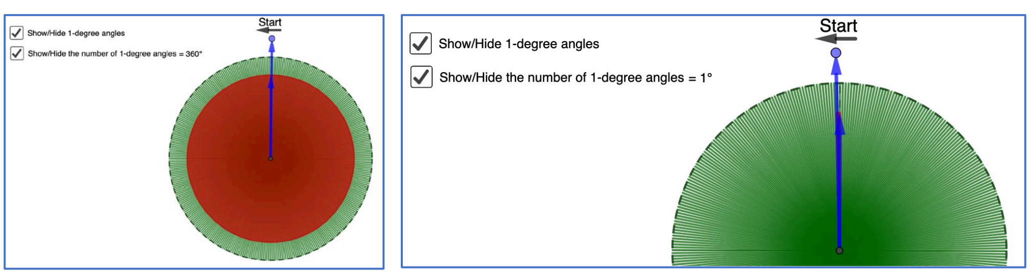
Note. A task where Axel conceived of creating many rays.

To connect his conception of "a lot" of rays to angles, I presented the "360 Angles" task in Figure 57a. As he rotated the blue ray, the "show angle measure" tool showed that one full turn equals 360°. When I asked Axel how the task was related to the other tasks he had explored, he stated, "360 degrees…it is showing all the angles from 1 to 360 [degrees]." When I probed

Axel to talk about the smallest angle he could make, he responded, “one-degree [rotated the ray to create a one-degree angle] (Figure 57b).” Subsequently, I asked him if there is a smaller angle than a one-degree angle and he said, “half a degree.” When I asked him if he could make a half-degree angle, he explained, “it won’t show on here [“show angle measure” tool], but probably yes.” I infer from his reasoning that he could mentally split a one-degree angle into two equal parts to create a half-degree angle. He then continued by saying “that’s why the other one [“Many Very Small Angles” task in Figure 56] was titled unlimited.” When I asked him if he could imagine unlimited angles on the previous task, he responded, “yeah, because there’s always an angle in. You can always cut a fraction [of a turn] in half. So, yeah, you can make unlimited angles.” Axel’s reasoning progressed from conceiving the discrete number of rays in the “Many Very Small Angles” task into conceiving angle as a continuous quantity in the “360 Angles” task as he explained that “there is always an angle in” between two rays. I infer that his mental action involved a continuous splitting of an angle into halves to “make unlimited angles.”

Figure 57

The 360 Angles Task Where Axel Reasoned about an Angle as a Continuous Quantity



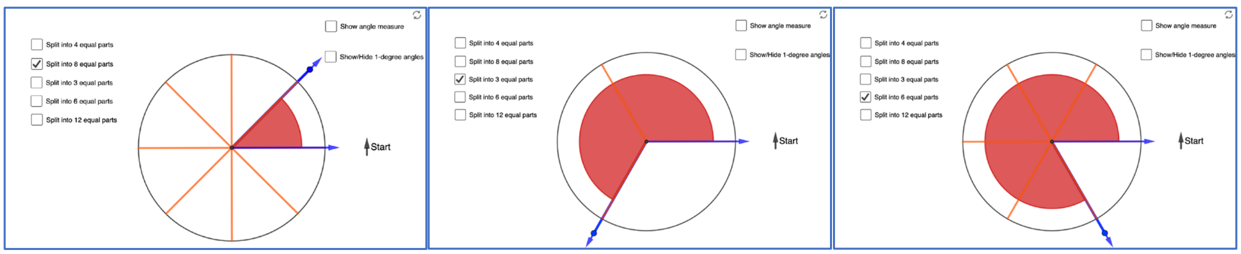
(a) (b)

Note. A task where Axel (a) recognized one-degree to 360 degrees angles in a circle, and (b) created a one-degree angle as his initial smallest angle.

In the “Splitting a Circle” tasks, Axel used his knowledge about fraction operations to reason about various angles he created by splitting a circle into parts and dragging a ray around in circle. For instance, he argued that an eighth of a turn (Figure 58a) is "45 degrees because it is half of 90 degrees." Then, he reasoned about the 4/8 of a turn by stating "4/8 is 180 degrees… because it is 2/4, which is 180 degrees." His responses illustrated a flexible understanding of associating the amount of turn, with the fraction of a circle, and the number of degrees.

Figure 58

The Splitting a Circle in Eighths, in Thirds, and in Sixths Task



(a) (b) (c)

Note. Axel split a circle (a) in eighths to reason about the degrees of an eighth of a turn, (b) in thirds to reason about the degrees of two-thirds of a turn, and (c) in sixths to reason about five-sixths of a turn.

Axel also illustrated a flexible understanding of using different fraction operations interchangeably to find the same amount of turn. For instance, when I asked Axel about finding 7/8 of a turn, he explained, "315, because 360 minus 45." I probed him to talk about why he subtracted 45. He reasoned, "because it would be easier to just subtract 45 from the whole than to add 45 seven times." Axel constructed 45° as an angle unit that he could multiply seven times. He also knew that he could subtract that angle unit from the total number of degrees and find the same result.

In a similar way he estimated the degrees of a 2/3 of a full turn as shown in Figure 58b stating "2/3 would be [rotating the ray to create 1/3 and then 2/3]. So, 360 minus 120 is 36 minus 12. So, 240." Although he used an iterative action to generate the angle, he used subtraction to estimate the degrees showing that he understands the two processes of iteration and subtraction of multiplicative parts to be reversible. When I asked him to find another way to get 240°, he stated, "multiply 120 times 2." Another example of his use of the two strategies was when he was asked to determine the degrees for 5/6 of a turn (Figure 58c) and he stated, "that would be 300 degrees, because 360 minus 60 is 300." When I asked him if there is another way to do it, he explained that he would multiply "five times 60."

At the end of the experiment, I asked Axel about what new things he learned. He stated that "if you have 60-degree angles in a half of a full turn, there would be six 30-degree angles in a half turn." Axel's response showed his new understanding about angles as a composition of a bigger angle.

5.3.1. Retrospective Analysis on the Third Iteration

Table 6 shows the progression of Axel's reasoning and the tasks that prompted him to express different forms of reasoning.

Table 6

Axel's Forms of Reasoning

Student Reasoning	Task
Angle Conception	
A. Angle as union of rays e.g., "An angle is two rays that point in certain directions." **B. Angle as transformation** e.g., Make an angle bigger by "stretching it, like pulling it out." **C. Angle as wedge** e.g., "This would be a part, that measuring, like right here [pointing at the blue wedge], it fills in the space between the two angles [referring to the angle sides], so you know this is in degrees."	A. Question: What is an angle? B. Triangle (Figure 49) C. Blue Wedge (Figure 52)
Multiplicative Comparison	
A. Fraction bigger than a whole e.g., "A full turn" and "another quarter," and 7/4 is "one full turn and three quarters."	A. Ferris Wheel (Figure 53) B. Modified Comparing the

B. Decomposition and composition strategies to compare two angles e.g., "maybe about two times bigger…, two or three." And he found that the blue angle was "three times bigger" than the red angle. **C. Additive iteration of the original angle to construct multiplicative changes in an angle** e.g., Double 2/12 angle, rotate two rays by 1/12 to make "4/12."	Openness between Two Angles (Figure 54) C. Modified Growing and Shrinking Angles (Figure 55)
Discrete/ Continuous Conception of Angle	
Conceived a circle as having a very large number of angles e.g., "because there's always an angle in. You can always cut a fraction in half. So, yeah, you can make unlimited angles."	Many Very Small Angles (Figure 56)
Numeric Multiplicative	
A. Composition of a full turn as 360° and decomposition into three hundred sixty groups of one-degree angles e.g., "360 degrees…it is showing all the angles…all the angles from 1 to 360 (degrees)." **B. Composition of 7/8 of a turn: compensation strategy or seven groups of 45** e.g., 7/8 of a turn is "315, because 360 minus 45…because it would be easier to just subtract 45 from the whole than to add 45 seven times." **C. Composition of 2/3 of a turn: compensation strategy or two Groups of 120** e.g., "2/3 would be [rotating the ray to create 1/3 and then 2/3]. So, 360 minus 120 is 36 minus 12. So, 240." **D. Composition of 5/6 of a turn: compensation strategy or two groups of 60** e.g., 5/6 "that would be 300 degrees, because 360 minus 60 is 300."	A. 360 Angles (Figure 57) B. Splitting a Circle in Eighths (Figure 58a) C. Splitting a Circle in Thirds (Figure 58b) D. Splitting a Circle in Sixths (Figure 58c)

Axel conceived an angle as a composition of two rays and a common point and that the rays point to different directions. I interpret that his reasoning showed two different conceptions of an angle that can be classified as angles as union of rays while at the same time incorporating the idea of angles as orientations. Students with this understanding are not limited to imagining typical examples of angles as described in the literature (e.g., Browning et al., 2007; Devichi & Munier, 2013). He was also familiar with some angle concepts such as the term "obtuse" being used to describe an angle that is larger than 90°, 90° is a quarter of a turn, 180° with a straight line or half a turn and that a whole circle has 360°.

Throughout the experiment, he was able to leverage his prior knowledge about degrees to reason about the size of angles and connect the degrees with fractions of a turn. It was also interesting to find that his reasoning was not dependent on the “show angle measure” tool. For instance, in the “Ferris Wheel” task, the perpendicular partitions and rotation motion of angle side probably prompted Axel to construct the connections between the quarters of a turn with the quarters of a whole, then with the equivalent degrees. In this task, Axel did not have access to the “show angle measure” tool that could show the degrees for every turn.

In the “Comparing the Openness Between Two Angles” task, Axel reasoned about one angle as “three times bigger” showing that he composed the bigger angle using the number of times he could estimate the iteration of the smaller angle and acted out this iteration to verify his estimate. He exhibited a multiplicative reasoning via the reversible processes of decomposition and composition of angles (Moore, 2012). When changing the size of an angle in the “Modified Growing and Shrinking Angles” task, Axel treated the original angle 2/12 as the angle unit (one group) to be multiplied two or three times (number of groups) when doubling or tripling the angle. His reasoning implied a coordinated measurement approach of Izsák and Beckmann (2019). In some interesting instances, Axel exhibited his prior knowledge about degrees by assigning 30° in every twelfth of a turn by dividing 360 by 12. However, he did not use this argument to reason about changing the 2/12 of a turn three times bigger.

In the “360 Angles” task, Axel initially reasoned about angles as a discrete quantity by describing that the task showed angles from one-degree to 360 degrees. However, my questioning about the angle smaller than a one-degree angle potentially prompted Axel to reason about half of one-degree angle and subsequently reason that he could “always cut a fraction [of a turn] in half” and there are “unlimited angles” because there is always an angle between an angle. His reasoning exemplifies the conception of an angle as a continuous quantity, illustrating continuous quantitative reasoning (Castillo-Garsow, 2012).

In the “Splitting a Circle” tasks, Axel’s prior knowledge about the degrees of angles potentially served as his springboard to connect the fractions of a turn with the fractions of a circle, and then with the degrees. He also exhibited a flexible understanding of operations on fractions of a turn by using the two processes of *multiplicative iteration* and *subtraction of multiplicative parts from a whole* to determine the degrees of a fraction of a turn. His reasoning showed that he could use these two operations interchangeably to find the same degrees for a fraction of a turn.

For the next iteration, I did not modify the tasks, tools, and their sequences. However, the researcher questioning was expected to be modified to follow student’s reasoning and prior knowledge. It seemed that Axel’s prior knowledge played a significant role in his reasoning given the same tasks with the other students but he was less dependent on the tools that show the angle measure in degrees. In the design experiment that followed, I further investigated the effect that student’s prior conceptions of angles have on her constructions of new knowledge.

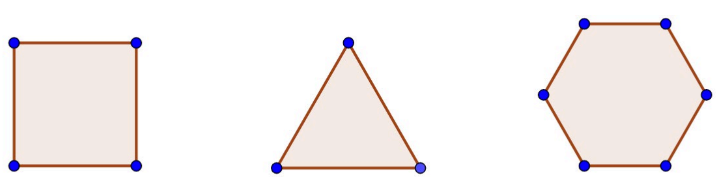
5.4. Case 4: Alicia

Alicia’s reasoning showed her prior knowledge about angles as corners of a shape. Then, her reasoning progressed to include two sides when creating an angle. This inclusion of the two sides to compose an angle illustrates a conception of an angle as a union of rays. Next, she reasoned about angles as the space between the two sides, the width, and the space within the arc closer to the vertex. Although Alicia initially expressed a prior knowledge that circles do not have angles, during the design experiment she exhibited multiplicative reasoning about angles as fractions of a full turn as illustrated by wedges. Then, she connected the fractions of a full turn to the fractions of a circle, then with degrees. I describe the progression of her reasoning in more detail in the following paragraphs.

At the beginning of the design experiment, I presented to Alicia the “Angles in Shapes” task (Figure 59) and asked her what an angle was for her. She explained, “an angle to me is a point at a shape. So, here is a triangle has three points, and it is, each corner of a shape.” I interpret Alicia’s initial definition of an angle as “a point at a shape” as referring to the corners where the sides meet. To further understand Alicia’s conception of an angle, I asked her what she needed to know to make an angle. She replied, “you need to know what shape you got, you want to make. But you can’t make a circle. Because circles don’t have angles or sides.” I infer from her response that she also considered the sides of a shape in creating an angle, although she argued that the corners of each shape were the angles. When I asked her whether the sides of a shape matter when we look at the angles, she explained, “yes, it matters because on each end of each line, it creates a corner, and it also creates an angle.” Her reasoning about angles as corners of a shape potentially influenced her to reason that circles do not have angles because circles do not have corners.

Figure 59

The Angles in Shapes Task



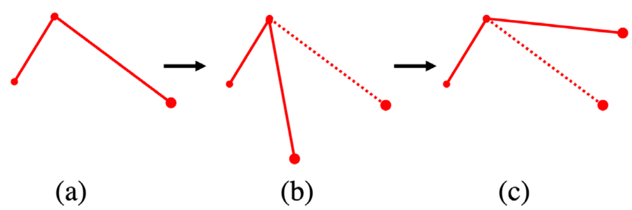
Note. Alicia referred to the corners of the shapes as angles in the Angles in Shapes task.

In the “Modified Three Pairs of Segments with Different Lengths” task, Alicia was asked how she could make an angle bigger (Figure 60a). She first rotated one segment closer to the other segment as shown in Figure 60b and explained, “because when it is like this, it is smaller.

And this is an angle itself. But if you move it [rotated the segment away from the other segment (Figure 60c)], I make more space in between and then makes the angle bigger." From Alicia's reasoning and action of rotating the segment closer or farther away from the other segment, I interpret that she conceived an angle as a space between the two sides.

Figure 60

The Modified Three Pairs of Segments with Different Lengths Task



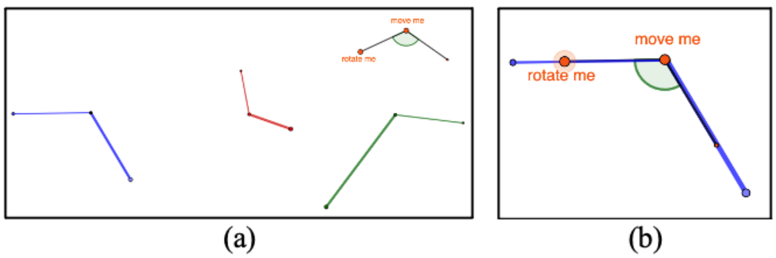
Note. Alicia modified angle (a) to create a smaller angle as shown in (b), and a bigger angle as shown in (c).

During this task, I asked her if the length of the sides also changed as she was making the angle bigger or smaller and she reasoned, "no, it does not get longer, or it does not get shorter. So, if I move it here [rotated the segment to make the angle bigger], it is the same height [length]." Alicia's reasoning and actions showed that she did not have the alternative conception of angle size as dependent on side lengths.

In the "Modified Comparing Angles with a Fixed Angle Object" task (Figure 61a), I asked Alicia to create angles using each pair of segments and then use the fixed angle object to compare their sizes. Alicia rotated the blue segment to the maximum size I set for each pair. Then, she dragged the fixed angle object and aligned its sides along the sides of the blue angle (Figure 61b). To follow up her understanding about the sizes of the angles, I probed her to talk about their openness. She explained, "they open the same, but the lines are shorter for the 'move me' line [the fixed angle object], instead of the blue [angle]." With my prompting, Alicia showed that she was able to talk about angle size in terms of openness. Similar to the "Three Pairs of Segments with Different Lengths" task, she did not consider side lengths to affect the openness of an angle in this task.

Figure 61

Modified Comparing Angles with a Fixed Angle Task

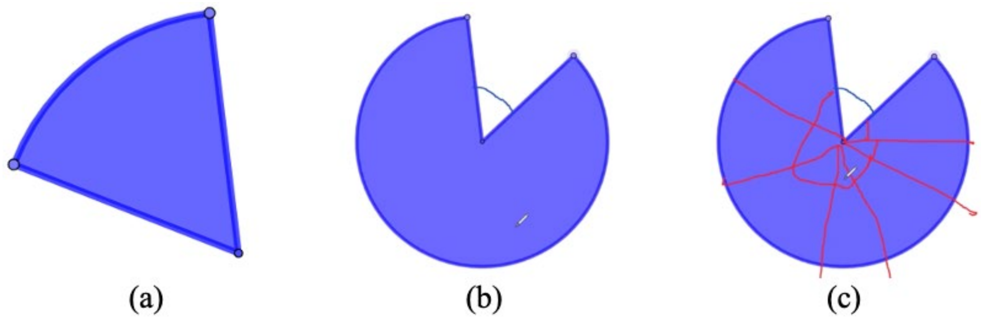


Note. (a) Alicia was asked to compare the openness between the three angles (a) and (b) she used a fixed angle object to verify their openness.

In the "Blue Wedge" task, Alicia was asked to rotate a pair of segments and discuss what she created (wedge) (Figure 62a). She rotated one side and created a circle arguing that there were "no" angles in circles. She later recognized an angle when I asked her to rotate a side that did not create a full circle. She called this shape a "fortune cookie." Then, I asked Alicia to draw the angle she could find on the "fortune cookie." She drew an arc in the empty space between the two segments as shown in Figure 62b. Later in the experiment, she argued, "The thing is, when you just have one plain circle, it would not be an angle because you don't have anything separated." Her reasoning showed that she considered the empty space as the angle and not the traced wedge. To prompt her thinking, I asked her to find another angle. She then created uneven partitions within the blue wedge and counted eight spaces, including the space with an arc (Figure 62c). Alicia explained, "it is like fractions of a fortune cookie" and that there were "eight spaces, and they can make, I'm just going to make a circle [drew red arcs within the blue wedge]. So, this is all an angle." I asked her the size of each angle she created, and she stated, "it's 1/8." Her thinking shows that she was able to define two angles in a wedge and also to talk about the size of an angle as a fraction of a circle.

Figure 62

The Blue Wedge Task

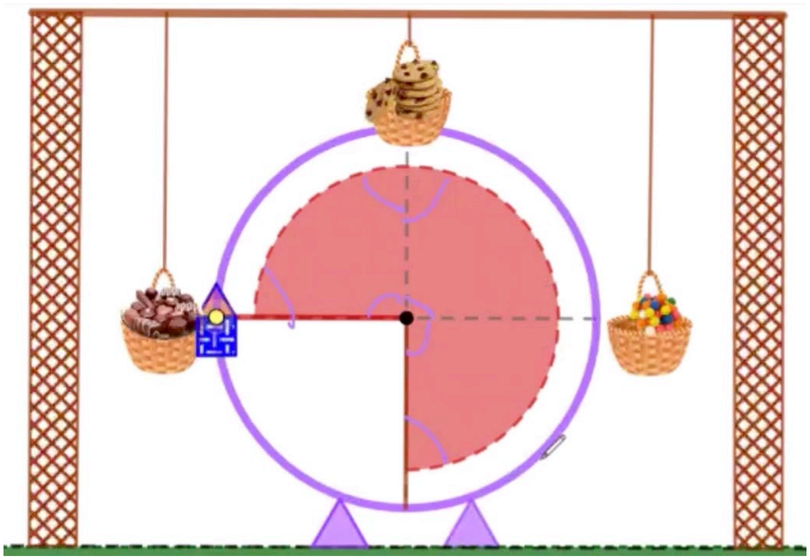


Note. (a) Alicia dragged the blue segment to create a wedge, (b) she identified the empty space of a wedge as an angle, and (c) she identified the wedge as a second angle and partitioned it to reason about size as a fraction of a whole circle.

In the "Ferris Wheel" task, I examined Alicia's reasoning about angles as fractions of a circle in more depth (Figure 63). While Alicia turned the Ferris wheel cart to the chocolates and created a 3/4 wedge, she stated that "this kind of looks like fractions." When I asked her what fraction she created with the cart at the chocolates, she replied, "It is 1/4. But if I took away these pieces [red wedges], it would be 3/4." I infer that Alicia was reasoning about both sides of the wedges with the missing 1/4 piece and the 3/4 shaded piece. Subsequently, I asked her to turn the cart from the Start to the cookies, she replied, "I'm making 2/4 of a turn." Then, I asked her the number of 1/4 turns she made from Start and go back to Start. Alicia reasoned that "I did four turns to go back to Start." When I asked Alicia on the size of the angle that she created, she stated that she had "four parts of an angle," exhibiting her understanding of fourths in a whole circle. Her statements show that she associated the fraction of a turn with the fraction of a circle. However, when I asked her if she could make 5/4 of a turn, she reasoned, "I don't know how."

Figure 63

The Ferris Wheel Task

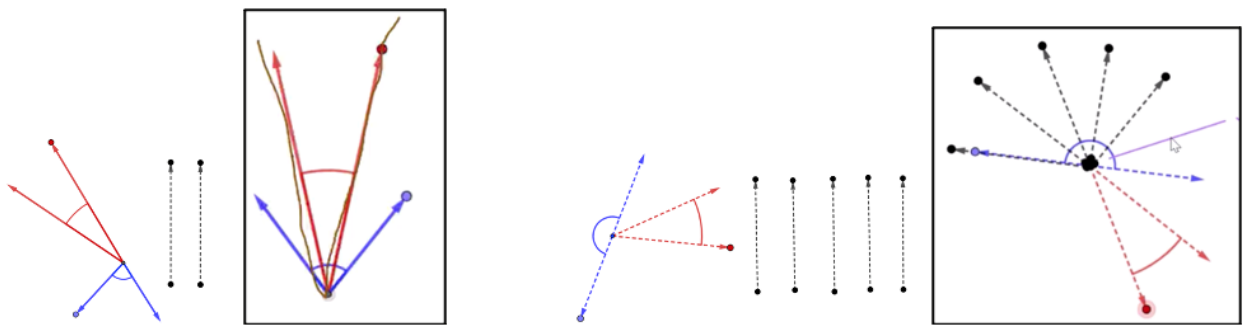


Note. The "Ferris Wheel" task where Alicia reasoned about the fractions of a whole turn and fractions of a circle.

When I asked Alicia to work on the "Comparing the Openness between Two Angles" task (Figure 64), she used a decomposition and composition strategy to reason about the multiplicative relationships between the two angles. For example, when I asked her to compare the pair of blue and red angles as shown in Figure 64a, she conjectured that the blue was "two fractions bigger" than the red angle. When she iteratively rotated the red angle into the blue angle, she reasoned that "the blue angle is three times bigger than the red" angle. Then, when I asked her to compare a different pair of blue and red angles as shown in Figure 64b, she conjectured that "the blue is three times bigger than the red angle." After she iterated the red angle within the blue angle and used the black arrows to mark her iterations, she reasoned that "the blue is six times bigger than the red, and then the red can fit six times into the blue." Although she incorrectly estimated the multiplicative comparisons between the two pairs of angles in Figure 64b, she seemed to envision the multiplicative decomposition of the bigger angle using the smaller angle. At the end of this task, she exhibited an example of a composition strategy by iterating the smaller angle within the bigger angle and stating the number of times she could fit the smaller angle into the bigger angle.

Figure 64

The Comparing the Openness Between Angles Task



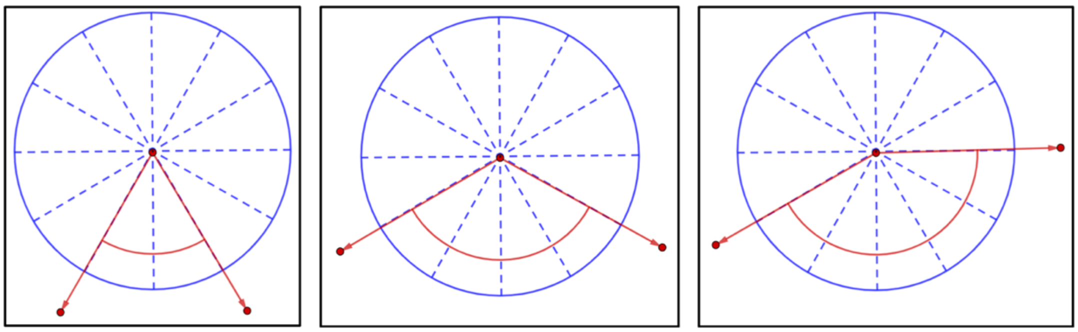
(a) (b)

Note. Alicia compared two pairs of angles where (a) in a pair of angles, the blue angle is three times bigger than the red, and (b) in a different pair, the blue angle is six times bigger than the other red angle.

In the "Modified Growing and Shrinking Angles" task (Figure 65a), Alicia determined the size of the red angle as "2/12." Then, when I asked her to double the red angle, she stated that "it would be 4/12…two times bigger" as shown in Figure 65b. I interpreted that she constructed the 2/12 angle as an angle unit that she grouped into two to double the original angle. However, when I asked her to triple the original angle, she added three 1/12 to the original angle 2/12 and created 5/12 (Figure 65c). I followed up her thinking by asking how she knew it was three times bigger. She reasoned, "because it was at these two lines, and I moved this one [referring to the left side], and I moved this one [referring to the right side] two times." Her reasoning shows that she did not group 2/12 as the unit angle to be tripled as opposed to my initial interpretation when she doubled the original angle. Instead, she considered 1/12 angle as the unit angle and added three 1/12 angles to be tripled. I decided to examine her construction of a unit more in the subsequent tasks.

Figure 65

The Modified Growing and Shrinking Task

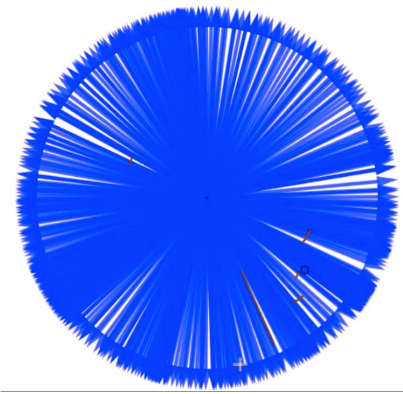


Note. Alicia changed 2/12 angle in (a) by doubling 2/12 into 4/12 in (b), and tripling 2/12 into 5/12 in (c).

Next, in the “Many Very Small Angles” task (Figure 66), I asked Alicia to drag the ray and describe what she was creating. She reasoned, “this looks like the fractions that we were talking about […] they’re cut off.” I followed by asking her how many “cut-off” pieces were there and she replied, “that is a lot, but I’m going to count it.” She counted the lines up to “52” while at the same time skipping some lines. Then, she reasoned that there would be “more because I didn’t count the lighter, the purplish, and the other parts.” She then added that “this looks like there’s angles [drew tiny arcs in between the gaps], there’s a lot of angles.” I infer that Alicia recognized that there were many very small angles in the circle because she intended to count the smaller angles when it was feasible for her. Her statement about “a lot of angles” illustrated an understanding of angle as a discrete quantity.

Figure 66

The Many Very Small Angles Task

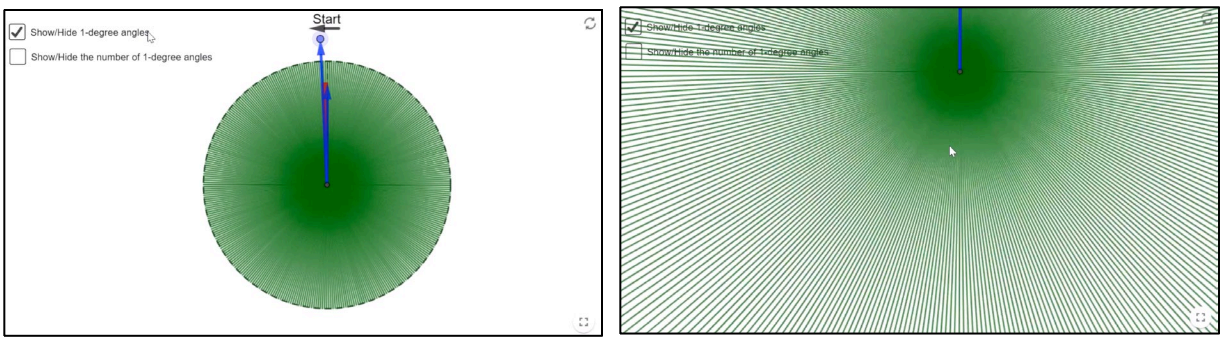


Note. Alicia reasoned about a lot of angles and drew tiny arcs in between the gaps.

Then, I asked Alicia to work on the "360 Angles" task (Figure 67a). At first, Alicia did not recognize any angles because the task looked like a solid circle. When I asked her to zoom into the center of the circle as shown in Figure 67b, she reasoned, "that's really hard to count." Her reasoning shows that she identified that there was a very large number of very small angles within a circle. This form of reasoning is consistent with what she illustrated in the "Many Very Small Angles" task. Subsequently, I decided to introduce the concept of measuring angles in degrees and 360° in a full circle. During our discussions about degrees, she raised a question "if we did a half-circle [she rotated one ray to create half a circle], how many degrees would it be?" I guided her to answer the question by asking her how many degrees the full rotation had. She replied that it was "360," but she could not determine the degrees for the half rotation because "you can't split the numbers. Three is an odd number. So that would be hard." I interpret from her statement that she struggled to split 360 into two equal parts because she probably intended to split the 3 in the hundreds place which was an odd number. Later, Alicia clicked on the "show/hide the number of 1-degree angles" and exclaimed that half of the circle was "180 degrees." She then became curious about the 1/4 of a turn and rotated the ray to create a quarter wedge. She stated, “I was thinking how many is 1/4… It is 90 degrees.” Although Alicia exhibited a numerical difficulty in dividing big numbers such as 360, her reasoning showed that she connected the degrees of an angle with the fractions of a circle.

Figure 67

The 360 Angles Task



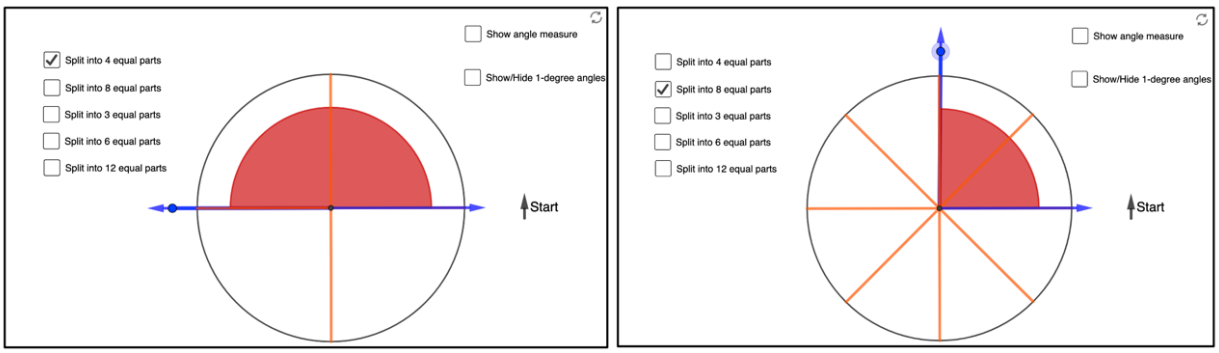
(a) (b)

Note. (a) The 360 Angles task, and (b) viewing the 360 1-degree angles by zooming in.

In the “Splitting a Circle in Fourths” task (Figure 68a), Alicia stated that the number of degrees for 2/4 is “180” because “I remember when we did it in the last one [the half of 360 in the “360 Angles” task], it was 180 degrees.” Her reasoning shows that she considers 1/2 and 2/4 to be equivalent. In finding the angle of 3/4 of a turn, she stated that “it is hard to do 180 plus 90.” So she used the “show angle measure” tool to determine that 3/4 of a turn was “270” degrees. When I asked Alicia why she thought it was 270°, she reasoned, “because when you add 90 plus 90 to get 180, and 90 plus 180… would equal 270.” Although she had difficulty in adding “180 plus 90” at the beginning of the conversation, her reasoning shows that she associated the 90° to 1/4 of a turn as an angle unit. Then, she iterated 90° three times to get 270° for 3/4 of a turn. She exhibited a similar form of reasoning when I asked her about the degrees for 4/4. She replied, “it’s 360” because “180 plus 180 would equal 360.” In this case, her reasoning shows that she was able to associate the 180° of 1/2 of a turn as an angle unit which she doubled to make 360°. One may also interpret that she reasoned in three levels of units by reasoning about a quarter of a turn as 90°, then reasoning for a half turn as 180° in terms of two 90-degree units, and then reasoning for a full turn as two 180-degree units.

Figure 68

The Splitting a Circle into Eighths Task



(a) (b)

Note. When splitting a circle (a) in fourths, Alicia reasoned about the degrees for 2/4 of a turn, and (b) in eighths, Alicia reasoned about the degrees for 1/8 of a turn using the 1/4 of turn.

Next, I asked Alicia to determine the size of each part in the "Splitting a Circle in Eighths" task. She rotated the ray and created a 1/4 wedge as shown in Figure 68. Then, she stated that "this is 1/4." I asked her to focus on the first partition, and she responded that each part was "1/8." When I asked how many degrees was 1/8, she replied, "it is 45…because I separated the 90 into two equal parts, it is 45." I infer that Alicia utilized the 1/4 wedge that she first created, then she used what she learned about 1/4 as equal to 90° to decompose "90 into two equal parts." This form of reasoning was similar to what Axel did at the same task in the previous iteration.

To further examine whether Alicia would use a similar strategy, I asked her to determine the number of degrees for 3/8 of a turn. She stated, "I'm thinking that 90 plus 45…it is 135." When I asked her about 4/8, she answered "180" degrees "because 4/8 is equivalent to 2/4." I infer that Alicia also had prior knowledge about equivalent fractions, and she utilized this knowledge to reason about the degrees for 4/8 of a rotation.

When I asked Alicia what she learned from our session, she explained, "I learned that angles can be very small and that angles can sometimes be in a circle… that 1/8 is equal to 45 degrees, and 1/4 is equal to 90 degrees. The whole circle is 360 degrees." I interpret Alicia's reasoning as illustrating her construction of the connection between the amount of rotations with fractions of a circle and their equivalent degrees. This form of reasoning must have been significantly shaped during her work with the "Splitting a Circle" tasks.

5.4.1. Retrospective Analysis on the Fourth Iteration

Table 7 shows the progression of Alicia's reasoning and the tasks that prompted her to express different forms of reasoning.

Table 7

Alicia's Forms of Reasoning

Student Reasoning	Task
Angle Conception	
A. Angle as union of rays e.g., "An angle to me is a point at a shape…, each corner of a shape." **B. Angle as rotation** e.g., Rotating a segment closer angle "is smaller" while rotating away "makes more space in between and then makes the angle bigger." **C. Angle as wedge prompted her to reason about angles in a circle** e.g., "The thing is, when you just have one plane circle, it would not be an angle because you don't have anything separated."	A. Angles in Shapes task (Figure 59) B. Modified Three Pairs of Segments with Different Lengths task (Figure 60) C. Blue Wedge task (Figure 62)
Multiplicative Comparisons	
A. Splitting a circle into four parts	A. Ferris Wheel task (Figure 63)

e.g., Three-fourth of a circle "this kind of looks like fractions." "It is 1/4. But if I took away these pieces [red wedges], it would be 3/4." **B. Decomposition and composition strategies to compare two angles** e.g., Conjectured that "the blue is three times bigger than the red angle." But she found that "the blue is six times bigger than the red, and then the red can fit six times into the blue." **C. Additive iteration of the original angle to construct multiplicative changes in an angle** e.g., The original angle is "2/12" and to double, "it would be 4/12…two times bigger."	B. Comparing the Openness between Two Angles task (Figure 64) C. Growing and Shrinking Angles task (Figure 65)
Discrete/ Continuous Conception of Angle	
Conceived a circle as having a discrete number of angles e.g., "There's a lot of angles."	Many Very Small Angles (Figure 66)
Numeric Multiplicative	
A. Decomposition of 360° in a circle using quarters of a circle e.g., The degrees in a circle is "really hard to count." **B. Composition of 3/4 as three groups of 90 degrees and the composition of 360 as two groups of 180 degrees.** e.g., For 3/4, "it is hard to do 180 plus 90" so she used the "show angle measure" tool to find "270." She explained, "because when you add 90 plus 90 to get 180, and 90 plus 180 would equal 270." **C. Decomposition of 90 degrees as two groups of 45 degrees to find the degrees of an eighth.** e.g., The degrees of 1/8 of a turn "it is 45…because I separated the 90 into two equal parts, it is 45."	A. 360 Angles task (Figure 67) B. Splitting a Circle in Fourths (Figure 68a) C. Splitting a Circle in Eighths (Figure 68b)

Following the progression of reasoning exhibited by Jordan, Angelie, and Axel about their conceptions of angles as union of rays, rotations, and wedge, I anticipated that Alicia could exhibit a similar development of reasoning. Alicia initially defined angle as a corner of a shape, but her statements did not include consideration of angles with different orientations. As Alicia experienced the dynamic rotation of rays, she progressed into reasoning about angles involving a corner and two sides. Alicia also reasoned about the angle size as the space between the two sides when prompted to change the angle bigger or smaller in the "Modified Three Pairs of Segments with Different Lengths" task.

When Alicia engaged with tasks involving angles as fractions in a circle generating a traced wedge, she developed the understanding of angles in a circle. The feature of a wedge that can be created into a full circle probably prompted Alicia to reorganize her thinking that a circle has angles. This was in contrast to her initial conception that there are no angles in a circle probably because she only conceived angles as corners of a shape and that circles do not have corners. This reorganization of her reasoning seemed to prepare Alicia in reasoning about angles as fractions of a circle in the “Ferris Wheel” task. While engaging in the task, she recognized the 3/4 wedge as a fraction of a circle. This reasoning showed her prior knowledge about quarters of a circle. She also showed in her reasoning that she understood the whole turn is decomposed into four equal parts.

When she worked with the “Comparing the Openness between Two Angles” task, Alicia reasoned multiplicatively about angle size. Her reasoning showed that she probably envisioned the multiplicative iterations of the smaller angle into the bigger angle to compare the sizes of the two angles. Her reasoning on the number of times she could decompose the bigger angle using the smaller angle and composing it back by a number of times that she could fit the smaller angle into the bigger angle illustrated the reversibility of the interiorized measurement processes (Moore, 2012).

In the “Modified Growing and Shrinking Angles” task, Alicia seemed to treat the 2/12 angle as the unit angle and added another 2/12 to double the original angle illustrating a coordinated measurement approach (Izsák & Beckmann, 2019). When tripling the original angle, Alicia did not use 2/12 as the unit. Instead, she used 1/12 angle. Then, she added three pieces of 1/12 to the original 2/12 angle to make it triple to create a 5/12 angle. Although I convinced Alicia to restart with the task and treat 2/12 as the angle being tripled, she was certain that she was tripling it the right way. I looked for further evidence about her thinking of the multiplicative relationships between angles in the succeeding tasks.

In the “Many Very Small Angles” task, Alicia exhibited an understanding of angle as a discrete quantity as described by Castillo-Garsow et al. (2013). When reasoning about the multiplicative relationship between angles in the “Splitting a Circle” task, Alicia relied on the “show angle measure” tool to determine the degrees of an angle. Although she heavily relied on the tool, she meaningfully decomposed the degrees of a fraction of a turn into smaller equal degrees. Then, she iteratively added the degrees of the smaller equal angles to justify the degrees of the bigger angle.

5.5. Concluding Remarks

In conclusion, students’ prior knowledge about angles influences the construction of a variety of angle definitions that they bring with them prior to formal instruction. Their angle definitions reflect one of the three angle conceptions: an angle as a union of rays, as a rotation, and as a wedge. Connecting the conception of angles as involving a common point and two sides and that angle size is the space between the two sides supports the bridging of Euclidean and topological relationships in the meaning-making about angles (Piaget & Inhelder, 1956). Bridging the two relationships offered a broader understanding of angles which has been useful for students in quantifying angles. For instance, conceiving angles at the center of a circle was found to be important for students to construct a connection between the fractions of a turn and fractions of a circle, then connect these fractions with degrees (Confrey et al., 2012).

The chronological accounts of each student presented my ongoing analysis of their reasoning about angles while they engage with the design of tasks, tools, and questioning. The findings suggest that the design prompted students to exhibit four distinct categories of reasoning about angles: angle conceptions, multiplicative comparisons, discrete or continuous conception of an angle, and numeric multiplicative reasoning. I summarized these categories in each design experiment as shown in Table 4, Table 5, Table 6, and Table 7, respectively.

The first level of retrospective analysis suggests that the design of the tasks, tools, and questioning may have led the progression of students' reasoning. However, students' prior knowledge about fractions and degrees significantly differentiated the reasoning in each task. The ways in which students responded or acted in each experiment characterize the modifications of the design for the succeeding design experiment. Specifically, students reasoning informed the exclusion of wedge in the exploratory tasks, the use of "show angle measure" tool, and questioning structures that potentially supported student reasoning in the subsequent design experiments.

In the next chapter, I will discuss the second level of retrospective analysis where I cross-compared students' reasoning after the four design experiments were completed. In response to *Research Question 1*, I will present the levels of sophistication of reasoning that the students made in each category of reasoning. In response to *Research Question 2*, I will discuss how the design of the tasks, tools, and researcher questioning might have supported students' different forms of reasoning about angles.

Chapter 6: Findings – Part 2 (Cross-Case Analysis)

In Chapter 5, I presented the ongoing analyses during each design experiment in the form of chronological accounts of students' reasoning while engaging in the design. I also presented a first level of retrospective analysis which I conducted at the end of every design experiment, where I reflected on each student's reasoning and the design that supported their reasoning. The reflection on how the design was changed after every design experiment responds to research question *(3)*. In this chapter, I present the second level of retrospective analysis in which I cross-compared students' reasoning after all the design experiments were completed. To respond to research question *(1),* I compared their forms of reasoning and discuss the four categories arising from their similarities and differences, namely forms of reasoning about angle conceptions, multiplicative comparisons between angles, discrete or continuous conceptions of angles, and numeric multiplicative relationships between angles. Within each category, I present students' forms of reasoning in subcategories that illustrate their qualitative differences and similarities. To respond to research question *(2),* after the discussion of in each category, I discuss how particular features of the design prompted and supported those forms of reasoning.

6.1. Angle Conceptions

Table 8 shows the subcategories of reasoning that students exhibited about angles as union of rays, wedges, and rotations. At the beginning of the design experiment all students illustrated a prior knowledge of an angle as a union of rays. By engaging with the dynamic rotational tasks, they then constructed a conception of angles as rotations. Students reasoned about angles as rotations in a similar way showing the impact of the task design on the formation of their reasoning. When they engaged with tasks that illustrated angles as wedges, they showed that they reorganized this reasoning to reflect both their prior knowledge and their experience with rotations as well as a conception of an angle as a wedge. The following paragraphs describe the cross-comparison of their reasoning in more depth.

Table 8

Students' Forms of Angle Conceptions

Angles as Wedges	Reasoning about a wedge as the amount of space between two angle sides, while incorporating degrees to describe the amount of space (Axel).
	Reasoning about a wedge as the amount of space covered as a fraction of a circle and that a quarter rotation has 90 degrees (Jordan).
	Reasoning about a wedge as a circular space created by rotations (Angelie).
	Reasoning about an angle as a wedge part of a circle (Alicia).
Angles as Rotations	Reasoning that rotating one angle side farther away from the other side makes the angle larger; otherwise, the angle gets smaller (Angelie).
	Reasoning that rotations of a segment away from the other segment create more space between angle sides without reasoning about the reversible relationship when the rotation is done at the opposite direction (Jordan and Alicia).
	Reasoning about creating an angle by *pulling* one side away from the other side but the reasoning did not imply reversibility of the result when the pulling is done at the opposite direction (Axel).
Angles as Union of Rays	Defining an angle as composed of two rays with a common end point and those rays can take different orientations (Axel).
	Defining an angle as an orientation of a line (Angelie).
	Defining an angle as a corner or vertex (point) of a shape (Jordan and Alicia).

To begin with, students exhibited a conception of an angle as a union of rays in three different ways: angle as a corner of a shape, as an orientation, and as a combination of these two. At the beginning of the design experiment, Jordan and Alicia reasoned about an angle as a corner of a shape. For instance, Jordan reasoned about an angle as a corner and Alicia reasoned that an angle is a point of a shape. This form of reasoning is the prevalent response among third-grade students (Clements & Battista, 1989). Therefore, I consider their reasoning to be a result of their prior learning of static angles as described in the literature (e.g., Devichi & Munier, 2013; Keiser, 2004; Smith et al., 2014). On the other hand, Angelie showed a conception of an angle as an orientation by referring to an angle as a single line with different orientations. For Angelie, the angle side could have different inclinations. Browning et al. (2007) found that students often experience difficulties in recognizing angles without a second side that lies horizontally. Angelie shows that she did not have this difficulty. Her reasoning also showed that she did not have a conception of the right-angle prevalence as described by Devichi and Munier (2013) where both angle sides should be perpendicular. Therefore, I considered her conception to be different from Jordan's and Alicia's. In contrast to these three students, Axel's reasoning showed a combination of conception of angles as composed of two rays with a common end point and that the angle sides can point to different directions. I interpreted his reasoning of pointing to different directions as involving orientations. I regarded Axel's reasoning as more sophisticated than the other three students because he showed a more complex prior knowledge about an angle.

As they engaged in tasks generating angles, students exhibited a conception of an angle as a rotation in three ways. First, Axel's reasoning about pulling of an angle side showed a partial conception of the relationship between the proximity of the two angle sides and the size of the angle. This kind of reasoning reflects a conception of angle size using the linear distance between the two connected sides (Keiser, 2004; Thompson, 2013). I refer to his reasoning as illustrating a "partial" conception because he did show that he understands the change in the proximity of the two sides when the rotation was done at the opposite direction. Jordan and Alicia reasoned that rotations of one side away from the other side create more space between angles sides, thus making the angle bigger. I consider their reasoning to be a bit different than Axel's because their reasoning illustrates the relationship between the amount of rotation and the amount of space between the sides. However, similar to Axel, their reasoning show a partial conception because they did not show how the rotation at the opposite direction could influence the change in the amount of space between the two sides and the angle size. Unlike the three students, Angelie showed that she constructed this reversible relationship between the change in direction and amount of rotation with the change in angle size.

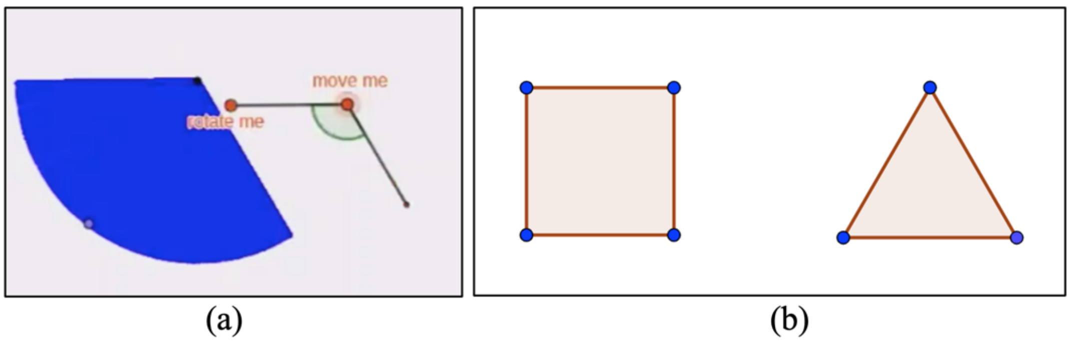
While engaging with the tasks involving wedges, students reorganized their reasoning to include different ways of expressing an angle as a wedge. For example, Alicia's prior understanding that there were no angles in a circular wedge was reorganized to recognize angles as wedges. Angelie's prior reasoning about making an angle bigger or smaller through a rotation was also reorganized to include the size of the wedge created during the rotation. Jordan showed a more complex reasoning when he associated angles with quarters of a circle and that a quarter has 90°. However, when he was asked to explain, he could not describe what 90° meant. On the contrary, Axel connected the size of a wedge with the amount of space between the angle sides, and that this space can be measured in degrees. Unlike Jordan, Axel's reasoning about degrees was not limited to 90° for a quarter of a circle. Instead, he successfully associated respective degrees with each quarter turns in a circle similar to what was described by Confrey et al. (2012).

6.1.1. Design that Potentially Elicited Students' Different Angle Conceptions

I observed that students' angle conceptions (Table 8) were elicited as they interacted with specific tasks, tools, and questioning. Angelie and Axel expressed their prior knowledge when they were asked "What is an angle?" before working on any task. For instance, when Angelie was asked "What is an angle?", she drew lines with different orientations. On the contrary, Jordan and Alicia expressed their prior knowledge about angles as corners as they were working with specific tasks. Tasks that illustrate angles as union of two segments or rays seemed to prompt these students to illustrate their prior knowledge about angles as corners of a shape. For instance, Jordan reasoned about an angle as a corner as he worked on the "Comparing Angles with a Fixed Object" task which involves a fixed angle tool with two line segments connected at a point (Figure 69a). Likewise, Alicia reasoned about an angle as a corner as she engaged with the "Angles in Shapes" task that showed shapes with different number of corners (Figure 69b).

Figure 69

Different Tasks that Potentially Elicited Conception of Angles as Union of Rays

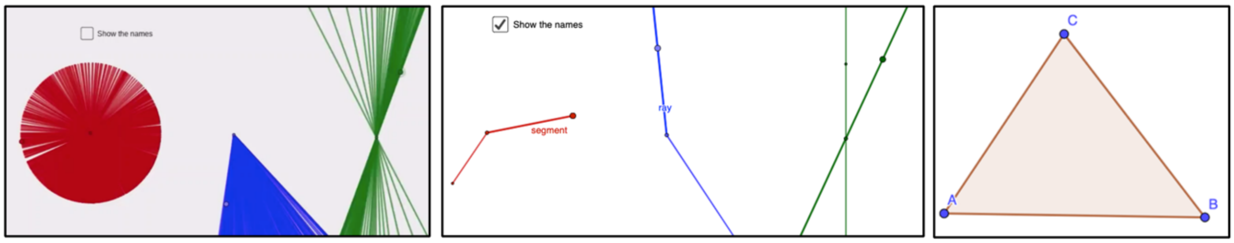


Note. The (a) Comparing Angles with a Fixed Angle Object and (b) Angles in Shapes tasks.

All the tasks in the design have the dynamic rotation feature and this supported students to rotate the angle sides and develop a conception of angles as rotations. For instance, the tasks in which students first reasoned about angles as rotations were the "Three Pairs of Different Objects" task (Figure 70a) for Jordan, the "Modified Three Pairs of Different Objects" task (Figure 70b) for Angelie and Alicia, and the "Triangle" (Figure 70c) task for Axel. In addition to the features of the tasks, my questions "What is changing?" and "How do you make an angle bigger/smaller?" probed them to focus on the quantities of the situation and reason about the change in rotations and the change in angle size. These dynamic explorations showed to elicit varied forms of conceptions of angles as rotations. For instance, when I asked students how they could make the angle bigger, Jordan and Alicia reasoned that they could create more space as they rotate one side away from the other side.

Figure 70

Different Tasks that Potentially Elicited Conception of Angles as Rotations



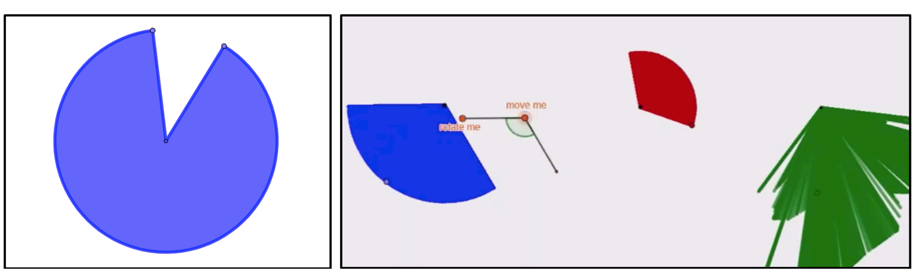
(a) (b) (c)

Note. The (a) Three Pairs of Different Objects, (b) Modified Three Pairs of Different Objects, and (c) Triangle tasks.

Additionally, I observed that the tasks involving wedges encouraged reasoning about angles as wedges. For example, while engaging with these tasks, Jordan reasoned about angles as fractions of a circle rather than conceiving angles in terms of openness. An example of tasks that have this wedge feature was the "Blue Wedge" task (Figure 71a) with a preconstructed wedge that students increase or decrease in size when they rotate at least one side. Another example was the "Comparing Angles with a Fixed Angle Object" task (Figure 71b) with the tracing tool that left traces of rotations when generating angles. Although the two tasks behave differently when a segment is rotated, they both involve generating wedges.

Figure 71

Different Tasks that Potentially Elicited Student Conception of Angles as Wedges



(a) (b)

Note. The (a) Blue Wedge and the (b) Comparing Angles with a Fixed Angle Object tasks.

It is worth mentioning here that in the early tasks with Jordan, the wedge tasks seemed to hinder him from focusing on the openness of the angle. Instead, he kept associating the size of an angle with the space within a wedge as described by Browning et al. (2007). Therefore, I modified these tasks to avoid showing a wedge during those early explorations. This modification showed to have worked for the students to not show this alternative conception. In the subsequent design experiments, when students were asked to explain what they were looking at to make judgements about the angle size, they stated that they looked at the amount of space between the angle sides, reflecting what (Moore, 2012) and Hardison (2019) studies reported. I interpret this reasoning to possibly illustrate their conception of an angle in terms of openness.

6.2. Multiplicative Comparisons Between Angles

In comparing students' reasoning about multiplicative comparisons between angles, I placed them in three distinct subcategories (Table 9): using their initial splitting strategy, using decomposition and composition when comparing angles, and reasoning about the multiplicative change in an angle. Each subcategory shows how students' reasoning qualitatively differed from one another. Similar to the previous form of reasoning, students' prior knowledge continued to play a significant role in their reasoning. The following paragraphs describe the cross-comparison of their reasoning in more depth.

Table 9

Students' Forms of Reasoning about Multiplicative Comparisons Between Angles

Multiplicative Change in an Angle	Doubling or tripling an angle by treating the original angle (2/12 of a circle) as a single group or angle unit and adding this angle unit two or three times (Jordan, Angelie, and Axel).
	Doubling or tripling an angle by treating the half (1/12 of a circle) of the original angle as the angle unit and adding this angle unit two or three times to the original angle (Alicia).

Decomposition and Composition When Comparing Angles	Estimating and imagining the number of times he could rotate the smaller angle within the bigger angle, then actually composing the bigger angle by iterating the smaller angle within the bigger angle (Jordan).
	Estimating the fractional part of the bigger angle where the smaller angle could fit without a clear illustration on how they split the bigger angle, then actually composing the bigger angle by iterating the smaller angle, while incorporating the closeness of the lines (Angelie and Alex).
	Estimating that the bigger angle was "two fractions bigger" than the smaller angle without a clear illustration on how the bigger angle was split, then actually composing the bigger angle by iterating the smaller angle within the bigger angle (Alicia).
Initial Splitting Strategy	Splitting a full rotation into quarters and associated the quarter turns with the quarters of a circle, then associated the number of degrees with each fraction of a turn (Axel).
	Splitting a full rotation into quarters and associated a quarter of a turn with each quarter of a circle, then only associated 90° with a quarter of a turn (Jordan).
	Splitting a full rotation into quarters and associated the split with each quarter of a circle (Angelie, and Alicia).

Students exhibited two forms of reasoning about the multiplicative comparisons between angles using the splitting strategy. Jordan, Angelie, and Alicia exhibited similar forms of reasoning when splitting a full rotation into quarters of a circle. Angelie, Alicia, and Jordan showed the connections between the quarters of a turn with the quarters of a circle, but only Jordan connected a quarter of a circle with 90°. However, his reasoning was only limited to angles in 90° resulting to not being able to connect between the other quarters of a turn and their degrees. This is not surprising as the conception of right angle is prevalent among elementary students (Devichi & Munier, 2013). Only Axel exhibited an understanding about the degrees for each quarter turn. His reasoning showed that he was able to measure rotations as fractions of a turn and then create a connection with the degrees (Clements & Sarama, 2014). I consider his reasoning as more sophisticated than the three students because Axel illustrated his prior knowledge about degrees without being prompted by my questioning.

The design also prompted the students to reason multiplicatively by decomposing and composing angles (Moore, 2012). In comparing the openness of two angles, students exhibited three qualitatively different ways of reasoning about angles using the decomposition of the bigger angle and its composition using the iteration of the smaller angle. Alicia conjectured that the bigger angle is “two fractions bigger” than the smaller angle, which I infer that she was estimating the bigger angle as being double the size of the smaller angle. Angelie and Axel illustrated similar reasoning when they conjectured that a smaller angle is a fraction of the bigger angle. While Alicia did not explain how she reached that conclusion, Angelie and Axel talked about the closeness between each pair of angle sides which potentially helped them to argue how one angle was bigger than the other. Among the four students, only Jordan reasoned that he imagined the number of times he could rotate the smaller angle within the bigger angle by comparing the lines that make the angles more open. Although all students did not always make accurate estimations about the openness of each pair of angles, they were able to reorganize their meanings to talk about the relationship between two pairs of angles by iterating the smaller angle to compose the bigger angle. Their reorganizations show the potential of the design to stimulate students in envisioning the decomposition of an angle and iterating back the smaller angle into the bigger angle resulting to multiplicative comparison between the two angles. Envisioning the composition and decomposition of angles are interiorized measurement processes as described by (Moore, 2012).

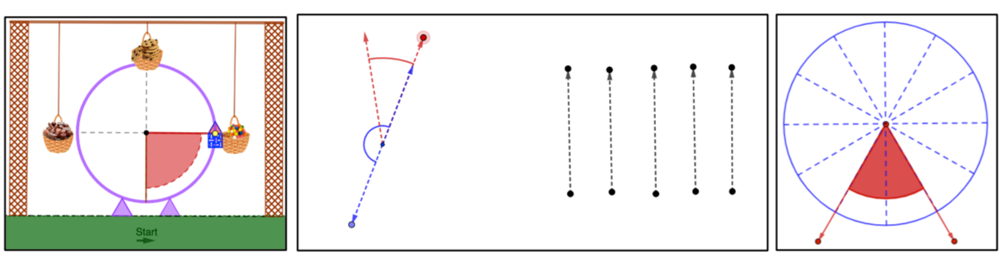
When students multiplicatively changed an angle, they showed two distinct forms of reasoning. These forms of reasoning differed on the ways they treated the original angle as a unit angle to be iterated. Alicia treated the 1/12 wedge as her unit angle instead of the given angle represented by the 2/12 wedge, showing that she only perceives unit fractions to be original angles. Conversely, Jordan, Angelie, and Axel treated the original angle 2/12 wedge as the angle unit, then multiplied this angle unit with a number of iterations (groups) to create the required multiplicative changes such as double or triple. Their strategy illustrates a coordinated measurement approach (Izsák & Beckmann, 2019) because they considered the product quantity (e.g., 4/12 wedge) as being simultaneously measured using two units, the angle unit that composed a single group (2/12 wedge) and the number of groups (2 groups) of the angle unit to make the product quantity.

6.2.1. Design that Potentially Elicited Students' Reasoning about Multiplicative Comparisons

The tasks that involved the equipartitioning of a full turn potentially elicited students to reason about the multiplicative comparisons between angles as shown in Table 9. These tasks were the "Ferris Wheel" task (Figure 72a), the "Comparing the Openness between Two Angles" task (Figure 72b), and the "Growing and Shrinking Angles" task (Figure 72c).

Figure 72

The Tasks that Prompted Reasoning about Multiplicative Comparisons Between Angles



(a) (b) (c)

Note. The (a) Ferris Wheel, (b) Comparing the Openness between Two Angles, and (c) Growing and Shrinking Angles tasks.

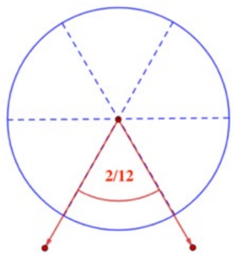
To elaborate, in the "Ferris Wheel" task (Figure 72a), the perpendicular diameters that split the circle into four equal parts potentially supported students' reasoning when splitting a full rotation. My questioning here focused on the amount of turn the cart made in reaching different objects around the Ferris wheel. This task prompted Angelie and Alicia to exhibit a prior knowledge about fractions of a circle and a lack of knowledge about degrees. Alicia's reasoning about the absence of angles in circles informed the modification of my questioning to prompt her to conceive angles in a full rotation, consequently, led her to reason multiplicatively about angles in a circle. This task might have also supported Axel to focus on the visual partitions that split the circle into four equal parts and utilize this prior knowledge with degrees to describe each quarter turn with 90°. This may imply that when the design of the tasks prompted students to quantify angles, reasoning about the multiplicative relationships was more intuitive, particularly for students who had prior knowledge about fractions of a circle and degrees.

In the "Comparing the Openness between Two Angles" task (Figure 72b), my questioning focused on comparing the bigger angle to the smaller one. Students' conjectures varied based on their prior knowledge about fractions and their attention to the closeness between angle sides. However, the design of the task with benchmark arrows that students could use to mark their iteration of the smaller angle within the bigger angle potentially supported Jordan to exhibit a sophisticated form of decomposition and composition of angles. The same design prompted the other three students to reorganize their thinking after they inaccurately conjectured about the openness of each pair of angles.

The "Growing and Shrinking Angles" (Figure 72c) task was the third task given to students at this stage. My questioning focused on how they could double or triple an angle. Jordan worked on the initial version of the task, while Angelie, Axel, and Alicia worked on the modified version that did not have a wedge. This was the task that Alicia iterated the half of the original angle, 1/12 as the unit angle. This might have been the result of showing the 2/12 angle as being split into two parts. A modified version that does not show this split may help her to consider the 2/12 as a unit angle (Figure 73). In a future iteration, it might be useful to explore this possibility.

Figure 73

A Modified Version of the Growing and Shrinking Angles Without Splitting the Original Angle



Note. A potential task modification that shows the angle inside the red wedge measures 2/12 which could be multiplicatively changed.

6.3. Discrete or Continuous Conception of Angles

Table 10 shows students' different subcategories of reasoning about angles as a discrete or continuous quantity. Conceiving a quantity as discrete involved thinking about discrete chunks while conceiving a quantity as continuous involves envisioning intermediate and infinite amounts (Castillo-Garsow, 2012). While most students conceived angles as a discrete quantity, Axel illustrated evidence of conceiving an angle as both a discrete and a continuous quantity.

Table 10

Discrete or Continuous Conception of Angles

Discrete or Continuous Conception of Angle	Reasoning about an angle as a continuous quantity by stating that there is always an angle in between and that you can make unlimited angles (Axel).
	Reasoning about angles in a circle as a discrete quantity by stating that there is a very large number of small angles but they are countable (Jordan and Alicia).
	Reasoning about angles in a circle as a discrete quantity by stating that it depends on the size of the wedge that could fit a number of times in a circle (Angelie).

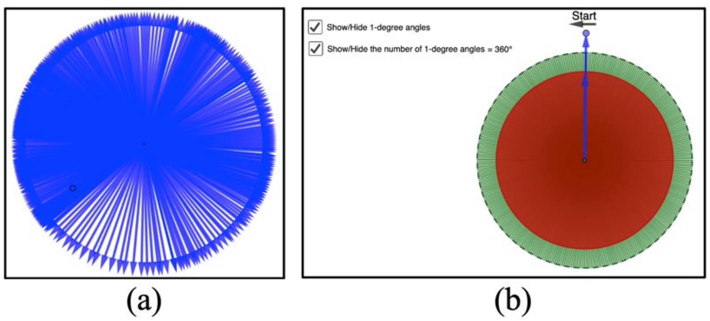
I classified students' reasoning about an angle as either a discrete or a continuous quantity in three subcategories showing different levels of sophistication. Angelie illustrated only a conception of an angle as a discrete quantity by referring to angles in chunks only. For instance, she reasoned that she would need four quarter wedges to cover the whole circle. Jordan and Alicia showed a more sophisticated form of conceiving angles as a discrete quantity by considering a very large number of very small angles. Although Jordan and Alicia explained that it would take too long to count the angles in a circle, their reasoning showed that they could count them when given the opportunity to do so. Similarly, in the "Many Very Small Angles" task, Axel reasoned that the multiple traces of a rotation show "a lot of" angles illustrating a reasoning about an angle as a discrete quantity. However, he then reasoned about unlimited angles and that there were smaller intermediate angles in between angles. His reasoning about unlimited angles and intermediate angles exemplifies a continuous and smooth thinking about a quantity (Castillo-Garsow, 2012).

6.3.1. Design that Potentially Elicited Discrete or Continuous Conception of Angles

There is evidence to show that the design of the tasks, tools, and questioning influenced students' reasoning about an angle as either a continuous or a discrete quantity. The "Many Very Small Angles" task (Figure 74a) was designed to show the continuous change in an angle during a rotation that could potentially elicit conceptions of an angle as a continuous quantity. However, students reasoned about an angle as a discrete quantity as they tried to count the discrete number of angles. My questioning which focused on how *many* angles there are may have also reinforced this kind of discrete reasoning. Meanwhile, the one-degree wedge in the "360 Angles" task (Figure 74b) and my questioning of whether there are angles smaller than one-degree prompted Axel to reason that there were intermediate angles in between angles. His reasoning about the number of one-degrees in a circle offered an opportunity for me to modify my questioning to examine his understanding of angles smaller than a one-degree angle. Specifically, he reasoned that he could always cut an angle into halves and make unlimited angles, illustrating his conception of an angle as a continuous quantity.

Figure 74

Different Tasks that Elicited Discrete or Continuous Conception of Angles



Note. The (a) Many Very Small Angles task and the (b) 360 Angles task.

Interestingly, the differences in students' conception of angles as either a continuous or discrete quantity did not seem to offer a significant contrasting conception of angles in any of the students' other forms of reasoning. In a future iteration, it might be important to explore further how students would reason about angles smaller than one-degree and if Axel's reasoning can be a form of a generalization for other students.

6.4. Numeric Multiplicative Relationships Between Angles

I classified students' reasoning about the numeric multiplicative relationships between angles in two subcategories as presented in Table 11. The first subcategory involves reasoning about the composition of a 360-degree angle and the second is about reasoning about the angles when a circle is split into a number of equal parts. In the following paragraphs, I present the different forms of reasoning that students exhibited in each category.

Table 11

Numeric Multiplicative Relationships Between Angles

Splitting a Circle	Decomposed and composed multiplicative groups of smaller identical angles to create an angle, while incorporating the knowledge about equivalent fractions and using a compensation strategy when finding the degrees of an angle (Axel).
	Decomposed and composed multiplicative groups of smaller identical angles to create an angle, while incorporating the knowledge about equivalent fractions when finding the degrees of an angle (Jordan).
	Decomposed an angle into multiplicative number of identical smaller angles and composition of multiplicative groups of smaller identical angles to create an angle and finding the degrees of an angle (Alicia).
	Composed multiplicative groups of smaller identical angles to create an angle and finding the degrees of an angle (Angelie).
Composition of a 360 Degree Angle	Reasoned about one-degree as the smallest angle and adding one-degree 360 times to create a 360-degree angle (Angelie).
	Reasoned that a quarter rotation or 90° as a benchmark angle to reason about the multiplicative composition of 360° by the number of quarters of a circle (Jordan)
	Reasoned about 360° as showing all the angles from one-degree to 360 degrees (Axel).
	Reasoned that degrees in a circle is difficult to count but understands that a full rotation has 360° (Alicia).

Students exhibited four different forms of reasoning when discussing the composition of a 360-degree angle. Alicia reasoned that the degrees in a circle would take her a long time to count. On the other hand, Axel showed that he conceived 360° as being composed of one-degree angles. It was not clear whether he perceived this composition additively or multiplicatively. Jordan illustrated a more sophisticated form of reasoning by considering a quarter rotation or 90° as a benchmark angle that can be multiplicatively transformed into four 90-degree angles. Similarly, Angelie conceived the smallest angle as one-degree and that she could add one-degree angles three hundred sixty times to create a 360-degree angle. Both the reasoning exhibited by Jordan and Angelie showed the coordinated measurement approach to multiplication (Izsák &

Beckmann, 2019). To elaborate, Angelie coordinated three hundred sixty groups of one-degree angle to create 360°. Jordan coordinated four groups of 90° to form the new angle of 360°.

Students also exhibited coordinated measurements (Izsák & Beckmann, 2019) when splitting a circle into fractional parts and finding the degrees of each part. To illustrate, Angelie coordinated multiplicative groups of smaller identical angles to compose an angle. She reasoned that an eighth of a full turn is 45°, so 45 plus 45 equals 90°. Alicia reasoned in a more sophisticated way by decomposing an angle multiplicatively into identical smaller angles in degrees. Then, she composed the multiplicative groups of smaller identical angles to recreate the decomposed angle. Jordan also expressed this reversibility by decomposing and composing angles using his knowledge about equivalent fractions to identify the degrees of the angle being asked. For instance, in splitting a circle into thirds and into sixths tasks, Jordan reasoned that he could "add 60 every time" for every multiple of 1/6 wedge. Later, he showed and reasoned that the 2/6 wedge is the same as 1/3 wedge each has 120°.

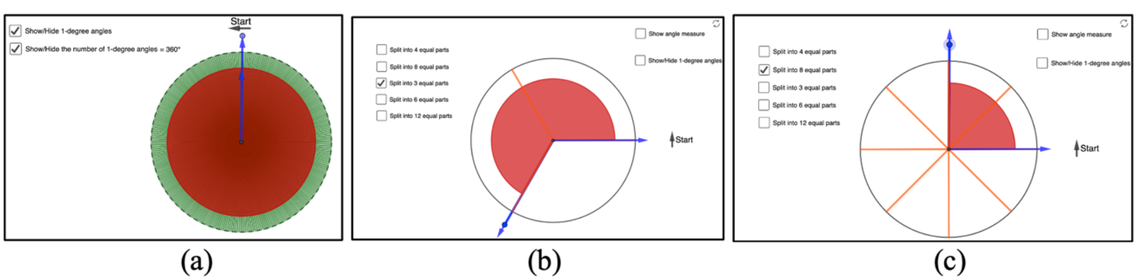
In contrast to the other students, Axel exhibited a variety of strategies in finding the degrees of an angle. His strategies included the decomposition and composition of multiplicative groups of smaller identical angles to create an angle, finding the degrees using equivalent fractions of a circle, and using a compensation strategy where he only subtracted the degrees of a unit angle from the whole instead of adding the degrees of the unit angle multiple times. For example, in splitting a circle into eighths task, Axel deductively explained that an eighth was "45 degrees because it is half of 90 degrees (1/4)." Then, he used this reasoning to explain that the 7/8 of a turn was "315 because 360 minus 45." For Axel, the compensation strategy was easier than to "add 45 degrees seven times." Axel's reasoning showed that he had these three forms of strategies as conceptual tools in finding the degrees of an angle in a circle. Among the reasoning of the four students, the decomposition and composition strategy and the coordinated measurement approach were the most prevalent forms of reasoning when finding the degrees of an angle in a circle. These forms of reasoning were also illustrated by the students when constructing multiplicative comparisons between angles as shown in Table 9 above.

6.4.1. The Design that Elicited Reasoning About Numeric Multiplicative Relationships Between Angles

The design of the "360 Angles" task (Figure 75a) illustrating the 360 one-degree angles in a full rotation and the "show angle measure" tool that shows the number of degrees potentially prompted students to exhibit different forms of reasoning about the composition of a 360-degree angle as presented in Table 11. Additionally, the splitting tools in the "Splitting a Circle" tasks (Figure 75b) and questioning about the degrees of a fraction of a full turn potentially prompted students to reason about numeric multiplicative relationships between angles.

Figure 75

The Splitting a Circle Task that Elicited Reasoning about Numeric Multiplicative Relationships Between Angles



Note. The (a) "360 Angles" and the "Splitting a Circle" (b) in thirds and (c) in eighths that potentially elicited reasoning about numeric multiplicative relationships between angles.

To elaborate, in the "360 Angles," Angelie and Alicia seemed to rely on the "show angle measure" tool to reason about the number of degrees composed in a full rotation. The tool may have helped Angelie to reason that she could add one-degree angles 360 times to compose the 360-degree angle. Meanwhile, Axel already had prior knowledge about 360° that he did not need to use the "show angle measure" tool. His reasoning offered an opportunity for me to modify my questioning to examine his understanding of angles smaller than a one-degree angle. Jordan also did not use the tool to reason about the number of degrees in a full rotation. Instead, he leveraged his prior knowledge that 1/4 rotation has 90° and added 90° for every 1/4 rotation.

Similarly, the splitting tools in the "Splitting a Circle" tasks (Figure 75b) helped students to envision the splitting of the circle into four, eight, three, six, or twelve equal parts, respectively. The "show angle measure" tool also potentially helped students to verify their conjectures about the degrees for each fraction of a turn. For example, Jordan, Angelie, and Alicia relied on the "show angle measure" tool to determine the degrees in an angle and this helped them to construct their reasoning about the decomposition and composition of an angle. The line segments that split the circle may have offered the students a constructive space to reason about the multiplicative groups of identical angles to compose or decompose an angle. In contrast, Axel did not use the "show angle measure" tool to determine the degrees of an angle. Instead, he used his prior knowledge about 360° in a circle and the equivalent fractions to exhibit a flexible understanding of using different fraction operations to find the degrees for the same amount of turn.

6.5. Concluding Remarks

In conclusion, in this chapter I classified students' forms of reasoning into four main categories: reasoning about the three conceptions of an angle (Table 8), reasoning about the multiplicative comparisons between angles (Table 9), conceiving of an angle as either a discrete or a continuous quantity (Table 10), and reasoning about the numeric multiplicative relationships between angles (Table 11). Each category was described based on the similarities and differences in students' reasoning that were presented in terms of subcategories within the main categories.

I found that the data offered empirical evidence that supports how the tasks, tools, and questioning could potentially elicit different forms of reasoning as they support a variety of prior knowledge that students bring in when engaging with the design. The framing of my questioning also played a vital role in stimulating students' prior knowledge and supporting them in reorganizing their reasoning. Furthermore, students' prior knowledge and reasoning influenced the evolution of my design. The next chapter discusses the implications that these findings might have for research and practice.

Chapter 7: Conclusions

In this concluding chapter, I first summarize the findings presented in Findings – Part 1 (Cases-by-Case Analysis) and Findings – Part 2 (Cross-Case Analysis). Next, I discuss the contributions of this study to mathematics research and practice. I then, outline the limitations of this study, and suggest directions for future research.

7.1. Summary of the Dissertation

This dissertation was an exploratory study to investigate how elementary school students may reason about angles as they engage in digital tasks that present angles dynamically. Specifically, this study focused on examining the following three research questions:

1. What forms of reasoning do students exhibit as they engage in dynamic digital tasks that aim to bridge the three conceptions of angles?
2. What characteristics of the design (e.g., characteristics of tasks, tools, and questioning) support the particular forms of students' reasoning for angles?
3. How did the design evolve to support students' reasoning for angles?

I followed a *design experiment* (Cobb et al., 2003) methodology to engineer particular forms of student reasoning about angles and to study these forms of reasoning as students engage with the design. The initial conjectures and design of the tasks were influenced by the existing research literature on angles, measurement, and multiplicative reasoning. To begin with, the exploration of the literature shows that static models of angles illustrating the union of two rays often encourage alternative conceptions of angles (e.g., Devichi & Munier, 2013; Smith et al., 2014). Research also suggests that a dynamic representation of angles through rotations could help students construct mathematical meanings of angles such as the amount of turn (Confrey et al., 2012; Moore, 2012). However, multiple rotations are difficult to measure. Research supports the use of wedges (Browning & Garza-Kling, 2009; Thompson, 2013) to help students quantify angles. Wedges allow the change in an angle size to be visible to students. Consequently, the research literature suggests that a combination of all three conceptions of an angle – as a union of rays, as a rotation, and as a wedge – may offer better opportunities for students to develop a robust understanding of an angle (e.g., Freudenthal, 1973).

To bridge these three conceptions, I followed the *dynamic measurement* (DYME) (Panorkou, 2021) approach to design digital tasks that illustrate how angles are generated and changed dynamically. Using a quantitative reasoning lens (Thompson, 2011), I conjectured that students could conceive the changing quantities during the generation of dynamic angles and reason about how the quantities were changing in relation to each other. By reasoning quantitatively about angles, students could unify the geometric and multiplicative nature of angles in their reasoning. To test my conjectures, I collected qualitative data from individual virtual interviews with four third-grade students as they engaged with the tasks, tools, and my questioning. I conducted two phases of data analysis. First, I conducted an ongoing analysis during each design experiment. Second, a retrospective analysis was conducted at the end of each design experiment to plan for the next design experiment (first level of retrospective analysis) and at the end of all design experiments, I cross-compared the students' reasoning (second level of retrospective analysis).

7.1.1. Students' reasoning about angles (Research Question 1)

In the Findings – Part 2 (Cross-Case Analysis) chapter of this study, I identified four categories of student reasoning about angles as students engaged in dynamic digital tasks that aimed to support students to bridge the three conceptions of angles (*Research Question 1*). In each category, I classified student reasoning into different subcategories. I found that by engaging with the task design, students were able to reason about all three conceptions of an angle – as a union of rays, as a rotation, and as a wedge – showing that they bridged the three conceptions of angles (*Category 1*). Students also constructed multiplicative comparisons between angles (*Category 2*). Specifically, students reasoned about the multiplicative change in an angle, decomposition and composition of angles, and the splitting of a full rotation to an angle. Students also reasoned about an angle as a discrete or a continuous quantity (*Category 3*). Particularly, students either reasoned that there was a very large number of angles illustrating a discrete conception or that there were unlimited angles showing a continuous conception of an angle as a quantity. I also found that students constructed numeric multiplicative relationships between angles (*Category 4*). In this category, students reasoned about the splitting of a circle into a number of equal parts and the composition of a 360-degree angle.

The findings of this study also showed that students' prior knowledge influenced the progression of their reasoning. For example, Angelie showed her prior understanding of an angle by sketching line segments with different orientations. In her succeeding reasoning on a task where she rotated a line segment, she described the result of the rotation as a "different angle line" illustrating again her definition of an angle as an orientation. Meanwhile, Jordan's reasoning showed his prior knowledge about an angle as a "corner" of a square. At the end of the experiment, he reasoned again about an angle as a corner of a shape and as a fraction of a rotation. Similar to Angelie and Jordan, each student constructed an understanding of angles by building on their prior knowledge. Even though each student reasoned about angles in sophisticated ways, traces of their prior knowledge was still evident in their reasoning through the end of the design experiment. This shows that students' prior knowledge is broadly important.

In addition to students' prior knowledge, the design of tasks, tools, and questioning also influenced the variations in students' reasoning. Examples of variations in their reasoning include their strategies of composing and decomposing angles and the nature of the connections they created with the amount of turn, fractions of a circle, and degrees. I describe the influence of the design in the next section.

7.1.2. Design that supported students' reasoning (Research Question 2)

The findings presented in Chapter 5 illustrated the characteristics of the design of tasks, tools, and questioning that supported students' reasoning about angles (*Research Question 2*). Specifically, a combination of tasks that illustrated angles as union of two rays, rotations, and wedges to explore the generation of angles supported students to bridge the three angle conceptions (*Category 1*). To elaborate, students showed their prior knowledge about angles in their response to my question "What is an angle?" For instance, Jordan reasoned about angles as corners of a shape. This reasoning was similar to what Clements and Battista (1989) found by asking the same question to their participants. The dynamic rotation feature of all tasks allowed students to rotate angle sides and this experience supported them to develop a conception of angles as rotations. For example, Jordan, who had the prior conception of angles as corners, reasoned that the rotation of one side farther away from the other made more space between the angle sides. The design of tasks with varying lengths of angle sides but the same size of an angle elicited this kind of reasoning among students and supported them to avoid the typical alternative conception of angle size as dependent on side lengths (Smith et al., 2014). Instead, students connected the angle size with the direction and amount of rotation. Tasks involving wedges, on the other hand, supported students to reason about angles as wedges. For instance, Jordan reasoned about a wedge as the amount of space covered between two angle sides, similar to what

Browning and Garza-Kling (2009) found. This study offers evidence to support the suggestion made in the research literature that a combination of the three angle conceptions can be powerful in eliciting a more robust understanding of angles (e.g., Freudenthal, 1973; Mitchelmore & White, 2000).

The overall design also prompted students to multiplicatively compare angles (*Category 2*). The rotation feature of tasks and the equipartitioning of a turn offered opportunities for students to iterate a smaller angle within a bigger angle and explore the relationships between the two angles. For example, Axel equipartitioned a full rotation into four quarters and associated each quarter turns with the quarters of a circle. Then he used his prior knowledge about degrees to reason about the degrees of a turn, a form of reasoning that was suggested by Confrey et al. (2012). The findings in this study also showed that tasks involving the iteration of an angle into the bigger angle together with my questioning about "How many times bigger or smaller was an angle compared to the other angle?" elicited students' reasoning about the decomposition and composition of an angle. For instance, Jordan first conjectured that a bigger angle was "four times more open" than a smaller angle. After iterating the smaller angle within the bigger angle and counting six iterations, he reorganized his thinking. Jordan's actions and reasoning showed that he could mentally decompose the bigger angle by estimating the number of times he could fit the smaller angle in (his initial conjecture), then he composed the bigger angle using the iterations of the smaller angle. Similar to Jordan, students were able to use decomposition and composition strategies as *measurement processes* (Castillo-Garsow et al., 2013) to quantify angles multiplicatively without the need to assign any numerical values to these angles.

Tasks that illustrated the traces of the rotations together with my questioning opened up opportunities for students to construct a conception of an angle as a discrete or a continuous quantity (*Category 3*). To elaborate, asking students to quantify the angles as they worked on tasks that left multiple traces of rotations prompted them to reason about an angle as a discrete quantity. For example, Axel reasoned that the multiple traces of a rotation showed “a lot” of angles illustrating a discrete conception of a very large number of angles. In contrast, asking whether there are angles that are smaller than the one-degree angle led Axel to reason about unlimited angles and intermediate angles in between one-degree angles. This kind of reasoning showed his continuous conception of an angle similar to what Castillo-Garsow (2012) referred to as continuous and smooth thinking.

Tasks asking students to split a circle and the use of the “show angle measure” tool prompted students to construct numeric multiplicative relationships about angles (*Category 4*). The design of the splitting tools supported the students to imagine the equipartitioning of a full turn into quarters, eighths, thirds, sixths, and twelfths. Students decomposed an angle and composed the smaller identical angles to compose the given angle. For instance, in the “Splitting a Circle” task, Axel decomposed a quarter of a turn into two eighths and composed the two groups of eighths to reason that an eighth was 45° because it was half of 90°. Axel who had prior knowledge about degrees illustrated a flexible understanding of the connections between the amount of turn, fractions of a circle, and degrees. While Axel had prior knowledge about degrees and division of large whole numbers such as 90 and 360 that supported his reasoning, the “show angle measure” tool aided students who did not have prior knowledge about degrees to reason about angles in degrees. For instance, Alicia relied on the “show angle measure” tool to reason about the degrees for one-quarter of a full turn. After learning that a quarter of a full turn has 90°, she stated that this time four quarters as 360° made sense to her. The findings in this study showed that students exhibited a coordinated measurement approach to multiplication (Izsák &

Beckmann, 2019) by coordinating the number of equal parts a circle has been split and finding the degrees of each part.

7.1.3. Evolution of design (Research Question 3)

In the Findings – Part 1 (Cases-by-Case Analysis) chapter, I presented the chronological accounts of students' reasoning and actions to illustrate how the multiple iterations of design experiments allowed me to refine both my theories about students' learning of angles and the design that elicited such reasoning. In other words, the findings showed how my conjectures and design co-evolved to support students' reasoning (*Research Question 3*). In this section, I discuss two levels of design modifications: at the micro-cycle level within each design experiment and at the macro-cycle level after each design experiment.

At the micro-cycle level, the findings showed that students' prior knowledge and in-the-moment reasoning about angles influenced the modification of the design to support student reasoning and amplify the construction of their meanings about angles. For example, the traces of rotations in the exploratory tasks created wedges which led Jordan in the first design experiment to reason about angles as the area of the wedge. This conception prevented him from reasoning about angles in terms of openness. Therefore, I removed the wedges in the exploratory tasks that followed. This modification helped Jordan reorganize his reasoning into angles as the amount of openness between the two angle sides. Another example was Axel who had prior knowledge that a full rotation has angles from one-degree to 360 degrees illustrating a discrete conception of an angle. His reasoning about the one-degree angles prompted me to modify my questioning to ask about angles smaller than the one-degree angle. This modification prompted Axel to reason about unlimited angles within angles, illustrating a conception of angles as a continuous quantity.

At the macro-cycle level of the design experiments, the findings show that modifications to the design after each design experiment elicited different ways of reasoning among students in the succeeding experiments. For example, the modifications of the exploratory tasks to remove the wedges in the first design experiment supported students in the later experiments to conceive the size of angles in terms of openness. Students in the succeeding experiments reasoned about angles as openness before they developed the conception of angles as wedges. Additionally, in the first design experiment, Jordan's definition of an angle as a corner of a shape prompted me to design the "Triangle" task to support other students who may have similar prior knowledge. This task showed angles as corners but it also allowed students to drag the vertices of the shapes, modify the size of an angle, and observe that the size of these "corners" could change.

Collectively, the ways in which the design prompted students to exhibit particular categories of reasoning and how the design also evolved to support such reasoning showed the *reflexive relationship* (Cobb et al., 2001) between the design and student reasoning.

7.2. Contributions to Knowledge

In this section, I discuss contributions to mathematics education research and teaching practice considering the key findings of the study.

7.2.1. Contributions for Research

By engaging with this study's design, students did not exhibit the typical alternative conceptions of angles reported in the literature. For instance, the "Three Pairs of Segments with Different Lengths" task that consisted of three angles of the same size but different side lengths might have supported students to avoid the common alternative conception that the angle size is dependent on side lengths (Smith et al., 2014). Additionally, in the current study, the tasks presented angles in different orientations and this might have supported students to develop a conception of angles not limited to right angles. Although students often used right angles to discuss the degrees of an angle or the fractions of a circle, they also did not show the alternative conception that all angles are right angles (Devichi & Munier, 2013). This study contributes to research by suggesting a design that supports students to avoid these alternative conceptions that they commonly exhibit when working with tasks involving static angles (e.g., Devichi & Munier, 2013; Smith et al., 2014). This is important because previous research has shown that students carry these alternative conceptions at higher grade levels of schooling and impede students from learning content from mathematical areas, such as trigonometry and geometry (Lehrer et al., 1998; Smith et al., 2014).

This study also illustrates the design of tasks that could prompt students to bridge the three common angle conceptions. As aforementioned, prior research has shown that students either reason about an angle as a union of rays, as a rotation, or as a wedge (e.g., Browning & Garza-Kling, 2009; Devichi & Munier, 2013; Smith et al., 2014). The literature also shows that individual conceptions of angles limit students into procedural thinking and memorizing terms rather than offering them opportunities for meaningful reasoning (Boston & Candela, 2018). On the contrary, this study shows that when students engaged with tasks that illustrate all three angle conceptions, they exhibited sophisticated reasoning about angles as shown in Table 12. Not only did students reason about the three angle conceptions, but they also constructed multiplicative comparisons, developed discrete and continuous conceptions of an angle as a quantity, and exhibited numeric multiplicative reasoning about angles. In other words, the findings of this study show the power of targeted design and questioning for achieving what Freudenthal (1973) suggested, that engineering tasks involving multiple angle conceptions could support students' construction of meanings of angles. Understanding these categories of reasoning in Table 12 is important for future studies to investigate how students' connection of the angle concept with other mathematical ideas could be supported. Further research can also examine other forms of reasoning about angles that might be possible when the three angles conceptions were bridged as a single construct.

Table 12

A Summary of Students' Categories of Reasoning

Student Reasoning
Category 1: Angle Conception
A. Angle as a union of rays B. Angle as a rotation C. Angle as a wedge
Category 2: Multiplicative Comparisons
A. Multiplicative change in an angle B. Decomposition and composition strategies when comparing angles C. Initial splitting strategy
Category 3: Discrete/Continuous Conception of Angle
A. Discrete conception of an angle B. Continuous conception of an angle
Category 4: Numeric Multiplicative Relationships Between Angles
A. Splitting a circle B. Composition of a 360-degree angle

Furthermore, this dissertation illustrates the interplay between students' geometric and multiplicative reasoning through the exploration of dynamic angles. This study offers empirical evidence of students' reasoning about dynamic angles by bridging the geometric nature of angles (e.g., union of two rays) and the idea of dynamic rotations with wedges that elicited students' multiplicative reasoning. Prior to this study, little was known about how students may reason about angles that are generated dynamically (e.g., Clements & Burns, 2000; Hardison, 2018; Smith et al., 2014). By exploring dynamic angles, this study illustrates two different ways of generating an angle: by iteration and splitting, and by rotations that relate to a full circle. Prior studies illustrate that students used these strategies separately showing one way to create angle over the other (e.g., Mitchelmore & White, 2000; Smith et al., 2014). The findings of the current study show that students could use both iteration and splitting, and rotation as a single construct. For example, in the "Splitting a Circle" task, Axel used the tool that split a circle into thirds to reason about the degrees of 2/3 of a full turn. The tool for splitting a circle prompted him to see the three partitions of a circle. He then used these partitions to guide his iteration of a 1/3 wedge as his unit angle. He exhibited this iteration via rotation of a ray with a goal to first create 1/3 of a full turn to explain 1/3 of 360°, and he showed a further rotation to create 2/3 of a full turn to explain the degrees of 2/3 of 360°. Axel's actions and reasoning exemplified the use of both strategies to reason multiplicatively about angles. The findings in this study show that bridging the two strategies of generating an angle dynamically could support students' multiplicative reasoning about angles. Future research could further examine this idea of bridging geometric and multiplicative thinking to describe angles in more advanced mathematical concepts, such as the angles of elevation and depression in trigonometry or solid angles in advanced calculus.

Students in the study exhibited the coordination of three levels of units (Steffe, 1992) when reasoning about angles. For instance, when reasoning about the degrees for fractions of turns, Alicia coordinated three levels of units by reasoning about a quarter of a turn as 90° [first level], then reasoning for a half turn as 180° in terms of two 90-degree units [second level], and then reasoning for a full turn as two 180-degree units [third level]. Students' construction of a connection between the amount of turn with the fraction of a circle, and the degrees (Confrey et al., 2012) can be foundational for them to coordinate the three levels of units. Although Alicia had problems with dividing a very large number (e.g., 360 divided by 2), she exhibited this coordination of levels of units without reference to numerical division. Similar to Alicia, students exhibited this form of reasoning because the design of splitting a circle showed how a full turn was split and guided the students to connect the amount of rotation with the fraction of a full turn. Then, the “show angle measure” tool helped students to recognize the degrees of an angle, which they further connected with each fraction of a turn. The design of tasks and tools supported students to coordinate three levels of units without the need to operate numerical division. This example showed that an angle can be a meaningful context in bridging the learning of geometry and rational numbers prior to students’ mastery of numerical operations. Clements and Sarama (2014) suggested that students must *connect* the idea of an angle as a geometric figure with its measurement. Future studies may find more evidence to support or contrast these claims on students’ coordination of levels of units in the context of angles.

Furthermore, the categories of reasoning that students exhibited in this study can be a starting point for creating a learning trajectory for dynamic angles. Research has shown that learning trajectories are important in teaching to support students in moving through levels of thinking by selecting appropriate instructional tasks for such levels (Clements & Sarama, 2004). Additionally, Clements and Sarama (2004) suggested that learning trajectories helped teachers in connecting learning goals, curriculum components, assessment, and teaching strategies. While Clements and Sarama (2014) presented a learning trajectory for angles and turn measures, the instructional tasks used were primarily involving static angles. Recall that static angles elicited alternative conceptions as discussed in the literature. Even when tasks included the measurement of dynamic turns in the existing learning trajectory (Clements & Sarama, 2014), there were no levels of students’ reasoning about dynamic angles similar to the categories and subcategories found in this dissertation. When dynamic tasks on angles were suggested to prompt the highest level of learning in that trajectory, students were expected to measure angles only in degrees. This level in the existing learning trajectory for angles did not aim for students to reason about how the degrees relate to fractions of a full turn and fractions of a circle. The categories and subcategories of reasoning in this dissertation can be an entry point for revising the existing learning trajectory by Clements and Sarama (2014) or even building a more fine-grained learning trajectory focusing only on dynamic angles. A learning trajectory for dynamic angles may offer research in mathematics education a platform to examine further the reasoning that students exhibited in this study and inform the development of instructional tasks to further support these forms of reasoning.

7.2.2. Contributions for Practice

The findings from this dissertation illustrated how teachers could use the design of tasks, tools, and questioning to guide students in exploring dynamic angles. Clements and Sarama (2014) argued that teaching angles while connecting the ideas of turns and their measure is a difficult task. This difficulty arises mainly because angles are presented statically throughout schooling. Teachers often use static illustrations to introduce angles as components of shapes starting from kindergarten and this continues when presenting static illustrations of angles as union of rays in higher grades. The design in this study could offer opportunities for teachers to explore angles dynamically and learn how to support students' conception of angles beyond the static definition.

While the literature has shown that there is no proper definition that can describe angles from all areas of personal experiences (Taimina & Henderson, 2005), introducing fragments of categories of angles in the curriculum may have resulted into confusion about what an angle is. Scholars like Freudenthal (1973) and Proclus (1970) suggested to consider all three angle categories for students to create a meaningful understanding of angles. This dissertation suggests potential curricular contributions to teaching and learning angles by bridging the three conceptions of angles as union of rays, as rotations, and as wedges.

The potential of the four categories of reasoning about dynamic angles in the development of a learning trajectory could also be useful for teachers for assessing and supporting student learning. Teachers could select the appropriate instructional task for students based on their prior knowledge. For instance, for students like Alicia who are not prepared to work with numerical operations involving bigger numbers, teachers could choose tasks that do not require numerical operations yet support students to compare angles multiplicatively.

The findings also showed that students coordinated the changes in rotation with the changes in the size of angles. This is an example of coordinating two quantities changing simultaneously – what research refers to as covariational reasoning (Confrey & Smith, 1995). The study of change such as covariation and multivariation has been a significant topic in research in mathematics education (e.g., Confrey & Smith, 1995; Panorkou & Germia, 2020; Thompson & Carlson, 2017). Most concepts that focus on the mathematics of change are often only in Calculus at higher education levels. This dissertation showed that angles can be used for teaching the mathematics of change which can be foundational for teaching higher mathematics levels. Future research could explore the connections between angle reasoning and covariational reasoning in more depth. Additionally, the teaching of geometry where angles are introduced is often left at the end of school year if time allows it. Teachers could use the study of change in angles while teaching multiplication and fractions and connect geometry to other concepts of mathematics taught in each grade level.

7.3. Limitations and More Suggestions for Future Research

Although this dissertation contributes to research on student reasoning about angles and other mathematical ideas, it has several limitations that offer suggestions for future research. First, this study was limited to a small sample size of four third-grade students. While this small sample allowed me to study in depth the students' reasoning and the design that supported it, the findings only showed a snapshot of how third-grade students could reason about dynamic angles. The small number of study participants could not offer significant and generalizable findings to represent third-grade students as a whole. Therefore, future studies need to validate and contribute to the findings by examining the reasoning of more research participants.

A second limitation was that the students worked individually and did not have opportunities for social interactions with other students to (re)construct their reasoning as it usually occurs in a real classroom setting. Research shows that students learning in dyads are more successful than individual students in working on tasks involving abstractions (Schwartz et al., 1991). In this study, I could not place students in dyads because of the COVID-19 pandemic. Further research with students working in pairs could create an environment for social learning where students have the confidence and freedom to work with someone of their age and build on each other's reasoning. This setting could also encourage a dynamic discourse between participants in every macro-cycle of the experiments enriching the limited interaction between the researcher and a participant. In other words, design experiments with small groups of students could "create a small-scale version of a learning ecology so that it can be studied in depth and detail" (Cobb et al., 2003, p. 9).

Additionally, design experiments are usually scaled up to whole classroom settings to examine whether the design is applicable in real classroom situations. In such studies, the design of tasks, tools, and questioning could elicit different conceptions and structures of reasoning which would have been interesting to explore. The design could also be refined to encourage collaboration between students and co-construction of knowledge.

A third limitation of this study was that the design experiments were conducted virtually because of the pandemic. This kind of setting did not allow the researcher to be physically present for the participants when digital or technological troubleshooting required an immediate attention. These included troubles with the spontaneity of the audio and video recordings and difficulties when some students required assistance in using digital tools on their computers. For example, most students were not able to use the annotation tool of Zoom to show the part of the task or tools they were focusing on or draw images to support their reasoning. There were also instances that I had to take turns in sharing the screen with the student to show them how to use the digital tools.

Additionally, interviewing students virtually from their homes resulted in multiple distractions during the experiments. For example, students seemed conscious of what they shared with me when their family members were next to them (Recall that guardians were allowed to be present in the interview). To help them become more comfortable, I encouraged them to share what they were thinking and emphasized that there were no right or wrong answers in our interview. Other distractions that halted their interactions from the tasks and questioning during the experiments were when students looked for their computer chargers or dealt with unnecessary background noise. Most of the distractions partly impeded students' attention and the spontaneity of their thinking. It would be more convenient and efficient in future studies to interview the participants in person to immediately respond to digital or technological issues, minimize distractions involved in virtual interviews, and immediately observe their in-the-moment gestures or illustrations that could have been used as evidence of their understanding about angles.

A fourth limitation of this study was the delay in modifying the design of tasks and tools when a student faced a difficulty. To elaborate, while students' prior knowledge, in-the-moment responses, and actions informed the modifications of the design, these modifications occurred an hour or a day later and may have not authentically prompted students to reorganize their reasoning at the moment. For instance, it took me a day to remove the wedges in the exploratory tasks after Jordan reasoned about angles as the space of the wedge. During this delay, the design experiment continued and other tasks, tools, and questioning might have already influenced Jordan to reorganize his reasoning about angles in terms of openness. A future iteration can have shorter and more spaced-out sessions to allow time for modifications. Additionally, a future iteration can test these new modifications on the task design. It can also test new task modifications such as a new version of the "Growing and Shrinking Angles" task without showing the split in twelfths to examine if it will support students like Alicia to consider 2/12 as the angle unit. Future research could further develop the design and examine student reasoning about angles. It could also explore further the reciprocal relationship between the forms of reasoning and the design.

A fifth limitation of the study was my lack of control over students' use of the "show angle measure" tool in the "360 Angles" and "Splitting a Circle" tasks. In some instances, students activated the tool before they made conjectures about the angle size in degrees. Although I designed the tool to hide the degrees by default, students were quick to unhide it and view the degrees before I asked them to create their conjectures. These instances of opening the tool prior to my instructions impeded students from creating their conjectures and reasoning about the size of angles in degrees by themselves. For instance, when I asked how they knew the degrees for two-quarter turns, some of them responded that they saw the degrees using the "show angle measure" tool. On the other hand, the availability of the tool was significantly helpful when students were not prepared to work with fractions or division of large numbers such as 360. Only one of the four participants did not rely on the "show angle measure" tool. In a future iteration of the design, it might be useful to have one version of the task without the "show angle measure" tool.

The sixth limitation of the study was the nature of my questioning in the micro-cycles and macro-cycles of the experiments. Before I conducted the experiments, I designed sets of questions and instructions for every task. Although the interview questions were intended to be semi-structured and open-ended, I utilized most of the questions and instructions I prepared. In many instances during the retrospective analysis, I noticed that students' responses required follow-up questions to further unpack their thinking that I did not ask. I noted these instances and in future iterations I would be more prepared to prompt the students when I encounter similar forms of reasoning.

Additionally, I made some modifications while working with students at the later design experiments that the students I interviewed earlier did not have the chance to explore. For example, when I asked Axel about angles smaller than the one-degree angle, he reasoned about angles as fractions of one-degree illustrating continuous reasoning. Axel was the third student I interviewed so I did not ask this question to the students I interviewed before him to examine if they could reason in a similar way. In a future iteration, I plan to ask students about the angles smaller than the one-degree angle to examine whether they could reorganize their reasoning from discrete into continuous. Also, I plan to revisit student interactions with prior tasks to examine other opportunities in the task design where this kind of continuous reasoning could be elicited.

7.4. Concluding Remarks

Research and mathematics curriculum standards have shown that a conceptual understanding of angles is important for students' mathematics learning. The review of the literature shows that presenting angles statically is primarily counter-productive to constructing meanings about angles by eliciting a variety of alternative conceptions. To offer a solution to this problem, I utilized the power of digital technology to represent the generation of angles dynamically and illustrate the change in the quantities involved in this generation. This dissertation illustrates how this dynamic design (tasks, tools, and questioning) supported students to avoid alternative conceptions and actively construct sophisticated forms of reasoning about angles. Findings from this study also provide evidence of students' reasoning bridging the three common angle conceptions and illustrate the interplay between geometric and multiplicative thinking. This interplay shows the reciprocal relationship between students' geometric and multiplicative reasoning in understanding the angle concept. The dynamic generation of angles and their composition and decomposition strategies support students' construction of sophisticated multiplicative reasoning. At the same time, by reasoning multiplicatively about angles, students developed a sophisticated understanding about the geometric nature of an angle as a union of two rays, as a rotation, and as a wedge. This reciprocal relationship together with the four categories of students' reasoning offer an entry point for further research on the development of theories on students' reasoning about dynamic angles and the construction of connections between angle reasoning and other mathematical concepts.

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Appendix

Letter to parents of student participants

April 20, 2021

Dear Parent/Guardian:

I am writing to let you know about an opportunity to participate in a research study about Investigating Elementary School Students' Reasoning about Dynamic Angle. The purpose of this study is to explore how elementary school students reason about angles as they work on digital tasks I designed.

If your child or dependent is a third-grade elementary student, your child or dependent may be eligible to participate.

Third-grade is the grade level when students begin to explore angles as an attribute of shapes, and this is the time to offer younger students opportunities to explore angles in ways that are meaningful to them. Understanding angles is essential in many areas of the mathematics curriculum. All activities related to the study will add to your child or dependent's usual learning.

If you agree to participate your child or dependent, I will observe and interview your child or dependent along with another third-grade student while they both work with my designed tasks in exploring angles.

This study will involve recording of video, audio, and screen sharing interactions with the participants. It will take about 5-8 sessions, 25-30 minutes each session, of their time. These sessions will take place in a virtual meeting platform (e.g., Google Meets, Zoom) during the time convenient to your child and another student. This time can be during break or after-school activity.

If you have any questions, please contact me at (201) 912-5304 or my email address germiae1@montclair.edu.

Thank you for considering your child or dependent's participation in this study. This study has been approved by the Montclair State University Institutional Review Board, Study no. FY20-21-2202.

Sincerely,

ERELL GERMIA
Doctoral Student
Department of Mathematical Sciences
Montclair State University