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# Using Teacher Noticing and Video-Mediated Professional Learning to Develop Preservice Teachers' Knowledge for Teaching the Derivative

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# Using Teacher Noticing and Video-Mediated Professional Learning to Develop Preservice Teachers' Knowledge for Teaching the Derivative

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DISSERTATION APPROVAL

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## Abstract

This study investigated how problem-solving videos can be used in video-mediated professional learning to support secondary preservice mathematics teachers (PMTs) in developing teacher knowledge for noticing student thinking in the context of the derivative concept in calculus. A model of the trajectory of PMTs' noticing was constructed as six PMTs viewed and analyzed videos of students' problem solving. At the same time, the nature of video-mediated interactions that were found to be productive in supporting this knowledge development was examined. A design experiment was used as the research methodology. Data was collected from video recordings of eight semi-structured teaching episodes for each of the three pairs of PMTs and analyzed through a grounded theory approach. Considering that the knowledge was video-mediated, developed collaboratively, and assessed in action, the study was grounded in situated and sociocultural perspectives, and the conceptual framework of professional teacher noticing guided the analysis. The constructed model of noticing development that emerged from the experiment entails the following four processes: *describing, interpreting, comparing and contrasting*, and *responding*. This model represents an approach to an ambitious professional vision that can support calculus educators in iteratively improving their practice through readily accessible, video-mediated professional learning. The study also identified interactions that were found to mediate the development of PMTs' noticing. The social interactions that supported PMTs' noticing of student thinking are *highlighting* (moments worth noticing) and *prompting* (PMTs to notice what was highlighted). Prompts posed to the PMTs generated centers of focus (Lobato et al., 2013) offered opportunities for their describing, interpreting, responding to student thinking. The material interactions refer to features of problem-solving videos that were found to support the PMTs' learning to notice: (1) The videos offer images of students' problem solving that could serve as an opportunity for decentering; (2) They are effectively curated; (3) They can be paused and re-viewed repeatedly, and (4) They provide a comprehensive and appropriate scope and sequence of derivative topics. These findings contribute to research in calculus education, with implications for the design of learning experiences in mathematics teacher preparation.

*Keywords*: teacher noticing, professional development, design experiment, calculus

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## Dedication

To my family: Doris, Liam, Robert, Erastus, and Monica Karei.

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# Chapter 1: Introduction

## Framing the Study

The use of videos in teacher education has recently gained much attention, as it is considered effective in supporting preservice teachers' development of knowledge and skills required for actualizing meaningful instruction in the classroom. However, as this literature review will reveal, there was a need to investigate further how videos can be used to develop authentic, situated, and contextualized domain-specific knowledge for teaching. For this reason, this study used the design experiment methodology to explore the knowledge for teaching the derivative that preservice mathematics teachers (PMTs) developed through their mediated engagement with videos of students solving mathematics problems. In these videos, undergraduate-level students were solving problems related to the concept of derivatives. PMTs were asked to view and analyze these *problem-solving videos* in order to develop their skills for noticing students' mathematical thinking and their knowledge for teaching the derivative.

In addition to analyzing the development of these skills and knowledge, I discerned the nature of interactions among the PMTs, the problem-solving videos, and the researcher that were seen to be productive for supporting the knowledge and skills development. To do so, I took both situated and sociocultural perspectives on knowing, as they honor the contexts in which knowledge is developed and the tools and interactions that mediate it. Moreover, focusing (Lobato et al., 2013) and teachers' professional noticing (Van Es & Sherin, 2002) frameworks were used to analyze these interactions in order to yield insights into how teacher knowledge may be developed. In my review of the literature, I reviewed the research on the use of video in teacher education, the current state of postsecondary calculus education, models of knowledge for teaching mathematics, and teachers' professional noticing in mathematics education. Through this review, I identified gaps in these bodies of literature and concluded by laying out a theoretical framework and methodology that this investigation into knowledge development employed. In my concluding remarks, I discussed the forms of knowledge PMTs might develop for teaching the derivative through their engagement in a design experiment mediated by problem-solving videos of students in a semi-structured interview setting. I used the review as a whole to inform a conjecture of the noticing skills and forms of knowledge that are needed to effectively teach the derivative, and the kinds of interactions that supported their development.

**Use of Videos in Teacher Education**

This study incorporated the use of problem-solving videos as a medium for developing prospective teachers' knowledge for teaching the derivative and their skills for noticing students' mathematical thinking. Videos have been used in preservice mathematics teacher education to support prospective teachers to develop the knowledge and skills needed to initiate and sustain student-centered teaching approaches in mathematics classrooms and also to support them as they learn to anticipate and address students' misconceptions and facilitate their learning of the target understandings of fundamental mathematical concepts. This review identified gaps in the research on the use of videos in preservice teacher preparation and proposed a means to address them. The research questions that framed this study will be presented immediately following this section.

Research has shown that records of practice (e.g., classroom videos, examples of student work, lesson plans, instructional materials) can support teachers in connecting the theoretical knowledge acquired in teacher education programs to the actual practices in a classroom context, such as "ways to introduce the task, questions to pose as students work on the task, and methods of managing the classroom discourse" (Borko et al., 2011, p. 184). In particular, as "sites for analysis [that] are situated in practice" (Doerr & Thompson, 2004, p. 180), videos can serve as "springboards for analysis and discussion about mathematics teaching and learning" (Borko, 2011, p. 184).

Borko et al. (2008) explored how teachers can use their classroom videos as a tool for enhancing productive analysis of their teaching practice and further develop insights for improving their teaching. They conducted a PD in which mathematics teachers developed a "supportive and analytical environment" within a period of 2-years. Teachers met once a month to examine their practice collaboratively as a community of practice by watching video excerpts of each other's instruction and focusing their discussions on students' thinking as well as the teacher's role in developing and implementing lessons. The study's findings reveal teachers' positive opinions about their participation in the PD: "Viewing footage from their own classrooms allowed the teachers to see what they were doing well and to identify areas for improvement" (p. 434), thereby developing richer models of responsive teaching. For example, they learned to "better appreciate their students' capacity for mathematical reasoning" (p. 434) and to develop an understanding that they and their colleagues are all grappling with similar problems of practice.

As teachers engage in video-mediated discussions of problems of practice, social, intellectual, and material resources (Vygotsky, 1978) mediate their learning, and negotiations of teachers' ideas contribute to the shared knowledge for teaching that emerges. Thus, by presenting the richness, complex reality, and authenticity of classroom interactions, videos of classroom practice offer a viable venue for the development of teacher knowledge with an eye toward an ambitious and responsive pedagogy. Indeed, research on teacher noticing (Sherin & van Es, 2002; van Es, 2011) reveals that videos can be an essential practice artifact. They allow teachers to systematically analyze and reflect on their teaching when viewed along with a structured protocol to mediate their analysis. Sherin (2004) adds that an affordance of videos is that teachers can pause and review them multiple times at their own pace to carefully analyze and reanalyze events as they unfold. Such an affordance can be invaluable, as it mitigates the overwhelming demand of observing an entire class where multiple events occur concurrently, all worthy of the observer's attention. In addition, videos are editable, which means that clips can be extracted that are specific to the goals of professional learning.

Moore-Russo and Viglietti (2011) used *animated* videos of modeling geometry classroom instruction in professional development with both novice and prospective teachers, and found them to be useful in helping teachers extend their content knowledge and get better at making instructional decisions based on student thinking. These animated videos are intentionally designed to focus on three "elements" of a learning environment: the chalkboard, the instructor, and the students. The camera views these elements from all angles, thus enabling a clear view of students' written work and their conversations, teachers' responses to students' inquiries, and even facial reactions of the participants.

Videos have also been used in teacher education programs to support preservice mathematics teachers learning to notice classroom events and teaching practice. For example, Star and Strickland (2008) used a pre-and-post-test design to determine the kinds of classroom events that preservice mathematics teachers noticed before and after completing a math methods course. Although they indicated limited noticing skills in the pre-test, in the post-test they had developed significant observation skills as revealed by their abilities to notice features of the classroom environment, tasks, mathematical content, and communication (teacher-to-student, student-to-student) that occurred during the lesson. Similarly, Wilson et al. (2011) used *statistics* problem-solving videos to teach teachers how to respond to student thinking by analyzing the problem solving of the students in the videos, characterizing what teachers noticed (Lobato et al., 2013), and proposing instructional decisions based on what they learned. Thus, resulting to a model of how teachers learn to notice student thinking.

Although research has explored the use of classroom videos in teacher education, as well as analysis of students' work as they solve mathematical problems, it has offered neither 1) the principles for the design of a video-based model of PMTs' knowledge development for teaching the derivative (e.g., What should those videos contain?), nor 2) an articulation of the nature of interactions among the PMTs, videos, and the teacher educator that can support that development (i.e., How are PMTs meant to engage with the videos?). Given that videos of professional practice have been shown to be useful in teacher preparation and professional development, new research can help us better understand just how it is that problem-solving videos in particular can be used to develop the noticing capacity and teacher knowledge for derivatives of future teachers of calculus. This study aimed at responding to that call for research.

## Research Questions

This literature review was organized around a study that sought to address the following research questions:

*1. How does teacher knowledge specific to noticing students' mathematical thinking in the domain of the derivative develop through video-mediated professional learning?*

*2. What forms of video-mediated interactions support this development?*

# Chapter 2: Literature Review

## The Current State of Postsecondary Calculus Education

Calculus is fundamental to STEM fields, and students take calculus courses at either the secondary or tertiary levels in the United States. Students in STEM-related programs such as engineering, medicine, economics, and mathematics take a sequence of calculus courses as its concepts have widespread applications in these fields. Thus, these courses tend to be prerequisites for much of their coursework, which means that they act as gateway courses.

Unfortunately, research has found that these courses too often rely on memorization and rote learning of calculus concepts (e.g., Bressoud et al., 2015), giving too little attention to their conceptual underpinnings. Such approaches to learning deny students a meaningful understanding of calculus, which contributes to the difficulties they experience in STEM courses, particularly in solving problems situated in dynamic and real-world scenarios (Moore & Thompson, 2015; Thompson, 1994). As a result, many STEM students eventually change their majors due to poor learning experiences in courses characterized by such an approach (Bressoud et al., 2015). Thus, it is crucial to undertake research to identify more viable approaches to the teaching and learning of calculus.

Since this study focused on constructing a model of PMTs' knowledge for teaching the derivative, I now present research related to students' learning of the derivative. This literature focuses on the ways students in calculus tend to think and reason about the concept of the derivative. Its findings can inform a study designed to support PMTs as they come to learn how to teach the concept of derivatives effectively. I present this research on students learning the derivative to achieve two goals: 1) Review what the research says about what students know about the derivative and the struggles they are likely to face, and 2) Review what the research tells us about how these misconceptions have been addressed in practice. These insights were invaluable as I developed a model of the knowledge that PMTs need to foster students' learning of the derivative concepts.

***Students' Knowing and Learning of the Derivative***

Research has shown that students tend to learn content with understanding when it is represented in an organized and meaningful way (Ball et al., 2008; Ball, 2003). Therefore, it is necessary to explore students' understanding of the mathematical concepts and ideas related to the derivative and how they are connected. I begin by presenting the literature on how students learn slope, as it is foundational to the concept of derivative

**Students' Conceptions of Slope.** Carlson et al. (2010) explored students' covariational reasoning, a mental action regarding "how one variable changes while imagining successive amounts of change in the other" (p. 117). They stressed the importance of understanding the covariational aspect of slope as a significant phenomenon for developing a meaningful understanding of the rate of change, a key component of the derivative. Moreover, Carlson et al. propose that covariational reasoning enables students to "unpack the notion of slope (steepness) by describing the relative changes of input and output values" and apply the same knowledge to "determine growth patterns and growth rates of various function types" (p. 117).

Studies (e.g., Lobato & Siebert, 2002; Lobato & Thanheiser, 2002) that investigated students' linkages of slope to real-life applications have reported their struggles to make those connections. For instance, Lobato and Siebert (2002) conducted a teaching experiment to support students in developing quantitative reasoning about linear functions and slopes. They engaged students in exploring the meaning of steepness in the context of hills and ramps. One student who was interviewed was asked how he would measure the steepness of a ramp. The students explained three methods: measuring the inclination angle, using a level, and making a series of height measurements taken at equal-width intervals. However, the student could not explain how the steepness of a ramp can be maintained by varying the height and the length, thus indicating that he had not considered a proportional relationship between the height and the length.

For a student to understand slope as it is applied in real-life situations and be able to engage in quantitative and covariational reasoning, Lobato and Siebert (2002) suggest an approach that involves three aspects: (a) isolating attributes in order to quantify them, (b) identifying how quantities reflect different measures of attributes, and (c) understanding relationships among quantities. Lobato and Siebert further discourage a focus on the slope as "a calculation used to determine the "tilt" of a straight line in a coordinate grid system (as is done in many traditional and some reform classrooms)" (p. 94). They do so because if this were a student's operating conception of the slope, they would find it difficult to use the concept to make sense of covarying quantities. Instead, they suggest "a focus on the slope as the construction of a ratio of the variation of one quantity to the associated variation of another quantity [where the two quantities covary] in such a way that the slope is a measure of some attribute" (p. 94). This would provide students with a knowing of slope that is general enough to connect with their experiences outside of math class.

Stump (2001) offers a standard for the conceptual knowledge of slope that students need to develop: "Understanding the relationships among the various representations of slope that typically appear in school (algebraic, geometric, trigonometric, and calculus), understanding slope as a measure of steepness and rate of change in real-world situations" (p. 210). Unfortunately, research reveals that students hold conceptions of slope much narrower than Stump's standard, undermining students' abilities to understand the derivative.

Nagle et al. (2013) conducted a qualitative study to explore college students' conceptions of the slope. Students responded to a paper-and-pencil 'quiz' with items that sought to assess their understanding of slope. The analysis revealed an almost exclusively procedural understanding. For example, slope as an indicator of the behavior of a graph (i.e., "considering slope as a gauge for determining increasing or decreasing trends of a line," p. 1500) and as a parametric coefficient (i.e., "m is the coefficient of x," p. 1503) were common in their responses. Furthermore, such narrow concepts of slope fail to capture it as a ratio of covarying quantities. Findings such as this help to explain why students might struggle to appreciate slope in the context of real-world applications where change is central.

Other evidence of students' struggles to conceptualize slope in real-world situations is provided in the literature (e.g., Nagle & Moore-Russo, 2013), and this maybe a byproduct of teachers' struggles to do so, as well. For example, Stump (2001) found that although preservice mathematics teachers deliberately designed lessons on slope with connections to real-world situations, they neglected to maintain those connections during their implementation. Stump stresses the need for teachers to instantiate and maintain these connections by incorporating scenarios involving ratios of changing quantities, such as velocity and acceleration, and by making connections to the slope of function graphs in order to support students' understanding of slope. Students should always be mindful, Stump writes, of the question, "What does the slope represent in the context of this situation?" (p. 87). The real-life situations provide a productive avenue in which students' meaningful understanding of slope can be developed. In conclusion, the research tells us that instructional approaches that represent slope as a ratio of covarying quantities and situated in real-life applications are productive for developing students' conceptual knowledge of slope.

**Students' Conception of the Derivative.** Research that explores student understanding of the derivative has found that students experience difficulties in learning just about every idea associated with the derivative concept—the derivative function, the slope of a tangent line, the rate of change, the difference quotient, and the limit (Park, 2013; Zandieh & Knapp, 2006). Park (2013) explored Calculus I students' understanding of the derivatives of a function, at a point, and over an interval. The students provided responses to tasks related to derivatives in a survey and participated in a follow-up, task-based, semi-structured interview to reveal their approaches to solving the problems. Park's findings reveal students' confusion between the derivative as a value (the slope of a tangent line at a particular point) and the derivative as a function (a new function that outputs the slope of the function graph at each point).

For example, one of the tasks assessed students' understanding of the relation between a function and its tangent line. One of the students indicated that $f\,'(1)$ equals the equation of the tangent line. That is, $f\,'(1) = \frac{1}{2}x + \frac{1}{2}$ and went on to explain that $f\,'(1)$ is a point and $\frac{1}{2}x + \frac{1}{2}$ is a function instead of realizing that $f\,'(1)$would give the slope of a tangent line at the point $x = 1$. This student's work implies that they had different views of the left-hand and right-hand sides of the equation; one as a value and the other as a function. The student did not even differentiate the function of the tangent line before thinking of evaluating $f\,'(1)$ to find the slope at $x = 1$. The student's thinking reveals a lack of understanding of the relationship between a function and its tangent line and also of the concept of slope at a particular point. Moreover, when finding the slope of $f(x)$ at $x = 1$, the student found the value of the function at $x = 1$ rather than the derivative function, further evidence of the common confusion.

Park (2013) further reports on students' over-reliance on differentiation rules to solve problems given in symbolic and graphical forms. For instance, to sketch a derivative graph, some students found its derivative and then evaluated it at several points to sketch a derivative function. They thought of derivative as discrete points and not as a function in its own right. In doing so, the students expressed difficulties in understanding that the derivative is the slope of a function at a particular point and that it may change over an interval, and that these understandings can be utilized to sketch a derivative function. That is, the connections they made were pointwise and not across an interval, in which case they would be able to conceive of and reason about change over time.

Researchers have utilized the notion of multiple representations of derivatives in various registers to describe what a deep and flexible understanding of the derivative should look like. For instance, Zandieh's (2000) framework provides a "description of a student's understanding of the concept of derivative" (p. 104), entailing a conception of derivative in four contexts (graphical, symbolic, verbal/rate, and physical), with each explained in all three layers (ratio, limit, and function). Jones and Watson (2018) extended Zandieh's (2000) framework to explore what the concept of derivative consists of, what student understanding of the derivative looks like, and how it evolves as students engage with tasks and activities related to the derivative concept in various contexts. They conducted task-based clinical interviews with first-semester calculus students to test their proposal that target understanding of the concept of derivative involves two features, the first of which is an understanding of all three key elements—ratio of objects, limit, and function (RLF in short). In this framework, these processes are referred to as process-object layers because each of them can be seen as a dynamic process that is reified to mathematical objects. For instance, for Zandieh (2000), the process that "takes two objects... and acts by division" (p. 107) is reified into ratio, a mathematical object. The next layer (limit) is obtained through passing (limiting process) infinite ratios to approach a limit value. Then, limit (as an object) is utilized in the next layer's process that "pass[es] through (possibly) infinitely many input values and for each determines an output value given by the limit... at that point" (p. 107). This process is reified into an object, which is a function.

The second feature of Jones and Watson's (2018) model of derivative understanding is the ability to explain the three layers in at least one context (i.e., representation). They found that students who were able to explain the three layers (i.e., RLF) in at least one context (graphical, symbolic, and physical) had a well formed RLF schema and were able to articulate concepts associated with the derivative. They assert that this framework offers insights into what learning of the derivative entails. They further posit that the developed understanding of the derivative through this model forms a foundation for advanced calculus courses and has implications for the design of instructional materials that support the attainment of target understanding (forming RLF schema of derivative in at least one context).

The literature reviewed in this subsection shows that students face difficulties as they strive to understand the concept of derivative, yet by presenting it to them through engagement with multiple representations, they are more likely to develop a meaningful understanding of the concept. The research further reveals that meaningful learning of the derivative depends on the knowledge that teachers hold and how they make it available to students through well-designed activities and other visual mediators.

***Instructors' Knowledge of the Derivative***

Nagle et al. (2013) provided a 13-item survey to college calculus students and instructors in order to assess their concept images of the slope, which is defined as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 152). They found that even though instructors provided several conceptualizations of slope, they did not provide slope as indicators of graph behaviors, a conceptualization that was elicited by the students. These discrepancies of the interpretation of slope "may be linked to the academic cultures and mathematical emphases found in secondary versus tertiary mathematics" (Nagle et al., 2013, p. 1492). This is a phenomenon of concern because "the first derivative test in calculus uses the sign of the derivative (the slope of the tangent line) to determine the increasing or decreasing behavior of a function" (p. 1509). When calculus instructors at the college level fail to represent the slope as a behavior indicator, the students are likely to have difficulties understanding the first derivative test of the function.

Nagle and Moore-Russo (2013) conducted a related study to explore preservice and secondary in-service teachers' concept of slope and their instructional intent. The participants were enrolled in a graduate-level course that focused on introducing digital tools that can be utilized to teach mathematics. They used Wordle software to create a word cloud so as to reveal their "individual's thoughts about a topic... and their understanding of slope" (p. 7), and the extent to which teachers "have developed an interconnected understanding of a topic and the likelihood that they will be able to mediate the concept to students" (p. 6). They also used Prezi zooming presentation software to create a concept map of slope that they would use to inform the design of instruction to teach the concept to secondary students. The tasks teachers wrote were used to assess what they know about slope and what they believe their students should know, thus enabling the researchers to discern a relationship between how teachers understand slope and how they think about presenting it. The findings revealed a gap between teachers' concept images of slope (revealed in the word cloud) and their instructional intent (indicated by concept mapping), implying that there exists a disconnect between teachers' understanding of a concept and the content they deliver to their students. In particular, to the researchers, “it appears that [pre-service and in-service] teachers emphasize an image of slope that is more procedural and less conceptual than their own image of the concept” (p. 14).

Nagle and Moore-Russo concluded from their findings that knowledge of a concept is necessary though not sufficient for meaningful instruction. Teachers may need more knowledge in choosing instructional materials for teaching and maintaining a commitment to their instructional objectives in their teaching. These findings should motivate researchers to engage in investigations that establish relationships between teachers’ concept images, instructional materials, and the dynamics of an enacted lesson. As I allude to later in this review, perhaps a solution to the disconnect between instructional intent and enactment can be found in developing a base of teacher knowledge through practice in the context of teaching.

## Models of Knowledge for Teaching Mathematics

In this section, I explore mathematics content knowledge (MCK), pedagogical content knowledge (PCK), and the relationship between the two. Since the development of knowledge for teaching calculus is central to the research undertaken in this study, I now provide a review of the literature on knowledge for teaching mathematics that inform the nature of knowledge that developed to teach calculus. I also include a rationale for why PMTs need an in-depth understanding of MCK and PCK to effectively teach derivative in calculus.

Research on teacher knowledge has been concerned with understanding the knowledge that teachers should possess and how they can make content available for students to learn (Ball et al., 2008; Rowland et al., 2005). Badillo et al. (2011) stressed the need in the particular case of teaching the derivative. Other efforts have led to various conceptualizations of teacher knowledge for teaching mathematics, such as Mathematical Knowledge for Teaching (MKT;

Ball et al., 2008; Hill et al., 2008), the model of 'proficiency' in the teaching of mathematics (Schoenfeld & Kilpatrick, 2008), and the knowledge quartet (Rowland et al., 2005). In this regard, some scholars have called for a shift in investigative attention from the structure of teacher knowledge (i.e., domains of knowledge that constitute teacher knowledge) to research that seeks to develop models of how such knowledge for teaching can be developed (Beswick et al., 2015; Scheiner, 2015).

**Mathematics Content Knowledge (MCK)**

Thompson (1984) asserts that teachers' conceptions of mathematical content influences how their mathematics is enacted in the classroom because teachers can only support students to learn the content they understand. Putman and Borko (2000) posit that teachers should engage with mathematical content in more meaningful ways in order to "understand the central facts and concepts of the discipline, how these ideas are connected, and the processes used to establish new knowledge and determine the validity of claims" (p. 6). Therefore, teachers need more opportunities – and maybe *different kinds* of opportunities – to learn mathematics content more deeply as they learn how to teach it.

Despite MCK being an essential component of teacher knowledge with a relationship to teaching and learning, research finds that many mathematics teachers lack the profound understanding of fundamental mathematics (Ma, 1999) that is necessary to actualize meaningful teaching in the classroom. For instance, Ball (1990) explored prospective elementary teachers' understanding of mathematics content at three colleges using questionnaires and interviews, and found that even though they had completed core content courses for mathematics majors, they still had shallow knowledge of elementary-level fraction concepts.

In the context of advanced mathematics, Pino-Fan et al. (2018) assessed prospective teachers' mathematical content knowledge related to the derivative through an analysis of questionnaires and found that the knowledge they possess is limited. For example, one student was able to provide meanings for the derivative, but given a function, they could not identify points where the tangent to the graph of that function was horizontal. Lew et al. (2016) conducted a case study and reported similar findings. They video-recorded the lecture of an experienced and well-respected professor and also interviewed him and his students to explore students' understandings of the central concepts within proofs in a real analysis class. Although the students followed the steps in the professor's calculations, they actually lacked the knowledge required to comprehend his explanations. If these findings can be generalized, they suggest that PMTs complete teacher education programs with narrow knowledge of the content they will teach, which means they're likely to encounter difficulties in their efforts to support students' meaningful learning of that same content.

Research has found that teachers' MCK is a significant predictor of their professional practice (Baki & Arslan, 2017), classroom instructional practice (Tsamir, 2005), curriculum implementation (Sherin, 2002), and student learning (Hill et al., 2005). For instance, Baki and Arslan (2017) worked with preservice mathematics teachers to explore the relationship between MCK and mathematics pedagogical content knowledge (MPCK), which is a proposed domain of knowledge that includes student content knowledge and knowledge of how to organize and present a mathematics lesson. Analysis of data collected through observations and interviews revealed that preservice mathematics teachers with weak MCK had difficulties interpreting students' responses and explanations and providing them with appropriate feedback, thus affecting the quality of their teaching. These studies are essential because they help us understand the quality of MCK held by the teachers and how such knowledge influences their mathematics teaching.

Furthermore, Baki and Arslan's (2017) finding of an interplay between MCK and MPCK led them to stress the need for teachers to develop in-depth and relational understandings of these domains of teacher knowledge. They expressed concern that professional development, although focused on strengthening teachers' conceptions of content knowledge needed for teaching mathematics, may not offer the kind of sustained engagement necessary to develop MCK and MPCK. In response, they recommended that teacher education programs provide opportunities for prospective teachers to develop MCK and PCK as well as their interplay in the work of teaching mathematics. Additional research that seeks to construct a theoretical model of teacher knowledge that develops in practice can inform this practical need.

Research has also found MCK to be a central component of teacher knowledge and an important influence on students' mathematical achievement (Ball et al., 2004; Hill et al., 2005). As a result, professional bodies of mathematics education (e.g., National Council of Teachers of Mathematics [(NCTM, 2000) have emphasized the need for teachers to develop the content knowledge they need to implement mathematics education reforms. However, teachers need more than MCK to teach mathematics meaningfully (Hill et al., 2008; Sherin, 2002). They also need to understand common student conceptions of mathematics, common errors they commit as they think and reason about mathematics, and knowledge of mathematics curriculum that can inform the design of instruction that meets the learning needs of students. In this regard, Sherin's (2002) observations and analyses of classroom lessons concluded that the meaningful teaching of mathematics entails an interaction of the following domains of teacher knowledge: subject matter, curriculum materials, and knowledge of student learning.

The obvious conclusion is that in teacher preparation coursework, mathematics teacher educators should teach the mathematics that PMTs will eventually teach. They should also support them in developing capacities to leverage that knowledge when teaching. However, the research on what and how much mathematics teachers should know is mixed, resulting in disagreements over the depth and breadth of mathematical knowledge that a teacher should know (e.g., Ball & Bass, 2000). Moreover, scholars do not even agree on the relationship between the number of courses prospective teachers take in college, the teacher knowledge that's developed, and the potential impact on students' achievement. Ball and Bass (2000), for instance, found that the number of advanced mathematics courses taken by a teacher has no association with their students' achievement. However, research finds that teacher knowledge does (Campbell et al., 2014; Darling-Hammond, 2000). What these findings repeatedly remind us is that MCK is necessary but not sufficient for enacting effective mathematics teaching, thus raising skepticism about any claims concerning a relationship between the number of mathematics courses a teacher has taken and the quality of their teaching. Therefore, still more research is needed to understand better how MCK, as a component of teacher knowledge, contributes to effective teaching. Perhaps it plays a stronger role than these findings would have us believe, but that the way it has been measured (in terms of the number of advanced math courses one has taken) is lacking in validity.

As teachers design instruction, they integrate their knowledge of the subject matter with their assumptions and beliefs about the teaching and learning of mathematics, what they know about students' prior knowledge and misconceptions, and the contexts in which the developed knowledge will be applied (Ball, 1988). These findings further emphasize that MCK alone is not sufficient to teach mathematics effectively. In response, some researchers have proposed looking to pedagogical content knowledge for answers. In the next section, I provide literature on how scholars have proposed a domain of pedagogical content knowledge and have suggested that both it and MCK are essential resources of teacher knowledge for supporting students' learning.

**Pedagogical Content Knowledge (PCK)**

In his 1986 Presidential Address to the American Education Research Association (AERA), Shulman introduced the concept of pedagogical content knowledge or PCK. PCK refers to knowledge at the intersection of content knowledge and pedagogical knowledge specific to the teaching of that content. Shulman (1986) offered PCK as a domain of teacher knowledge that distinguishes the "understanding of the content of a specialist from that of the pedagogue" (p. 8). He went on to describe PCK as a "special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding" (p. 8). Furthermore, he saw PCK as a form of knowledge that "goes beyond subject matter per se to the dimension of subject knowledge for teaching" (p. 9). That is to say, Shulman saw PCK as a form of knowledge relevant to the specific work of teaching.

PCK is the knowledge acquired as a result of the transformation of the substantive and syntactic structures of an academic discipline into curricular materials and pedagogical representations (Shulman, 1987). This transformation involves preparing and critically interpreting curricular materials, representing ideas in the form of analogies and metaphors, selecting appropriate teaching methods and models, and adapting representations to the characteristics and needs of children being taught. In the transformation process, teachers seek to reorganize and represent the subject matter of the discipline in a form "that is appropriate for students and peculiar to the task of teaching" (Grossman et al., 1989, p. 32). Thus, according to them, PCK can be understood as a constellation of "forms [of knowledge] that are pedagogically powerful and yet adaptive to the variations in ability and background presented by the students" (p. 15).

To accomplish this transformation and represent materials in ways that students can comprehend, Shulman (1987) explained that the teacher should know instructional strategies, appropriate ways to represent content to the students, and common student misconceptions. Instructional strategies include ways of organizing instruction, specific actions teachers might take, activities for student learning, and the materials they can use in instruction. Scholars have taken up Shulman's notion of transformation and applied it to their specific disciplines. For example, in mathematics education, the transformation has been described as a means of unpacking mathematics content into forms that students can access and understand (Adler & Davis, 2006).

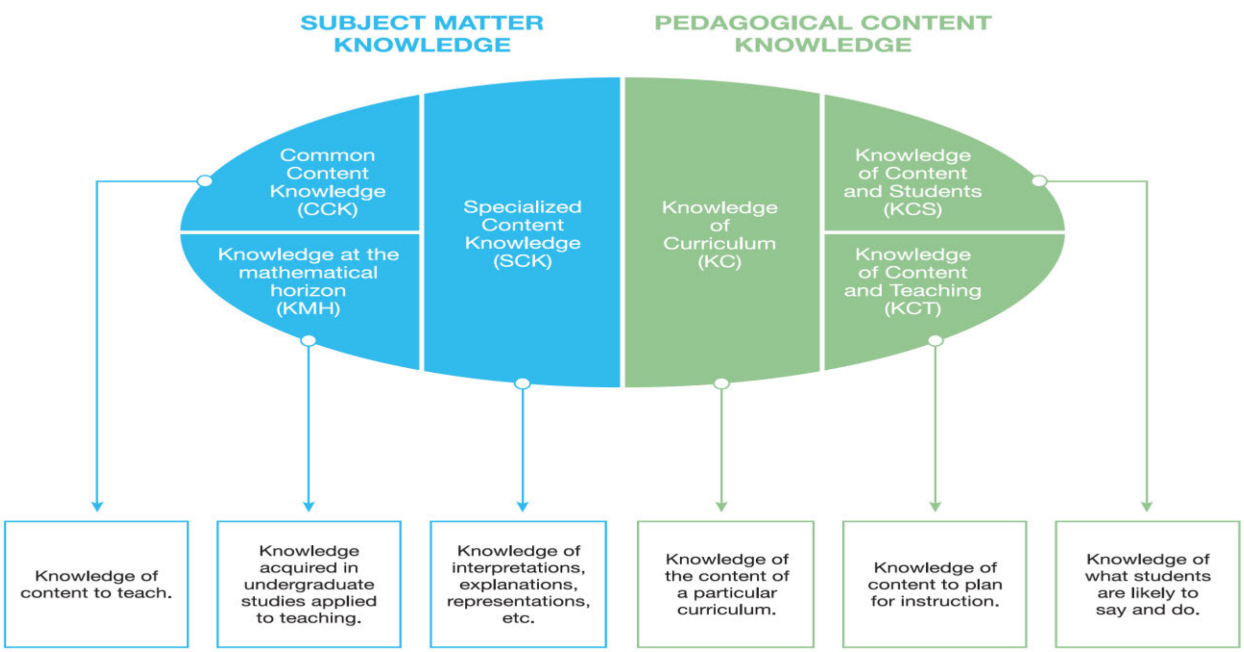
There is a need for teacher professional learning that can support PMTs to develop and represent mathematics with the complex realities of classrooms in mind (Cochran-Smith, 2003). These professional practices require that mathematics teacher possess a profound grasp of MCK and the ability to unpack and meaningfully represent it to facilitate students' learning. Thus, PCK cannot be viewed as a sub-category of SMK or a generic form of knowledge, but a new form of knowledge "that preserves the planning and wisdom of practice that the teacher acquires as a result of repeated planning and teaching of, and reflection on the teaching of, the most regularly taught topics" (Hashweh, 2005, p. 290). That is to say, as PCK is transformed from MCK in the manner described by Shulman (1987), Hashweh (2005), and others, it becomes increasingly aligned to the work of teaching because both the content and pedagogy – essential constituents of teacher knowledge – are integrated and interact in the enactment of classroom teaching. As such, PCK is an invaluable component of teacher knowledge that should be given full attention to professionalizing teachers in teacher preparation.

**Mathematics Knowledge for Teaching (MKT)**

Ball et al. (2008) define mathematical knowledge for teaching (MKT) as “mathematical knowledge that teachers need to carry out their work as teachers of mathematics” (p. 4). An adaptation of their original representation of MKT appears in Figure 1 below.

**Figure 1**

***Ball et al's (2008) Mathematical Knowledge for Teaching (MKT)***




A reproduction of Ball, Thames, and Phelps’s (2008) conception of Mathematical Knowledge for Teaching (MKT).

MKT is a framework that emerged from analysis of the work of teaching and that focuses on the ways in which teachers use their knowledge to teach mathematics in the classroom. Principally, Ball and colleagues (2008) posit that teachers should be able to discuss the models, concepts, and procedures of mathematics in the classroom. This kind of discussion constitutes reasoning about mathematical concepts, establishing their meanings, and making sense of the relationships among them. Further, Hill and colleagues (2005) offer that the work of teaching involves providing justifications for mathematical statements, making connections among ideas and procedures within mathematics, as well as providing exemplars for mathematical concepts, strategies, and proofs to the students. Moreover, teachers should not only share strategies for solutions to mathematical problems, they should also attend to the meanings of concepts that are implicated in those strategies.

Ball et al. (2008) introduced constructs of teacher knowledge specific to the teaching of mathematics by building on two domains of teacher knowledge introduced by Shulman (1986) – pedagogical content knowledge (PCK) and subject matter knowledge (SMK). Specifically, as shown in Figure 1 above, PCK is parsed into three components: knowledge of content and students (KCS; knowledge about mathematics and students in terms of their conceptions and their misconceptions of mathematical concepts as well as interpreting their mathematical thinking); knowledge of content and teaching (KCT; knowledge of the design of instruction, such as the knowledge governing the choice of examples to introduce a concept and then take students deeper into it); and knowledge of content and curriculum (KCC; which refers to the understanding of instructional materials and programs).

Similarly, Ball et al. (2008) also parse subject matter knowledge (SMK) specific to mathematics into three domains: specialized content knowledge (SCK; mathematical knowledge and skills tailored for teaching, not typically held by any well-educated adult, and not used in settings other than teaching mathematics), common content knowledge (CCK; mathematical knowledge possessed by any well-educated adult, and certainly by all mathematicians), and horizon content knowledge (HCK; teachers' knowledge regarding how various mathematical topics link up and relate to each other). Finally, Ball et al. (2008) portray MKT as a domain of teacher knowledge beyond a mere grasp of content knowledge to include representations of content, being familiar with typical students' misconceptions, and making in-the-moment decisions during instruction to achieve stipulated learning objectives. Thus, MKT is conceived as an integrated base of knowledge for teaching mathematics.

Ball and colleagues (2008) propose that MKT is a practice-based theory because it evolves from on-the-job analysis of teaching. However, while they appreciate Shulman's PCK as a point of departure for discussions related to teacher knowledge in discipline-specific areas, they critiqued it for several reasons. One of the critiques is the lack of an empirical grounding to justify that PCK is a special and distinct form of teacher knowledge. Here, Ball and colleagues (2008) recognize the transformed nature of SMK (Shulman, 1987) by recognizing that it also includes knowledge of how to unpack SMK and represent it in ways that students can come to understand. Still, they regard the transformation process to be theoretical (as opposed to empirical) because it occurs in teacher preparation, independent of authentic teaching situations.

Ball et al. (2008) further critiqued Shulman (1986) for holding a static view of PCK. A static view of knowledge alludes to the notion that knowledge for teaching is held internally and awaiting application in the classroom. Ball and colleagues (2008) raised concerns about the prospect that this knowledge could be robust yet removed from the practice of teaching in a classroom context. They argued that the transformation may fail to capture the complex reality of classroom situations and influence student learning. Moreover, other scholars propose to broaden Shulman's conceptualization of PCK beyond the knowledge of instructional strategies, representations, and student (mis)conceptions to also incorporate beliefs about the nature of mathematics (Friedrichsen et al., 2011), knowledge about how emotions and affects influence teachers' and students' educational experiences (Zembylas, 2007), and knowledge of media for instruction (Mark, 1990).

Ball (2003) proposes three core principles that typify the nature of teacher knowledge that PMTs should develop in their teacher preparation. The first principle underpins the need for PMTs to develop "more understanding of the insides of ideas, their roots and connections, their reasons and ways of being represented" (p. 8). The second principle holds that PMTs should learn specialized knowledge for teaching mathematics. On that ground, they are expected to resolve problems that are inherent to the work of teaching, such as "interpreting someone else's error, representing ideas in multiple forms, developing alternative explanations, [and] choosing a usable definition" (p. 8). Finally, the third principle upholds the need for PMTs to learn mathematical knowledge related to the mathematics they will teach in the future and meet the demands of teaching. Such knowledge should support teaching in a raft of ways, "from offering clear explanations, to posing good problems to students, to mapping across alternative models, to examining instructional materials with a keen and critical mathematical eye, to modifying or correcting inaccurate or incorrect expositions" (p. 8). These principles are important because they tell us about the kind of knowledge that teacher education programs should provide to support prospective teachers to develop to meet the demands for teaching mathematics. To address the need for a model of PCK that accounts for the complexity of classroom contexts, Ball and colleagues (2008) developed MKT so that it is dynamic and grounded in empirical evidence from observations and assessments of actual classroom teaching and student learning. The argument that MKT is dynamic implies that this kind of teacher knowledge develops as it is applied in teaching; its development is contingent on the teaching context.

**Teaching Knowing: Dynamic and Static Views of PCK**

Before continuing my review of the research on teacher knowledge domains, I take a pause in this section to elaborate on the distinction between static and dynamic views of teacher knowledge. This distinction is critical because it informed my intentions of developing a model of how PMTs acquire knowledge for teaching mathematics through video-mediated methods.

Building on Shulman's (1986, 1987) work, Gess-Newsomen's (1999) imagined PCK being both *integrated* and *transformative*. An integrated model holds that PCK does not stand as a distinct domain of teacher knowledge. Instead, knowledge from across domains such as subject matter, pedagogy, and the context (i.e., situations and activities embedded in the teaching and learning) are integrated and interact in the act of teaching. As mentioned above, MKT is conceived as an integrated model. It is also a *static* model. This means that it can be developed independently of the contexts in which it will be applied. In other words, it can be acquired outside of teaching situations and held it in ready-made form awaiting application in teaching. Shulman's (1987) model of teacher knowledge is also statically held, because he omitted—or at least under-emphasized—a constant dialog and a dynamic interaction of these domains with actual learners and within the context in which knowledge is developed and applied.

Gess-Newsomen's (1999) view of integrated knowledge has the additional feature that the knowledge is contingent and adaptive to the immediate context in which the teacher interacts. The supposed divisions between subject matter, pedagogy, and the context wither away as these interdependently blend to form a new, more highly integrated body of knowledge hypothesized to be more useful in teaching than its constituent domains operating in parallel. Thus, as Hashweh (2005) proposes, such knowledge is relevant to teaching precisely because we can inquire about its use in authentic teaching contexts. This notion of professional knowledge coming into being through the immediate context in which it is enacted (i.e., being adaptive to the situations and classroom complexities) has been described as a *dynamic* view of teacher knowledge (Gess-Newsomen, 1999; Scheiner, 2015). Putnam and Borko (2000) underpin the need for such a dynamic view, as they emphasize the value of developing "professional knowledge in context, stored together with characteristic features of classrooms and activities, organized around the tasks that teachers accomplish in classroom settings, and accessed for use in similar situations" (p. 13).

Cochran et al. (1993) share a dynamic view of teacher "knowing." This use of "knowing" as opposed to "knowledge" is intentional. As a critique of the static view of teacher knowledge, Cochran and her colleagues introduced a form of knowing called "pedagogical content knowing" (PCKg) that is dynamic, evolving, contingent, and more consistent with the tenets of constructivism. From this perspective, they propose that teacher educators need to simultaneously construct "a teacher's integrated understanding of four components of pedagogy, subject matter content, student characteristics, and the environmental context of learning" (p. 266). They should use that understanding to develop "teaching strategies for teaching specific content in a discipline in a way that enables specific students to construct useful understandings in a given context" (p. 266). Here, the "students" are preservice teachers, and they construct knowledge in the immediacy of a teaching context. Thus, Cochran et al. took an explicitly constructivist view of teacher knowledge. They held that "teachers must develop their pedagogical knowledge and subject matter knowledge in the context of two other components of teacher knowledge: teachers' understanding of students and of the environmental context of learning" (p. 266). Thus, PCKg refers to knowledge that is continually developing and considerate of a teaching context as it iteratively attends to the particularities of that context.

Exploring ways of knowing how to teach is essential to scholars like (Scheiner et al., 2019), who attribute the notion of specialization of mathematics teacher knowledge to knowing how to teach or a "style of knowing" (p. 168), rather than an acquired body of knowledge. For this reason, specialized content knowledge (SCK, presented next) for teaching mathematics is not accounted for by the mathematical content knowledge that teachers know but by how "teachers' knowing comes into being" (p. 168). In a phrase, a dynamic view holds that the knowledge is "not [in] a state of being but a process of becoming" (p. 167). That is, the knowledge for teaching is seen to develop in the context of teaching or simulations of teaching. In the context of design, (Schön, 1992) describes this phenomenon as "knowing in action" (p. 5). Consistent with Scheiner et al.'s (2019) perspective on knowing – and Schön's (1992) view of "knowing in action" (p. 5), this dynamic view of teacher knowledge is one that took in this study.

**Common Content Knowledge and Specialized Content Knowledge**

Having reviewed the literature on the broader domains of teacher knowledge above (i.e., MCK, PCK, and MKT), in this section, I present research-based conceptions of their subdomains to gain insights into the nature of knowledge developed in the study on which this review is framed. Specifically, I explore two components of SMK (Ball et al., 2008; see Figure 1 above), common content knowledge (CCK) and specialized content knowledge (SCK). Then, I review the literature on how they are defined, how they are advocated for in teacher preparation and professional development, and the nature of their interactions and the relationship between them. Also, I will explore how they are positioned as subdomains of mathematics teacher knowledge. It is expected that exploring PMTs' understanding of these subdomains of knowledge for teaching mathematics will help clarify their distinction, reveal insights for developing them during teacher preparation, and hopefully illuminate and inform the design of a video-mediated study of professional learning.

Ball and colleagues (2008) extended earlier notions of SMK (see Ma, 1999; Shulman, 1986) by devising CCK and SCK as essential constituents of SMK. As such, it seems that CCK is Ball et al.'s way of referring to MCK for teaching. However, as I understand CCK and MCK, there are distinctions between them. The latter is more extensive than the former, going beyond conceptual and procedural mathematics to a more elaborate model of mathematical proficiency, such as the one provided in "Adding it up" by the National Research Council (2001), which includes conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. Research on teacher knowledge (e.g., Ball, 2003; Council & Committee, 2001; Hill et al., 2008) finds that CCK is not sufficient for effective mathematics teaching. Thus, MCK and/or some other construct(s) of content knowledge specialized for teaching mathematics must be identified.

Ball et al. (2008) conceived of specialized content knowledge (SCK) as a dimension of mathematical content knowledge. Hill et al. (2005) describe it as involving "explaining terms and concepts to students, interpreting students' statements and solutions, judging and correcting textbook treatments of particular topics, using representations accurately in the classroom, and providing students with examples of mathematical concepts, algorithms, or proofs" (p. 373). Hill and colleagues (2008) further considered SCK as pure content knowledge "tailored in particular for the specialized uses that come up in the work of teaching" (p. 436) and the promotion of students' mathematical reasoning and thinking. It is described as "pure" knowledge because it is "not mixed with knowledge of students or pedagogy and is thus distinct from the pedagogical content knowledge identified by Shulman and his colleagues" (Ball et al., 2008, p. 396).

**Relationship Between Mathematical and Pedagogical Content Knowledge**

This section reports on research that has illuminated how mathematical content knowledge (MCK) and pedagogical content knowledge (PCK) as essential domains of mathematics teacher knowledge are related. The determined interactions between the two domains can provide us with insights to inform their curricular emphasis in teacher preparation and professional development. Therefore, I draw from the literature to explore the intertwined nature of the MCK and PCK and their importance and position in the teacher knowledge base required for teaching mathematics.

Research has found that MCK plays a role in PMTs' development of PCK (Baki & Arslan, 2017; Kilic, 2011). Baki and Arslan (2017) posit that MCK is a predictive prerequisite for the development of PCK. For instance, Baki and Arslan (2017) found from their study that PMTs with limited mathematical content knowledge (MCK) had difficulties developing and leveraging mathematical pedagogical content knowledge (MPCK) to present instructional materials to students in ways they could comprehend and learn from. This was evidenced as PMTs struggled to assess students' understanding of the concepts and their deficiencies in addressing students' responses and explanations appropriately. Further, Baki and Arslan found that even PMTs with strong MCK but insufficient MPCK enacted ineffective lessons. They showed difficulties in making the tasks accessible while maintaining cognitive demand, selecting appropriate and suitable representations, and leveraging knowledge already known to the students.

Kilic (2011) also found a direct relationship between MCK and PCK. He suggested that teachers need a vast and in-depth understanding of the subject matter and the capacity to represent instructional materials appropriately, assess students' learning, and make sound instructional decisions associated with presentations and sequencing of the instructional delivery. Kilic conceives PCK in terms of four constituent components with a “reciprocal relationship between them” (p. 3): knowledge of subject matter (teachers’ procedural knowledge and conceptual understanding of mathematics), knowledge of pedagogy (teachers’ ability to choose appropriate tasks, examples and representations for a particular group of students and their repertoire of teaching strategies), knowledge of students (involves teachers’ knowledge of students’ conceptions, misconceptions, and possible difficulties about a particular topic and their ability to diagnose and eliminate such misconceptions and difficulties effectively), and knowledge of curriculum (knowledge of learning goals for different grade levels and knowledge of instructional materials). Kilic investigated the development of this knowledge among secondary preservice mathematics teachers through their participation in the activities and discussions in their methods course and field experiences. The participants were interviewed and also asked to respond to content-specific questions in order to assess their PCK. The researcher participated in the course, took notes, and accessed written materials provided in the class. Kilic found from the analysis of observations, interviews, and written artifacts of students’ work that the participants’ knowledge of the subject matter influenced their development of PCK. And in particular, those participants with in-depth subject matter knowledge elicited “a rich repertoire of teaching strategies, and [were] able to critically select tasks, examples, representations, and instructional materials to promote students’ understanding of a particular topic, and to diagnose and eliminate students’ errors and misconceptions effectively” (p. 8).

For Scheiner (2015), PCK and MCK intertwine to “build the knowledge bases that constitute the particular kind of knowledge that is considered as specialised for the purposes of teaching mathematics” (p. 568). This intertwining is evident in his model of PCK, which has three domains in different dimensions: (1) knowledge of students' understandings (KSU, cognitive perspective), (2) knowledge of learning mathematics (KLM, epistemological perspective), and (3) knowledge of teaching mathematics (KTM, didactical perspective). It is worth mentioning that Scheiner's model is unique among the other models presented here in the sense that it proposes that teachers should possess a model of how learning works.

Research underscores the need for mathematics educators to develop the ability to manage the classroom, organize lesson activities, develop a good lesson plan, motivate students, and assess students' understanding of the mathematical content as they enact student-centered instruction in the classroom (Fennema & Franke, 1992). Shulman (1987) suggests that the knowledge that a teacher needs for teaching involves a "blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction" (p. 8). Other scholars (e.g., Ball, 2000) shared the same assertion that teachers need to know about the students, curriculum, educational goals, and instructional materials to design instructions that meet the needs of the learners. Park and Oliver (2008) further emphasized the importance of understanding the intellectual needs of students, mathematical topics and their organization, and utilization of curriculum materials as essential components of PCK that support teachers to enact effective teaching.

What emerges from this body of literature is that MCK and PCK are inextricably intertwined and that their interactions result in an integrated base of knowledge needed for the teaching of mathematics. This insight informs the proposed study framed around this literature review regarding what domain of mathematical knowledge is more crucial and should receive more attention. In an earlier conception of this study, I had focused my attention on PCK. My focus shifted later because the literature revealed that neither of the two components of mathematical knowledge alone can provide sufficient affordances to teach mathematics effectively. Instead, a model of knowledge that integrates MCK and PCK is needed.

**Preservice Teachers' Mathematical and Pedagogical Content Knowledge**

PMTs seek to develop MCK and PCK as they progress through teacher preparation coursework. Researchers (Ball, 2000; Piccolo, 2008) argue that as preservice teachers go through the program, they gain knowledge and experiences interweaving MCK and PCK to develop teacher knowledge that they can draw on to support students as they learn mathematics. For example, Ball and Bass (2009) posited that PMTs should go beyond understanding mathematical facts, algorithms, and procedures. In addition, they should develop in-depth knowledge regarding the connections between mathematical concepts and know how to break big ideas down into concepts that students can understand. Thus, the expectations are bold, as PMTs require a high level of preparation to develop skills of drawing insights from the interaction of MCK and PCK to inform their classroom instruction (Cooney, 1994).

Several studies reveal a close relationship between MCK and PCK of mathematics (Ball et al., 2008; Ding et al., 2014; Krauss et al., 2008). For instance, Krauss and colleagues (2008) examined the relationship between the MCK and PCK of preservice mathematics teachers. They found that those who had a strong background in MCK also had some PCK for that content too. This finding is consistent with the work of Ding and colleagues (2014), which employed a video-based approach to investigate the relationship between preservice math teachers' MCK and PCK associated with ratios. Their findings reported that preservice math teachers with a rich understanding of MCK selected a variety way of representing mathematical concepts related to ratios to their students. These findings suggest that preservice teachers with strong MCK are already on their way to developing PCK, thus acknowledging the value of strong MCK for teaching. These findings are rather unexpected and have implications for teacher preparation regarding the need for mathematics coursework in addition to methods courses.

**Concluding Thoughts on Teacher Knowledge**

Research reviewed on teacher knowledge speaks to knowledge that should be acquired in teacher education programs as a resource for enacting meaningful instruction in classrooms. Unfortunately, when teachers lack the requisite knowledge for effective teaching, they tend to abandon research-based strategies of teaching acquired in teacher preparation and instead fall back into ineffective lecture-oriented models of teaching they experienced in their classrooms as students (Cohen, 1990; Lortie, 1977). Thus, there is a need to identify just what this requisite specialized knowledge for teaching is and understand how it can be developed.

To meet the demanding task of teaching mathematics, scholars have developed various knowledge models for teaching mathematics. These models have informed the field of mathematics education regarding the conceptions of teacher knowledge domains. However, although these knowledge models help teacher educators understand the relationships between content and pedagogy, they have not been specific about that content. For that reason, some scholars (e.g., Ball, 2003; Pino-Fan et al., 2018) have called for research that focuses on content-specific specialized knowledge and how such knowledge may be developed. The study tethered to this reviewed literature aimed to align with such calls, and it did so in the context of calculus.

Shulman's notion of PCK has been central to a large body of research in mathematics education. The literature reviewed in this section identified efforts to specify Shulman's (1986, 1987) domain-general conceptualization of PCK to mathematics teaching. However, some scholars take issue with the static nature of Shulman's PCK and have stressed the need to model teacher knowledge that is dynamically emergent in response to what (Brown, 1992) describes as the "blooming, buzzing confusion" (p. 141) of classrooms. Thus, there is a need to extend Shulman's PCK to include other essential aspects deemed relevant to teaching, such as the curriculum, students, and the context, to provide an integrated knowledge base that can only be observed in the enactment of instruction. For that reason, specialized knowledge for teaching mathematics has been operationalized as an integrated form of teacher knowledge in the discipline of mathematics embedded in the situations, mathematical activities, and interactions among teachers and students.

The literature that has been reviewed in this section further reveals that PMTs' level of MCK is a strong determinant of their ability to develop the knowledge needed for the effective teaching of mathematics. Such knowledge results from reorganizing, describing (Scheiner, 2015), transforming (Shulman, 1987), and facilitating a high degree of integration (Scheiner, 2015) of MCK and other components of teachers' knowledge bases. Therefore, it has become clear that there is a need to ensure that PMTs have a flexible and rich knowledge of the calculus concepts in order to be able to develop a concrete and an integrated knowledge for teaching calculus.

This review of the literature on teacher knowledge focuses almost exclusively on research that assumes static conceptual models defined in terms of their constituent bases of knowledge (e.g., MCK, PCK, MKT). However, there is little research operating from a dynamic model of the development of teacher knowledge, particularly in calculus education. This study aimed to address this gap by providing a model of knowledge development that secondary mathematics methods instructors could utilize to support PMTs in developing a dynamic and responsive body of teacher knowledge, specifically for teaching calculus. In addition, this research aimed to offer the field an understanding of how PMTs come to *notice* (e.g., Lobato et al., 2013) student thinking, a construct whose literature I review next.

## Teachers' Professional Noticing

In this section, I review the literature on teachers' professional noticing. I begin by defining teachers' professional noticing and address the way it has been used as a framework for supporting PMTs as they learn to attend to the substance of students' ideas and, in turn, broaden their knowledge of students' mathematical thinking. Reviewing the research on teachers' professional noticing brings together two main foci of this literature review: teacher knowledge and teacher professional development. I explore the affordances of the professional noticing framework (also referred to as "teacher noticing") and its potential to contribute to a constructive process of building teacher knowledge and a capacity to notice student thinking. I conjectured that this framework could support PMTs learn to attend to and center students' ideas at the heart of their teaching practice. Additionally, through a review of studies that demonstrate how critical capacities of teacher noticing can be learned from practice, I further examine how such developed noticing capacities can inform teachers' instructional decisions.

Teachers' noticing is an evolving body of research that has been applied in teacher education and professional development programs through deliberate, focused, and systematic analyses of artifacts of practice (e.g., classroom videos of teaching or problem solving; Corwin et al., 1996) and microteaching experiences (Amobi, 2005). Van Es and Sherin's (2002) model of teacher noticing involves three sequential and interrelated steps: (1) attending to and noticing noteworthy events of classroom teaching, (2) interpreting and making sense of the identified classroom events, and (3) responding based on their interpretations of those events. Jacobs and colleagues (2010) built on van Es and Sherin's (2002) model to explore teachers' noticing in the particular domain of children's mathematical thinking and in the particular context of their problem solving. As problem solving was also the context for this study, it was the model that I used. Jacobs et al.'s (2010) structure includes: "[a] attending to children's strategies, [b] inferring children's understandings from those strategies, and [c] deciding how to respond based on those inferences" (p. 172). Thus, Jacobs and colleagues (2010) applied van Es and Sherin's (2002) model of teacher noticing in the particular domain of children's mathematical thinking. Mason (2011) builds from van Es and Sherin's model, as well, by naming three processes within the psychological realm of noticing: 1) a mental preparedness and readiness to notice, 2) reflecting on the past to create an awareness of what to notice, and 3) noticing in the moment (attending to, interpreting, and responding to student thinking face-to-face and in real-time). Importantly, for Mason, these processes are inseparably linked and co-occur "as if constituting a single, integrated teaching move" (p. 173).

**Research on the Development of Teachers' Professional Noticing**

Jacobs et al. (2010) and van Es (2011) have applied the framework of teachers' professional noticing to develop teacher knowledge in video-mediated professional development. Leveraging that framework has been a means by which to nurture teachers' professional vision for responsive instruction that leverage knowledge of student thinking to inform and support their learning. Sherin (2007) characterized this professional vision according to two interrelated features: selective attention (the ability to identify essential classroom events that represent broad principles of teaching and learning) and knowledge-based reasoning (a cognitive process of making sense of the identified noteworthy classroom events). Thus, teachers' professional noticing, in the context of this study, was concerned with the systematic analysis of student thinking to support PMTs learn to identify and interpret relevant instances of student thinking, reflect upon them, and develop competency and expertise for making instructional decisions.

Sun and van Es (2015) designed a video-based course to support preservice teachers (PSTs) to develop a vision for an "ambitious pedagogy" (p. 202) through noticing student thinking. PSTs aimed to establish how particular teacher moves in the classroom are likely to influence student learning. They engaged in analyzing teaching videos by attending to, interpreting, and describing student thinking. To further develop their noticing capacities, the researchers engaged PSTs in "reflective cycles" (Rodgers, 2002) that involve "the process of learning to see, describe, and contemplate noteworthy events and interactions that occurred during teaching" (Sun & van Es, 2015, p. 204). Later, the researchers analyzed teaching videos that PSTs had submitted after the course for California's portfolio-based assessment of preservice teachers' preparation. As they analyzed these videos, researchers looked for "evidence of reform-oriented mathematics practices, such as eliciting student thinking, exploring and probing student ideas, and comparing and contrasting student-generated strategies" (p. 205). The researchers found that the video-based course participants were more likely to enact noticing practices than those who did not participate in the course. In particular, they elicited, attended to, and responded to student thinking during instruction. Sun and van Es used these findings to argue that these particular practices enable teachers to enact responsive teaching and examine whether a lesson's objectives have been achieved.

Jacobs and colleagues (2010) also investigated prospective and practicing teachers' noticing, but they did so in the context of children's mathematical thinking (Carpenter et al., 1999). Their purpose was to model how noticing develops among participants with varying levels of teaching experience, teacher leadership (e.g., visiting other teachers' classrooms and mentoring them, sharing mathematical problems with colleagues, and giving presentations at faculty meetings and conferences), and exposure to PD focused on understanding children's mathematical thinking. The participants were sorted into the following groups: Prospective teachers (no experience, no PD), Initial participants (experienced, no PD), Advancing participants (experienced, PD), and Emerging teacher leaders (experienced, PD, having leadership skills). Teacher leaders are those teachers who organized professional development for their colleagues around instruction that is responsive to children's mathematical thinking.

All participants in the study solved mathematical problems, read research related to student thinking, analyzed classroom videos, and characterized students' mathematical thinking through analyses of their written work. Then, depending on artifacts that teachers submitted (either teaching video clips or students' written work), teachers either watched students' problem-solving video clips or examined written student work. Their analyses were guided by prompts such as, "Please describe in detail what you think each child did in response to this problem" (p. 178), and "Please explain what you learned about these children's understandings…[and] pretend that you are the teacher of these children. What problem or problems might you pose next?" (p. 179). Analyses of post-PD assessments revealed that the PD had supported teachers to assess children's thinking and respond to it in ways that support children's understanding. Moreover, the difference in the levels of experience, PD, and leadership skills was associated with varying levels of expertise related to noticing. Specifically, the effect was stronger for those teachers with more substantial experience. That is, teachers' growth in ability to attend and respond to children's strategies correlated with years of teaching experience. This study demonstrates not only that teacher noticing can be learned (and that learning is associated with teaching and leadership experience), it also yields examples of activities (those incorporated in the PD) that have been shown to support its development.

As mentioned above, learning to notice involves teachers refining what they attend to and how they reason about what they see. With an objective similar to Jacobs and colleagues (2010), who sought to understand the nature of the development of teacher noticing, Sherin and Han (2004) studied how such teacher learning occurs through the use of video clubs. These are "meetings in which groups of teachers watch and discuss excerpts of videotapes from their classrooms" (p. 163). Four middle-school mathematics teachers participated in the study. In each meeting, they watched videos of themselves teaching and discussed what they saw. To direct their attention to student thinking, the researcher asked teachers to explain a particular student's statement or action. For example, a researcher asked a teacher to interpret what a student was saying about a particular graph they were interpreting. The teacher repeated exactly what the student had said about the graph: "It is not realistic." In the discussion that ensued, the teachers agreed that the student's words were "not very descriptive" (p. 169) of the graph. Thus, those words did not enable the teacher to attend to the student's ideas. Had the student's analysis of the graph been more elaborate, the teacher would be provided with resources to attend to the student's ideas and then interpret and respond to them. In other instances, a researcher would ask teachers to "clarify or expand upon a comment that he or she made or to explain the connection between a particular comment and what was viewed in the video" (p. 167). Over time, the researchers found that these "focusing" discussions supported teacher-participants' shifts in attention from focusing exclusively on students' actions to focusing on the mathematical substance (i.e., their thinking about content) of those actions. Further, the teacher-participants' analyses of student thinking became more sophisticated over time, from shallow statements of students' ideas to substantive analyses and evidence-based interpretations, and eventually, to generalize a trajectory of the development of students' thinking. Over time and through a similar intervention, teachers shifted from focusing on what students do and mere descriptions of their actions to attending to and analyzing students' strategies and discussing their mathematical thinking in detail.

Sherin and van Es's (2002) study yielded similar findings. They conducted a study in which preservice and in-service teachers in a video club discussed excerpts of classroom videos. Their objective was to understand how videos can support teachers to learn to notice and improve their practice. In-service teachers discussed videos of their teaching only, while preservice teachers discussed their own videos and those of others' classrooms. Teachers focused on three aspects of the videos: student thinking, the teacher's role, and classroom discourse. For each of these aspects, teachers were asked to respond to the generic prompt "What did you notice?" In addition, they wrote narrative essays of what it was about the videos that they noticed and discussed. Teachers' responses to the prompts were analyzed and revealed that two fundamental changes occurred over time: 1) what the teachers noticed and 2) how they described and interpreted what they noticed. Their analyses shifted over time from focusing on the classroom teacher's practice to focus on the students and their mathematical thinking. Also, they moved from making evaluative statements to interpreting classroom events and providing evidence from the videos to support their stances. Importantly, these changes in what teachers notice and reason about as they analyze practice indicate growth of knowledge-based reasoning. They also point to a viable means by which teacher learning occurs in this context, as evidenced in the increasing sophistication of teachers' analyses of students' thinking and proposals for alternative teaching moves that can better support meaningful classroom instruction.

Van Es's (2011) study with elementary teachers yielded similar findings. Van Es explored how elementary teachers learn to notice children's mathematical thinking over time in a video club. She purposed to develop a model of how such learning develops over time. The teachers met regularly to share and discuss videos of their lessons. Initially, teachers' assessments of student thinking were relatively shallow. They tended to provide general strategy descriptions and evaluative comments and offered little or no evidence from the video to support their claims. To help them learn to notice, the researcher provided general prompts that focused their attention on noteworthy events and specific prompts that directed their focus to student thinking. For example, a prompt such as "Let us take a look at how Lindsey solved that problem" (p. 137) is a specific prompt that directs the teacher to focus on the mathematical thinking of a particular student. Other prompts that the researcher used are "Why do you think she chose that method?" (p. 137) and "Where do you see that in the transcript?" (p.137). These specific prompts supported teachers to focus on mathematical details, interpret identified student thinking, and provide evidence from the videos to support their claims.

Learning to notice student thinking is a continual process. Teachers progressively engage in it even after the video-mediated professional development programs. Franke and Kazemi (2001) explain why this learning is sustained as teachers leverage student thinking in their instruction. To do so, they characterize teacher knowledge as learned with understanding if it is generative (Carpenter & Lehrer, 1999; Greeno, 1988), connected (Hiebert & Carpenter, 1992), and driven by one's own inquiry. It is generative in the sense that teacher learning is sustained in daily classroom dynamics that enable teachers to continue to further develop their understanding of teaching, even beyond formal professional development. It is connected because new knowledge is integrated into existing networks of knowledge and provokes a reorganization resulting in "rich integrated knowledge structures" (Franke & Kazemi, 2001, p. 105). Finally, it is driven by one's own inquiry when learners "perceive their knowledge as their own" and understand "that they can construct knowledge through their own activity" (p. 105).

Franke et al. (2001) used this framing of learning with understanding in a follow-up study that followed teachers' participation in a Cognitively Guided Instruction (CGI; Carpenter et al., 1999) professional development program. CGI is a professional development program designed "to help teachers develop an understanding of their own students' mathematical thinking, its development, and how their students' thinking could form the basis for the development of more advanced mathematical ideas" (Fennema et al., 1996, p. 404).The objective of that study was to examine the degree to which teachers who had learned practices central to noticing were later engaged in continuous professional growth. A classification scheme was used to analyze teachers' levels of engagement with children's mathematical thinking. Remarkably, the analysis revealed that all continued to implement the principles of teaching they had learned in the PD four years earlier. Some teachers were found to be at the same level of engagement as when they left the PD, while others moved to higher levels of enacting more responsive teaching, indicating that they had experienced continuous learning in practice. Thus, their learning was generative.

Franke and Kazemi (2001) also studied practices related to noticing in the context of a CGI workshop. Researchers had worked with teachers for a year— teaching them a university-based mathematics methods course and spending time with each of them in their school. Then they explored how these teachers, in their classrooms, implemented knowledge they had gained in the course about strategies that children are likely to use to solve particular problems, how these strategies develop over time, and how they could assess students' understandings by making inferences based on these strategies. The teachers had given their students mathematical problems to solve and then worked in groups with other participating teachers to analyze students' solutions using CGI principles collaboratively. The researchers found that teacher participants had used this knowledge to assess (or notice) their students' thinking formatively and that this knowledge had served them as a tool for the continual development of teacher knowledge, even a year after the PD. Thus, the teachers learned this knowledge with understanding because it was generative and self-driven. This review of some of the CGI literature informs my approach to students' thinking about the derivative, as it stresses how valuable it is for PMTs to be able to anticipate common misconceptions and strategies that students are likely to use to solve problems and to be able to assess their understanding by making inferences from those strategies.

The literature reviewed in this section revealed that efforts to develop teachers' professional noticing can be productive for understanding and embracing the complex task of teaching and learning and improving classroom practice. A profound contribution of the teachers' professional noticing research is that through video analysis and support, teachers can develop cognitive awareness of noteworthy aspects of instruction they can attend to and make sense of to develop knowledge-based reasoning that iteratively informs the design of classroom instruction. The notion of teachers' in-the-moment noticing of students' thinking and reasoning has been emphasized. Teachers operating from an inquiry orientation to teaching and learning are engaged in an ongoing learning process to elicit and respond to student thinking during instruction. Thus, analyzing teaching artifacts is a promising venue in which to support teachers in developing knowledge for teaching mathematics. As teachers' attention to student thinking is nurtured, they develop dynamic knowledge that emerges from the context of teaching and learning as they transform their content knowledge into knowledge that is accessible to students. In short, they develop PCK (Shulman, 1986) and MKT (Ball et al., 2008).

## Concluding Remarks on the Literature Review

The literature reviewed on teacher knowledge and teachers' professional noticing provides insights into how these two constructs are linked and how they can complement one another to support PMTs in developing knowledge for teaching. Models of teacher knowledge that presuppose that knowledge is transformed, integrated, and developed in the context of teaching are proposed to be more relevant to the core work of teaching than statically held knowledge of teaching. Such knowledge is dynamic and contingent on classroom interactions and is made visible in instruction, particularly when a teacher orchestrates instruction around students' thinking. To develop the capacity to enact such an instruction, I conjecture it can be developed in and from practice based on the literature reviewed in this study.

The review of research on teacher noticing emphasizes the critical role that artifacts (e.g., classroom videos, problem-solving videos) have played in this body of work. They are "springboards" (Borko et al., 2011, p.184) that mediate discussions about teaching and learning and that have the potential to represent classroom realities faithfully. Therefore, they offer opportunities for an analytic deconstruction of practice to help teachers develop insights around designing and enacting instruction that is responsive to the needs of students. Thus, this study employed problem-solving videos and a semi-structured interview protocol for supporting PMTs to attend to and make sense of student thinking as it unfolded in the video. Similar to Jacobs and colleagues' (2010) approach, prospective teachers of calculus watched videos of students solving problems to capture situations of interest, reflect upon them, and interpret them to understand students' thinking. That way, we could access PMTs' development of knowledge for teaching to enact instructional practices that are responsive to students' thinking. Developing a responsive pedagogy from practice yields the kind of ambitious teaching (Kazemi et al., 2009) advocated by reform efforts (NCTM, 2000) for the sake of improving mathematics education.

The core task involved in the episodes of this study was to investigate and advance how PMTs attended to, interpreted, and reflected upon aspects of teaching the derivative. As PMTs developed knowledge for teaching the derivative, the process culminated in a model in which such knowledge was developed in video-mediated professional development. As has been shown above, such a model could make a significant contribution to the research on calculus education with implications for the design of learning experiences in secondary mathematics teacher preparation.

Lastly, this body of research also demonstrates that teacher knowledge developed in video-mediated professional development can be generative. Teachers acquire a model, an orientation to professional vision that can support them in an iterative and ongoing manner across the trajectory of their practice. This study aimed to address the following research questions:

*1. How does teacher knowledge specific to noticing students' mathematical thinking in the domain of the derivative develop through video-mediated professional learning?*

*2. What forms of video-mediated interactions support this development?*

## Theoretical Perspectives

### Sociocultural Perspective

According to a Vygotskian (1978) sociocultural perspective, one's cultural, historical, and social interactions significantly impact an individual's cognitive and social growth. Vygotsky stresses that learning does not reside solely within an individual. Instead, it is distributed within a broader social context such that one's cultural interactions impact how and what one thinks. The culture models beliefs and attitudes within the community that children learn from, thus influencing learning. Children learn within the community's culture as they interact with peers, adults, teachers, and other mentors.

Vygotsky identified language as a powerful mediator for learning, a means for not only expressing thoughts and ideas but for first forming them. It is a psychological tool that enables humans to develop higher mental functions. Through language, cultural norms and practices are passed to on children through social interactions that differentiate and generalize words that once had more narrow meanings. This is a two-step process: “Every function in the child’s cultural development appears twice. First, on the social level and, later on, on the individual level … [that is] between people (interpsychological) and then inside the child (intrapsychological)” (p. 57). For the child, this is a process of internalizing what was once external, thereby promoting cognitive development. Artifacts such as instruments, tools, and signs mediate this process. They are the cultural tools for intellectual adaptation. Thus, for Vygotsky, thought and language promote (rather than precede) cognitive and social development. Thus, learning is the end product of those culturally situated, social interactions. This is in contrast to a Piagetian (1970) perspective, where the development of cognitive structures precedes and thus enables individuals to construct their understanding of the world.

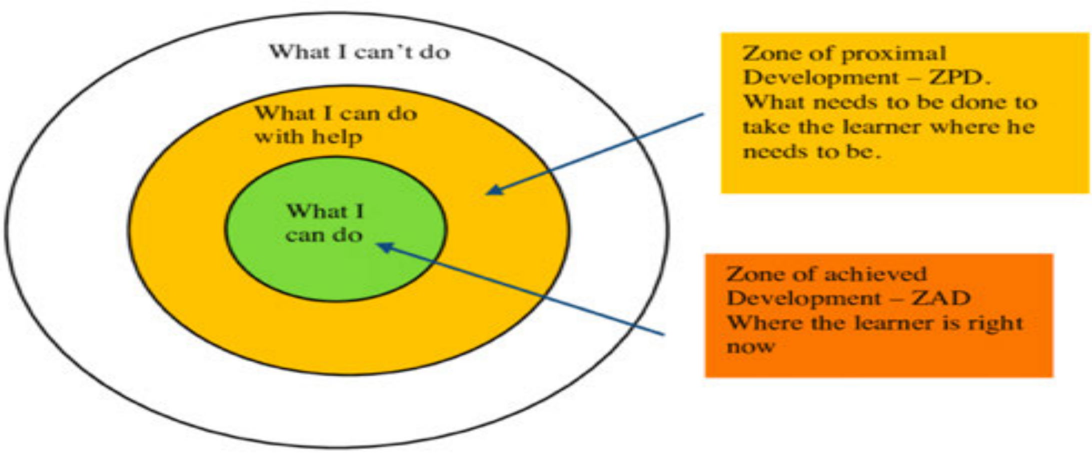
***Vygotsky’s Zone of Proximal Development and Scaffolding***

Vygotsky (1978) defines the zone of proximal development (ZPD) as *"the distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem-solving under adult guidance or* *in collaboration with more capable peers."* (p. 38). ZPD refers to the difference between one's current level of cognitive development and their potential level of cognitive development.

Working with a teacher or some other "more knowledgeable other" enables a learner to operate at that potential level of development. It is at that potential level where learning occurs, where the learner moves from "what I can only do alone" (i.e., what has already developed) to "what I can do with help" (i.e., what is currently developing; see Figure 2 below). ZPD can be thought of as the "sweet spot" of targeted intervention. Any instruction targeted below it will not be useful as development has already occurred there; any instruction outside of ZPD will also not be useful, because learning there is out of reach of the learner. Thus, ZPD involves two critical components: the potential level of development and also the interactions with others that support it.

**Figure 2**

***Vygotsky's Zone of Proximal Development***




Note: Vygotsky's zone of proximal development, adapted from *Educational Technology,* by S. Kurt, 2020, retrieved from https://educationaltechnology.net/vygotskys-zone-of-proximal-development-and-scaffolding/.

When learners come to complete tasks independently without the instructor's help, the learner moves from what Kurt refers to as his zone of current development (ZCD) through into ZPD and eventually towards achieving the learning goal (the zone of achieved development, ZAD), as represented in Figure 2. The change from "what I can't do" to “what I can do with help” to "what I can do alone" indicates that learning has occurred, and the learner has achieved cognitive development.

Vygotsky (1978) describes scaffolding as a critical concept for considering the actions that facilitate learners’ growth through purposeful and meaningful interactions with others. Through scaffolding, a more knowledgeable other (instructor or a more knowledgeable peer) works collaboratively with a learner to provide the support they need in order to enable them to operate within their ZPD so that learning may occur. In the beginning, the instructor offers more facilitation and closer guidance, but over time, as learners become increasingly independent in their problem solving, scaffolding is removed. This process continues until their development is no longer potential but achieved.

***The Value of a Sociocultural Perspective for this Study***

This study was interested in the instructional experiences of PMTs as they developed teacher knowledge through video-mediated discussions and in constructing a model of this knowledge development. As PMTs viewed videos of students' problem solving, they were asked to share their opinions, ideas, claims, and justifications about what they saw in relation to content, pedagogy, and curricular materials. Alongside the PMTs, the researcher, as the more knowledgeable other, scaffolded their engagement by facilitating discussions that directed their attention and elicited their conceptual and pedagogical thinking. Mediated by artifacts such as problem-solving videos, the researcher collaborated with PMTs to orchestrate a productive discourse as they analyzed student thinking. Through the researcher's interventions, PMTs would progress toward robust understandings of content and a reform-oriented pedagogical model that could support effective teaching of the derivative.

**Situated Learning Perspective**

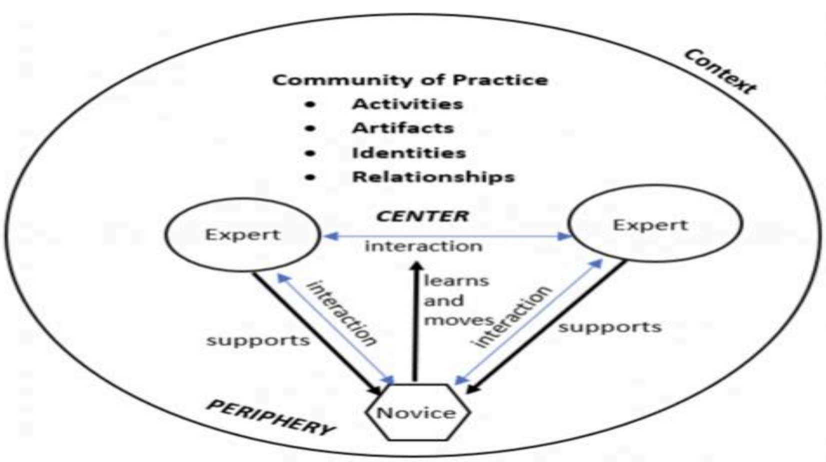
Situated cognition theory, also referred to as situated learning, is a learner-centered instructional design model developed by Brown et al. (1989). As a learning theory, it rejects the proposition that knowledge is “an integral, self-sufficient substance, theoretically independent of the situations in which it is learned and used” (Brown et al., 1989, p. 32). Instead, this perspective posits that there is no division of knowledge and practice. Simply stated, knowledge is embedded in the context, the activity, and the culture in which it is developed (Brown et al., 1989; Lave, 1991).

From a situated perspective, knowing involves doing. Novices are immersed into a community of practice, and through observing, doing, and reflecting, they gradually develop situated expertise (Brown et al., 1989; Lave, 1991). Professional learning communities operate as a community of practice when they share a concern or a passion for something they do and learn how to do it better as they interact regularly” (Wenger, 2009, p. 1). Through their participation in a community of practice, novices, or ‘newcomers’ (Lave, 1991), interact and relate with oldcomers (experienced practitioners) as they participate in activities through which they master knowledge and skills. Thus, they acquire new identities as full participants in the community of practitioners. As an example of a community of practice, the mathematics teachers in a department at one school may gather regularly after school to have conversations and reflections around the student-centered teaching of mathematics. As they do so, they develop their professional knowledge and improve their teaching practice. Borko et al. (2008) posit that professional development activities need not only occur in K-12 classrooms: “An alternative is to bring ideas and events from the classroom into the professional development setting through the use of tangible artifacts such as lesson plans, curricular materials, student work, and video of lessons" (p. 418).

From a situated perspective, cognition is fundamentally understood to be a product of individual-environment interactions within a context. Figure 3 below depicts these interactions and their role in moving apprentices along a trajectory of expertise from legitimate peripheral participation as novices to more central participation in the community of practice through their interactions with experts (practitioners) of a given field. Thus, situated learning involves not only knowing through participation and collective social practice, but also identity formation to become experienced and reliable members of a community of practice. Learning through a contextualized trajectory of participation is thus a process of becoming, as learning and identity account for each other.

**Figure 3**

***Interactions in a Situated Learning Theory***




Note: A situated learning theory, adapted from *Educational Technology,* by S. Kurt, 2020, https://educationaltechnology.net/vygotsky-zone-of-proximal-development-and-scaffolding/

***Contribution of Situated Learning to this Study***

This study explored PMTs' development of knowledge for teaching the derivative concept by analyzing students' thinking as it unfolded in the video of their problem-solving. Thus, framing learning as knowing through participation in the discourse of a community of practice, and as a process of becoming along a contextualized trajectory of participation (Lave & Wenger, 1991), enabled us to imagine how PMTs could develop knowledge for teaching the derivative through such a learning trajectory. The situated perspective informed this image by forefronting the intellectual power of spaces in which people engage in discourse as a community as they share their ideas and refine their ways of thinking (Lave & Wenger, 1991).

Instructional design informed by a situated perspective situates learning in the context of relevant and authentic experience. Learning in this case is conceived as the development of practice, particularly professional noticing and the knowledge and skills relevant to it. Movement along a trajectory of participation is facilitated by the researcher, positioned as the expert, who apprentices the PMTs – the novice participants – to develop their noticing skills. Mediated by interactions among the researcher, the problem-solving videos, and the PMTs, their practice evolved over time. So does the way they participated, from legitimately peripheral participation – like noticing superficial qualities of students' problem solving – towards more central participation characterized by sophisticated analyses of student thinking and responsive practice associated with ambitious teaching.

Finally, given the situated nature of learning and the researcher's intention to support the development of PMTs' dynamic and integrated base of teacher knowledge, I considered the authenticity of what they would observe in the problem-solving videos. In that regard, these scenes *were* authentic, because they depicted actual students solving typical calculus problems.

As Sherin (2004) asserts, "Video allows one to enter the world of the classroom without having to be in the position of teaching in-the-moment" (p. 13). As such, those participated in analyzing students' problem-solving videos had "the opportunity to develop a different kind of knowledge for teaching—knowledge not of 'what to do next, but rather, knowledge of how to interpret and reflect on classroom practices" (p. 14). Thus, if the researcher scaffolded PMTs' video analysis "as if" they were interacting with the students in the videos (as noticing work tends to do), then the researcher's claims about the PMTs' movement along a trajectory of knowing become more viable. This is because the videos depicted approximations of practice (e.g., Howell & Mikeska, 2021), and a situated perspective held that learning that occurs in one context is more likely to be transferred to another when the two situations are similar (Brown et al., 1989; Lave, 1991). A trajectory of participation accounted for contextual elements (as is warranted by a situated and sociocultural perspective), such as the mediating roles of arguably authentic tools and tasks in students' problem solving and both the researcher's and PMTs' inferences about the students' knowledge. Thus, as PMTs developed contextualized knowledge to become more expert-like, to teach more and more meaningfully. Hence the gap between theory and practice is bridged.

In conclusion, situative theorists (e.g., Greeno, 1998; Lave & Wenger, 1991) perceive the context and activities in which individuals engage within a learning milieu as critical and constitutive determinants of their learning. And while the situated perspective is sensitive to the range of participant interactions with the social, material, and conceptual resources of a learning environment, the sociocultural perspective offers a lens with which to analyze the nature of those mediating interactions. Thus, the particular elements of a context in which the researcher mediated PMTs' analyses of video recordings of students' calculus problem solving were accounted for, and viable claims could be made about a PMTs' situated and constructed knowing and its actual relation to teaching. For that reason, situated and sociocultural perspectives were suitably selected for a study that aimed to analyze PMTs' learning while accounting for the video-mediated interactions that supported it.

## Chapter 3: Methodology and Methods

This chapter will describe the methods and methodology I used to investigate PMTs' development of knowledge for teaching the concept of the derivative and a theory of how such knowledge developed. In particular, I begin by describing the problem that this research aimed to address and then I describe the methodology. Next, I explain the methods I used to collect and analyze data and ultimately answer my research questions. Finally, I address how trustworthiness of this study was established.

Research shows that mathematics teachers lack sophisticated skills for noticing student thinking (Sherin & Han, 2004). This is a concern because teachers need to learn how to assess student thinking so that they can design or redesign instruction in response to their students' understanding. Furthermore, static models of teacher knowledge developed outside of actual classrooms – or of approximations of actual classrooms – to be theoretical and thus may not address the authentic challenges of teaching mathematics. In contrast to these static models, a dynamic model of mathematical knowledge for teaching developed by analyzing teachers' practice in naturalistic settings may prove to be more relevant to classroom teaching.

For the sake of convenience, I remind the reader that this study aimed to 1) develop a model that supported PMTs to enact the knowledge for teaching and develop expertise to notice students' understanding of the derivative, and 2) identify the forms of interactions that supported their development in the video-mediated professional learning setting. These are the questions that guided this research:

*1. How does teacher knowledge specific to noticing students' mathematical thinking in the domain of the derivative develop through video-mediated professional learning?*

*2. What forms of video-mediated interactions support this development?*

To answer these questions, I conducted a design experiment (Cobb et al., 2003) together with semi-structured interviews (Clement, 2000; Merriam & Tisdell, 2015).

**Design Experiment Methodology**

The Design Experiment (Brown, 1992) is a research methodology that aims to engineer a particular form of learning and develop domain-specific theories about how such learning develops and how its development can be supported (Cobb et al., 2003). As such, I used it to answer research questions related to the development of PMTs' noticing of students' mathematical thinking and to theorize how such learning occurs in a video-mediated setting.

A crosscutting feature of design experiments is "to develop a class of theories about both the process of learning and the means that are designed to support that learning" (Cobb et al., 2003, p. 10). Cobb et al. (2003) provide an example: In the case of a one-on-one design experiment, for example, "the broader theoretical goal might be to develop a psychological model of the process by which students develop a deep understanding of particular mathematical ideas, together with the types of tasks and teacher practices that can support that learning" (p. 10).

As such, design experiments entail both a "pragmatic bent"– in that they entail engineering a form of learning, and a "theoretical orientation"– in that they entail developing domain-specific theories through the analysis of those forms of learning and how they can be supported (Cobb et al., 2003, p. 9). These theories are humble (Brown, 1992) in the sense that they are entirely accountable to the activity of the design. This means that they are also pragmatic because they speak directly to problems that teachers experience in practice. Thus, design experiments have the "dual goals of refining both theory and practice" (Collins et al., 2004). Unlike other philosophical orientations, such as constructivism, humble theories provide clear pathways for designing and organizing instruction.

Researchers undertaking design experiments test and refine their conjectures through iterative cycles of "progressive refinement" (Collins et al., 2004, p. 18). A conjectured design "is put into the world to see how it works. Then, the design is constantly revised based on experience until all the bugs are worked out" (p. 18). These theories are thus justified when their designs impact learning, when they work in the real world. That is, a design is effective if it impacts theory and practice. As such, both practice and theory are mutually informing as conjectures are subjected to an “iterative design process featuring cycles of invention and revision” (Cobb et al., 2003, p. 10). Through an iterative process, conjectures are generated, tested, and maybe refuted. Consequently, new conjectures are framed and tested.

Design experiments are highly interventionist in engineering and studying new forms of learning (Collins, 1992). Design research takes place within the naturalistic setting (Brown, 1992) of a learning ecology whose elements may include “the tasks or problems that students are asked to solve, the kind of discourse that is encouraged, the norms of participation that are established, the tools and related material means provided, and the practical means by which classroom teachers can orchestrate relations among three elements” (Cobb et al., 2003, p. 9). The inquiry is not purely naturalistic, though. The researcher yields some control in engineering a specific form of learning.

**Research Setting**

The study was conducted at Ntiba University (a pseudonym), a public Research Doctoral University and Hispanic-Serving Institution (HSI) in the Northeastern United States. This site was selected because "it is not in any way atypical, extreme, deviant, or intensely unusual" (Patton, 2015), and thus it follows the typical structure of an undergraduate mathematics education program that may be found at other various universities. These considerations were essential because I intended mathematics teacher educators to use the instructional model for developing teacher knowledge that emerged from this study in similar settings.

During the Fall 2021 semester, the Department of Mathematics offered a math methods course (MATH 470) for secondary preservice mathematics teachers (PMTs). MATH 470 aims to prepare PMTs to teach mathematics meaningfully by exploring how to remain committed to mathematics throughout planning and instruction, providing cognitively demanding learning opportunities, supporting PMTs as they engage in inquiry to develop teacher knowledge, and determining effective methods for assessing and evaluating students. PMTs taking this course are mostly in their senior year. The mathematics of the course is consistent with middle- and high-school content.

Meetings with research participants were held outside of class and spread throughout the week, depending on PMTs' availability. As stated above, I hypothesized that PMTs would develop their capacity for noticing student mathematical thinking and further develop their understanding of derivatives as they viewed and analyzed videos of students solving derivative problems. Thus, this project complemented what PMTs learned in this class while also offering them additional opportunities to develop their knowledge of content and pedagogy. These opportunities served as incentives for their participation in the study.

### ***Participants***

The size of the research team and the participants in a design experiment depends on its type and purpose (Cobb et al., 2003). When the number of participants is relatively small, a single researcher can conduct and record the teaching sessions in a one-on-one design experiment. I was the sole researcher in this study, and I recruited six participants to form three pairs of students. The rationale and criteria for selecting six participants are provided below.

As the study aimed to support the development of the content and pedagogical knowledge of prospective teachers of secondary mathematics, the participants should possess knowledge foundational to those knowledge domains. Therefore, the selection criteria ensured that the purposefully selected participants were information-rich (Patton, 2015) and "from which the most could be learned" (Merriam & Tisdell, 2015, p. 96). In this regard, I considered preservice mathematics teachers currently enrolled in a secondary math methods course.

Since the study explored the development of knowledge for teaching the derivative, I selected participants who had completed Calculus I, which would ensure that they were prepared to analyze videos of students solving problems related to the derivative. In addition, students in this course were enrolled in a Mathematics degree program leading to K-12 teaching certification, implying they had already made up their minds to become math teachers. The final criterion was their tendency to express their ideas aloud, since I would need to rely on those expressions to analyze the trajectory of their participation in the design experiment sessions.

### ***Recruitment***

I recruited participants from those enrolled in a secondary math methods course in the Fall 2021 semester. Eleven PMTs were enrolled in the course. Students in this class must spend fifteen hours on an independent project outside of class. The course instructor had agreed that these students could participate in my study to satisfy that requirement. Therefore, I recruited six students to participate in the study. The remaining five non-consenting students would complete a similar project but through an asynchronous, unfacilitated sequence of modules. In the subsequent paragraphs, I provide the criteria and rationale for recruiting six participants.

To recruit participants, I visited the class early in the semester and explained the purpose of my research and the activities I would like them to perform. Then I gave them time to ask me questions. Next, I distributed the consent forms. Since my research depended on being able to elicit students' mathematical thinking, I sought to recruit participants who tended to share their ideas in class. To meet that criterion, I attended the first two in-class sessions to monitor and document their participation. I ensured that the six students who consented to participate in my study were viable based on my observations.

I recruited six participants for this study. Patton (2002) asserts that since qualitative research centers on describing, understanding, and interpreting how individuals view the world around them instead of aiming to make generalizations, the recruitment of only a small number of participants is appropriate. That said, it was reasonable to expect the kinds of variation in these students' ways of thinking that would allow me to develop a viable theory of knowledge development. To manage the efficacy of tracking each participant's learning during the discussions, I divided the six participants into three pairs based on their availability during the week. Accordingly, I conducted three teaching experiments, each with one pair of students whom I met each week. In total, I met with each pair eight times during the study.

**The Episodes of a Design Experiment**

I conducted eight design experiment sessions with three pairs of PMTs, with each episode lasting around 45 minutes. I met each pair of PMTs weekly via videoconference. I had identified eight videos of students solving problems related to the derivative. My criteria for video selection are provided in the next section. During each session, pairs of PMTs viewed an entire video uninterrupted in order to achieve a broad overview of the arc of students' problem solving.

A second rationale for the initial viewing was to determine how PMTs noticed student thinking without the researcher's facilitation and scaffolding. That enabled me to discern a full breadth of the trajectory of PMTs' noticing of student thinking over time. Next, we viewed brief segments of the videos that were determined according to what I had deemed ripe opportunities for noticing. Following each segment, the PMTs were asked to respond to the prompts in my semi-structured interview protocol, as provided in Table 1 below. I also paused and even rewinded the video as they wished so that I could assess what they found worthy of attending to. In addition, I asked PMTs to take notes as they viewed videos of students solving problems to maintain their attention and hold onto their in-the-moment reflections. I made copies of these notes and added to the data corpus.

**Semi-Structured Interviews**

I conducted semi-structured interviews (Clement, 2000) with the PMTs during the design experiment sessions. Semi-structured interviews involved "a subject (the problem solver) and an interviewer (the clinician), interacting with one or more tasks (questions, problems, or activities) introduced to the subject by the clinician in a preplanned way" (Clement, 2000, p. 519). Then, I analyzed these interactions (verbal, non-verbal actions, and behaviors) to make inferences about students' mathematical meanings and their problem solving (Clement, 2000; Goldin, 2000). The value of these interviews is that they provided the researcher with an opportunity "to expose hidden structures and processes in the subject's thinking that could not be detected by less open-ended techniques" (Clement, 2000, p. 547). In my case, the semi-structured nature of interviews afforded me the opportunity to facilitate participants' noticing-related engagement with the problem-solving videos during the design experiment episodes. I say more about what occurred in these episodes in the following section.

The questions of a semi-structured interview were determined ahead of time, and they tended to be rather open-ended (Merriam & Tisdell, 2015). The order of questioning was not predetermined. Instead, they were asked flexibly in response to the dynamics of the research situation. With the same intention, follow-up questions were determined *in situ*. Merriam & Tisdell (2015) refer to these responsive questions as the “probing questions” of a semi-structured interview, since they enable interviewees to elaborate on their thoughts and ideas.

The capacity to responsively interact with participants by asking follow-up questions made the researcher an instrumental tool of real-time formative assessment during the design experiment episodes. For example, when an idea contributed by a participant is unclear or incomplete, the researcher may ask the participant to clarify or elaborate. Also, in the case where a participant hits an impasse, the researcher may branch the question sequence, offer heuristic strategies, or return the discussion to earlier, relevant points.

In general, my intent was to explore and advance the ways that PMTs attended to student strategies, interpreted them, and responded to them. These moves served to discern the interrelatedness of the attend-interpret-response components of one’s noticing skills. Table 1 shows some of the prompts that appeared in the semi-structured interview protocol to guide the interactions during the sessions. A purpose for each prompt is also included.

**Table 1**

***Excerpts from the Protocol for Semi-structured Interviews***

| Prompts | Purpose |
|---|---|

| | |
|---|---|
| • Would you tell me how these students are solving the problem?<br>• What strategies did they use to solve the problem?<br>• Would you explain to me how *you* would solve this problem? | • Help to understand PMTs' ability to attend to student thinking<br>• Focus PMTs on attending to episodes in which student thinking is elicited |
| • Given what you said about students' approaches, what mathematics do they understand?<br>• What mathematics do you think the students *do not* understand? What makes you think so?<br>• Let's take a look at how Alex (pseudonym) solved this problem. Would you explain why you think she chose that method? | • To help understand how PMTs interpret students' thinking |
| • Given what the student did there, if this were a student in your class, how would you respond?<br>• What would you hope to accomplish with that response?<br>• You have said that the student does not understand something about the mathematics (maybe a specific concept). How would you support the student to learn it? | • Access PMTs' ability to respond to the students' thinking<br>• Determine their ability to orchestrate instructional strategies likely to support students' learning |

A table showing the prompts that were used to facilitate a discussion during the design experiment.

Broadly speaking, my objective in using these prompts was to support the use of problem-solving videos to develop PMTs' capacities for professional noticing, that is, their abilities to attend to, interpret, and respond to student thinking (Jacobs et al., 2010). As an example, after PMTs watched a segment of a video of students solving a problem related to derivatives, I would ask questions such as these: *Would you summarize what the student(s) did? Would you explain why you think they did that? What strategy did they use? What do you think that strategy says about what they know? If you were their teacher, what might you do next? Why?* Other scholars (e.g., Sherin & van Es, 2002) have used more general prompts, like "What did you notice?" These would be useful early on as I assessed what the PMTs attended to, however, I conjectured that generic prompts may not direct PMTs' attention to the students' ideas, thoughts, and actions. Instead, they may focus on other events (e.g., the tasks) unrelated to the phenomena under investigation in this study.

To assess PMTs' ability to interpret students' thinking, I asked them to re-watch video segments to explore their responses further or focus on a particular moment that I thought was worthy of their attention. Then, I would use the essential "Tell me more about that” prompt to ask them to explain what they thought one or both students understood. I also used “funneling questions” (Herbel-Eisenmann & Breyfogle, 2005) to shift PMTs’ analyses towards interpreting students’ mathematical actions. Funneling questions developed depending on the in-the-moment responses of the participants. Probes encouraged PMTs to elaborate if I sensed they had more to learn or that their responses could benefit from further clarification (Patton, 2015). For instance, to direct their attention to specific episodes that were rich in student thinking, I said, “Let's take a look at how Alyssa solved this problem. Would you explain why you think she chose that method?” Such prompts helped me to elicit responses that revealed how PMTs made sense of the students' conceptions of the derivative. Moreover, when a PMT explained an episode of student thinking without citing evidence from the video, I asked: "Where do you see that [in the video]?" (van Es, 2011, p.137). The reviewed literature shows that such probes support PMTs to shift from making evaluative statements to providing evidence from the videos supporting their claims and engaging in knowledge-based reasoning (Sun & Van Es, 2015).

To assess how PMTs' responded to student thinking based on the inferences they made from their problem-solving behavior, they were asked, “Based on your understanding of the student thinking, describe some ways you might respond to this student. Would you explain how you chose to respond in those ways?” Other possible prompts included, “Given what the student did there, if this were a student in your class, how would you respond? What would you hope to accomplish with that response?” Explaining why they chose to respond in a particular way allowed me to assess how that response attended to aspects of student thinking. For example, PMTs suggested productive ways to respond to a misconception or how to scaffold the problem solving to students who seemed to be stuck.

**Interview Protocol Adjustments**

In this subsection, I describe some of the adjustments that I included in the interview protocol based on the observations that I made as PMTs analyzed students’ thinking in the experiments. In Session 1, I followed the protocol as provided in Table 1. As the results reveal in Chapter 4, the PMTs’ analysis, in terms of describing, interpreting, and responding to student thinking was superficial. That is, their analysis did not fully capture the thinking of the students as they problem solved. As such, in Session 2 I made a few changes to the protocol. I played shorter clips and I replayed them until I was convinced that the PMTs understood the students’ problem solving, which required more reviews than I had offered them in Session 1. As evidenced by the greater depths in the PMTs’ analyses, this change to the protocol made the analysis more manageable for them. For instance, in Session 2, I chose the video segment in which Alyssa explained the concept of an increasing rate of change (timestamp 1.18-2.14). I played that segment three times until I was assured that the participants understood Alyssa's reasoning. I had analyzed her reasoning prior to the episodes and found her arguments to be mathematically justifiable. After repeatedly watching and discussing the same clip three times, one PMT (Mia) said, "I think I understand what she is saying now." Her subsequent descriptions, interpretations, and responses to the student's problem solving confirmed that she did indeed understand "what she is saying." As a result, I applied this same contingent "review and discuss" strategy in the subsequent sessions.

I made a second modification to the protocol in Session 3 when I observed that the participants were not as responsive as they had been in Session 2. I had asked them to discuss the students' thinking as they solved a problem related to the instantaneous rate of change (see Video 3 in Appendix C). In Episode 1 of Session 3, the discussion was characterized by intermittent moments of silence as the PMTs tried to remember the concepts. Leah commented, "It's been so long since I did this [kind of] math." In response, I found myself providing more scaffolding to maintain participation and engagement than I was used to, because the PMTs were struggling with the content underlying the posed problem. This hadn't been an issue before. Accordingly, I adjusted the protocol, and in Episodes 2 and 3, I played the video and then engaged the students in a discussion focused exclusively on the relevant mathematics and disconnected from the students' problem solving. These brief mini-lessons were somewhat successful in reminding students of the mathematics they had learned and developing it to the extent that they could at least attend to the students' thinking about that mathematics. I then re-played the video and asked them to analyze the students' problem solving. The instances in which these side discussions of the mathematics enabled PMTs' subsequent participation confirmed the value of those discussions and the modification to the protocol.

As a final point, I note that by Sessions 7 and 8, the frequency with which I replayed video segments and my need to provide prompts and highlights had diminished. At that point the PMTs assumed greater control over the analyses of students' problem solving and it became less necessary for me to facilitate. The discussions were continuous (unpunctuated by moments of silence) and more productive than they had been in Session 1, 2, and 3. The steady withdrawal of my scaffolding is evidence that the PMTs had achieved greater fluency in noticing by the end of the experiment.

**Sequence of Design Episodes**

In the very first session of each design experiment, I began by explaining to PMTs the tasks they were expected to accomplish before viewing the video of students solving those tasks. Then, I had them view an entire video uninterrupted for the first time in order to get a sense of what the video is about. Then, they would analyze student thinking using prompts such as those that appear in Table 1 above. Next, we collaboratively discussed the video as PMTs responded to the probing questions.

Finally, I would allow PMTs to re-watch segments that I felt were worthy of further analysis. In the "Problem-Solving Video" section below, I explain what I mean for segments to be worthy of the PMTs' analysis. Repeated viewing of the problem-solving video or re-viewing a segment of the video helped me direct the PMTs' attention to student actions suitable for making inferences about their thinking so that we could take the time to reflect on them more deeply. Table 2 below provides a tentative schedule of activities for the episodes of the design experiment.

**Table 2**

***Schedule and Activities for PMTs and the Researcher***

| ***Session*** | ***PMTs Activities*** | ***Researcher Activities*** |
|---|---|---|
| Session 1 | • Watch video1: Constant Rate of Change<br>• Discuss episodes of student thinking<br>• Respond to follow-up questions and probes | • Welcome the participants<br>• Administer video 1<br>• Facilitate the discussion, ask follow-up questions and probes<br>• Record and collect the session data |
| Between sessions | | **One-week break after sessions 1, 2**<br>Review previous recordings and PMTs' written work.<br>• Light transcription<br>• Retrospective video analysis<br>• Make any necessary revisions to protocols of subsequent sessions in light of analysis.<br>**One-week breaks after sessions 3, 4, 5**<br>• Review previous recordings and PMTs' written work<br>• Moderate transcriptions<br>• Retrospective video analysis<br>• Make any necessary revisions to protocols of subsequent sessions in light of analysis.<br>**Session 6, 7, and 8**<br>• Moderate transcriptions<br>• Retrospective video analysis |

Table 2 above shows the protocol for session 1. PMTs performed these same activities in each of the other seven sessions. I conducted sessions weekly except for at several points when the breaks between sessions were longer. The one-week breaks between sessions allowed me to transcribe salient moments of the data, conduct a retrospective video analysis of the previous sessions, and make modifications to the interview protocol in light of preliminary findings. I also clarify that since I conducted design experiments with three pairs of students, I conducted each of these eight sessions three times per week, for a total of 24 data-collection sessions. I explain what these problem-solving videos were about later in this chapter.

Cobb et al. (2003) underscore the importance of both prospective and reflective aspects of a design experiment. By "prospective," Cobb et al. mean that the imagined design of the learning process is a hypothetical one. Thus, it's critical that the researcher remain open to "other potential pathways for learning and development by capitalizing on contingencies that arise as the design unfolds" (p. 10). The "reflective" aspect implies that the design is an initial conjecture that can be revised and refined – or even refuted and replaced –as data suggests during the experiment sessions. Thus, one should expect that the protocols for subsequent sessions were revised to some extent throughout the experiment. Such is my rationale for breaks between sessions that would allow me to conduct retrospective analyses of collected data. Findings from these analyses were used to inform and reform the subsequent sessions. Next, I describe the methods for collecting data for this study.

**Data Collection**

All interactions during the design experiment episodes were video recorded using Zoom and with one external video camera. I shared my screen to present the problem-solving videos to PMTs, and the Zoom recording served as screencasting software. I also placed a microphone on my table to obtain a quality audio recording. I provided each student with a google doc in order to capture the notes that PMTs took as they viewed the videos. I also asked them to do sketches on paper when problems call for it and then digitally scan them at the end of the session.

I completed a Contact Summary Form (Walters, 2017) after every session (see Appendix A). A contact summary form is a set of reflective prompts that l used to record my reflections about what seemed to be the main ideas and events that occurred during each episode of the experiment. I also took field notes (Maanen, 1988) during the sessions, although I expected that doing so could distract me from the flow of the discussion. The contact summary form facilitated a routine for documenting "detailed descriptions of activities, behaviors, [and] actions" (Patton, 2015, p. 14) that emerged as PMTs attempted to make sense of students' thinking. In addition, I kept a research journal to document and reflect on salient moments that contributed to theory development, such as connections between meaningful moments for the PMTs and what in the videos may have provoked them.

## Problem-Solving Videos

Although I have already alluded to them above, in this section, I more deeply describe the problem-solving videos that I had hypothesized would mediate PMTs' learning to notice. These videos show students engaged in mathematical problem solving on problems related to the concept of derivatives. The videos come from the NSF-funded project, The Calculus Video Project (https://calcvids.org), which investigates student learning and sense-making from instructional calculus videos. Their purpose is to conduct design research to generate knowledge about how students engage with, make sense of, and learn from videos that address foundational calculus concepts. The project also provides animated instructional videos explaining how students should approach the problems that appear in the videos to help students make sense of and understand the core concepts of calculus. They are available for free so that calculus instructors can use them in their teaching. I used them to provide opportunities for noticing interactions.

In contrast to the objectives of my study, which proposed to use these videos as resources for *teacher* learning by attending to, interpreting, responding to the thinking and reasoning elicited by Julian and Alyssa, *The Calculus Video Project* intended for them to be used as a resource for *student* learning of calculus concepts using principles of quantitative and covariational reasoning. I refer to them as problem-solving videos, alluding to the activities involved in these videos and setting them apart from other forms of videos (e.g., classroom videos) described in Chapter 2. In the following paragraphs, I provide my rationale for their selection by describing the features that I conjectured would support PMTs in developing teacher knowledge.

The videos I chose from the entire collection showed two students, Julian and Alyssa, as they worked through various problems related to the derivative. The problems covered the big ideas of calculus including constant rate of change, average rate of change, instantaneous rate of change, the limit definition of the derivative, and Riemann sums. In animated videos that demonstrated strategies for solving the calculus problems that Julian and Alyssa attempted to solve, the developers drew from a quantitative reasoning framework relative to coordinating amounts of change of co-varying quantities (Carlson et al., 2002; Moore et al., 2009).

I selected eight problem-solving videos in their entirety to be used in the design experiment. Two criteria were used to make these selections. First, I selected videos that could be viewed and analyzed within the 15 hours available to each participant. I conjectured that the number of videos should be reasonable so as not to overwhelm the participants, since each could be explored in depth. Secondly, I chose videos that could stimulate meaningful discussions and provoke thoughtful reflections during the design experiment sessions. To do that, I wrote up a descriptive overview of every problem-solving video. I identified what I deemed to be opportunities for noticing work, which is when students in the videos are either crafting and implementing a strategy to solve a problem or they were explaining how one could be used (see

Appendix B for an example). I referred to these opportunities as "rich segments" in the clips that I conjectured offered the material resources for a productive "noticing" conversation regarding the student thinking elicited in the videos.

This overview is provided in the form of a table with timestamps and brief descriptions of what is happening in the clip (see Appendix B). To ensure that the videos were serving their intended purpose, I added to the "Interactions" column of that table after every session with notes and reflections in response to the actual discussions during the sessions. The participants in this study were preservice mathematics teachers who I assumed to be novices at noticing student thinking based on where they were in their coursework. Therefore, I believed that viewing eight problem-solving videos with rich segments for students' thinking in interaction with a knowledgeable researcher would provide the social, material, and intellectual resources needed to investigate how one's noticing of student thinking seemed to change over time in a video-mediated situation.

**Problem-Solving Videos used in the Design Experiment Sessions**

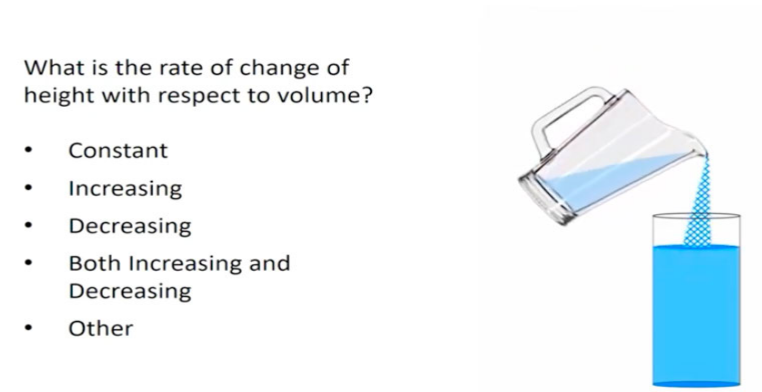
In this section I describe the problem-solving videos that PMTs analyzed during the design experiment episodes. As I do so, I reveal my conjectures about how the eight videos could potentially offer the PMTs the opportunity to enact knowledge related to teaching the derivative as they analyzed the reasoning of the student problem solvers. The results concerning how PMTs enact knowledge to teach the derivative and their development of noticing in all eight sessions are presented in Chapter 4.

***Session 1: The Constant Rate of Change***

In this session, PMTs worked in pairs as they viewed and collaboratively analyzed the video, "Constant Rate of Change: Student Problem Solving" (see Video 1 in Appendix C), which features two students, Julian and Alyssa, solving a mathematical problem related to the concept of constant rates of change. The participants were asked to notice (i.e., attend to, interpret, and respond to) the students' mathematical thinking as they solved the problem. The video provided an animation in which water is being poured into a cup. The students were asked to determine the relationship between the height and volume of water as it is poured into the cup. A screenshot of the animation and the task for Session 1 is provided in Figure 4 below.

**Figure 4**

***A Screenshot of the Task in Session 1***

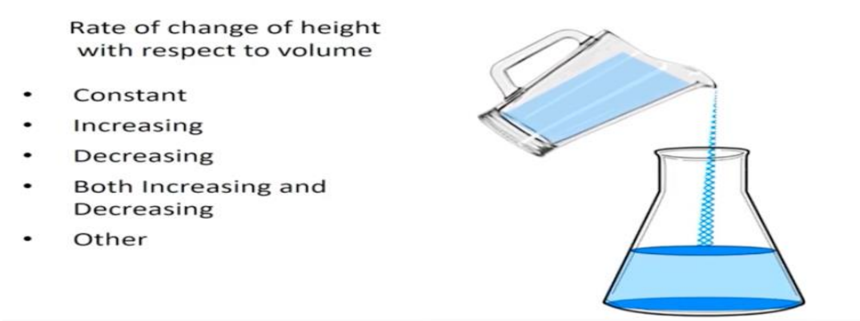



***Session 2: Increasing Rate of Change***

The participants in this session watched a 3-minute video, "Increasing Rate of Change: Student Problem Solving" (see Video 2 in Appendix C). In this problem-solving video, two students, Julian and Alyssa, attempt to describe the relationship between the volume and height of water being poured into a beaker. The students problem-solving activity illuminated ideas about whether the rate of change of height with respect to volume is constant, increasing, decreasing, both increasing *and* decreasing, or changing in some other manner. The screenshot below comes from the animation that accompanied the task that was given to the students.

**Figure 5**

***Screenshot of Session 2 Task***

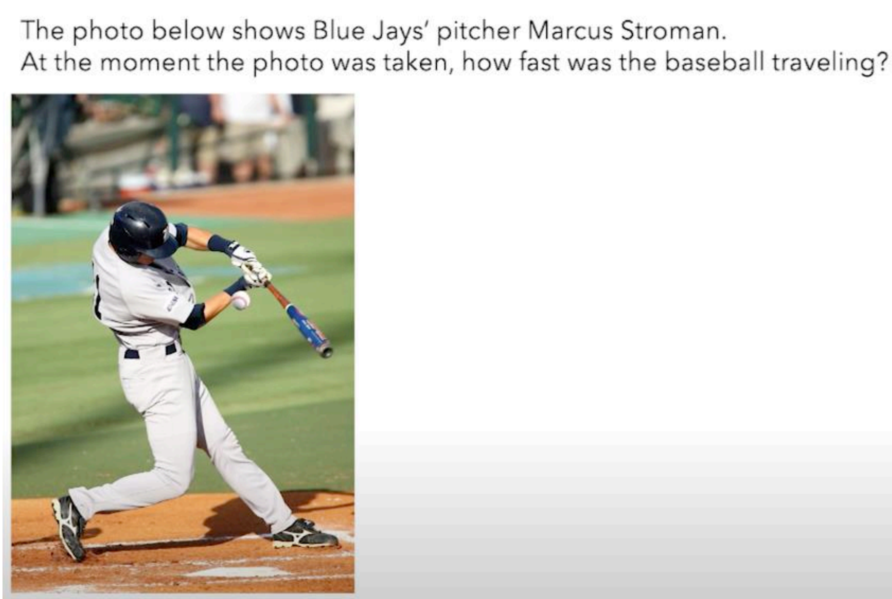



***Session 3: Approximating Instantaneous Rate of Change***

The PMTs viewed the video, "Approximating Instantaneous Rate of Change: Student Problem Solving" (see Video 3 in Appendix C), centered on ideas about the instantaneous rate of change in this session. In the video, the two students, Julian and Alyssa, were shown a photo of Blue Jays' pitcher Marcus Stroman (see Figure 6) and asked to approximate baseball speed when the photo was taken.

**Figure 6**

***Screenshot of the Mathematical Task***

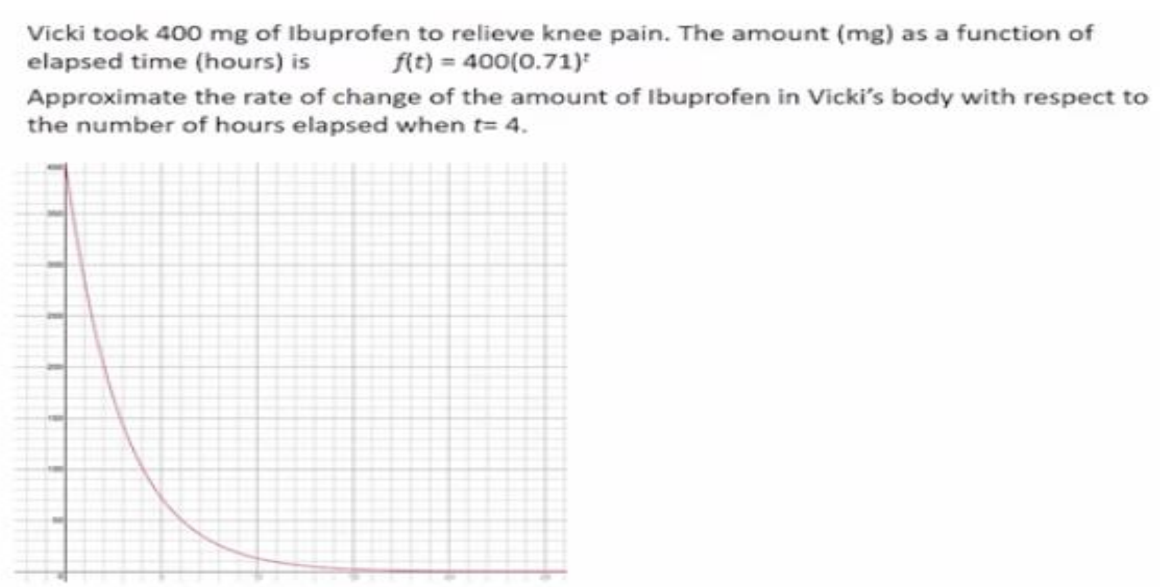


***Session 4: Limit Definition of the Derivative***

In this session, the participants watched and analyzed the video, “Limit Definition of Derivative: Student Problem Solving” (see Video 4 in Appendix C), in which the two students, Julian and Alyssa, were solving a problem related to the limit definition of the derivative. A screenshot appears in Figure 7. The task involves approximating the rate of change in the amount of ibuprofen in a person’s body with respect to elapsed time at a particular point in time.

**Figure 7**

***Screenshot of a “Limit Definition” Task***

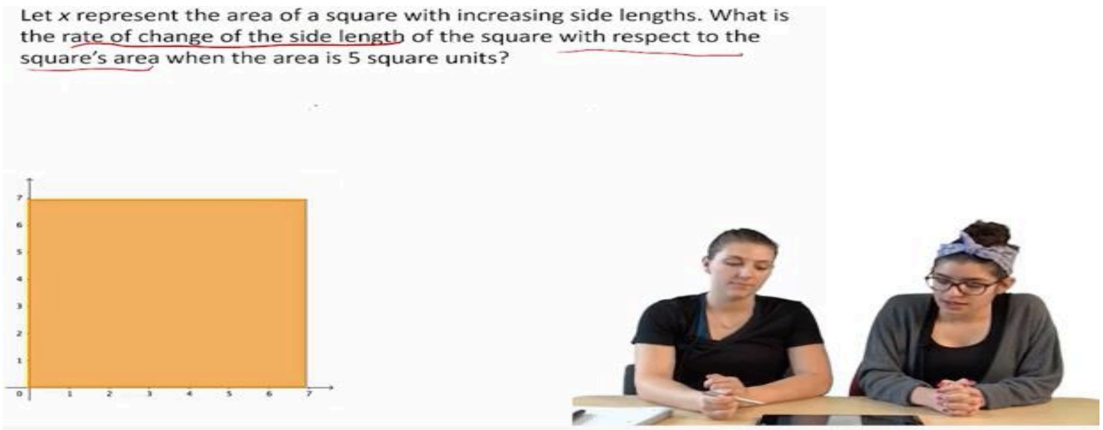



***Session 5: Using Limits to Compute Derivatives***

In this session, the PMTs watched the video titled, “Using Limits to Compute Derivatives: Student Problem Solving” (see Video 5 in Appendix C). In the video, two students, Kelly and Maria, use the limit definition of the derivative concept to solve a problem related to the rate of change. I asked participants to consider the students’ thinking and reasoning about the problem and articulate ways the students could be supported to solve similar problems. A screenshot of the problem appears in Figure 8.

**Figure 8**

***Rate of Change Task for Episode 5***

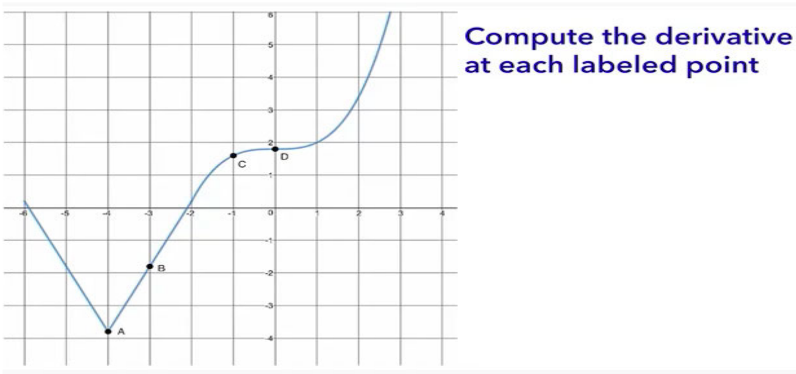



***Session 6: Slopes of Secant and Tangent Lines***

In session 6, I showed the 4-minute problem-solving video, “Slopes of Secant and Tangent Lines: Student Problem Solving” (see Video 6 in Appendix C). Julian and Alyssa are tasked with computing derivatives using slopes of tangent lines. As they sketch tangent lines at various points on a graph (see Figure 9), their problem-solving elicits their thinking about what a tangent line is. As the PMTs viewed the video, I asked them to consider how the students expressed their understanding of tangent lines and their slopes and deliberate about the validity of their reasoning.

**Figure 9**

***A graph for which Julian and Alyssa must Find Derivatives at Each of the Labeled Points Using Tangent Lines***

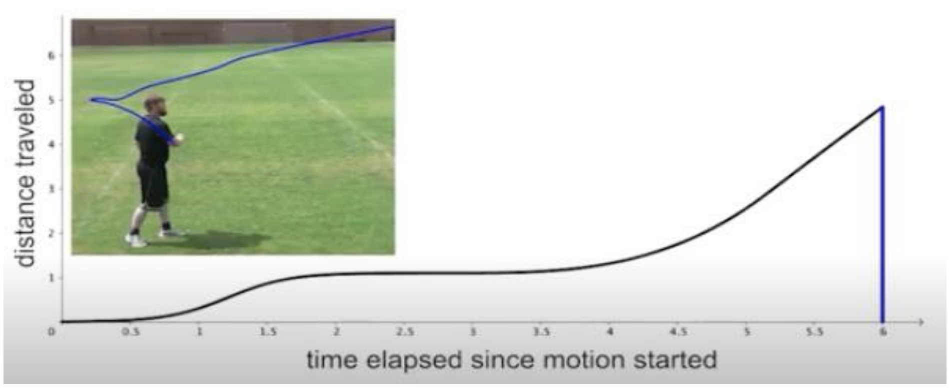



***Session 7: Graphing the Derivative***

In this session, Kelly and Maria appeared in the 5-minute video, “Graphing Derivative: Student Problem Solving” (see Video 7 in Appendix C). They were shown an animation of a football being thrown and were then asked to graph its speed over time. The participants were asked to examine the students’ thinking, assess its validity, and interpret the obstacles they encountered as they constructed the graph. The screenshot in Figure 10 shows the height of the thrown football (blue trace) and a graph of the vertical distance (or height) of the football over time.

**Figure 10**

***Graph Showing the Height of a Football Over Time***




***Session 8: Interpreting Derivatives***

In this session, two students, Kelly and Maria, worked together to solve the following problem in the nearly 6-minute problem-solving video, "Interpreting Derivatives: Student Problem Solving" (see Video 8 in Appendix C):

> *Courtney is going on a road trip. The function $f(t)$ measures the amount of fuel Courtney's car has consumed (in gallons) where t is measured in hours since Courtney left her hometown. Explain what $f'(5) = 2.5$ means.*

The students' problem-solving illuminated ideas about interpreting the derivative in context. As PMTs viewed the video, I asked them to pay close attention to students' reasoning.

## Data Analysis

In the context of this study, three integral elements of the learning ecology interacted to engineer an opportunity for teacher learning. These were the researcher, the PMTs (participants), and the problem-solving videos. Figure 11 depicts the possible interactions among these three elements. As I explained within the presentation of my theoretical framework, situated and sociocultural perspectives offered viable lenses with which to analyze these mediating interactions. In particular, I analyzed the enactment of teacher knowledge and noticing skills as they were mediated by social (e.g., researcher-PMT interactions), material (e.g., problem-solving videos), and conceptual (e.g., mathematics) artifacts.

**Figure 11**

***Three Elements of the Teacher Learning Ecology***

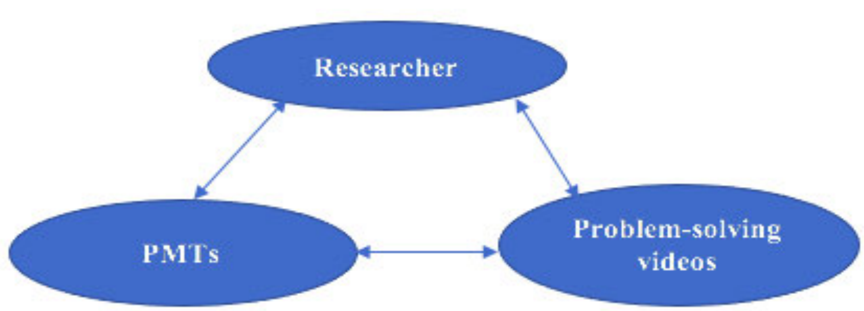


Figure 11 depicts the possible interactions between the elements of learning ecology in my design experiment.

As depicted in this figure, the researcher interacted with both the problem-solving videos and the PMTs. The PMTs interacted with the problem-solving videos and the researcher, as well. These interactions were conjectured to facilitate the PMTs' enactment of knowledge for teaching the derivative and mediate the development of noticing related to it. Thus, understanding the nature of these interactions provided an opportunity to account for how the PMTs' knowledge for teaching and noticing student thinking was developed.

Whereas Research Question 1 is concerned with the kind of knowledge for teaching the derivative that was evoked and *how* it was enacted, Research Question 2 is concerned with the forms of interactions through which that knowledge was further developed. In other words, I was not only interested in features of problem-solving videos that mediated the focusing interactions of noticing (Lobato et al., 2013), I was also interested in how focusing interactions directed PMTs' attention to specific segments of the videos and provided them with opportunities for knowledge development. Next, I provide the analytic methods in relation to each of these two questions.

**Data Analysis for Research Question 1**

Each of my research questions called for different analytic approaches. In this section, I provide the analytic methods I used to answer Research Question 1, which is as follows: *How does teacher knowledge specific to noticing students' mathematical thinking in the domain of the derivative develops through video-mediated professional learning?*

The study employed qualitative methods and entailed collecting and analyzing data in order to yield detailed descriptions of how PMTs developed their noticing abilities, a key feature of responsive teaching. Bearing in mind that RQ1 was concerned with the kind of knowledge for teaching the derivative that PMTs enacted as they observed student thinking, the intention was to examine the process through which they assessed and modeled their thinking and discern a trajectory of knowledge generation across the episodes of the design experiment. Specifically, I explored the knowledge for teaching associated with noticing students' mathematical thinking in the derivative domain that PMTs leveraged and enacted, along with the contingent processes that explain the emergence of this knowledge.

Since I intended for the analysis to be open so as to develop a theory grounded in the data, I used a grounded theory analytic approach (Glaser & Strauss, 1967) to respond to Research Question 1. Grounded theory is a method for analyzing qualitative data to develop new theories and concepts based on data (Corbin & Strauss, 1990). Thus, this method was compatible with the aims of this study, which were to 1) engineer the design of professional learning in which PMTs develop knowledge for teaching the derivative, 2) construct a theory that explains how that knowledge is developed, and 3) identify forms of interactions that mediate those developments. This theory and those forms were inductively derived from the data corpus through open, axial, and selective coding, which I describe below. For now, I provide a summary overview of my analytic intentions. Then, I return to the generic features of a grounded theory analysis.

After transcribing audio- and-videotaped data verbatim, I started with open coding (Glaser & Strauss, 1967; Strauss & Corbin, 1990) through an initial pass of the data. Open coding is "the process of breaking down, examining, comparing, conceptualizing, and categorizing data" (Strauss & Corbin, 1990, p. 61). This process occurred during breaks between sessions, which means that analysis was ongoing throughout data collection.

The specific aim of my open coding process was to identify noticing phenomena and then label, categorize, and describe them (Glaser & Strauss, 1967). In my first pass through the data, I looked for moments where PMTs enacted knowledge for teaching the derivative and applied descriptive codes to label the process involved (i.e., what accounts for the enactments). The reviewed literature on models of teacher knowledge provided base codes for the forms of knowledge for teaching that PMTs brought to bear on their video analyses (e.g., knowledge of content and students, or KCS; Ball el al., 2008). In addition, I used the components of the professional noticing framework (Jacobs et al., 2010) as base codes (i.e., attend, interpret, and respond), as well. Of course, I also remained open to emergent codes. This *ad hoc* analysis was particularly necessary for coding the *ways* in which that knowledge was enacted was developed. Then, I combined these codes conceptually to characterize PMTs' noticing.

In addition, given my dynamic perspective on knowing, I expected to identify forms of enactments that don't have their groundings within the literature review. By remaining unconstrained by the forms of teacher knowledge identified in the literature, this approach allowed me to remain open to discerning emergent patterns in how PMTs assessed and modeled student thinking. These newly identified knowledge-eliciting processes were given new codes and eventually collapsed to form themes. For example, the 'attending' and 'describing' codes were collapsed to form the *describing* category. These themes denoted the processes through which PMTs modeled student thinking and suggested ambitious instructional moves to support student learning. I used them to construct a theory of knowledge development related to noticing students' derivative thinking that was abstracted from the design experiment.

In general, my analytic process for Research Question 1 was as follows. Once first-pass codes related to enacting knowledge for teaching were labeled in the transcript, I conducted a constant comparative analysis (Glaser & Strauss, 1967), which involved "(1) comparing incidents applicable to each category, and (2) integrating categories and their properties" (Kolb, 2012). Specifically, I grouped together data (e.g., quotes from the transcript) labeled with the same code and recorded them in a codebook, or an inventory of codes, along with their descriptions and exemplars from the transcript. These were important during the analysis, because they ensured the application of codes was consistent, thus ensuring their integrity, by which I mean that they helped me demonstrate that the results and conclusions were grounded in the data. Table 3 provides this analytic process:

**Table 3**

***The Coding Process for Research Question 1***

| Levels | Analytic Process | Activities |
|---|---|---|
| 1 | • Open Coding | • Read the transcript<br>• Interpret each piece of data (words, sentences, paragraphs)<br>• Label pieces of data with codes (e.g., attending to students' problem solving)<br>• Lines that express the same idea to be labelled with the same code |
| 2 | • Axial Coding | • Identify connections between level 1 codes<br>• Group codes that express same meanings together (e.g., attending to student thinking)<br>• Develop broader categories based on their conceptual connections |
| 3 | • Selective coding | • Establish the connection between categories<br>• Group the connected categories together<br>• Develop overarching categories (e.g., responding to students' derivative understanding)<br>• Establish distinct categories<br>• Review categories and codes<br>• Remove those that lack enough supporting data<br>• Write narrative to describe overarching categories (models of teacher knowledge) |

A table explaining the codes in an analytic process based on the principles of grounded theory (modified from Strauss & Corbin, 1990, 1998).

In the next phase of analysis, I conducted axial coding (Corbin & Strauss, 1990). Open coding breaks down the data into smaller components whereas axial coding collates them back together at a conceptual level—based on how ideas and concepts are related. Then, I examined the relationships between codes in open coding through the constant comparative method (Glaser & Strauss, 1967) to form categories (Corbin & Strauss, 1990). To do that, I looked across the knowledge codes to discern relations among them in order to form categories. Codes expressing similar ideas and having a conceptual relationship were assigned to the same category.

The last step of the analytic process was selective coding (Glaser & Strauss, 1967). Here, the categories developed were connected to form themes. In developing these themes, one may consider the categories that emerged through axial coding or they may construct new themes by connecting those categories. Finally, to respond to Research Question 1, I described these themes as the four-element framework that supported PMTs' learning to notice student thinking. The four-element framework is presented in Chapter 4.

Next, I describe how I analyzed data to respond to the second research question, which was as follows: *What forms of video-mediated interactions support such development?* I adapted Lobato et al.'s *focusing framework* (2013) as an analytic tool to answer this research question. Before stating the analytic methods that I used to answer the question, I present key elements of that framework.

**Focusing Framework**

Lobato et al. (2013) developed their *focusing framework* as both a conceptualization of students' mathematical noticing and as a tool to analyze its development. In this section, I present their framework. I use it in an adapted form as a tool with which to analyze the development of PMTs' mathematical noticing. Lobato et al.'s (2013) focusing framework has four components:

centers of focus, focusing interactions, mathematical tasks, and nature of mathematical activity. I will explain each of these components and further demonstrate how they were adapted to analyze data in this study.

*Centers of Focus* are "properties, features, regularities, or conceptual objects that students notice" (Lobato et al., 2013, p. 814). Since what students notice cannot be accessed directly, centers of focus are identified in their verbal responses, gestures, and written work. In the context of this study, I considered centers of focus as aspects of students' thinking (conceptual objects) that PMTs attended to as they analyzed videos of students' problem solving. These included the students' strategies, concepts, heuristics, and misconceptions. The remaining three components of the focusing framework helped to discern what contributed to the emergence of these centers of focus during the design experiment.

*Focusing Interactions* are teachers' "discourse practices (including gesture, diagrams, and talk) that give rise to particular centers of focus" (p. 814). One such discourse practice is highlighting (Goodwin, 1994), which includes labeling and annotating particular features so as to make them prominent, thus shaping how they are perceived (Lobato et al., 2013). I extended focusing interactions to include the discourse practices of the researcher in the design experiment. For example, the researcher used prompts to draw PMTs' attention to observable moments of student thinking in the video. In so doing, the researcher highlighted that segment of the video to draw PMTs' attention to it. These examples aim to reify the conjecture that highlighting directed PMTs to attend to students' reasoning and thereby generated centers of focus for their collaborative consideration and contemplation. The researcher's focusing interactions were also used to center PMTs' focus on pedagogical moves, like deliberating how they could support a student to resolve a misconception or become unstuck.

*Mathematical tasks* are the material resources that afford or constrain what the students attend to (i.e., their centers of focus) and thus their features affect what the students notice (Lobato et al., 2013). For example, a task that calls on students to graph the relationship between two quantities provides them with an opportunity to attend to how those quantities co-vary. In the context of this study, videos of students solving problems formed the central concern. They formed "the backdrop of discourse practices," (p. 814) because they provided occasions for PMTs to notice students' thinking. In addition, the affording and constraining *features* of these problem-solving videos were also subject to analysis here, since this study aimed to discern the forms of video-mediated interactions that supported PMTs' learning. In particular, I was interested in analyzing the interplay between the discourse practices (focusing interactions) and the moments of problem-solving video that offered PMTs opportunities for noticing students' mathematical thinking.

The *nature of mathematical activity* refers to the norms for student participation in problem solving and in classroom discourse more broadly. The expectations for the way students participate influence what become their centers of focus (Lobato et al., 2013). In the same way, in this study, the centers of focus for PMTs were determined through the norming of discourse practices in each design session. For example, these norms may regulate how and when the videos are paused, what gets re-viewed, and what counts as evidence for claims, thereby shaping PMTs' noticing.

In summary, these four elements of Lobato et al.'s (2013) focusing framework provided analytical tools in the open coding process: focusing interactions, mathematical tasks, the nature of mathematical activity, and centers of focus. Using these analytic tools, I established how an increasing prevalence of PMTs' centers of focus gave rise to *shifts in their noticing capacity* and *knowledge for teaching the derivative*.

Research Question 2 aimed to discern the forms of interactions that informed PMTs' noticing skills and their knowledge for teaching the derivative. With nominal modifications to the framework, l accounted for who was noticing and what they were asked to notice. I used the four elements of a focusing framework as analytic tools in the analysis in order to account for how artifacts and interactions (e.g., features of videos, features of tasks, discourse practices, norms) within the video-mediated environment fostered PMTs' noticing skills and knowledge for teaching the derivative. In addition, I identified how noticing skills and teacher knowledge are linked to moments of student thinking that PMTs attended to.

**Data Analysis for Research Question 2**

In this section, I provide the analytic methods I used to answer Research Question 2, which is as follows: *What forms of video-mediated interactions support such development?* Specifically, this question was concerned with developing a theory of how mediated interactions with problem-solving videos supported the development of noticing skills in a video-mediated professional learning context. Thus, the nature of those interactions and the particular features of those videos were relevant in the construction of that theory.

To address my second research question, I again engaged with the transcribed data. I used open coding (Strauss & Corbin, 1990) in the first phase of the analysis to code for the four components of Lobato et al's. (2013) focusing framework. As these initial entering codes got refined, I not only identified centers of focus, for example, I also coded for how they were identified, in other words, the conditions that prompted their emergence. The distinction between a center of focus identified by the researcher and one identified by a PMT was key because the latter signified more central participation (Lave, 1991) in professional learning. This analysis constituted an attempt to empirically assess the extent to which the centers of focus that PMTs attended to, and even shifts identified in what they attended to, were a product of the mediated discussions that had taken place within the socially situated environment of the experiment.

It was crucial to identify the focusing interactions as well as the mediating interactions associated with each of the centers of focus in order to discern how those focusing interactions contributed to the emergence or shifts in centers of focus. Thus, it was important to code for these elements simultaneously because there was an interrelationship among them. Accordingly, in my next analytic pass at the data, I applied tripartite codes to capture 1) centers of focus, 2) the focusing interactions that gave rise to each one, and 3) the contributing role of problem-solving video interactions. The simultaneous and coordinated coding of these three elements enabled me to develop a conceptual connection between them that could illuminate the mediating interactions that supported the development of noticing.

To analyze the focusing interactions that developed at the emergence of each center of focus, I used open coding (Strauss, 1987) to identify the discursive practices (e.g., gesture, diagrams, and talk) that enabled the centers of focus to emerge. The goal was to describe the form of interactions that seemed to influence the number and nature of the candidate centers of focus that emerged. Whereas for Lobato et al. (2013), *centers of focus* are “properties, features, regularities, or conceptual objects that *students* notice” (p. 814, emphasis added), in this study, the “students” were the PMTs and I was the teacher-researcher. And instead of considering the *nature of mathematical activity* as a component of the noticing framework, here the “activity” is viewing a video and analyzing student thinking.

In order to determine how the problem-solving videos mediated PMTs' knowledge development, I applied codes to the video segments in which new centers of focus were identified or other moments that mediated learning and aligned them with the coded focusing interaction(s) they corresponded to. That way, I could coordinate the three constructs – centers of focus (or other moments that mediate learning), focusing interactions, and the corresponding features of videos – to realize how they interacted and facilitated PMTs' noticing of student thinking over time. As with all other codes, the names given to these codes conveyed meaningful descriptions of these interactions. By identifying patterns in the interactions among these three constructs, I developed a theoretical claim about the nature of interactions by which teacher noticing was developed. The results that respond to this second research question are presented in Chapter 5.

## How Trustworthiness was Established

In this final section of this chapter, I provide four means by which the trustworthiness of this study was established: credibility, dependability, confirmability, and transferability. Establishing trustworthiness is crucial for demonstrating that the data analysis has been conducted rigorously and transparently so that researchers can confirm the findings and practitioners can confidently act on them.

### Credibility

This is an important criterion in qualitative research for establishing the trustworthiness of the research findings. Thus, it is important to demonstrate how it was established in this study. Credibility is concerned with the congruence of the participants' actual perceptions of the phenomena under investigation and the researcher's presentation of their perceptions and perspectives. Essentially, credibility entails pointing out the correspondence of the study's findings with the reality to make it possible to determine the truth of the findings.

I used triangulation and prolonged and substantial engagement of PMTs to establish credibility (Lincoln & Guba, 1985). Triangulation involves using multiple methods, data sources, and theoretical perspectives to enhance a robust understanding of the phenomenon under investigation and ensure the study's findings are well-developed (Mertens, 2005). I used multiple data collection methods and sources, including video and audio recordings of semi-structured interviews, participants' written work, and reflections that recorded at the end of every session. I coded these pieces of data in order to assess their consistency in terms of how PMTs developed noticing skills and knowledge for teaching. In addition, during the data analysis phase, my supervisor, Dr. Greenstein, reviewed my findings to determine if there were blind spots in the analytic process. This use of multiple analysts contributed to the robustness of the findings thus making them more credible. Furthermore, I employed two theoretical perspectives – the sociocultural and situated perspectives – to inform the research design, including the research setting, the data that was collected, and how the data was analyzed.

Next, prolonged and substantial engagement with the PMTs is yet another means I used to establish credibility. Several design studies informed the duration of my study and the number of problem-solving videos that should be analyzed (e.g., Sherin & van Es, 2002; Steffe & Thompson, 2000). This study involved three pairs of participants, and each pair analyzed eight problem-solving videos, for a total of twenty-four sessions over fifteen weeks. This prolonged engagement provided an opportunity to explore the PMTs' trajectory of participation and their development of professional noticing skills.

**Dependability**

In qualitative research, dependability attempts to respond to the question, “How can one determine whether the findings of an inquiry would be consistently repeated if the inquiry were replicated with the same (or similar) subjects (respondents) in the same (or similar) context?” (Guba, 1981, p. 80). By doing so, dependability aims to establish “the consistency and reliability of the research findings and the degree to which research procedures are documented, allowing someone outside the research to follow, audit, and critique the research process (Moon et al., 2016).

To ensure dependability, Guba and Lincoln (1989) suggest a detailed documentation of research design, methodology, and methods. Similarly, Moon et al. (2016) add that the researcher should provide detailed information about data collection and analysis, such as memos and field notes. To establish dependability, I provided details about the methodology and methods I used, and I followed up by supplementing these with details about what actually happened as the data was collected and analyzed. To provide some examples, field notes were taken to document my ongoing reflections, a form was completed after every session to document thoughts and insights that developed during the session, and analytic memos was recorded to document how the codes, categories, and themes evolve throughout the analysis stage.

**Confirmability**

In qualitative research, confirmability addresses the question, “How can one establish the degree to which the findings of an inquiry are a function solely of the subjects (respondents) and conditions of the inquiry and not of the biases, motivations, interests, perspectives and so on of the inquirer?” (Guba 1981, p. 80). It is concerned with ensuring that the results are linked to the conclusions, thus eliminating the researcher’s bias or influence. In other words, the results and their interpretations should reflect participants' experiences and preferences, not those of the researcher.

Lincoln and Guba (1985) suggest that a researcher can establish confirmability by keeping a reflexive journal, which is what I did. In this journal, I recorded reflections about PMTs' learning during the sessions and how my theoretical model of their learning evolved. These reflections played a critical role in data analysis and in the writing up of the study's findings.

**Transferability**

Transferability is concerned with how the results from a given study can be applied to other situations or populations (Lincoln & Guba, 1985). However, it is important to emphasize that in qualitative research there is no intention to generalize the results. Instead, sufficient details about participants and settings are provided so that the reader can determine whether the findings can be transferred to another setting due to their shared characteristics. Accordingly, transferability is established by providing a vivid description of the research setting so that a reader can determine its similarity with other settings. Furthermore, transferability of the models is established through purposeful sampling (Patton, 2002) and a thick description of the study (Geertz, 1973). As such, in this study, the setting, participants, and the procedure for data collection and analysis were precisely described and thoroughly elaborated.

## Chapter 4 Results: Trajectories of Knowledge Development

In this chapter, I respond to Research Question 1, which is as follows:

> *How does teacher knowledge specific to noticing students' mathematical thinking in the domain of the derivative develop through video-mediated professional learning?*

Results for Research Question 1 express how I discerned the trajectories of PMTs' learning knowledge for teaching the derivative and of noticing students' mathematical thinking across the eight sessions of the design experiment. As the results will show, the participants developed situated skills and teacher knowledge for noticing in a researcher-supported, video-mediated ecology of professional learning.

A framework of four processes depicts how PMTs' noticing developed over time as they explored and made inferences about students' understanding of mathematics and their own understanding of how to teach the derivative. These four processes are: *describing, interpreting, responding,* and *comparing and contrasting.* I describe these processes in this chapter in addition to the analytic means by which they were determined. Excerpts from the data are included as evidence that these processes supported the development of PMTs' noticing.

As I will illustrate, the participants' processes of modeling student thinking incorporated various domains of knowledge for teaching the derivatives that PMTs enacted in their noticing. The two constructs, learning to notice and enacting knowledge for teaching, are intertwined and interdependent, thus making it impossible to consider them separately. Therefore, as I present the processes through which PMTs analyzed the students' mathematical understanding, I also describe the forms of teacher knowledge that these processes entailed, as well as the ways in which they supported the PMTs' learning to notice.

I begin by presenting the coding that I used to identify the four processes through which the PMTs' noticing developed. Then, I show how the participants engaged in these processes to notice student thinking as they analyzed students' problem solving in the experiment episodes. Finally, I conclude the chapter with a proposal based on these findings for how PMTs can learn to notice students' thinking in a video-mediated environment of professional learning.

**Coding for the Learning Trajectory of Participants**

Chapter 3 presents the analytic methods for Research Question 1 along with a rationale for their selection. That presentation provides a general overview of the analytic approach. In this section, I get more specific about the approach as it actually played out.

I initiated the analysis of data through open coding (Glaser & Strauss, 1967; Strauss & Corbin, 1990). I read the transcribed video data to familiarize myself with it. I also examined the tasks given to the students in the videos, their work and their discursive actions, and their thinking as I inferred it from their written work and their discourse. As I proceeded through open coding, I looked for instances where the PMTs enacted mathematical and pedagogical knowledge to attend to, model, and respond to the students' mathematical thinking. Across these instances, PMTs leveraged knowledge for teaching the derivative in various ways. For example, they enacted knowledge of content and students (Ball et al., 2008) to *describe* (action) the students' (observable) work. It is important to emphasize, though, that I was not so concerned with *what* types of knowledge for teaching were enacted, as that was not a focus of Research Question 1. Rather, it is *how* knowledge specific to noticing students' derivative thinking developed. This is why answers to this question are in the form of processes, not knowledge domains.

As I began to open code the data, I used the base codes of *attending, interpreting,* and *responding*, which are derived from the professional noticing framework (Jacobs et al., 2010). Concurrently, I remained open to the possibility of other codes emerging (Strauss & Corbin, 1990). I assigned the same code to segments of data that expressed the same idea. In addition, I assigned new codes to the newly identified ways of enacting knowledge for teaching and described them. At the culmination of the initial round of analysis, the codebook consisted of seven codes: *attending, interpreting, responding, comparing, describing, contrasting, and situation modeling.*

*Attending* codes were applied to segments of the data that were the objects of the PMTs' attention (e.g., student misconceptions, heuristics), while *interpreting* was applied to PMTs' inferences about students' thinking from their observable discursive actions. For *responding,* I coded segments in which PMTs provided pedagogical moves that they proposed would help the student better understand a concept, make progress in their problem solving, or resolve a misconception. *Describing* is about narrating or recounting what the students did or the words they used at some point in their problem solving.

Next, *comparing* involves identifying similarities in two students' strategies, whereas *contrasting* involves discerning differences. Lastly, *situation modeling* was applied to segments of data where PMTs considered the context of a given task or proposed a context for a conceptually related task that might further support the problem solvers' learning.

Next, I conducted axial coding (Corbin & Strauss, 1990). Through constant comparative analysis (Glaser & Strauss, 1967), I analyzed the relationships among the Level 1 codes (i.e., those codes that emerged from open coding) and collapsed those with the same conceptual connections into categories. Five categories emerged from this process: *describing, interpreting,* *responding, comparing, and contrasting*. As the *attending* code was concept-generic, it was merged into the *describing* code, because both relate to what the PMTs pay attention to as students problem solve. Similarly, the *situation modeling* code was merged into the *responding* category, as they both relate to designing instructional interventions for students in response to assessments of their thinking.

Lastly, I applied selective coding (Glaser & Strauss, 1967). Comparing the conceptual connections among the developed categories (Kolb, 2012) enabled me to form overarching themes. These themes convey *how* PMTs can develop noticing skills by enacting knowledge that supports the teaching of the derivative. Although by definition, *compare* and *contrast* refer to distinct phenomena, these phenomena were not sufficiently distinguished in the data so as to warrant that distinction. On many occasions, PMTs both compared and contrasted students' ideas and approaches in a single response. As a result, I combined them into a single theme called *comparing and contrasting*. I also considered the other categories that emerged from axial coding and designated them as themes since they did not warrant further collapsing.

At the end of the analysis, four themes emerged that depict the processes through which the PMTs' knowledge relevant to noticing developed: *describing*, *interpreting*, *responding*, and *comparing and contrasting*. These four processes of learning to notice provide a third-order model of how PMTs describe and interpret student work, how they respond to it, and what they think about mathematics and mathematics teaching. A *third-order model* refers to a PMT's perceptions of a student's thinking from the perspective of a researcher through analyses of the PMT's interactions with the student's problem solving (Wilson et al., 2011; a *second order model* refers to the PMT's perception of the student's thinking). The third-order model, then, is my description of the way PMTs modeled student thinking. In the next section, I describe each of the four themes in detail. Then I present exemplars of the ways in which the participants enacted each of these processes in their participation across their eight sessions.

## The Emergent Framework of Learning to Notice Students' Thinking

In this section, I present the four-element framework for developing noticing through video-mediated learning that emerged from this study. I also present the learning-to-notice trajectories for three pairs of PMTs that was enabled by their participation in the processes of that framework. These presentations depict the PMTs' increasing sophistication in applying their mathematical and pedagogical knowledge for teaching the derivative to notice student thinking about the derivative.

### Learning to Notice through Describing

By observing and analyzing videos of students' problem solving, the participants used *describing* as a process to explain, recount, or detail aspects of the students' mathematical activity. In describing, participants "listen" closely to students' discourse. Discourse can include students' written work, their verbal expressions, the analogies they use, the representations they make, and their justifications for conclusions. All six PMTs enacted knowledge for teaching by describing students' mathematical activity. Next, I provide instances of "describing" activity from the data.

#### *Superficial Descriptions*

In the early episodes of the experiment, the PMTs evidenced their novice noticing skills as they demonstrated a relatively unsophisticated ability to describe the features of students' problem solving. Their summative accounts of students' strategies and their attention to the mathematics in the words they used were understandably shallow. Several examples follow.

In Session 1, PMTs viewed a video (see Video 1 in Appendix C) in which students were given an animation of water being poured into a beaker. They were asked to describe the rate of change of the height of the water with respect to its volume (see Figure 4 on p. 80). The point of the task was to illuminate ideas related to the constant rate of change. After watching the video, PMT Mia commented, “It was not clear what [students Julian and Alyssa] were trying to accomplish.” She added that she was “confused for the first part of what they were discussing… [and was] unsure what they were actually trying to figure out.” These comments do not provide a response to the students’ thinking for which knowledge for teaching the derivative could have been enacted. This comment is characteristic of Mia’s novice noticing skills at the beginning of the experiment.

Recall that I deliberately chose videos that offered opportunities for deep engagement with students’ mathematical thinking. Thus, due to the opportunities to notice that were offered to the PMTs, I argue that the length of time they spend describing their noticing is a measure of the depth of their engagement with mathematical thinking that can be observed in the videos, and thus an indication of their noticing capacities. Mia took only 38 seconds to describe what she had observed. Mia’s partner, Leah, had even less to say. She spoke for only six seconds when she commented that the students were “confusing themselves” and that she herself was “even getting more confused.” She remarked, “[Alyssa] did not even know the answer.” Such noticing is regarded as less sophisticated, because it is purely commentary and evaluative (van Es, 2011); it is an evaluation of whether the student was right or wrong and not a description of the thinking that Alyssa’s problem solving entailed.

At a similar depth of sophistication, Amelia (who partnered with Nova) shared, "I think a lot of the times I see this with students when they're trying to say something but can't find the words, you know." This comment is meaningful in that it leverages Amelia's familiarity with what students often do (i.e., her knowledge of content and students). At the same time, it does not describe what the students were discussing or the mathematical reasoning they demonstrated during problem solving. Doing so would have been regarded as more sophisticated, because it would have leveraged Amelia's mathematical knowledge for teaching.

Similarly, Nova, Amelia's partner, limited her analysis to a description of a behavioral issue. She shared that she was confused by the students' representations: "I did get a bit confused with the diagrams, and I think they were confusing themselves, too." Also, as she attended to the students' thinking, her description could not be confirmed by the data: "They both felt free enough to talk to each other because [Alyssa] was saying, oh, like constant, and [Julian] was like constant." Actually, both students did not agree on the constant rate of change. Alyssa identified a constant rate of change, while Julian identified a rate of change, which he said was "both increasing and decreasing." Nova's noticing is therefore regarded as novice, given that while she did attend to aspects of students' thinking, her descriptions weren't based on evidence from the video.

These comments indicate that when the PMTs entered the experiment, they could not draw on their knowledge for teaching the derivative, which would have enabled them to describe specific moments of student thinking during the design experiment. They are also indicative of PMTs struggles to describe the students' problem-solving in Session 1 when I invited to do so by the researcher. These results indicate that PMTs entered the experiment with novice abilities to describe student mathematical thinking. However, the quality of their describing improved over time, as I demonstrate next.

***Describing Students' Problem-Solving Strategies***

I now demonstrate that as a result of the discursive nature of the video-mediated discussions in the experiments, the quality of PMTs' responses shifted from superficial descriptions of students' mathematical activity to detailed and evidence-based descriptions of features of their problem solving. One instance in which a PMT's response allows for a claim of such a shift was when Mia and Leah watched a video of students discussing the increasing rate of change in Session 2 (see video 2 in Appendix C). Mia commented, "Julian is more so talking about the water that's being poured from the pitcher. He's saying that the water is being poured at a constant speed. And therefore, he's equating that to the rate of change of height with respect to volume." She also described Alyssa's approach to solving the problem, "She's talking about putting in the same amount of water each time. At the bottom, it would fill up a smaller height, but towards the top, it would fill up a larger height." In both comments, Mia describes how both students deliberated about the problem. From them, I inferred that Mia drew on both knowledge of content and students and common content knowledge to describe what students did to solve the problem. Julian and Alyssa's problem solving elicited their thinking about the concept of rate of change and provided Mia with an opportunity to assess it and then reflect on how they can be supported to learn that concept more meaningfully.

In addition, Mia took 60 seconds to describe the students' work here, which is much longer than the time she took to do the same in Session 1. As she became more familiar with the process by which she was asked to engage with the students' problem-solving, it would appear that she became more attentive to their thinking. Compared to her comments from Session 1, which leveraged a much shallower depth of teacher knowledge, these comments show growth in Mia's describing of the students' discursive mathematical activity. My conjecture is that the facilitated discussion of content knowledge related to the rate of change in Session 1 – and Mia's assimilation of a scheme for noticing first introduced in Session 1 – supported her changes in noticing student thinking by Session 2.

A second instance that also occurred in this session offers evidence to justify the claim of shifts in the PMTs' describing over time. Liam commented, "Like Alyssa said, the same amount of water will fill a different amount of height, essentially, as you go up, higher and higher… If I added another liter of water for the second time, the height would be a greater increase." As Liam described Alyssa's approach, he was enacting both CCK (relative to an increasing rate of change) and KCS (relative to Alyssa's thinking about that rate of change).

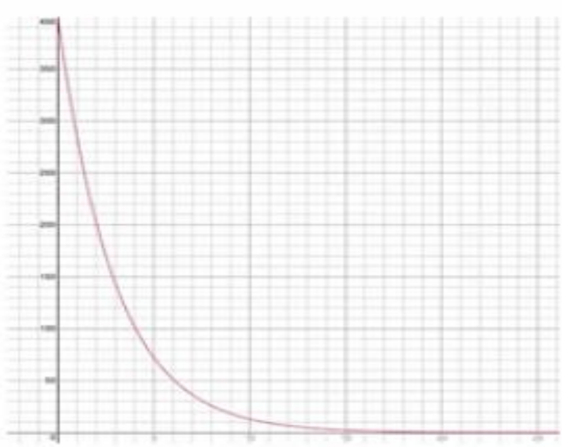
A third instance comes from Nova in Session 4. She had watched a video in which a graph was given and the students were asked to approximate the instantaneous rate of change at time $t = 4$ as shown in Figure 12 below:

**Figure 12**

***Screenshot of a "Limit Definition" Task***

Vicki took 400 mg of Ibuprofen to relieve knee pain. The amount (mg) as a function of elapsed time (hours) is $f(t) = 400(0.71)^t$

Approximate the rate of change of the amount of Ibuprofen in Vicki's body with respect to the number of hours elapsed when $t = 4$.



Nova described the students' strategy, "At first, they were looking at time $t = 3$ and $t = 5$. Then, they were starting to say, stay as close to $t = 4$ as possible, so they chose values $t = 3.9$ and $t = 4.1$." She added that they decided to use these values to get the “best approximation at $t = 4$." Although Nova only attends to what the students said and did without providing her interpretations, her descriptions can be seen as a resource she draws on in an inference she makes later about the students' thinking. This inference entails a conclusion she made about the relationship between the students’ strategy and the thinking that must have informed it. At this moment, she describes students' observable actions with greater sophistication than she did in Session 1. Based on the examples I have provided, it appears that the participants' description of the students' work is more detailed in Session 2 than in Session 1. This suggests a change in their ability to analyze students' problem solving. A detailed account of the conditions that supported the participants to improve their description of students' work is presented in Chapter 5.

***Using Students' Own Words in Descriptions to Support Claims***

Instances in which the PMTs used the very same words and phrases as the students in the video to support their claims occurred during later sessions, indicating that the PMTs' sophistication in describing students' mathematical activity increased over time. [In using the students’ own words to support their interpretive claims, I acknowledge an overlap between describing and interpreting.] The PMTs describe what happened during problem solving using students' own words, which I find sophisticated because they used these words to make inferences about their understanding. Warrants for this claim of increased sophistication come from researchers who have conducted similar studies of noticing and argue that using evidence from a video to notice students’ thinking is a marker of noticing abilities that are more sophisticated than claims made that are lacking such referents (e.g., Sun & van Es, 2015, van Es, 2011). As an example, in Session 6 the PMTs analyzed a video (see Video 6 in Appendix C) of two students, Julian and Alyssa, as they computed derivatives using slopes of tangent lines at various points on a graph (see Figure 9 on p. 84). In the video, Julian asked Alyssa questions about how to draw the tangent lines that Alyssa was unable to answer. Using Julian's own words, Nova invoked Julian's question (e.g., "What about if later on in the graph [the tangent] touches it?") to inform and substantiate her interpretation that both students lacked some conceptual understanding of the role of slopes of tangent lines in relation to the derivative. By doing so, Nova provides a more vivid image of the students' thinking as she uses their own words to describe and make inferences about their problem solving.

A second instance occurred in the same session, when (student) Julian drew multiple lines at Point *A* (see Figure 13 below) that (PMT) Amelia considered as she argued that Julian had some misunderstanding about differentiability at a cusp

**Figure 13**

***The Artifact of Students' Work on the Tangent Line Task***


To corroborate her claim, Amelia recalled the same question that Julian had asked his partner, Alyssa, "Don't these other lines work, as well?" Mia, too, referred to Julian's questions as he wondered about all different possible ways a line could hit Point *A*: "*Why does it have to be just this horizontal line through [point] A? Why can't it be a diagonal line?*" Mia used these to support her claim that Julian was considering alternative approaches to drawing a tangent line at Point *A* as he reasoned through the problem. I consider using students' own words to describe features of students' problem solving to be vital to the construction of second-order models of mathematical thinking (Cobb & Steffe, 1983). Indeed, other researchers (Jacobs et al., 2010; van Es, 2011) also interpret these instances as a manifestation of an increased capacity to notice student thinking. In addition, the participants began describing students' work in Session 6 using the exact words they used in the video. This indicated their progress in describing students' problem solving as they continued to participate in the noticing activities.

## Learning to Notice through Interpreting

In this section, I present an analysis which reveals that the PMTs moved beyond describing students' words and actions to using those words and actions to interpret the thinking that underlies them. As one process in the four-element framework for learning to notice, I demonstrate that the PMTs used *interpreting* as a process to develop their capacity to model students' conceptions of mathematics related to the derivative.

### *Using Analogies to Make Interpretations*

As PMTs' interpretive activity became more sophisticated, they began to use analogical reasoning to support their analyses of students' thinking. By analogical reasoning, I mean that the PMTs used situations analogous to the one in the given problem to support their claims or justify their arguments. Being able to provide a rate of change situation analogous to a given one is evidence of a conceptual understanding of the rate of change (Carlson et al., 2010), and thus I argue that this ability is tied to the sophistication of one's capacity for interpreting. Given the link between teacher knowledge and noticing, enacting concepts of mathematics or pedagogy to support inferences of students' thinking from their problem solving constitutes a new form of participation and thus a new moment of learning. An exemplar of this phenomenon follows.

In Session 1, Mia enacted content knowledge of quantitative reasoning in a response about Julian's reasoning: "It does not matter how fast the water is being poured, the height is still going to go up, and the volume is still going to go up, unless it's one of those problems where it's like you're pouring water into a pitcher that has a hole in it." Even though the analogous situation is not isomorphic to the given one, it works to validate the quality of her interpretation of Julian's thinking. In addition, Mia then provided her own example of an analogous situation involving a constant rate of change of height with respect to volume: "If you had four cubic inches of volume and then you gained one inch of height, then for the next four cubic inches of volume you get [a total of] two inches of height." Here, Mia uses this analogy to further depict a mathematical situation of a constant rate of change. Earlier at the beginning of this session, Mia had demonstrated what I regard as novice noticing skills, she neither described nor interpreted the student thinking: "It was not clear what [Julian and Alyssa] were trying to accomplish." At the same time, in contrast to her initial struggles to understand how the students were supposed to solve the problem in the beginning of the session, at this moment Mia leverages common content knowledge relevant to the problem situation. Using analogies to elucidate the substance of the problem and the students' interpretations of it, it shows a change in Mia's noticing abilities.

Another example of analogical reasoning is found in Session 3. The PMTs were shown a picture of a ball in motion and were asked to approximate its instantaneous speed. As Amelia observed, Alyssa (student) had difficulty understanding the context of the problem. Alyssa had indicated that since the ball is not moving (as in the picture), there is no instantaneous speed. To facilitate the understanding of the problem in context, Amelia used the analogy of a running person: “Think about you running fast. If someone took a picture when your body is moving forward, what happened while that photo was being taken?” She used that analogy to demonstrate that the picture shows a “snapshot of like, a time frame, but the ball [body] is moving.” By interpreting the problem through an analogy, in my view, may introduce a new perspective on the problem and better understand its context.

The significance of this moment lies in the fact that it reflects a change in her participation as she interprets student thinking. This shift in participation (i.e., in noticing) is made evident from a situated perspective. The guided inquiry that I provided the PMTs (e.g., prompts and re-viewing the videos) enculturated norms that moved them along a trajectory of noticing. Analyses of the mediators of the PMTs’ learning are more fully provided in Chapter 5.

***Considering the Design of Tasks in the Interpretation of Elicited Thinking***

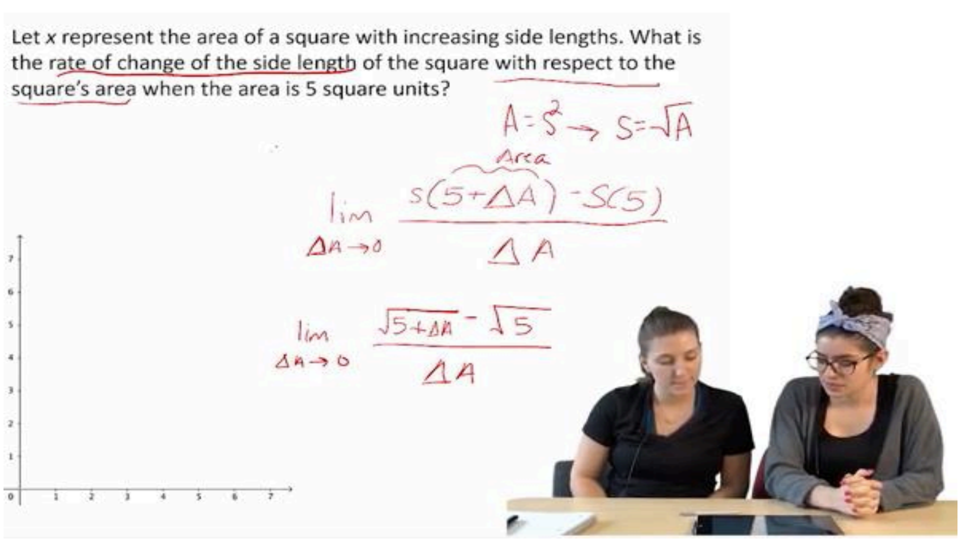
Although the focus of the PMTs’ analyses was on students’ problem solving, the PMTs also considered the design of the tasks that the students were solving to account for the mathematical thinking that the tasks made visible. According to the PMTs, task design can inform – and misinform – students' problem solving. Two exemplars of this phenomenon follow.

In Session 1 (see Video 1 in Appendix C), Mia attended to the design of the mathematical task the students were solving and proposed that Julian was being misled by the animation that accompanies the problem statement. Instead of discussing how the height changes with volume as water is poured into a cup, Julian talked about how the height of water in the cup increases over time. Mia reasoned from her analysis of the animation that it probably accounted for Julian's misconception that time is a factor when considering how changes in height covary with changes in volume. Mia suggested that water being poured “faster” and then “slowly” into the cup in the animation is "what made him start thinking about speed." Relying on instances such as this one, I propose that interpretations of students' thinking during problem solving should include considerations of the designs of the problems being solved, since it is the contingent strategies that students generate to solve these problems from which interpretations are made. This finding further establishes the value of assessing students' interpretations of problem statements even before they embark on their problem solving.

As a second example of PMTs' considerations of task design in their interpretations, in Session 5, Gray and Liam watched students Kelly and Maria solve a rate of change problem using the limit definition of the derivative concept (see Video 5 in Appendix C). The students were asked to find the rate at which the side length of a square increases as its area increases at the point when the square's area is 5 square units (see Figure 8 on p. 84). The students' work, which the participants analyzed during their discussions, is shown in Figure 14 below.

**Figure 14**

***Screenshot of the Students' Work on a Rate of Change Problem***



Gray considered the design of the task given to the students and wondered aloud, "Why would [the problem writers] label area, $x$? Like $x$ is … $x$-axis." Liam added, "Usually, when I see area, I just assume it's $A$. Like, that just make sense to me." These comments were unprovoked by prompts or highlighting and therefore speak to the participants' knowledge of students and curriculum as they suggested that using the variable $x$ to represent area could confuse students. Next, Gray argued: "They shouldn't be using $\Delta A$, they should be using $\Delta s$. And then the same goes for like where the limit is. They have $\Delta A$ is approaching zero. So that's a mistake they made, I think." Liam's interpretation coincided with Gray's, as he added the justification for Gray's argument: "Because whatever you have in the parentheses (see the students' work in Figure 14 above) should match whatever you have as your variable on the right side." The students should have used "$f(A)$, which is equal to $\sqrt{A}$." [The PMTs' response to the identified students' error is discussed in the following section.]

I observed a number of noticings by Gray and Liam related to the students' problem solving in this session. In addition, while undirected, they also attended to the problem statement and discussed how the choice of variables might be the source of the students' confusion. Contemplating sources of students' thinking beyond their own mathematical activity may be a marker of Gray and Liam's skills in noticing and designing learning activities for the students.

### ***Leveraging Content Knowledge to Construct Interpretations***

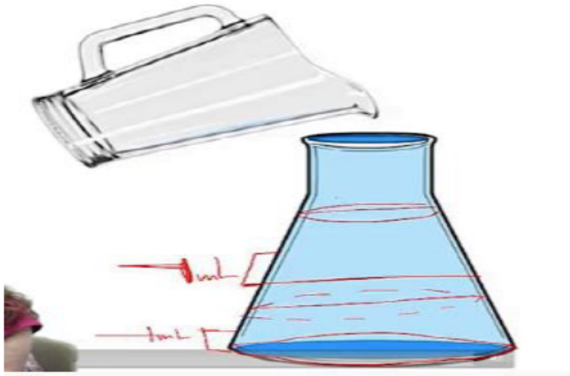
To construct their interpretations of students' mathematical thinking, the participants drew on their content knowledge of concepts related to *rate of change* and *change in a rate of change*. As a case in point, in Session 1 I replayed a clip (from Video 1 in Appendix C) in which Alyssa claimed that for every 1 ml of water added to a cup, the height of water in the cup increases by the same amount. This covarying relationship constitutes a constant rate of change.

(PMT) Liam attended to her argument, leveraged his content knowledge to interpret it, and then made this comment in light of that interpretation: “The height changes as the volume changes… If I add more water… the volume increases, and the height of that cylinder, like where the water line level is, also increases.” Gray, Liam’s partner, drew a conclusion based on the same interpretation he made of Alyssa’s claim: “Changes of height and volume are a ratio of 1:1,” so the rate of change is constant. Reasoning quantitatively in this way is important in this problem, because in order to establish a rate of change, one needs to first determine the relevant quantities, then find out how they are changing, and then figure out the relationship between them (Carlson et al., 2010; Lobato & Siebert, 2002). Liam and Gray’s comments depict conceptions of quantitative and covariational reasoning that are essential to understanding the concept of the rate of change. Indeed, it was their grasp of this content knowledge that enabled them to notice Alyssa’s thinking as they interpreted and analyzed (Jacobs et al., 2010, van Es, 2011) the validity of her argument.

Then, soon afterward, the participants applied their understanding of an *increasing* rate of change to assess Alyssa’s understanding. This occurs in Session 2 when Alyssa explains that because the beaker in the problem is narrowing at the top (see Figure 15 below), for equal amounts of water added to the beaker, the *change in height* of the water is increasing. In her drawing on the beaker as shown in Figure 15 below, the second vertical segment representing a change in height is longer than the first one.

**Figure 15**

***Alyssa’s Illustration of an Increasing Rate of Change***



The following are interpretations of Alyssa's thinking made by three of the PMTs as they enacted their content knowledge of the rate of change:

Mia: She's talking about putting in the same amount of water. At the bottom, it would fill up a smaller height, but towards the top, it would fill up a larger height … She's thinking about the same amount of volume and how the height is going to change.

Liam: Like Alyssa said, the same amount of water will fill a different amount of height, essentially, as you go up higher and higher … That means… as the volume is increasing, the rate of change in height would also be increasing at a greater rate.

Gray: Height increases with respect to volume. Like, as equal amounts of water are being poured, the rate of change of height increases.

What is also significant in these interpretations is that as the PMTs leveraged their knowledge of the covarying relationship between the height and volume of the water, they demonstrate the need to keep changes in one of the variables constant (volume) in order to determine the rate of change in the other (height). This, too, is a way of thinking that Lobato and Siebert (2002) suggest is foundational to an understanding of the rate of change. In sum, these occasions to notice Alyssa's thinking provided the PMTs with opportunities to develop their content knowledge and noticing for teaching the rate of change, a concept fundamental to the derivative.

By leveraging their content knowledge of graphing the derivative in Session 7, participants developed fundamental mathematical ideas and valid problem-solving approaches. Students (Kelly and Maria) were shown an animation of a football thrown and asked to graph its speed over time (see Video 7 in Appendix C). The students drew a rough sketch of what the speed-time graph should look like (see Figure 16 on p. 132), and then plotted a speed-time graph together after discussing the problem (see Figure 17 on p. 142).

Mia drew from her pedagogy and considered sketching a rough graph as "a great student strategy to start solving a problem." Leveraging on her specialized content knowledge, Mia interpreted the students' plotted speed-time graph: "Derivative starts increasing, and then it decreases [as the ball] comes to a full stop. And then it starts increasing again… when the ball is thrown." An interpretation of the speed-time graph that the students plotted was essential to my understanding of her thinking, specifically how the height varies with time. Mia also sought to make sense of the students' rough sketch (see Figure 16). Mia noted that the sketched graph was "increasing the whole time," yet the ball stopped at some point. Her interpretation reveals a flaw in the graph (Figure 16) in that the ball stopped at some point and, therefore, the speed cannot be increasing continuously on that interval. Leah, her partner too focused on the behavior of the graph at time $t = 4s$: "The distance traveled is much more drastic. He let off the ball and now it's traveling in the air." Mia added that "the slope is positive… It's a much steeper slope because the change in the distance traveled is greater than the change in time." From these comments, Leah and Mia leveraged their mathematics to collaboratively interpret the distance-time graph drawn by the students, leading to the construction of fundamental mathematical idea, for example, "positive slopes" "increasing and decreasing derivatives" and so on.

Liam too leveraged his content knowledge to interpret the students' sketched graph and then described it as "less informative," indicating that critical, explanatory features of the graph are missing when one constructs the graph in a point-by-point fashion. He noted that instead of calculating slopes and then plotting them point by point, the students should have analyzed the distance-time graph more globally to "figure out where there is constant speed, where there is no speed, where there is a change in speed… just to visualize the whole slope, rather than just trying to figure out numbers that represent the slope." From Liam's comments, we develop a conceptual understanding of how to plot a derivative graph, and thus indicating the way content knowledge was leveraged to develop mathematical ideas and problem-solving approaches.

***Identifying and Analyzing Misconceptions (in an Applied Context)***

In this section, I demonstrate how misconceptions revealed in students' problem solving provided the PMTs with a unique opportunity to interpret student thinking, since these interpretations tended to be accompanied by conjectures made by the PMTs as to how those misunderstandings might have arisen. This excerpt comes from Session 3 when PMTs viewed a video in which the students, Julian and Alyssa, were shown a photo of Blue Jays' pitcher Marcus Stroman (see Figure 6 on p. 81) and asked to approximate the speed of a pitched baseball at the instant the photo was taken.

In the video, Alyssa reasoned that "It's instantaneous. Like, there's no time really happening [when the picture is taken] and, like, for speed… it's like distance over time, but if we don't have any time, we can't really be moving." When Julian shared that when the ball is pitched and "the batter's swinging, there *is* movement and there *is* a change in time," Alyssa responded to rebut Julian's claim and argue that "We're only seeing, like, [an] instantaneous moment." I asked Mia what she could infer from Alyssa's argument about her understanding of mathematics. In response, Mia suggested that Alyssa has a misconception about the meaning of instantaneous rate of change: "She's still confused with what instantaneous means. She thinks it has to be no time." Leah, Mia's partner, added, "If you divide [distance] by zero [in the formula, speed = distance divided by time], you're not going to get an answer. It’s undefined.” In another session, Gray provided a similar interpretation of Alyssa’s reasoning: "She is assuming time is zero, which is wrong because there is a shutter speed. It doesn't take zero seconds for a camera to take a picture. And then, under that assumption, she assumes, well, if time is zero, then the ball is moving at zero mph, which we know is not true because we know a pitcher threw it. And balls don't float in the air." With these comments, Gray interprets Alyssa’s thinking and then leverages knowledge of cameras and baseball that he apparently believes “we” all know to justify that her reasoning is faulty.

In these comments, the PMTs drew from their content knowledge to interpret Alyssa’s understanding of the concept of instantaneous rate of change in light of her efforts to determine the speed of a pitched baseball at the moment a photograph was taken. They also leveraged their everyday knowledge to challenge Alyssa’s claim that the ball’s instantaneous speed was zero because no time had passed. In doing so, they asserted that Alyssa must possess the common misconception that instantaneous speed is equivalent to no speed, because no time passes in an instant. Because the derivative is precisely an instantaneous rate of change, the PMTs’ interpretations of Alyssa’s thinking would be critical if they were her teacher. Thus, the knowledge they brought to their analyses demonstrates the value of the opportunity they were provided to notice a common misconception in order to develop their knowledge of content and students. In addition, this opportunity demonstrates the value of engaging PMTs in noticing student thinking in an applied context. The context of taking a picture of a moving ball (as time elapsed and the ball traveled some distance) was a productive venue for the students’ reasoning about well-connected derivative concepts. It also elicited quite a lot of everyday and mathematical thinking that the PMTs could use as they’re learning to construct reliable models of students' misconceptions that would make their instructional responses more viable. As an example, Amelia suggested using the analogy of a running person in order to help the students understand that in a snapshot of a moving body, time elapses and distance is covered. According to her, students need to understand that "the body moves while the photo is taken" and that "the distance and time taken cannot be zero." The understanding of these mathematical ideas is fundamental to the learning of the rate of change (covariation of quantities), as well as the broader subject of differential calculus. Thus, the PMTs' activity in this instance suggests the educative value of doing noticing work in the context of students' struggles in an applied problem-solving situation.

### ***Collaborative Analysis of Derivative Reasoning***

In this section, I depict the contribution that collaborative analysis afforded the PMTs as they did their noticing work in pairs. Although collaborative analysis is common to all of the vignettes presented here, its particular value has yet to be forefronted in these results. Thus, I share a prolonged vignette to demonstrate its value for the development of the PMTs' noticing. This vignette was also chosen because the analysis is centered on the conceptual context of covariational reasoning, which researchers have argued is crucial for understanding the derivative as a rate of change (Carlson et al., 2010; Moore & Thompson, 2015). As the reader will observe, in this vignette the PMTs develop a shared understanding of covariational reasoning that they subsequently rely on to interpret the students' actions and model their thinking.

This vignette occurs in Session 4 as the PMTs are analyzing students' problem solving on a task about the limit definition of the derivative. At this point in the students' problem solving, they are given the task and the graph that appear in Figure 12 above. The task involves approximating the rate of change in the amount of Ibuprofen in a person's body at a particular point in time. In order to determine how Mia and Leah would interpret the graph, which would give me an indication of their capacity to engage the students' graphical and covariational thinking, I asked them to look at it and describe how the two quantities are changing in relation to each other. Specifically, I asked them to describe how "the amount of Ibuprofen is changing with respect to time." Mia responded, "As time increases, the amount of Ibuprofen in the body decreases. So, at time $t = 0$, it's 400 mg, and then right away, it starts decreasing." Then, Mia further shared that the slope would be "negative." Leah agreed with Mia: "That's right. That causes a negative correlation between the two [variables]." Then, Mia elaborated: "Yeah, one variable is increasing and the other variable is decreasing." What this exchange reveals is that the intersubjective (Vygotsky, 1978) responses in the PMTs' collaborate discourse yielded conceptual material (the covariational reasoning entailed in an interpretation of an indirect relationship) that they leveraged to construct a richer interpretation of the students' problem solving than they may have achieved alone.

In another episode with Liam and Gray, I also asked for their interpretation of the graph. Liam responded, "It is an exponential decay. From the moment you put Ibuprofen in the body, it is quickly deteriorating… The body is absorbing Ibuprofen over time." He then added that the rate of change of Ibuprofen in the body was "decreasing." I then asked, "So, is the rate of change positive or negative?" Liam said it was, "Negative, because you would always have a [decreasing] amount." Gray elaborated Liam's analysis by referring to the interval on the graph from $t = 0$ to $t = 4$ as he demonstrated that the "Ibuprofen is getting lost out of the body faster and faster." Then Liam added that there is an "inverse relationship" between the amount of Ibuprofen and the time, because "the more time that has gone by, the lower amount of Ibuprofen is in the body." These exchanges between Gray and Liam reveal what appears to be a shared understanding of the covariational relationship between the two quantities, the negative rate of change between them, and the representation of that negative relationship on the graph.

In both Sessions 1 and 2, the pair of students in the video was given a task to determine how the height of water being poured into a container changes as its volume changes. Water is poured into a cylindrical beaker in Session 1 (see Figure 4 on p. 80), while in Session 2, the beaker narrows toward the top (see Figure 5 on p. 81). In response to Nova and Amelia’s own analyses of the rate of change in the narrowed beaker context of Session 2, I followed up: "You're saying the volume is increasing and the height is increasing. But the height is increasing even more rapidly as you increase volume because of what? Why?” In her response, Nova contrasted the task situations in Sessions 1 and 2: "I know since the original beaker was a cylinder [in Session 1], everything was fine because it's straight all the way through. But this one [in Session 2], since it does have like a wide part at the bottom and then it gets narrower, I do feel like it does have an impact on [the rate of change]." For Nova, this explains why, in Session 1, the rate of change in height with respect to volume was constant, while in this session, it was increasing. Amelia then added a clarification: "Because of the shape of the beaker." This *co-constructed explanation* conveys Nova and Amelia’s understanding of the differential impact of the shapes of the beakers on the rates of change in the two tasks. Moreover, it is grounded in a connection they made between the design of the tasks in Sessions 1 and 2 and the mathematics represented in the contrasting situations (constant rate of change for the cylinder, increasing rate of change for the narrowed beaker). This unprompted cross-task analysis undertaken by Nova and Amelia is sophisticated. It leverages their knowledge of content and curriculum (KCC, Ball et al., 2008) to interpret the students’ thinking in relation to the design of each task. More importantly, I argue that it gets its particular power from the collaborative means by which it was constructed.

The participants' collaborative application of their quantitative and covariational reasoning to interpret graphs and evaluate the students' thinking during problem solving seemed to support their enactment of knowledge for teaching the derivative. The knowledge they each brought to their consideration of the shape of the beaker and its relation to the rate of change in the height of water with respect to volume enabled what at this point in the experiment was an extraordinary analysis of the students' problem solving. Other participants brought other knowledge to their analyses, as well, including ideas about the connection between covarying quantities and their representation as slope. Although I emphasize the collaborative analysis of student thinking here, I would be remiss if I didn't also emphasize that their content knowledge played a crucial role in their discussions of the underlying mathematics ideas and informed meaningful assessments and interpretations of student thinking.

***Making Sense of Students' Struggles in Problem Solving***

This section presents one example of an opportunity provided to the PMTs to make sense of struggles that the students encountered in their problem solving. In this example, we observe a variety of ways that the PMTs interpreted students' thinking and made conjectures about how that thinking might account for the struggles they experienced as they sought to solve the problem posed to them.

In Session 6, the participants discussed ideas related to secant and tangent lines as they watched a video of students solving a problem (see Video 6 in Appendix C) where these ideas were central. They discussed the students' understanding of a tangent line through the students' discourse and the sketches they made of tangent lines on a graph. Figure 13 above (on page 110) shows the students' work at the conclusion of the video.

At one point in the discussion, Mia considered the students' definition of a tangent line: "A tangent line touches a graph at only one point." She said that the students "understand that the derivative is the slope of tangent line, but they don't understand exactly how to draw tangent lines" and they seem to be "confused." She elaborated on her assessment by suggesting that the students seemed not to know whether they could "extend the tangent line and hit other points of the graph" or if the "tangent line could intersect the graph" at any other point. Leah, her partner, agreed with Mia and followed up: "But I don't think they would be able to then answer any further questions about how a tangent line relates to the derivative, and how the slope of the tangent line relates to the derivative." Thus, both Mia and Leah inferred that the students were unclear about what a tangent line is and how it is related to the derivative.

Next, I replayed a video segment in which the students were drawing tangent lines at Point *A* on the graph (see Figure 13 above) in order to focus the PMTs' noticing on Julian's thinking. Mia responded, "I just remembered something about derivatives and tangent lines that I don't think they know. You can only draw a tangent line on a curve. There is no derivative at a sharp point. *That's why they're struggling with drawing a tangent line at Point A*" (emphasis added). Mia's comment emphasizes the value of this study's use of the dynamic perspective on knowing. One could assess Mia's knowledge for teaching the derivative through some sort of decontextualized, summative assessment, but it was her in-the-moment enactment of knowledge in response to student interactions that revealed the pedagogical power of her knowing as she sought to make sense of the students' problem solving. Making sense of those struggles leveraged multiple forms of knowledge for teaching the derivative, because it required that Mia consider how the students' problem solving could have gone better than it actually did. Thus, I argue that Mia's noticing at this moment is at least somewhat predictive of what her noticing could look like in the "blooming, buzzing confusion" (Brown, 1992, p. 141) of an actual classroom.

Moving on, Mia commented on some of the wonderings Julian posed: "You know, asking, 'Why does it have to be just this horizontal line through [point] *A*? Why can't it be a diagonal line?'" Mia seems to interpret Julian's questions as his efforts to justify and validate his reasoning, which she considers a crucial process in problem solving. She says "Julian is thinking about all possible ways a line could be drawn at point *A*. Even though his thinking is incorrect, mathematically, I think it's good that he's analyzing the problem and trying to think of different approaches." In addition, her appreciation for Julian's "why" questions may also indicate her own affinity for an inquiry-based learning (Franke & Kazemi, 2001), a form of pedagogy that supports students' learning as being driven by their own knowledge and curiosity.

In a similar moment in another session, Nova also considered Julian's questions: "But what is a tangent line?" "What about if later on in the graph it touches it [another point]?" To clarify, Julian is interrogating Alyssa's definition of a tangent line as one that "touches a curve at only one point" and wondering whether a line would still be considered a tangent line if it were tangent to a curve at more than one point. Nova pointed out that Alyssa was unable respond to these questions and concluded that something was missing from Alyssa's understanding of a tangent line: "Drawing lines doesn't mean she knows why she's drawing them." For Nova, Julian's questions were "valid… A teacher would have to explain [them] so that misconceptions aren't there." Nova seems to believe that the Julian's misconceptions and Alyssa's inability to respond to them may stem from derivative knowledge that is entirely rote and procedural (i.e., knowing *how* to draw a tangent line but not knowing “why she’s drawing them.”).

Gray and Liam noticed flaws in the students’ definition of a tangent line, as well. Gray commented, "The first thing they did is, obviously, define a tangent line. But they did this incorrectly. And then they go ahead and experiment with that definition to see what their drawings of tangent lines would look like." Realizing that the students’ conception of a tangent line was incomplete, Gray added that in addition to understanding that a tangent line touches a graph at only one point, as it was defined by the students, the students should also understand that a tangent line "represents slope at a specific point of the graph." Liam, Gray’s partner, had the impression that the students "weren't quite sure how to draw a tangent line." Gray and Liam not only described what the students did, which is a viable approach for gaining insights into student thinking, they went further to infer from the students' work (of sketching tangent lines) that they may have struggled because of the way they defined a tangent line. This was the case for Mia, Leah, and Nova, as well. The quality of their noticings of the students’ thinking must have been sufficiently sophisticated to enable them to make conjectures about the source of the students’ struggles by leveraging common and specialized content knowledge and knowledge of content and (its relationship to) teaching. Understanding why students solve problems in a certain way provides PMTs with an opportunity to find out why they think in a certain way. This involves knowing what students do or do not understand. Through such understanding, as I demonstrate in Chapter 5, PMTs were able to orchestrate instruction that would eliminate the misconceptions identified and augment the students' understanding of specific concepts, for example, the differentiability at a cusp.

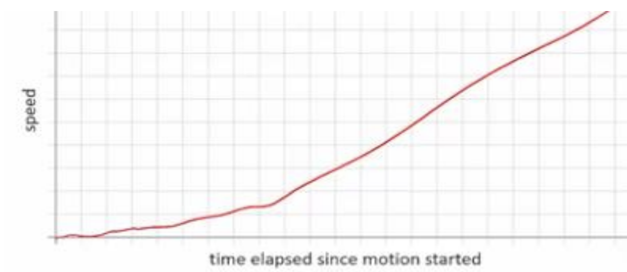
***Making Inferences about Students’ Thinking from (Multiple) Artifacts of their Work***

In this section, I discuss how PMTs analyzed artifacts of students' work in order to infer their understanding of mathematics. I provide examples from the data to demonstrate how artifacts can provide a site in which PMTs develop their noticing skills through their interpreting activity.

In Session 7, PMTs viewed and analyzed a video of students (Kelly and Maria) exploring derivative ideas as they constructed and interpreted graphs relating the distance, speed, and time of a thrown football. For their first task, they analyzed how the vertical distance (height) of the football changed over time and sketched the speed-time graph shown in Figure 16 below.

**Figure 16**

***Kelly and Maria's Sketch of Speed-Time Graph***




The PMTs' first move was to attend to and interpret the graph. Mia noted that it indicated that the ball was "increasing [in height] the whole time," yet the ball stopped at some point (as shown in Figure 10 on page 85), which means that the speed cannot be increasing continuously as the sketch graph indicates. This is the first instance in which Mia makes an inference about a student's thinking from artifacts of their work, as opposed to their in-the-moment discourse. It was a first for Nova, as well. She noted that the sketched graph "looked similar to the [*distance*-time] graph that was presented to them" earlier (in Figure 10 on p. 85) and concluded that the students must have "confused" them. This comment is significant, because it is an indication that Nova's ability to notice features the capacity to iteratively construct models of students' thinking from interpretations based on artifacts that are both inside and outside of her immediate perceptual awareness. Although it was not my intention to encourage the PMTs to consider multiple moments in students' problem solving in order to develop more robust models of their thinking, Nova's spontaneous act of doing so convinced me that the opportunities for sustained noticing over time (as opposed to making inferences from a single statement or action) would be useful for others, as well. Indeed, Gray and Liam exploited such an opportunity later in this session when their sustained attention to Kelly and Maria's discourse about a speed-time graph enabled them to make sense of how the students understood the relationship between instantaneous rate of change, the slope of a tangent line, and the derivative. This moment is presented next.

Gray's first comment after viewing the video was to suggest that "[Kelly and Maria] seem a bit confused." He noted that they did not deliberate about how the speed-time graph should look by considering pertinent questions, such as, "Where should it be decreasing? Where should it be positive? Where should it be on the *x*-axis?" Liam's attention was similarly placed. He described the students' sketched graph as "less informative," meaning that critical features of the graph were missing. He suggested that a pointwise graphing approach could have helped the students determine how the plotted speed-time graph could foreground how the quantities of speed and time are changing. This, I argue, is a rather sophisticated act of noticing, as it leverages intertwined knowledge about covarying quantities, discrete and continuous change, and their representations in graphing contexts in order to interpret (and respond to) gaps in students' thinking solely from artifacts of their work.

This concludes the presentation of findings related to the second of four kinds of opportunities for PMTs to learn how to notice student thinking, which is through *interpreting*. Next, I present findings related to the third kind of opportunity, which is through *responding*.

## Learning to Notice through Responding

As demonstrated in the previous sections, as they *described* and *interpreted* students' mathematical activity, the PMTs developed new understandings of their mathematics, which were integrated into their own knowledge, including their specialized content knowledge and their knowledge of content and student. As Wilson et al. (2011) propose, this *restructured knowledge* (p. 58) subsequently becomes available to them "in the ways [they] anticipate, interpret, and respond" (p. 58) to students' mathematical activity. In this section, I will show how the PMTs' enactments of their restructured knowledge enabled them to offer viable suggestions for how to further advance the students' derivative understandings by *responding* to their in-the-moment interpretations of the students' current understanding. I will also emphasize that in contrast to the PMTs' responses earlier in the experiment, which were rather superficial and based on general principles of teaching, these responses are relatively sophisticated and relevant to the specific content under consideration. As such, they offer evidence of the development of PMTs' noticing, because they are grounded in students' understandings of the derivative.

### *Superficial Instructional Responses*

The responses provided by PMTs in the first and second sessions of the experiment were generic and lacking in connection to student thinking. These I refer to as *superficial instructional responses* (SIRs). However, instances of SIRs diminished over time as the PMTs progressively learned to model the students' understandings of the derivative and to make pedagogical decisions about how to respond them. Next, I present some exemplars of superficial instructional responses and contrast them with later responses, which demonstrate the PMTs' improvement in responding to student thinking over time.

In Session 1, PMTs watched a problem-solving video of students illuminating ideas related to the constant rate of change and discussed what they noticed about the students' thinking. The students were examining how the height of water in a cup changed with volume as water was being poured into the cup (see Figure 4 on p. 81). Leah noted that the students focused on the "speed of water" rather than how height changed with volume and inferred from their focus that the students may not have understood the problem. She suggested that if she were their teacher, she would ask them to "watch the animation again to understand the problem." Gray said he would go back to the problem, underline the statement "height with respect to volume," and then ask the student to "read the question again." He explained that these moves could prompt the students to think about height and volume as the covarying quantities without considering time. Amelia said she would ask them to look at the question again: "I would say, let's go back to our question and see if there's maybe something we're missing. Is there something we're forgetting about?" She added that the intent of these questions is to get the students to come to the realization, "Oh, we have not talked about volume." I regard these responses as *superficial* because they direct the students to repeat what they've already done rather than helping them move their thinking forward. Further, these moves are generic; they are not responsive to student thinking.

Other instances of superficial responding arose in Session 2. In this session, the PMTs watched a video of students solving a problem related to an increasing rate of change in the context of water being poured into a flask (see Video 2 in Appendix C). The students were asked to determine how the height of the water changed with respect to volume. In the video, Alyssa explained, "Water would fill up more in the beaker because it is getting narrower at the top." Julian (her partner) took her statements to mean that "there is less volume to fill at the top" and that "volume is decreasing" as the beaker becomes narrower. To assess the quality of PMTs' responsive noticing, I asked Gray and Liam how they might respond to Julian's claim that the rate of change of height with respect to volume is "decreasing" in order to shift his focus to the quantities called for in the task. Liam commented, "I would kind of, like, reiterate the question, especially I would be like, so does that mean the rate of change of height with respect to volume is increasing or decreasing?" Liam added that his reasoning for asking this question would be to assess Julian's interpretation of the problem and to determine which quantities Julian was referring to when he determined that the rate of change of those two quantities is "decreasing." I would deem Liam's instructional move as superficial because, rather than exploring Julian's elicited understanding, he suggested reiterating the question. His idea that "volume decreases" as more water is added to the beaker, for instance, would have been responsive to Julian's thinking and therefore worth exploring. Taking into account that these superficial responses were elicited in earlier Sessions 1 and 2, I infer that PMTs had limited noticing skills when they began the experiments. This finding is consistent with findings from other research on noticing (e.g., Sherin & van Es, 2002; Sun & van Es, 2015).

***Conceptual Instructional Responses***

I give the designation, Conceptual Instructional Responses (CIRs), to *responding* comments that are based on the models PMTs have constructed of the students' understanding of a concept associated with the derivative. In other words, CIRs are teaching decisions specific to a particular concept. Sánchez-Matamoros et al. (2015) use *conceptual actions* in a similar sense, as pedagogical moves that *PMTs* suggested would augment students' learning and/or resolve their misconceptions.

CIRs were given by PMTs in every one of their eight sessions. And when PMT provided a CIR, they coupled it with a rationale for *how* it could potentially support the students' learning. This added rationale is what's missing from SIRs. Contrary to SIRs, CIRs address students' thinking as it is interpreted by the PMTs and foreground the meanings of mathematical concepts. Next, I present exemplars of CIRs from the data to demonstrate how PMTs learned to notice students' thinking about a concept associated with the derivative by responding to the students' thinking about that concept.

**CIRs Related to the Rate of Change.** In the video viewed in Session 1, students Julian and Alyssa were given a task to determine how the height of water being poured into a container changes as its volume changes. Rather than considering these covarying quantities, Julian instead considered how the volume of water changed over time. The PMTs noticed this in his discourse and inferred from it that Julian was focused on the speed at which water was being poured. So, I asked the participants about the instructional strategies they would use to support Julian to reason covariationally with height and volume if he was their student. CIRs emerged in response, which I present next.

Amelia replied, "I would say something like, I see what you're saying: the water is being poured in at a constant rate. How does that impact how the volume is changing? Or what does that tell us about the height? What do we know about the volume?" Amelia suggested that these questions might shift Julian's attention to how the intended quantities are changing in relation to each other. The instructional response suggested by Amelia's partner, Nova, was similar: "I would probably say, 'I see that the pour [of] water into the beaker, but we have to look at what is happening *in* the beaker'… I think I would probably just get his attention closer to the beaker, because I think he's just looking at the water." Both Amelia and Nova's responses accounted for Julian's thinking as well as the tasks' intended quantities involved in a rate of change. This is what makes their responses conceptual instructional ones. Next, I provide a second CIR related to the rate of change. In this instance, the CIR is related to the *instantaneous* rate of change and a PMT's response entails a concretizing experience for that abstract concept.

In Session 3, the PMTs viewed the video of students solving a problem centered on ideas about the instantaneous rate of change. In the video, two students, Julian and Alyssa, were shown a photo of Blue Jays' pitcher Marcus Stroman (see Video 3 in Appendix C) and asked to approximate the speed of the baseball over the small interval of time (1/2000 sec) in which the picture was taken. At this moment in the video, Alyssa asserted that the time it took for the camera to take the photo was zero, meaning that no time passed at the instant the photo was taken. Apparently, Alyssa has the common misconception that a measure of instantaneous speed requires that time is zero rather than that the elapsed time approaches zero (i.e., its limit is zero; it tends to zero).

Gray interpreted Alyssa's reasoning as follows: "She is assuming time is zero… And then, under that assumption, she assumes, well, if time is zero, then the ball is moving at zero mph, which we know is not true because we know a pitcher threw it. And balls don't float in the air." I asked Gray and Liam, "As a teacher, and having noticed Alyssa's mistake about the time duration, what would you do?" Gray said he would ask her to, "Take out your phone and take a picture." Then he would ask, "how fast that picture just happened. Is it zero seconds? … She'd be, like, well, I'm assuming it's *not* zero." Gray's responsive noticing entailed the enactment of specialized content knowledge to help Alyssa recognize (or re-cognize) her conclusion about the duration of time at a particular moment. It also offered evidence of Gray's capacity to orchestrate an experientially real (Gravemeijer & Doorman, 1999, p.11), conceptually relevant instructional intervention in response to a student's misconception.

In a separate episode, Amelia engaged in analogical reasoning to provide a similar response. Whereas Gray had contextualized instantaneous rate of change in the situation of using a phone to take a picture, Amelia suggested the context of a person running, which is analogous to the thrown baseball context: "I would say, think about you running fast. If someone took a picture when your body is moving forward, what happened while that photo was being taken?" She added, "Hopefully, if she is like, 'my body would keep going forward,' [I would ask her if it would] be reasonable to assume that this ball is still going, and even though we're just taking a snapshot of like, a time frame, that the ball is still moving, there is motion there. We just don't see [that] in a snapshot of it." Like Gray, Amelia suggested a conceptually relevant instructional intervention in response to Alyssa's misconception that she apparently hypothesized would be more experientially real to Alyssa than the thrown baseball situation. Thus, in this instance, both Gray and Amelia enacted conceptual instructional responses that demarcate a noticing practice entailing capacities to describe and interpret students' thinking, and then provide a conceptually relevant, experientially real instructional response to that thinking that aims to further support the students' learning.

**CIRs Related to the Derivative as the Slope of a Tangent Line.** In this section I demonstrate how PMTs provided CIRs based on their interpretation of students' understanding of the derivative at a point as the slope of a tangent line. In this instance, the PMTs observed that the students had difficulty defining a tangent line and sketching them at various points on the graph. In response to their observations, the PMTs were given the opportunity to determine an intervention to support the students' learning of these concepts.

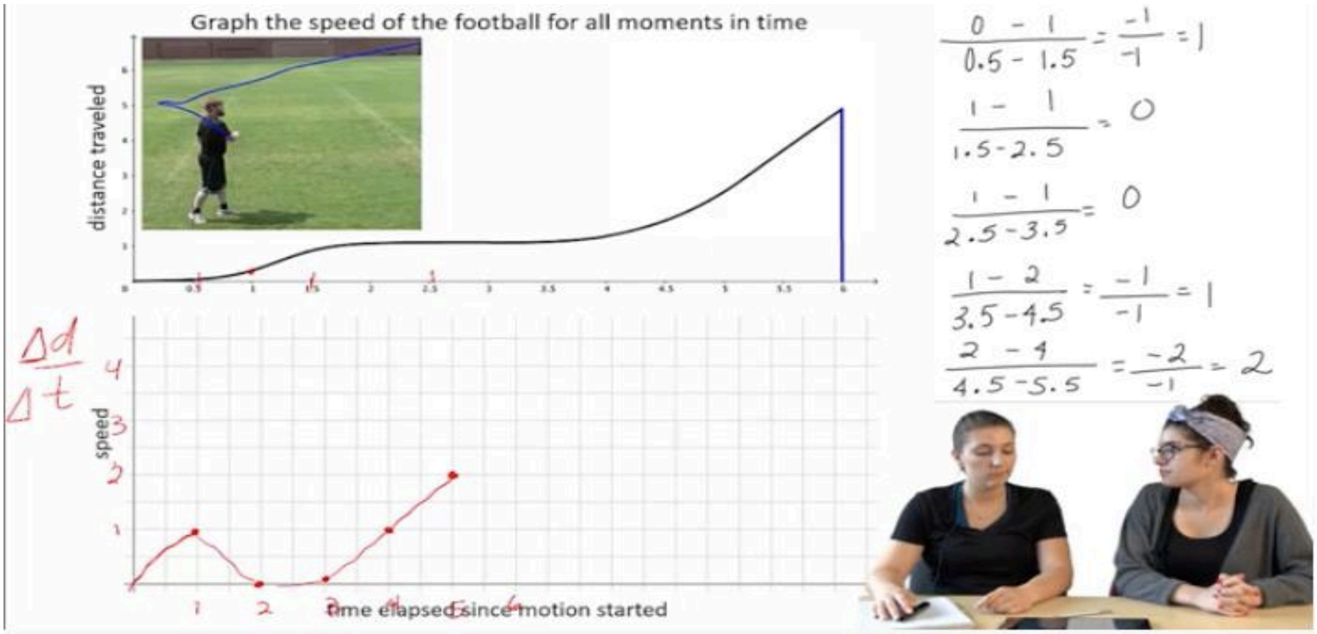
In Session 6, the PMTs watched Julian and Alyssa computing derivatives using slopes of tangent lines (see Video 6 in Appendix C). As the students sketched tangent lines at various points on a graph (see Figure 13 on p. 111), their thinking about the meaning of a tangent line was elicited. I asked Mia and Leah how they would help the students understand that because the graph contains a cusp at Point *A*, a tangent line cannot be drawn there, which means that the derivative – or the slope of a tangent line at Point *A* – cannot be determined. Leah suggested asking the students, "If you can't draw a tangent line at Point *A*, or if it's not working out the way you think a tangent line should, what do you think that might imply about the derivative?" She proposed that this question could help the students realize what it means for the differentiability of a function at a point if a tangent line cannot be drawn to its graph at that point.

At this point in the presentation of PMTs' enactments of conceptually relevant responses to the students' mathematical thinking, it's worth pausing to offer a brief commentary on the responsive moves presented thus far. What's worth noting is that the PMTs' responses feature instructional moves that don't involve telling students what to do (e.g., telling them to reason with other quantities, showing them how to evaluate a difference quotient, or providing them with an alternate problem-solving strategy). Instead, their responses are of a scaffolding nature that resonates with an inquiry pedagogy. This is more likely due to the instruction they've experienced in their coursework (where constructivist-oriented teaching is endorsed) than their experiences in the teaching experiment. The point is, responses of this kind are essential to a practice of teacher noticing if one takes the perspective that learning through inquiry develops mathematical knowledge that is both generative (Carpenter & Lehrer, 1999; Greeno, 1988) and connected (Hiebert & Carpenter, 1992). If it had been the case that the PMTs' proposals for responsive instructional moves had lacked this quality, I would have accounted for its absence in my facilitation schemes (e.g., prompting, highlighting).

**CIRs Related to the Graphical Representation of Instantaneous Speed.** In Session 7, the PMTs watched a video in which students, Kelly and Maria, were shown an animation of a football being thrown as its corresponding graph (distance over time) was traced. They were asked to graph its speed over time (see Figure 10 on p. 86), which Kelly realized was "the derivative" graph. At this point, Kelly and Maria were constructing the graph by finding the speed of the football at times $t = 1, 2, 3, 4$, and 5, and then plotting those speeds as points with which to complete the graph (see Figure 17 below). To approximate the speed at time $t = 1$, they found the change in distance over what they referred to as a "small time interval" from $t = 0.5$ to $t = 1.5$. This gave them an average speed of 1 over that interval. They were aware that their calculations gave an average speed at a particular point, but they say they "don't know about the points in between." This statement (as well as their facial expressions) suggests that they believe their plotted points for speed are correct (they are indeed values of the derivative), but they're concerned that their graph might not be accurate over the entire domain.

**Figure 17**

***Speed-Time Graph Plotted by Students and Their Work***




Mia expressed concern over how Kelly and Maria constructed their graph: "They're assuming that because 1 is the [speed] between 0.5 and 1.5, they'll just put the derivative there." She emphasized that the speed is "over the whole interval" from 0.5 and 1.5, not just at the point $t$ = 1. Leah made the same observation and regarded this as an error in the students' work. Then she proposed a CIR. She suggested that students would develop a "better idea of [speed] if they picked [shorter intervals] of distance and time." I inferred from this instructional response that Leah believes that if Kelly and Maria had calculated the football's speed over increasingly shorter time intervals, they would have seen that the average speed over those intervals comes to approximate the instantaneous speed, which is what they were struggling to represent.

Amelia and Nova noticed these struggles and were aware of the students' concerns about the behavior of the graph between their plotted points. Nova commented that the students "were supposed to find the derivative at a point." Yet, instead of finding the instantaneous speed, "they were thinking about the average speed." Then Amelia added, "[Alyssa] does say it later, 'We have three points, do we actually know what's happening in between each of these points?'"

Nova said that if she were their teacher, she would "bring them back to that." In order to find better approximations of the instantaneous rate of a change at a point, she would offer the same suggestion that Leah offered, which is that they use smaller intervals. The ones the students are currently using are "too large," Nova added.

Mia, Leah, Amelia, and Nova attended to and interpreted the students' thinking about the average and instantaneous rates of change as they sought to construct a derivative, speed-time graph from a given distance-time graph. They then offered CIRs (leveraging specialized content knowledge and knowledge of content and students) to respond to the struggles the students were experiencing as they contemplated how to graphically represent instantaneous speed over an interval, not just at a finite set of points. These suggestions are responsive to the students' thinking about the instantaneous rate of change. Moreover, they build from the graphing approach already taken by the two students, which means they're accessible to the students and likely to be taken up by them.

Gray and Liam's interpretations were somewhat more sophisticated, and this enabled them to propose a CIR that was even more responsive to the students' mathematics. Gray attended to the students' initial sketch of a speed-time graph (see Figure 16) and lamented the fact that they hardly deliberated about the shape of the graph: "Where should it be decreasing? Where should it be positive? Where should it be on the *x*-axis?" In response, he suggested an instructional approach that might meet the students where they are, one that is "less numerical and more just visual… [By] sketching, like looking at the slope." In other words, rather than constructing the graph point-wise by computing several values of the speed, Gray thought it would be useful for Kelly and Maria to take a more global approach by investigating how the slope of the graph changes over time. Liam agreed. He suggests that a more global approach to constructing the graph by "figur[ing] out where there is constant speed, where there is no speed, where there is a change in speed" would give them the "visual" impression of the graph that a collection of points could not. Liam also suggested that the students could have "drawn tangent lines" on the distance-time graph to visualize the changing speed over time.

According to Liam and Gray, and in contrast to the CIRs provided by the other PMTs, a pointwise sketch of the speed-time graph can only provide values of speed at certain points; it cannot convey the covarying relationship between speed and time the way that the array of tangent lines could. These comments leverage Gray and Liam's common content knowledge to assess the students' thinking. Then they leverage specialized content knowledge and knowledge of content and teaching to suggest a method that any student could use when sketching a derivative graph.

As a final point, what's unique about Gray's noticing in this instance is that he interpreted the students' work and then responded, yet he did not feel the need to *describe* the work before doing so. This finding could indicate that as PMTs develop noticing expertise, they may skip over the phase of describing students' actions when doing this work in collaboration with others.

**Learning to Notice through Comparing and Contrasting**

In his section, I demonstrate how the PMTs developed their capacity to notice by *comparing and contrasting* multiple students' mathematical thinking as they sought to respond with instructional moves that they hypothesized would help one or more of those students resolve their misconception or learn the derivative concept more meaningfully.

In Session 2, as the students were contemplating the rate of change in the height of water in a beaker with respect to its increasing volume (see Video 2 in Appendix C), Julian argued that the flow of water into the beaker is constant, so the rate of change of height with respect to volume is constant, as well. Mia attended to his thinking and thought it would be worthwhile to compare it to how he was reasoning in Session 1. She commented, "I think there is a similarity to the last video clip we watched. Julian is focused on how fast the water is being poured." This comment refers to Julian's struggle to understand the rates of change in both Sessions 1 and 2. In Session 1, Julian considered changes in the height of water with respect to time instead of volume. In Session 2, Julian confuses an increasing rate of change with a constant rate. Mia further contrasted his thinking with Alyssa's: "Alyssa was trying to bring [Julian's] attention more to height versus volume… to determine exactly what it is they are trying to figure out." From her comment, I infer from this comment that Alyssa understood that the question called for finding the relationship between height and volume. Mia's attention to the particulars of Julian's thinking across multiple sessions, her interpretations of it, and her comparisons of it to what she had seen earlier enabled her to document changes in Julian's thinking over time and discern a trajectory of his mathematical thinking. As a result of comparing and contrasting the work of the two students (Julian and Alyssa), Mia constructed a more robust model of their thinking that could form the basis of orchestrating instruction that meets the students' specific needs.

At another moment, in Session 3, after the PMTs watched a video in which the students were approximating the speed of a baseball when its photo was taken (see Figure 6 on p. 83), they noticed that the students had "two contradicting ideas." Liam explained, "Alyssa is like, oh, if I look at just this moment, at this pitcher, nothing is happening. The ball is not moving. … whereas Julian is like, well, if I was at a baseball game, and I saw this happening, at that point, the ball is still moving." With these comments, Liam explains that Julian and Alyssa reasoned about the problem differently, and these reasonings are what he relied on to assess and contrast their thinking. Contrasting Julian and Alyssa's ideas seemed to be useful to Liam as he assessed how each of the students approached and solved the problem differently. Although the explicit link to a claim about the development of Liam's noticing is absent here, given the link between teacher knowledge and noticing, I would argue that contrasting two students' thinking leverages Liam's knowledge and thereby provides an opportunity to develop it. Then, any subsequent act of noticing that leverages this newly developed knowledge is an act that is enhanced by that development.

I replayed a video segment that I deemed worthwhile for offering the PMTs an opportunity to notice the contrast between two students' thinking. In the short video segment, Julian responded to Alyssa's argument that the time the ball traveled is zero by explaining to her that since the ball was in midair when the photo was taken, it must have covered some distance in that moment and therefore some time. Mia then described and interpreted Julian's thinking: "This example of a baseball is a real-world example. [Julian] was able to conceptualize and be like, well, I know that the ball has to be moving. It's in midair. From knowing that about the baseball, he was able to work towards a more productive solution." Leah followed up on Mia's interpretation of Julian's thinking: "And [Julian] was convincing [Alyssa], trying to explain why time couldn't be zero… because it's a moving ball." These comments indicate that Mia and Leah realized that it was Julian's contextualization of the problem that enabled him to conclude that since the ball was in motion, the time it took to travel cannot be zero. They also provide a contrast between his thinking and Alyssa's claim that the ball's travel time is zero. Apparently, contrasting students' thinking can offer insights into their reasoning that can support a teacher as they construct models of their students' mathematical thinking.

## Conclusion to Chapter 4

Through the analysis presented here, this chapter responds to Research Question 1, which is concerned with the development of teacher knowledge related to noticing students' mathematical thinking in the derivative domain through video-mediated professional learning. The research intention was to discern how PMTs develop their skills of noticing student thinking by analyzing problem-solving videos in a facilitated, situated, social learning environment. Over eight design experiment sessions, as the PMTs viewed these videos and were supported to notice the problem solvers' thinking, I examined their acts of noticing and inferred the teacher knowledge that was enacted as they attend, interpret, and respond to students' thinking. My purpose in doing so was to construct a trajectory of their learning to notice over the course of the experiment. I concluded from that analysis that the PMTs' noticing became more sophisticated over time, and this conclusion prepared me to propose a theoretical model of teachers' learning to notice through their participation in video-mediated professional learning. I call this model *the four-element framework of learning to notice*. The four elements are PMTs' opportunities to develop their noticing through describing, interpreting, responding, and comparing and contrasting students' mathematical thinking in the domain of the derivative.

The process of *describing* involves explaining, recounting, or detailing aspects of students' mathematical activity. As PMTs describe elements of students' problem solving, they orient themselves to students' discourse (e.g., verbal, gestural, and written expressions), and these descriptions of their observations provide the source material for their subsequent interpretations of that discourse. Thereafter, they are prepared to ponder instructional responses to support the students' learning.

In the earlier sessions of the design experiment, PMTs tended to demonstrate novice skills in describing student thinking. Their responses were often superficial and evaluative and not contingent on the idiosyncratic heuristic and conceptual particulars of the students' in-the-moment actions. However, it wasn't long before they began to draw on their common and specialized content knowledge and their knowledge of content and students to enrich their descriptions of students' strategies using the students' own words to support their claims. Eventually, some of the PMTs no longer felt the need to describe what they noticed about the students' thinking and skipped ahead to offering their interpretations of it.

*Interpreting* is the second component in the four-element model of learning to notice. In order to make sense of the students' mathematical work, the PMTs relied on their descriptions to interpret the meanings of the students' actions. Guided by a disposition to wonder what the students would have to know or not know in order to act as they did, the PMTs sought to determine why they were solving the problem (or not making progress) as they were.

At the beginning of the experiment, PMTs were more likely to misinterpret what the students were thinking. Then, supported by the researchers' prompting and highlighting, they began taking more into account in order to construct more viable interpretations. They considered the design of the tasks that were posed, the requisite mathematical knowledge for solving them, the students' strategies, and the mis/conceptions of that knowledge that were elicited as they carried out their problem solving. Leveraging multiple facets of their mathematical knowledge for teaching (MKT) in the derivative domain enabled the PMTs to construct those more robust interpretations. Importantly, the PMTs' use of these facets of knowledge to interpret student thinking demonstrated the interconnectedness of the enactments of knowledge for teaching and the development of noticing skills. In addition, as they integrated the students' understandings into their own understandings, they became better prepared to propose an instructional response to further support the students' learning of the derivative concepts that were central to the problem they were solving. That their proposals were conceptually relevant to these concepts indicates the PMTs' increasing proficiency in noticing. Indeed, the responses they provided earlier in the experiment were not contingent on their emerging understandings of the students' mathematics. Rather, they were based on general principles of teaching and learning. Thus, I gave the label "Superficial Instructional Responses (SIRs)" to those proposals.

Soon after, the PMTs' responses started to shift from superficial to more substantive ones, the latter of which I labeled "Conceptual Instructional Responses (CIRs)." These refer to conceptual actions proposed by a PMT and based on their constructed model of student thinking. They include CIRs related to the rate of change, to derivatives as slopes of tangent lines, and to the graphical representations of instantaneous speed. As another indication of the interrelationship between teacher knowledge and noticing, the PMTs enacted knowledge of content and teaching (KCT) and specialized content knowledge (SCK) to prepare and propose these CIRs.

Lastly, PMTs developed their capacity to notice by *comparing and contrasting* the thinking of multiple students to refine their models of each student's understanding of the derivative. In the instances where the two students took a different approach to solving the problem or where there were variations in their conceptual understandings, the PMTs were provided an opportunity to nurture their capacity to notice student thinking as they made contrasts and comparisons of the different ways of thinking and solving a problem.

In conclusion, the findings presented in this chapter indicate and explain the broadened scope of PMTs' enactments of knowledge for teaching the derivatives, as well as the progressive changes in the four processes they employed to notice student thinking. Moreover, these findings establish the validity of the four-process framework that emerged from the analysis relative to the hypothesis that PMTs' noticing would be nurtured through their engagement in those processes. Thus, these findings can legitimately be regarded as an effective response to Research Question 1, which seeks to determine how teacher knowledge and noticing in the domain of the derivative can be developed through video-mediated professional learning.

## Chapter 5 Results: Mediators of Knowledge Development

This chapter focuses on the findings for Research Question 2 (RQ2), which asks, *what forms of video-mediated interactions support the development of teacher knowledge specific to noticing students' mathematical thinking in the domain of the derivative?* This question is concerned with developing a theoretical model of how the social interactions and the features of students' problem-solving videos contributed to the development of knowledge related to noticing in a video-mediated professional learning setting. As I explained in Chapter 2, I take a situated (Brown, 1989) and sociocultural (Vygotsky, 1978) perspective on the analysis of data, as these perspectives account for the social, cultural, and historical interactions that mediate a learner's social and cognitive development. As I will demonstrate with these findings, the mediating roles of participant-researcher interactions and the features of the problem-solving videos played a fundamental role in the construction of a model of this development. Below is Figure 11 (also found in Chapter 3 on P. 87) demonstrating the mediating roles of social (e.g., participant-researcher interactions), material (e.g., problem-solving videos), and conceptual resources (e.g., mathematics) in teacher learning.

**Figure 11**

***Three Elements of the Teacher Learning Ecology***

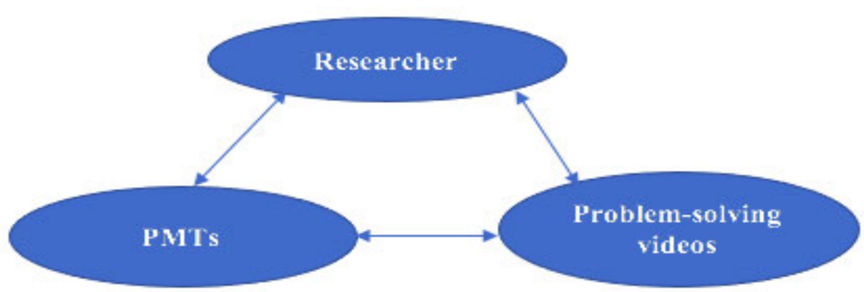


Figure 11 depicts the possible interactions between the elements of the learning ecology in my design experiment.

To respond to RQ2, I took an open coding approach (Strauss & Corbin, 1990) to code the four components of the *focusing framework* (Lobato et al., 2013) as they are provided in Chapter 3. These components include: the centers of focus, focusing interactions, features of the mathematical tasks (which I refer to as features of problem-solving videos), and the nature of mathematical activity (which I refer to as the nature of a noticing activity). Thus, I organize the findings in this chapter according to these four constructs and include descriptions of the nature, processes, and patterns of interactions among them. The descriptions will provide a theoretical basis for how teacher noticing can be supported and achieved. Since these four components are interrelated (Lobato et al., 2013), I coded them concurrently.

I initiated the analysis by looking for *centers of focus* (Lobato et al., 2013). Centers of focus refer to what the participants noticed in the videos as they discussed the students' problem solving. In my presentation of the findings of RQ1 in Chapter 4, I provided detailed accounts of what the participants noticed, although I did not label them 'centers of focus.' I also documented changes in participants' centers of focus (what they noticed), which I attributed to their increased level of participation in the act of noticing (Lobato et al., 2013; van Es, 2011). In response to RQ2, in this chapter I characterize the interactions that mediated the participants' learning to notice and that contributed to the emergence of centers of focus that were presented in Chapter 4.

**Centers of Focus**

Lobato et al. (2013) describe *centers of focus* (CoFs) as the "properties, features, regularities, or conceptual objects that students notice" (p. 814). These are simply the mathematical features that attract students' attention in a mathematics classroom. In taking into account the purpose of this study, I made PMTs the subjects of noticing (rather than students) and I extended the definition of CoFs to include other elements that I expected to emerge as the objects of PMTs' analyses in the design experiment episodes. In the context of this study, therefore, CoFs refer to what PMTs attended to, described, and interpreted, and to the instructional decisions they made based on those interpretations. Among them are the mathematics that underlie the tasks, students' mathematics, features of the problem-solving videos, the design of the problems that structure those videos, and the pedagogy (i.e., the instructional decisions proposed by PMTs to support student learning).

In the first analytic pass of the data, I identified and coded the centers of focus that were the objects of PMTs' noticings that were presented in Chapter 4. As such, determining the conditions that gave rise to the PMTs' development of noticing constituted the objective of the analysis for RQ2. As I conducted open coding, I identified other centers of focus related to task design and pedagogy, and coded them, as well. As I identified and coded these centers of focus, I also coded the corresponding focusing interactions, features of problem-solving videos, and the nature of the noticing activity that contributed to their emergence. By coding these four aspects of noticing together, I was able to construct a conceptual link between them that helped me to understand the mediating interactions that contributed to the development of PMTs' noticing.

In the second analytic pass, I used the constant comparative method (Glaser & Strauss, 1967) to group the codes related to the centers of focus into five categories: describing, interpreting, responding, task design, and the mathematics underlying those tasks. Respectively, these categories relate to PMTs' descriptions of students' work, their interpretations of students' thinking, their responses to student thinking, how the framing of a task influenced students' problem-solving strategies, and the mathematics underlying each task.

Three categories of CoFs (describing, interpreting, and responding) are related to the aspects of students' understanding of the derivative that PMTs modeled. These CoFs are PMTs' inferences about students' thinking and the instructional suggestions they provide in response to these inferences so as to foster their learning. The concept of noticing student mathematical thinking (attending, interpreting, and responding) as presented in the literature review in Chapter 2 underlies this broad category of CoFs. The other two categories of CoFs (the task design and their underlying mathematics) are related to the mediating role of the tasks in eliciting the students' understandings. Table 4 below provides codes and examples for each of these five categories of the broad categories (i.e., super-categories) of centers of focus and mediating resources.

**Table 4**

***Codes and Categories for Centers of Focus (CoFs)***

| Super-categories | Categories | Codes (CoFs) | Examples |
|---|---|---|---|
| CoFs related to student thinking | Describing CoFs | Student misconception | "The problem is asking them about the height and the volume of water in the glass. But Julian's [a student] first response is to talk about the speed at which the water is being poured." |
| | Interpreting CoFs | Cause of a misconception | "I think what makes it hard is that it might have been the animation for them that mess it up. They're still thinking about how fast and slow." |
| | Responding CoFs | Instructional response to misconception | "I would ask, if you're pouring in more water, volume is increasing, right? What does that say about height?"<br><br>"Maybe give a little more detail in the actual asking of the problem. Be more explicit like the rate of change of the height of the water in the cup with respect to the volume of the water in the cup." |

| | | | |
|---|---|---|---|
| CoFs related to mediating resources | Mathematics CoFs | Content: constant rate of change | "In this case, it's increasing. It's the same for every increment of volume. So, for like one increment of volume, we've got two increments of height. The next, one increment of volume, same two increments of height." |
| | Task design CoFs | Suggestion to modify the task | "I think the animation really threw them off. Maybe I would actually have them like take a bottle of water and pour it into a cup like see it for themselves, instead of the animation, because I don't think that was helpful." |

The centers of focus shown in Table 4 were the noticings that emerged through the PMTs' participation in video-mediated professional learning. Because calculus students' problem solving in those videos occurred in the context of the derivative, changes in the PMTs' mathematical knowledge for teaching the derivatives and their noticing of students' derivative thinking (presented in Chapter 4) can be attributed to PMTs' engagement with these centers of focus.

Next, I describe the roles of the three other components of Lobato et al.'s (2013) focusing framework in understanding the development of PMTs' knowledge and noticing in the context of the derivative.

**Focusing Interactions**

Lobato et al. (2013) refer to *focusing interactions* as the "discursive practices (including gesture, diagrams, and talk) that can give rise to particular centers of focus" (p. 814). My analysis in this subsection demonstrates how focusing interactions (like highlighting a moment in the video) contributed to the emergence of CoFs, which are manifestations of socially structured teacher noticing. I describe how I coded for focusing interactions and then how the facilitator and participants' discursive practices contributed to the emergence of CoFs.

In a subsequent analytic pass of the data, I coded for focusing interactions, which I connected to centers of focus that were coded and categorized in prior passes (and appear in Table 4). These connections elucidate enabling conditions that may have supported the development of PMTs' noticing, which I presented in Chapter 4. This analysis confirmed the value of sorting discursive practices that give rise to centers of focus into two categories that appear in the literature: *highlighting* (Goodwin, 1994, as cited in Lobato et al., 2013) and *prompting* (van Es, 2011). I then took a constant comparative approach (Glaser & Strauss, 1967) to sort instances of prompting into three categories: prompting related to (1) describing; (2) interpreting; and (3) responding. I provide evidence next to substantiate the claim that the use of these prompts promoted PMTs' ability to describe, interpret, and respond to the students' observable actions, respectively.

### ***Highlighting and Prompting***

According to Goodwin (1994), *highlighting* "makes specific phenomena in a complex field salient by marking them in some fashion" (p. 606). For Lobato et al. (2013), it refers to "visible operations upon external phenomena, such as labeling and annotating, which can shape the perceptions of others by making particular features prominent" (p. 814). And for Mason (2002), it is a *marking* (p. 33) or *calling out* (p. 64) through verbal, visual, or physical means to make specific features of the domain of scrutiny relevant and more visible. I used *highlighting* in this study to direct PMTs to salient video segments that would draw their attention to the students' spoken and written expressions, thereby making them more prominent.

By *prompting*, I simply mean asking PMTs questions to nurture their noticing ability by deepening their analyses of students' problem solving. In this section, I provide exemplars from the data of the three categories of prompting in order to explain the emergence of CoFs and the evolution of PMTs' noticing. Whenever I highlighted a video segment of a worthwhile moment, I would also use prompts to direct PMTs' attention to students' thinking in that moment. For this reason, instances of highlighting and prompting are presented simultaneously. Thus, the exemplars presented here are of highlighting and prompting relating to (1) describing, (2) interpreting, and (3) responding.

**Highlighting and Prompting Relating to Describing.** Prompts related to describing are those that ask PMTs to recount what students did or how they solved a problem. These prompts were usually posed to PMTs after they watched a highlighted video segment, because those were segments that I believe offered rich opportunities for productive discussions of student thinking. As I aim to demonstrate using this first example, the joint use of highlighting and prompting was critical for directing PMTs to salient moments of student thinking as an opportunity for them to model student thinking and generate knowledge for noticing in the process.

In the following excerpt, which occurred in Session 1 (see Figure 4 in Chapter 3), I highlighted video segments in which Julian explained that the rate at which water was poured into a cup slowed down at the top of the cup, and thus the change in height was decreasing. Then I posed what I regard to be the *essential prompt*, "What do you think [student X] is trying to explain here?":

Facilitator: [Highlighted segment timestamp 1.28-1.38]. So, what is Julian trying to explain here?

Liam: He's saying that, because [the height] was getting slower with the pour, the rate of change for [the height] was decreasing. Like it's slower. That's what I picked up.

Next, I replayed a clip (timestamp 2.18—4.49) in which Alyssa was explaining that for every 1 ml of water added to the cup, the height increases by the same amount. Again, I followed with the essential prompt:

> Facilitator: [Highlighted segment timestamp 2.18-4.49]. So, what do you think Alyssa is trying to explain here?
>
> Liam: She is saying that… if I add more water to it, the volume increases, and that the height of that cylinder, like, where the water line level is, also increases.

I highlighted the video segment and posed these prompts to direct Liam and his partner Gray's attention to the concept of the constant rate of change as it was embedded in Alyssa's explanation. I conjectured that doing so would provide them with an opportunity to deepen their understanding of that concept, since Alyssa's explanation was mathematically accurate. In each of these excerpts, Liam responded to my essential prompt by describing what Julian and Alyssa said, although he did not elaborate on the mathematical meanings embedded in those statements. Thus, mediated by my prompt and by 'rich' video segments that were hypothesized as worthy of the PMTs' attention, *describing* CoFs related to the students' problem solving emerged. Although their potential was not realized in these instances, such centers of focus can offer insights into the students' understanding of the rate of change that would enable the PMTs to model their thinking and propose an instructional response to it.

This same prompt was used in Session 2 in the context of students solving a problem related to an increasing rate of change (see Figure 5 on p. 82). To direct Mia and Leah's attention to Julian's reasoning, I replayed a video segment in which Julian is arguing that the water poured from a pitcher is being poured at a constant rate and thus the rate of change of height with respect to volume is constant. Then I asked the participants, "What is Julian trying to explain here?" Mia *described* Julian's thinking as follows: "Julian is more so talking about the water that's being poured from the pitcher. He's saying that the water is being poured at a constant speed. And therefore, he's equating that to the rate of change of height with respect to volume." As is characteristic of a *describing* CoF, Mia relies almost entirely on Julian's words to form her response. Other instances of a similar phenomenon (*describing* prompts and CoFs) appear in the data for Liam in Session 3 and Nova in Session 4.

I should emphasize that I do not mean to fault Liam or Mia for not elaborating in their responses. I only mean to characterize the nature of a typical noticing response from PMTs early in the design experiment. Liam and Mia did precisely what I asked of them when I welcomed them into the study in their first session: "You will be watching videos of a pair of students solving mathematical problems and then try to analyze their mathematical understanding. I encourage you to think aloud, talk to your partner, share ideas, and so forth." I also reminded them that we could pause and rewind the video in order to "understand what the student is trying to explain." In their responses, Liam and Mia (and the other PMTs, as well) shared what they thought the students were 'trying to explain.'

**Highlighting and Prompting Relating to Interpreting.** Prompts related to interpreting are those that help PMTs assess student thinking and offer evidence for their inferences. In the following excerpt, which occurred in Session 3, Mia and Leah viewed a video (see Video 3 in Appendix C) in which students were tackling a problem and sharing their ideas about the instantaneous rate of change. The two students were shown a photo of a pitcher hitting a baseball and were asked to determine the baseball's speed at the time of the photo (see Figure 6 on p. 83). Alyssa argued that the ball photo was taken in zero time. This excerpt comes from my conversation (as a facilitator) with the PMTs and illustrates how *interpreting* CoFs emerged.

Facilitator 16.09: I would like us to watch some segments of this video and then try to unpack what these students are trying to say. [I played a short video segment to highlight a moment in Alyssa's problem solving]. So, what is Alyssa trying to explain in this case? What do you think she understands, or does not understand, in this case?

Mia 17.35: She's trying to say that she doesn't think the ball is moving at all, because she thinks that instantaneous is representing zero time. So, she's saying if you don't have any time, the ball can't be moving any distance. She doesn't think, consequently, that there can't be any speed of how fast the ball is traveling.

Leah 18.43: She wouldn't understand, like, just what the question's asking. When she hears the word instantaneous, I feel like she thinks it's like the split-second kind of thing. But then we see that she later does, she is thinking about, like, the difference between the distance and the time, and that there needs to be a difference in the time. Because we know that the ball is moving, there's never a point when the ball is not moving until it's like caught.

In this excerpt, I replayed a video segment in order to highlight Alyssa's thinking and prompted the PMTs to share what they understand about Alyssa's conception of the mathematics. The prompts served multiple purposes: asking the PMTs to (1) describe what Alyssa was saying, (2) explain the mathematics that is fundamental to it, and (3) determine from their interpretations what she does or does not understand. In response to these prompts, both Mia and Leah offered their assessments of Alyssa's understanding. Their assessments were imperfect, but nonetheless, they were still grounded in Alyssa's discourse. The more important point is that *interpreting* CoFs emerged for Mia and Leah in response to *prompts relating to both describing and interpreting*. In Mia and Leah's responses, they did not describe what students did. Rather, they began by interpreting what students were thinking, a trend that emerged as participants continued to participate in noticing activities. These findings lend further credence to the claim that Mia and Leah's movement along this trajectory is attributable in part to the researcher's focusing of the PMTs' attention to worthwhile moments in the students' problem solving. It should also be stated that by Session 3, the moments hypothesized by the researcher-facilitator as worthwhile are emerging as actually worthwhile. Other instances of this same phenomenon appear in the data for Amelia and Nova in Session 5, and for Mia and Leah in Session 6.

In the cases above, I paired *interpreting* prompts with highlighted video segments in order to generate opportunities for interpreting on the part of PMTs. At other points in the experiment, I paired *interpreting* prompts with brief encapsulations of moments in the students' problem solving. On some of these occasions, I replayed the video segment associated with an encapsulation. For example, in Session 6 with Amelia and Nova, I sought to deepen their noticing of Julian and Alyssa's thinking, so I provided them with a summary of what Julian and Alyssa had discussed:

> Going by Alyssa's definition, a tangent line touches the curve at only one point. Julian seems unsure about that definition in the case of Point *A*, where multiple tangent lines could be drawn to touch that point. So, he's like, 'Are all these tangent lines?' And that's what the question or rather the problem is. Let's focus on how they are finding a tangent line at each of the points and then we'll talk about it.

My objective in focusing the PMTs on this aspect of Julian and Alyssa's activity was to explore how their definition of a tangent line informed their noticing actions. This would allow me to assess the knowledge that Amelia and Nova could bring to their interpretations. I replayed video excerpts at each instance where Julian and Alyssa were drawing a tangent line at a different point on a curve and then asked Amelia and Nova, "What do you think about their reasoning there?" In response, Nova pointed out that Julian had asked, "But what is a tangent line?" Her subsequent interpretations suggest her conclusion that Alyssa did not possess a conceptual understanding that tangent lines represent the slope on a curve at a particular point: "Drawing lines doesn't mean she knows why she's drawing them... She can't make a connection."

Nova added that Alyssa did not respond to Julian’s earlier question about a tangent line, nor did she respond to one he posed later about whether a tangent to a graph at one point could also intersect the graph at other points: "What about if later on in the graph it touches it?" For Nova, these questions were "valid… A teacher would have to explain that so that misconceptions aren't there." With these comments, Nova suggests that students' misconceptions and their inability to respond to conceptual questions stem from their exclusively procedural understanding of mathematics (i.e., knowing *how* to draw a tangent line but not knowing “why she’s drawing them.”).

Amelia had also suggested that the students’ struggles with tangent lines arose “from what they defined a tangent line to be.” She pointed out that whereas Alyssa defined a tangent line as “a line that touches a curve at only one point… Julian drew multiple lines at Point *A* and then asked her, ‘Don’t these other lines work, as well?’” Just as Nova had done a moment earlier, Amelia uses Julian’s own words as evidence to substantiate her analysis. By using Julian’s own words as evidence from the video to support their ‘interpreting’ claims, Nova and Amelia’s participation in noticing is regarded as more sophisticated than noticing acts consisting of interpretations without evidence (and these are regarded as more sophisticated than acts consisting solely of ‘describing’).

By pairing interpreting prompts with highlighted video segments and capturing brief moments of student thinking that correspond to those segments, PMTs have more opportunities to assess student thinking and generate interpreting CoFs. In addition, providing the PMTs with a brief summary of their discussions serves to bring their ideas together and help them develop a connected understanding of the students' thinking. The pairing of prompts with these summaries gives PMTs an opportunity to further interpret students' thinking based on their shared ideas and understandings. It appears that combining prompts with video segments or brief summaries can help PMTs pay selective attention to significant aspects of student thinking and engage in knowledge-based reasoning to model it.

**Highlighting and Prompting Relating to Responding.** Prompts related to responding are those that direct the participants to determine an instructional move in response to their in-the-moment understanding of a student's understanding as it was elicited in their problem solving. Determining a responsive move entails reflecting on a student's current understanding and making a conjecture about the kind of learning experience that could revise or advance that understanding. It is necessary to leverage one's knowledge for teaching mathematics in order to construct viable conjectures and enacting this knowledge in novel conceptual situations may also develop that knowledge further (Ball et al., 2008).

The excerpt below provides an example of a *responding* prompt and the CoFs that emerged from it. This excerpt comes from Session 6, in which the PMTs viewed a video of students drawing tangent lines to various points on a graph and justifying their mathematical actions. I had just replayed a video segment in which Julian is contemplating the meaning of a tangent line by considering the case of Point *C* (see Figure 13 on p. 111). As he did so, he asked Alyssa, "Wouldn't it touch the graph later on down here, and wouldn't it touch it later on up here?" Then I used a responding prompt to maintain the focus of their noticing on Julian's thinking and provoke them to contemplate how they would respond to him.

Facilitator 40:59: So, what could you do to help these students understand the concept of a tangent line? What strategies would you employ to help in their learning about a tangent line?

Amelia 41:32: I would probably want to stop [them] and [instead] start with a simple kind of parabola. You know, having points on it and getting tangent lines at each point or like some class activity that has [students] do that. And then, like, class discussion about what's happening at each point, and why the tangent line at each point is different to kind of bring back that idea. And then maybe we could then go back to introducing these weird instances, like, when there's a cusp or absolute value or something like that.

Amelia 42:37: I also think it would be important to go back and just make sure they understood exactly what a tangent line is. Because that seems like prior knowledge that's needed for this lesson.

The *responding* prompt I posed to Amelia gave her a chance to imagine instructional moves she believed would enhance the students' learning, thereby leading to the emergence of *responding* CoFs. In particular, the prompt directed Amelia's attention to the students' struggles with tangent lines and called on her to respond to them by proposing a pedagogical move that could help them gain a better understanding. Evident in Amelia's responses is her appreciation for the need to understand what a tangent line is in order to apply that understanding in a graphing context. The activity she suggests would have the students explore and contrast tangent lines drawn at a variety of points on a parabola. Doing so, Amelia proposes, would enrich their understanding of a tangent line, which she suggests is too narrow or superficial to solve the problem they're working on. Thus, the *responding* highlights and prompts led to the emergence of a *responding* CoF for Amelia, which centered on the design of an exploratory activity conjectured to develop the students' thinking about tangent lines. The richness of Amelia's proposed instructional response is worth foregrounding, even though that richness cannot be attributed directly to the *responding* highlights and prompts that actualized its emergence. I would argue that common content knowledge, knowledge of content and students, and knowledge of an inquiry pedagogy are among the forms of knowledge enacted in Amelia's response. Thus, this finding speaks to the generative power of opportunities to respond to student thinking for developing one's capacity for noticing, and to highlighting and prompting as invitations to engage in such opportunities.

I used a similar responding prompt back in Session 1 with Gray and Liam. At this moment, the two students in the video were asked to examine how height changed with respect to volume as water was being poured into a cup. Julian, one of the students, analyzed the rate of change in height with respect to time rather than volume. Gray and Liam interpreted his thinking to be a misconception. Below is the conversation that prompted their suggestions of instructional moves that could resolve what they deemed to be a misconception.

Facilitator: Considering how the students are solving the problem, if they were your students in your class, what strategies would you use to support them to understand the concept of the rate of change of height with respect to volume?

Gray: I would probably just walk over and underline 'with respect to volume' in the problem. I would just point at that and be like, I want you to, like, to read the question again.

Liam: I would point out that [Julian] has a good analysis. I would clarify, you're analyzing both volume and height with respect to time. Now, can you analyze them with respect to each other? I would definitely start out by clarifying what he just did out loud, to hopefully make him realize that he's analyzing with respect to time, and now he has to read the question again, and see that he needs to compare the two [variables].

The *responding* prompts I posed allowed the participants to imagine how they might step in and support Julian based on their assessments of his understandings. Through these prompts, the PMTs are given a real opportunity to contemplate a viable instructional response to a student's mathematics. Eventually, they did provide instructional suggestions, and thus *responding* CoFs (i.e., restating the problem, validating the student's work) emerged. That said, the responses they provided featured generic teaching moves that were not responsive to Julian's thinking. Instead, they hoped to help Julian dismiss his current approach as they suggested one *they* deemed more viable. This is in contrast to the quality of the responsive moves that appear in the previous example, which occurred later in the experiment. I suggest that the difference in sophistication between the two responses is evidence of the mediating role of CoF prompts and highlights in supporting the development of the PMTs' noticing. In addition, I note that participation in this responsive exercise was distributed across the two PMTs, as Liam's response consists of an implicit agreement with Gray's response as well as an elaboration of it. This moment was not unique, as intersubjective meaning making through collaboration is an essential aspect of situated learning. What's significant about it is that it offers an example of the mediating role of social resources – in the form of collaborative analysis within each pair of PMTs – in the development of noticing and of the knowledge that the practice entails.

**Features of Problem-Solving Videos**

This study aims to shed light on how videos of students' problem solving can support PMTs in developing skills for noticing student mathematical thinking and knowledge useful in the teaching of derivative. So, in the first analytic pass through the data, I examined how the features of problem-solving videos were associated with changes in centers of focus. This involved looking for moments in problem-solving videos that were conjectured to invite and sustain PMTs' participation in analyses of student thinking.

During the design of the study, I had not considered features of the mathematical tasks in these videos to be the objects of PMTs' analyses during the episodes. My focus was only on students' discussions as they solved those problems. However, PMTs attended to features of the tasks and argued that the students' strategies could be attributed to these features. In addition, in the course of the study, I came to realize that the PMTs' content knowledge was constraining their noticing. In response, I would discuss the mathematical tasks with them to understand the mathematics relevant to it in order to enable the PMTs to attend to and make sense of the problem solvers' mathematical thinking. Essentially, the videos and the tasks that appear in them were "springboards" for analyzing and discussing mathematics teaching and learning (Borko, 2011, p. 184). In Chapter 3, some of the features of these videos are described in greater detail.

As my analysis will demonstrate, the result of these efforts was that PMTs became more engaged in the collective noticing activity because they found new opportunities in the problem-solving videos to do so. Next, I share some of these moments and explain their contributions to the emergence of CoFs.

***Analyzing Students' Problem Solving as an Opportunity for Decentering***

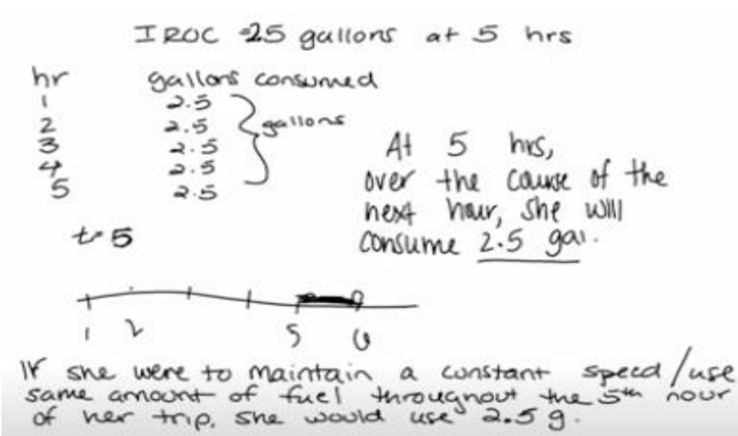
In this section I describe how the problem-solving videos offered a venue in which PMTs could increase their content knowledge through discussions of the mathematics underlying the tasks, as well as their pedagogical content knowledge (PCK) through discussion of students' problem solving. Next, I use the concept of *decentering* to elucidate how PMTs considered the perspectives of the students as they analyzed and discussed their problem solving independent of how they themselves would think about it. Piaget (1955) introduced the concept of *decentering* to refer to an act of contemplating a child's ideas from the child's perspective and not their own. I also give examples from the data to corroborate the claims about the learning potential enabled by these videos.

Each of the videos analyzed by the PMTs features a task and presents the students' problem solving for that task. In their analysis, PMTs could attend to the students' problem-solving strategies and make inferences about the students' understanding from those strategies. As such, their noticing activity is akin to what a teacher might do as they analyzed their students' problem solving and their explanations about it. Piaget (1955) used the idea of decentering in this sense. Decentering is about attempting to understand the actions, thoughts, and perspectives of another irrespective of one's own actions, thoughts, and perspectives.

As the PMTs engaged in noticing the students' mathematical thinking, they set aside their own understanding of the derivative in order to model the students' understanding of the concept (i.e., by decentering). For example, as the students were interpreting a derivative problem in Session 8, Gray commented, "They are misunderstanding [the expression $f'(5) = 2.5$] for some sort of average rate of change. They think that 2.5 is the number of gallons being consumed per hour [at every hour]. From that table (see Figure 18 below), that is what they are showing." This is just one of the many instances in which a PMT decentered by assuming the point of view of a student in order to interpret and then characterize that student's understanding of some derivative concept. Consequently, through decentering the PMTs further developed their knowledge for teaching.

***Figure 18***

***Students' Slope Calculations and Plotted Points***



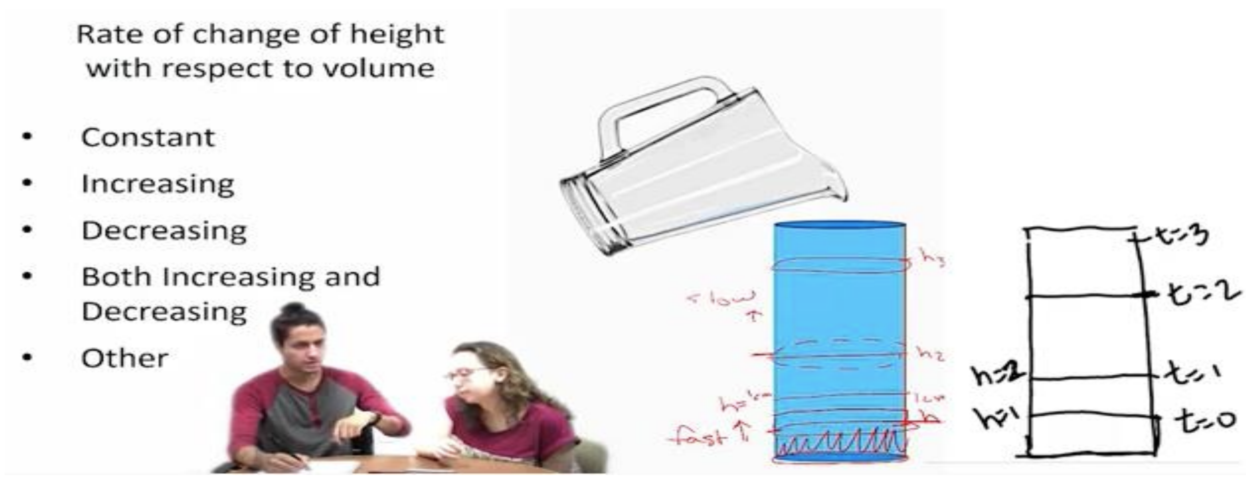

On other occasions, the PMTs analyzed a posed problem, determined the mathematics relevant to it, and shared a first-order model (Steffe & Thompson, 2000) of their own understandings of that mathematics. For example, as the PMTs analyzed student work associated with finding the instantaneous speed of a ball in Session 3 (see Video 3 in Appendix C), Mia explained how she would solve the problem: “When I think about rates of change, I think about, like, an initial point and a final point. What I would do is I would take, like, the whole length and then the starting point and the ending and do something with it.” In this instance, Mia has taken a personal perspective (as a mathematician) as opposed to a student’s to share the strategy she would take if she were the problem solver. As I have argued in Chapter 4, the PMTs' conceptions of mathematics played a critical role in their noticing. Operating from a personal perspective is also an opportunity for generative learning (Franke et al., 2001) as one reflects on their own understanding in the context of a problem that is new to them. On this point, Teuscher et al. (2016) argue that decentering and the construction of both first and second-order models (Steffe & Thompson, 2000) is essential to developing knowledge for teaching. In this study, videos of students’ problem solving offered PMTs that opportunity for professional learning.

***The Videos are Effectively Curated***

The problem-solving videos used in the study are short, with an average duration of 3 to 5 minutes. Each one shows two student problem solvers, the mathematical task that has been posed to them, and written artifacts of their work. The background is white, as all other elements of the context in which they are doing their problem solving have been erased. As the pair of students solves each problem, we are able to observe their discussions. By editing the videos to present what their authors regarded as only the most essential elements of students’ problem-solving activity, viewers of the videos are not distracted by elements deemed irrelevant to their problem solving. This editing made these videos useful for the purposes of this study, as well. In tandem with the researcher-facilitator’s uses of highlighting and prompting, the characteristic sparseness of the video contexts approximates practice (Howell & Mikeska, 2021) by constraining the focus of PMTs' attention to students and their problem solving so that the students' mathematics becomes the main subject of investigation. None of the other elements one is likely to encounter in the "blooming, buzzing confusion" (Brown, 1992, p. 141) of a typical classroom offer opportunities for distraction. The screenshot in Figure 19 is typical of what appears in these videos.

**Figure 19**

***Students' Reasoning About Constant Rate of Change***




***The Videos can be Paused and Re-viewed Repeatedly***

As we viewed a video within each session, I was able to pause it and allow PMTs time to reflect on what they observed. I was also able to replay video segments several times so that the PMTs could do their best to understand the students' mathematical activity. During Session 2, for instance, the PMTs analyzed a video of students discussing the increasing rate of height with respect to volume (see Video 2 in Appendix C). I replayed a short clip in which Alyssa explained that since the flask of water gets narrower with height as each addition 1 ml of water is added to it, the change in height increases (see Figure 15 for her illustration). PMT Leah was unable to understand Alyssa's thinking, especially when Alyssa was using specific values to demonstrate the increasing rate of change in height. As a result of replaying the video segment two more times, both Mia and Leah were able to provide their interpretations of Alyssa's reasoning about an increasing rate of change. This finding substantiates prior research (e.g., Sherin, 2004; Wilson et al., 2011) that finds that professional learning of noticing can be enhanced by problem-solving videos that can be paused and re-viewed repeatedly.

***The Videos Provide a Comprehensive and Appropriate Scope and Sequence***

The Calculus I topics that appear in the videos are organized sequentially based on how they build conceptually on each other. They range from the rate of change to the interpretation of derivatives. Eight of these videos were selected for analysis and were presented in the prescribed order. I deemed these worthwhile, particularly in terms of the sequenced building up of topics and the opportunities they presented for noticing. Every PMT participated in eight experiment sessions, which enabled them to cover the entire conceptual scope of the derivative at the undergraduate level. The longitudinal nature of this professional learning experience satisfies calls from research for PD that is sustained over time (Garet et al., 2016).

The order of topics in these videos reproduces that standard order that appears in conventional Calculus I textbooks. This sequencing proved helpful for developing PMTs' understandings of the mathematics associated with each problem beyond their current understanding and also for making new conceptual connections among them. As a result, the PMTs developed content knowledge that enabled them to increase the sophistication of their participation in noticing.

For example, in Session 1, the participants discussed students' problem solving related to the constant rate of change (see Figure 4 on p. 81). Then they leveraged elements of these discussions and the ideas that they discussed in this session in their analyses of students' problem solving related to an increasing rate of change in Session 2 (see Figure 5 on p. 82). After Alyssa made her argument in Session 2 that there was an increasing rate of change in height with respect to volume in the case of the narrowing beaker, her partner, Nova, elaborated: "I want to make a concluding sentence. You know, the rate of change of height with respect to volume is increasing at an increasing rate." With this statement, Nova enacted her understandings of both the *rate of change* and the *change in the rate of change*, concepts associated with the first and second derivatives in calculus. She continued, "I know since the original beaker [in Session 1] was a cylinder, everything was fine because it's straight all the way through. But this one, since it does have like a wide part at the bottom and then it gets narrower, I do feel like it does have an impact on [the rate of change]." Nova's analysis conveys an understanding of the differential impact on the rates of change of the shapes of beakers in the two tasks. In Session 1, the cylinder is "straight and the same all through." In Session 2, near the top of the beaker "is a smaller space." For Nova, this explains why, in Session 1, the rate of change in height with respect to volume was constant, while in this Session 2, it was increasing. This example illustrates how the sequencing of topics in the videos could mediate enactments of PMTs' mathematics to support the development of their noticing.

## Concluding Chapter 5

This study examined how PMTs develop teacher knowledge related to noticing students' mathematical thinking in addition to the design of a video-mediated professional learning environment that can facilitate this development. Since this study aimed to engineer and support teacher learning through collaborative participation in a social learning context, I applied sociocultural and situated learning perspectives to the analysis of the data. This chapter addressed Research Question 2, which concerns the nature of interactions that help PMTs learn to notice. This question is a follow-up to Research Question 1 (addressed in Chapter 4), which seeks to explain how PMTs' noticing developed across the design experiment sessions.

Findings presented in this chapter describe the nature of interactions that appeared to have a mediating role in the development of PMTs' professional noticing. I drew upon the focusing framework (Lobato et al., 2013) to respond to Research Question 2. This framework provides the social and situated conceptualization of professional noticing and, along with research that uses this framework, offer a means to analyze it. To understand how the PMTs in this study developed their noticing, I analyzed their enactments of the components of the focusing framework (i.e., describing, interpreting, responding) using the principles of grounded theory (Strauss & Corbin, 1990). This approach enabled me to identify mediators of knowledge development in the data. Then, using constant comparative analysis (Glaser and Strauss, 1967), I collapsed the coded components into two super-categories of CoFs (see Table 4 on p. 147): (1) CoFs related to student thinking and, (2) CoFs related to mediating resources.

These super-categories of CoFs represent what PMTs noticed, including the students' understanding (or lack thereof) of the derivative, the forms of knowledge they enacted in their noticings, proposals for how they would support student learning, and resources that mediated students' learning during problem solving (e.g., concepts relevant to tasks). Other components of developing noticing that were analyzed (e.g., focusing interactions, features of problem-solving videos) accounted for how PMTs developed these CoFs through their participation in the experiment sessions.

Two categories of social interactions that mediated PMTs' noticing of student thinking emerged from the analysis of the focusing interactions: *highlighting* and *prompting*. In my role as facilitator of their professional learning, I would highlight video segments for PMTs to focus on that would provide them with salient noticing opportunities. As a follow-up to highlighting, prompts are categorized according to the CoFs that they intended to generate: (1) prompts relating to describing CoFs, (2) prompts relating to interpreting CoFs, and (3) prompts relating to responding CoFs. As demonstrated through the results presented in this chapter, this variety of prompts facilitated social interactions that lead to the emergence of CoFs.

This chapter also presented four features of problem-solving videos that were found to mediate the PMTs' noticing and the emergent CoFs: (1) The videos offer images of students' problem solving that could serve as an opportunity for decentering; (2) They are effectively curated; (3) They can be paused and re-viewed repeatedly, and (4) They provide a comprehensive and appropriate scope and sequence of derivative topics. Through their noticing activity in contexts offered to them by eight videos featuring these four properties, the PMTs' existing networks of knowledge for teaching the derivative were elicited and newly developed knowledge was integrated into these existing networks to restructure (Franke & Kazemi, 2001) it. As a result, their capacity to notice student thinking was developed.

# Chapter 6: Conclusion of the Study

## Introduction

This study aimed to: (1) investigate how teacher knowledge related to noticing students' mathematical thinking develops and construct a model of how teachers can learn to notice, and (2) examine the nature of interactions that can support teacher learning in a video-mediated professional learning environment. The following research questions guided the inquiry: (1) *How does teacher knowledge specific to noticing students' mathematical thinking in the domain of the derivative develop through video-mediated professional learning?* (2) *What forms of video-mediated interactions support this development?*

In order to discern how PMTs developed knowledge for noticing and teaching the derivative, my response to Research Question 1 depicts the ways in which knowledge about teaching the derivative manifested itself in PMTs' discussions of students' problem solving and how they leveraged that knowledge to model students' mathematical thinking. My response to Research Question 2 depicts how video-mediated interactions between the participants, the researcher, and the problem-solving videos supported this development. I draw on these interactions to construct a theory of how a video-mediated professional learning context can be used to develop teachers' professional noticing.

## Summary of the Findings

This study investigated how PMTs enacted their knowledge for teaching the derivatives through processes that supported them to model and learn to notice students' mathematical thinking in a video-mediated professional learning environment. The study further examined the interactions constituent to these processes. Using a design experiment methodology, data was collected from preservice secondary mathematics teachers (PMTs) through semi-structured, noticing interviews as they analyzed videos of calculus students' problem-solving. That data was analyzed from situated (Brown, 1989) and sociocultural (Vygotsky, 1978) perspectives that seek to understand how social, material, cultural, intellectual, and historical interactions shape a learner's social and cognitive development. In addition, the teachers' professional noticing framework (Jacobs et al., 2010) was used to structure the PMTs' noticing experiences and orchestrate a productive discourse about them (as they attended to, interpreted, and responded to students' thinking).

In this concluding chapter, I summarize the findings in relation to Research Questions 1 (presented in Chapter 4) and 2 (presented in Chapter 5), establish their significance, and propose implications for research and practice. These are listed in Table 5. In Chapter 4, I presented a *four-element framework* I constructed through my analysis of the data. This framework illustrates how knowledge for teaching the derivative and how skills of professional noticing developed in a video- and socially mediated learning environment. In Chapter 5, I shared the social, material, and conceptual resources that mediated the PMTs' learning of teacher knowledge (related to noticing student thinking).

**Table 5**

***Summary of the Findings for Research Questions 1 and 2***

| **Research Question** | **Findings** |
| --- | --- |

| | |
|---|---|
| RQ 1: Chapter 4: Trajectories of knowledge development | Four-element framework<br>• Includes 1) describing, 2) interpreting, 3) responding, and 4) comparing and contrasting<br>Trajectories of knowledge development<br>• PMTs come into the experiment with less sophisticated noticing skills.<br>• PMTs increase their ability to analyze students' problem solving over time.<br>• PMTs leverage content knowledge to notice student thinking.<br>The skills of noticing (attending to, interpreting, and responding) are interconnected. |
| RQ 2: Chapter 5: Mediators of knowledge development | Centers of focus (CoFs)<br>• CoFs related to student thinking<br>• CoFs related to mediating resources |
| | Focusing Interactions<br>• Highlighting and prompting |
| | Features of problem-solving videos<br>1. The videos offer images of students' problem solving that could serve as an opportunity for decentering.<br>2. They are effectively curated.<br>3. They can be paused and re-viewed repeatedly.<br>4. They provide a comprehensive and appropriate scope and sequence of derivative topics. |

To be specific, in Chapter 4, I respond to Research Question 1 by describing the processes through which PMTs enacted teacher knowledge and developed their capacity for noticing in terms of a *four-element framework for learning to notice* through 1) *describing,* 2) *interpreting,* 3) *responding,* and 4) *comparing and contrasting.* This framework proved useful for understanding how PMTs' skills for noticing student thinking in a video-mediated professional learning context could be developed. Specific to this context, it illuminates an approach to helping PMTs learn to attend to students' conceptions of mathematics as they are elicited in real-time, helping them learn to assess and characterize those conceptions, and helping them to learn to respond to those assessments by devising instructional strategies to further support student learning. Also included in this chapter are vignettes from the data that depict the trajectories of PMTs' learning to notice student thinking through their problem solving across the episodes of the design experiment.

In Chapter 5 I respond to Research Question 2 by discussing the social, material, and conceptual resources that contributed to the emergence of centers of focus for the PMTs' noticing. Next, I provided a summary of my findings in relation to each of the two research questions.

***Responding to Research Question 1: Trajectories of Knowledge Development***

To respond to RQ1, I present a four-element framework for learning to notice. These elements are the processes of describing, interpreting, responding, and comparing and contrasting. By describing, the PMTs gave the details of features of the students' problem-solving strategies and their discourse. In so doing, their engagement with the students' problem-solving then enabled them to interpret and respond to it. In the early sessions, PMTs had novice describing skills, but their sophistication increased as they participated in the noticing activity. In the later experiments, they leveraged mathematical knowledge for teaching (e.g., KCS, CCK) to provide evidence-based descriptions that were more substantial and sophisticated than earlier descriptions, which were more shallow and superficial. Eventually, the PMTs offered their interpretations of students' thinking without even bothering to describe it. In Chapter 4, I argued that this behavior – this lack of a felt need to provide the sources of the PMTs' interpretations in the students' problem solving – constituted advanced noticing and was evidence of its development.

The PMTs used *interpreting* to make justified, evidence-based inferences about student thinking from their problem-solving activity. Although they entered the experiment with rudimentary skills for interpreting student thinking, they accumulated experiences that nurtured their tendency to leverage and apply their knowledge for teaching mathematics. These experiences included analyzing students' problem-solving discourse, strategies, and work artifacts; assessing their mis/conceptions of the derivative; and examining the designs of the tasks that were posed and the mathematics behind them. By taking up these participatory opportunities, PMTs developed robust interpretations of students' thinking, which then became the basis for responding to it.

The PMTs restructured (Wilson et al., 2011) their knowledge for teaching as they described and interpreted students' thinking. Then they leveraged this knowledge to suggest instructional moves that would take students where they area and help them learn. Proposing instructional moves in response to what one notices about student thinking is called *responding*, the third component of the four-element framework.

The responses PMTs proposed earlier in the experiment were based on general principles associated with teaching and learning, and not on students' elicited thinking about derivatives and their associated concepts. Responses like these are what I call *Superficial Instructional Responses* (SIRs). Later on, in contrast, they enacted *Conceptual Instructional Responses* (CIRs). CIRs are suggestions that are specific to a student's thinking and that are grounded in a constructed model of that student's thinking. The CIRs that were identified in the analysis are related to the rate of change, derivatives as slopes of the tangent lines, and the graphical representations of instantaneous speed. As they suggested these conceptual actions, the PMTs enacted knowledge of content and teaching (KCT) and specialized content knowledge (SCK) in regard to the students' problem solving and the task they were posed.

The fourth process of the learning to notice framework involves *comparing and contrasting* students' thinking to illuminate similarities and differences between them in the process of modeling their understanding of derivatives. For example, in the case where two students solved a problem using different strategies, PMTs could compare and contrast their strategies and the inferences they made from them to build more robust models of each student's thinking. Consequently, comparing and contrasting provides PMTs with an opportunity to develop their knowledge for teaching and thereby develop their ability to notice.

**Evidence of Increasing Noticing Skills.** As the analysis demonstrates, PMTs entered the experiment with less sophisticated skills for noticing. Their initial assessments of the students' mathematical thinking and their initial reasonings about the phenomena they observed were rather shallow. Similar results have been reported in other studies (e.g., Jacobs et al., 2010; van Es, 2011). Nevertheless, the results also indicate that as the PMTs applied their conceptual and pedagogical knowledge to interpret and respond to students' thinking, the extent of their analyses increased over time as they moved toward more central participation (Lave, 1991) in noticing. Vignettes that offer evidence of PMTs' movement along a trajectory of noticing student thinking about the derivative are provided in Chapter 4. The social, material, and conceptual resources that mediated them are discussed in Chapter 5.

At one point along their trajectory of participation, the PMTs began to attribute features of the students' interpretations of the problems they were posed and their strategies for solving them to the design of those problems. This was the case for their interpretation of (student)

Julian's problem solving and his preoccupation with the rate at which water was being poured into a container at the expense of the need for him to consider the covarying relationship between the height of the water and its volume, as the task required. As the PMTs noticed Julian's thinking, they attributed his confusion to the design of the animation accompanying the task and not to any qualities of Julian's mathematical knowledge. This finding of teachers' attributions of features of problem-solving activity to the design of the problem is consistent with Lobato et al. (2013). What is significant here is that the videos offered PMTs opportunities to come to this way of thinking and realize that a key consideration in making informed instructional decisions is ensuring the appropriateness of the mathematical tasks that appear in the curriculum and understanding the interactions between them and the ways that students go about solving them.

Another finding concerns the role of common and specialized content knowledge in noticing student thinking. This was evident for (PMT) Mia, who initially struggled to understand (student) Alyssa's argument about a constant rate of change in Session 1. It wasn't until she leveraged her own knowledge of the derivative to understand an *increasing* rate of change in the context of another problem in Session 2 that she was able to make sense of that argument. This interaction between knowledge and noticing was evident at one point for (PMT) Amelia, as well. In Session 6, she observed students' struggles to draw tangent lines, but her lack of knowledge about the concept of nondifferentiability at a cusp prevented her from interpreting and thus responding to those struggles. Other studies have yielded similar findings regarding the relationship between noticing competence and knowledge for teaching (e.g., Jacobs et al., 2010; Magiera et al., 2013; Thomas et al., 2017). Thus, it was a benefit to PMTs to come to this realization through their participation in this professional learning experience and be supported by a knowledgeable facilitator to negotiate it.

Professional noticing occurs when enacting mathematical knowledge for teaching results in sophisticated descriptions, interpretations, and responses to students' mathematical thinking. However, as the results presented in Chapter 5 demonstrate, acquiring this (static) knowledge is necessary, but not sufficient for PMTs to enact (i.e., participate in) professional noticing. They must be provided with authentic opportunities to leverage this knowledge in order to cultivate their capacity to enact it. Their researcher-facilitated engagement with the curated selection of problem-solving videos provided them with those opportunities. I say more about the features of these videos in the next section.

The findings also revealed the interconnected nature of the noticing skills (i.e., attending, interpreting, and responding; Jacobs et al., 2010), and therefore the need to offer professional learning opportunities that assume these interconnections so as to develop them in tandem. Attending to and describing the actions of students generates an authentic instructional opportunity to interpret those actions (i.e., make inferences about them) and respond accordingly and appropriately. The results show that the PMTs were provided with these opportunities. In the many instances in which PMTs attended to features of students' problem solving and interpreted their understanding of the derivative in light of their actions, the PMTs proposed conceptual actions designed to foster those understandings. The quality of their decision making about these teaching strategies relied on the viability of their descriptions and interpretations of students' understandings.

***Responding to Research Question 2: Mediators of Knowledge Development***

To respond to RQ2, I explored the shifts in PMTs' enactments of knowledge (RQ1, presented in Chapter 4) and their development of noticing skills, as well as how elements of the learning ecology contributed to those shifts. I examined what PMTs noticed as well as the discursive practices that elicited them as they viewed the problem-solving videos. I used Lobato et al.'s (2013) focusing framework to support my analysis of what PMTs attended to and how the social and conceptual components of their noticing activity revealed their knowledge and supported their learning. That framework also helped me conceptualize a mediating relationship between the instances of student thinking the PMTs observed (i.e., their centers of focus) and what conditions enabled them to do so. Next, I elaborate on the components of the focusing framework and how they mediated PMTs' learning of noticing.

**The Centers of Focus.** I identified two super-categories of centers of focus (CoFs) that seemed to capture PMTs' attention during the design experiments: (1) CoFs related to student thinking and (2) CoFs related to mediating resources. As PMTs notice CoFs that are new to them, they develop their knowledge about noticing and mathematics teaching. Discussions around the observed CoFs and instructional decisions provided an opportunity for the PMTs to enact mathematical and pedagogical knowledge pertaining to the teaching of the derivative.

The CoFs related to student thinking are divided into three categories: *describing* CoFs, *interpreting* CoFs, and *responding* CoFs. These categories of CoFs emerged as PMTs described, interpreted, and responded to moments of student thinking. The PMTs further pointed to the design of the problems that students were given and to the mathematics that underlies those problems as mediating resources for students' discourse while solving problems. These became CoFs for the PMTs' noticing discourse throughout the experiment. As I elaborated in Chapter 4, task design informs the mathematics that students discuss and as well as the strategies they devise to solve those tasks. As such, the CoFs represent the knowledge that PMTs developed about teaching mathematics through the experiment episodes. In addition, the focusing interactions, features of the problem-solving videos, and the nature of the noticing activity contributed to the emergence of CoFs (seen as manifestations of teacher knowledge). They shed light on how participants' noticing was socially orchestrated and situated within the context of analyzing students' problem solving. I discuss the *focusing interactions* next.

**Focusing Interactions**. When PMTs were asked to view a selected video in its entirety in the earlier episodes, essential prompts (e.g., What do you notice?) elicited general strategy descriptions and evaluative comments without evidence from the videos. This finding is consistent with other research that has reported similar findings when general prompts were used in video clubs (e.g., Sun & van Es, 2015, van Es, 2011). To support their learning to notice, more specific prompts were used by the researcher to help the PMTs get better at describing the students' strategies, making inferences about their understandings, and deciding how to respond. Three categories of highlighting (of videos and specific student thinking) and prompting were implemented to support this learning: (1) highlighting and prompting relating to describing CoFs, (2) highlighting and prompting relating to interpreting CoFs, and (3) highlighting and prompting relating to responding CoFs.

Highlighting and prompting were coupled for the purpose of directing the PMTs' attention to moments in replayed video segments where student thinking was salient with respect to the students' problem solving. These prompts called on the PMTs to describe, interpret, and respond to student thinking, a tripartite network of professional moves that was new to them. Over time, the scaffolding supports of highlighting and prompting were progressively removed as pairs of PMTs were able to sustain their discussion about student thinking in the later episodes without the need for so much support. One example that stands out is in Session 7 when Nina and Amber watched the video once and then engaged in a sustained and substantive discussion without my prompting. This finding is critical as it signifies that the video-mediated professional learning experiences moved the PMTs along a trajectory of noticing from peripheral to more central participation in the community of practice, where central participation is characterized by sophisticated and unprompted analyses of student thinking and a responsive practice associated with ambitious (Kazemi et al., 2009) and responsive teaching.

**Features of the Problem-Solving Videos.** The features of the problem-solving videos selected for the design experiment seemed to play a mediating role in fostering the emergence of CoFs. They served as a springboard for PMTs to analyze students' problem solving and model their understanding of the mathematics related to the derivative. By enabling the PMTs to shift their vantage point from a student (in preservice teacher preparation) to a teacher in service, videos of real students solving real problems enabled the PMTs to approach problem solving more as a teacher of mathematics than as a student of mathematics (Stylianides & Stylianides, 2010). As such, the videos provided opportunities for decentering (Teuscher et al., 2016), by which PMTs put their perspectives aside and assumed the students' perspectives. In so doing, the PMTs developed an empathic stance that contributed to their capacity to make sense of students' actions, model it, and respond to it.

Grounded theory enabled the identification of four features of the videos that contributed to the emergence of CoFs for the PMTs' analyses. First is the composition of the videos (e.g., the task presentations and an exclusive focus on students' problem solving). Secondly, the videos are effectively curated, showing only the interactions between two students and the real-time construction of artifacts of their work. The curation of videos is important so that they do not provide extraneous and non-essential information that may distract or overwhelm the PMTs' attention (Walters, 2017). Third, since the videos are fluid—in the sense that they can be paused and re-viewed repeatedly—PMTs were able to re-watch "worthwhile" video segments and give their sustained attention to the reasoning of the students in order enrich their analyses of it. Finally, the videos provide a comprehensive and appropriate scope and sequence of topics that build conceptually on each other. As such, they supported the PMTs to re/structure their knowledge for teaching mathematics by developing a parallel base of specialized content knowledge. Two moments in the PMTs' experiences come to mind in this regard. One PMT leveraged an understanding they developed about the constant rate of change in Session 1 in order to understand an increasing rate of change central to the task in Session 2. And several PMTs used something they learned about the limit definition of the derivative in Session 3 in their analyses of students' problem solving in Sessions 6, 7, and 8. This finding demonstrates the value of professional learning experiences that simultaneously rely on enactments of both common and specialized content knowledge in teaching situations that call for rich and focused analyses of student thinking. Table 5 below provides a summary of the findings of Chapters 4 and 5.

## The Significance of the Findings

The expertise that practicing teachers bring to their noticing activities is often lacking in prospective teachers (Krupa, 2017; Wilson et al., 2011). As professional noticing is a critical element of ambitious teaching, it is worthwhile to develop this practice among prospective teachers. Research finds that noticing is a learnable skill (Jacobs et al., 2010; Krupa et al., 2017) and that teachers can make progress on learning to notice even within a single semester (e.g., Star & Strickland, 2008). Simply learning the mathematics, how students tend to think about the mathematics, and a number of ways to respond students through instruction it is not sufficient. Supported by the research on professional noticing, I hypothesized that prospective teachers could develop their knowledge for noticing by formatively assessing students' understandings through analyses of their problem solving in real-time. The videos of students' problem solving provided the venue for this work and the results support the claim that my hypothesis could be supported.

As the participants in this study engaged in noticing activity over the course of four months, with each pair of PMTs meeting once a week, they viewed these videos and analyzed the problem solving of students that appear in them. The problems posed to those students spanned the entire conceptual scope of the derivative. Through their participation, not only did the PMTs develop their capacity to notice, but the longitudinal extent of their engagement suggests that their capacity is likely to endure (Garet et al., 2016; Tunney & van Es, 2016). As a concluding remark on the results presented in Chapters 4 and 5, teacher noticing can be developed through video-mediated professional learning when particular social and material supports are provided.

## Limitations of the Study

The study has three limitations that could be addressed in future research. The first is a lack of research on how to develop knowledge for teaching specific domains of mathematics (such as the derivative) that could provide a stronger foundation for this study. The literature on models of knowledge for teaching mathematics is extensive, but not specific to the teaching of derivatives. As such, there is no theoretical trajectory of knowledge development in video-mediated professional learning that could ground this study. In light of this limitation, more research that focuses on developing knowledge for teaching specific domains of mathematics is warranted.

Since I conducted this study during the Covid-19 pandemic, I collected data via Zoom to avert the spread of the virus, and this presented a second limitation. For example, I could not see how the participants were representing or modeling situations, like graphing, on a piece of paper. Although they scanned and emailed the copies to me, I had missed an opportunity to seek clarification "in the moment" as they worked. To overcome this challenge, it may be more effective to conduct such a study in person, so that the researcher can interact with the participants in real time, seek clarifications, observe gestures, and make inferences from them.

Lastly, I cannot say whether the participants in this study are representative of the population of undergraduate student who are pre-service high school mathematics teachers. Although the purpose of qualitative research is not to generalize the results to this population, future research with other participants could determine whether the themes and patterns that emerged from this study are consistent.

## Implications and Directions for Future Research

This study has implications for both research and practice in mathematics education. Among the contributions of this study is the empirical development of a four-element framework that explains how PMTs learn to notice in a video-mediated learning environment. The framework may be useful in teacher education in precisely the same format as the one I used in my study. Alternatively, smaller-scale versions could also be useful. For instance, a teacher educator may provide PMTs with a one or more problem-solving videos in the conceptual context of their teaching along with highlighting cues and probing questions to guide PMTs in describing, interpreting, and/or responding to the problem solver’s thinking. This approach could address persistent problems of practice in calculus education by providing opportunities for teacher learning in contexts that approximate authentic practice. Calls for reform in calculus education advocate for a shift in pedagogy from a sole reliance on lecture (that dominates calculus teaching) to centering students' thinking as the basis for designing instruction.

Cultivating a calculus educator's capacity for noticing would support such a shift, and the framework presented in this study would provide an empirically supported structure for that work.

Despite the findings being primarily relevant to understanding how *problem-solving* videos can be used to facilitate teacher learning, they have implications for research on the use of video recordings in teacher education more broadly, and they contribute to the literature on frameworks of knowledge development through analyses of videos: *learning to notice* (van Es, 2011), *professional noticing of children's mathematics* (Jacobs et al., 2010), and the *professional noticing of student thinking in the context of proportionality* (Fernández et al., 2013). The designs of these studies differ from mine, but their findings have at least one thing in common: their participants made substantial progress in moving along a trajectory from superficial descriptions of students' mathematical activity to substantive analyses of their meanings that formed the basis of more expert-like responsive instructional decision making.

Problem-solving videos have been used in research and practice for a variety of purposes, including learning to notice (van Es, 2011; Wilson et al., 2011) and the deconstruction and interrogation of professional practices (Grossman et al., 2009). Researchers and teacher educators can use their analyses of learners' engagement with these videos to make determinations about their learning. In my case, I found from my analysis of the PMTs' analyses of students' problem solving that the PMTs experienced changes in both their knowledge of the derivative and in their capacity to notice other's thinking about the derivative. These changes enabled the future teachers to enact knowledge of the derivative that demonstrates an image of how it should be understood and how it could be taught. This image can be seen through a review of the PMTs' discourse in Chapter 4. Since this study contributes to the development of noticing and knowledge for teaching derivative among pre-service teachers, its findings have implications for teacher preparation.

My analysis of the PMTs' engagement with the videos identified features of the problem-solving videos that provided PMTs with the opportunities to experience these changes in knowledge and noticing. Findings from this analysis can be shared in the form of principles for the design of problem-solving videos that are likely to induce similar changes. It also offers insights into the nature of interactions associated with the analysis of those videos in which knowledge for teaching related to noticing can be developed. The principles of task design, facilitation, and the associated interactions that contribute to the development of mathematical knowledge for teaching may be useful to researchers as they design related studies. Researchers could then use empirically generated trajectories of noticing competence and contexts developed in this study to examine the knowledge needed to teach specific mathematical concepts, and how that knowledge can be supported.

This research has methodological implications, as well. Lobato et al. (2013) used the focusing framework to examine how students notice mathematics. I extended this framework by including pedagogical considerations, since knowledge for teaching has both conceptual and pedagogical dimensions. In addition to this modification to the framework, I also modified the definitions of both focusing interactions and features of the tasks to account for the pedagogical features that participants noticed and to also capture the goals of the study. With these modifications to the framework, I could account for both mathematical and pedagogical features and regularities in students' problem solving that captured the participants' attention and became the focus of their noticing interactions.

I suggest that when researchers analyze PMTs' discussions in noticing activities using the focusing framework, they can better understand how noticing as a socially situated practice occurs. They can also come to understand how it can be supported. By extending the focusing framework to include pedagogical elements, I was able to account for how noticing experiences influenced the development of knowledge for teaching around students' mathematical reasoning. This elaborated framework is now available to other teachers and researchers who have similar objectives. My process for elaborating it is also available, should teachers and researchers with not-so-similar objectives wish to elaborate Lobato et al.'s framework or mine to suit their needs.

Studies focusing on developing knowledge around specific areas of mathematics are few, and this study aimed to close that gap. Further research is warranted to identify other additional benefits that may be realized from developing teacher knowledge around specific domains of mathematics (e.g., geometry, algebra). In order to develop a pattern of knowledge development for teaching across the domains of mathematics, future research could focus on developing local theories that explain what knowledge and how it develops around specific mathematical ideas.

By analyzing students' problem solving over time, the participants developed their understanding of derivatives, their knowledge for teaching the derivative, and students' likely understandings of derivatives. They also developed a professional vision for orchestrating pedagogical moves based on their interpretations of students' problem solving. Such a professional vision has implications for PMTs' future practice. As an example, the participants may design instructional activities based on their interpretations of students' thinking as they solve mathematical problems. Accordingly, a follow-up study could examine how the participants put their professional vision into practice. That study could help us assess the impact of the PMTs' experiences in this study on their practice.

**References**

Adler, J., & Davis, Z. (2006). Opening another black box: Researching mathematics for teaching in mathematics teacher education. *Journal for Research in Mathematics Education*, *37*(4), 270-296.

Amobi, F. A. (2005). Preservice teachers' reflectivity on the sequence and consequences of teaching actions in a microteaching experience. *Teacher Education Quarterly*, *32*(1), 115-130.

Badillo, E., Azcárate, C., & Font, V. (2011). Analysis of mathematics teachers' level of understanding of the objects f'(a) and f'(x). *Enseñanza de las ciencias*, *29*(2), 191-206.

Baki, M., & Arslan, S. (2017). Effects of Mathematics Content Knowledge on Mathematics Pedagogical Content Knowledge. *Journal of Teacher Education and Educators*, *6*(1), 53-68.

Ball, D., Bass, H., & Hill, H. (2004). Knowing and using mathematical knowledge in teaching: Learning what matters. *Proceedings for the 12th Annual Conference of the South African Association for Research in Mathematics, Science and Technology Education.* Durban. SAARMSTE,

Ball, D., Hoover Thames, M., & Phelps, G. (2008). Content knowledge for teaching: domains of mathematical knowledge for teaching. *Journal of Teacher Education*, *59*(3), 389-407.

Ball, D. L. (1988). *Knowledge and reasoning in mathematical pedagogy: Examining what prospective teachers bring to teacher education* (Vol. 1). Michigan State University. Department of Teacher Education.

Ball, D. L. (1990). The Mathematical Understandings That Prospective Teachers Bring to Teacher Education. *The Elementary School Journal*, *90*(4), 449-466. https://doi.org/10.2307/1001941

Ball, D. L. (2000). Bridging practices: Intertwining content and pedagogy in teaching and learning to teach. *Journal of Teacher Education*, *51*(3), 241-247.

Ball, D. L. (2003). What mathematical knowledge is needed for teaching mathematics. *Secretary's Summit on Mathematics, US Department of Education*.

Ball, D. L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. *Multiple Perspectives on the Teaching and Learning of Mathematics*, *4*(1), 83-104.

Beswick, K., Muir, T., & Wells, J. (2015). Proceedings of the 39th Conference of the International Group for the Psychology of Mathematics Education. The 39th Conference of the International Group for the Psychology of Mathematics Education,

Borko, H., Jacobs, J., Eiteljorg, E., & Pittman, M. E. (2008). Video as a tool for fostering productive discussions in mathematics professional development. *Teaching and Teacher Education*, *24*(2), 417-436.

Borko, H., Koellner, K., Jacobs, J., & Seago, N. (2011). Using video representations of teaching in practice-based professional development programs. *ZDM*, *43*(1), 175-187.

Bressoud, D. M., Mesa, V., & Rasmussen, C. L. (2015). *Insights and recommendations from the MAA national study of college calculus*. MAA Press.

Brown, A. L. (1992). Design experiments: Theoretical and methodological challenges in creating complex interventions in classroom settings. *The Journal of the Learning Sciences*, *2*(2), 141-178.

Brown, J. S., Collins, A., & Duguid, P. (1989). Situated Cognition and the Culture of Learning. *Educational Researcher*, *18*(1), 32-42.

Campbell, P. F., Nishio, M., Smith, T. M., Clark, L. M., Conant, D. L., Rust, A. H., . . . Choi, Y. (2014). The relationship between teachers' mathematical content and pedagogical knowledge, teachers' perceptions, and student achievement. *Journal for Research in Mathematics Education*, *45*(4), 419-459.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352-378.

Carlson, M., Oehrtman, M., & Engelke, N. (2010). The precalculus concept assessment: A tool for assessing students' reasoning abilities and understandings. *Cognition and Instruction*, *28*(2), 113-145.

Carpenter, T., Fennema, E., Franke, M., Levi, L., & Empson, S. (1999). *Children's Mathematics cognitively guided instruction.* Postmouth, NH. In: Heinemann & Reston, VA: The National Council of Teachers of Mathematics, Inc.

Carpenter, T. P., & Lehrer, R. (1999). Teaching and learning mathematics with understanding. *Mathematics classrooms that promote understanding*, 19-32.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. *Handbook of research design in mathematics and science education*, 547-589.

Cobb, P., Confrey, J., DiSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher*, *32*(1), 9-13.

Cochran, K. F., DeRuiter, J. A., & King, R. A. (1993). Pedagogical content knowing: An integrative model for teacher preparation. *Journal of Teacher Education*, *44*(4), 263-272.

Cochran-Smith, M. (2003). Assessing assessment in teacher education. In: Sage Publications.

Cohen, D. K. (1990). A revolution in one classroom: The case of Mrs. Oublier. *Educational Evaluation and Policy Analysis*, *12*(3), 311-329.

Collins, A. (1992). Toward a design science of education. In *New directions in educational technology* (pp. 15-22). Springer.

Collins, A., Joseph, D., & Bielaczyc, K. (2004). Design research: Theoretical and methodological issues. *The Journal of the learning sciences*, *13*(1), 15-42.

Cooney, T. J. (1994). Research and teacher education: In search of common ground. *Journal for Research in Mathematics Education*, *25*(6), 608-636.

Corbin, J. M., & Strauss, A. (1990). Grounded theory research: Procedures, canons, and evaluative criteria. *Qualitative sociology*, *13*(1), 3-21.

Corwin, R. B., Price, S., & Storeygard, J. (1996). *Talking mathematics: Resources for developing professionals*. Heinemann.

Council, N. R., & Committee, M. L. S. (2001). *Adding it up: Helping children learn mathematics*. National Academies Press.

Darling-Hammond, L. (2000). How teacher education matters. *Journal of Teacher Education*, *51*(3), 166-173.

Ding, L., He, J., & Leung, F. K. S. (2014). Relations between subject matter knowledge and pedagogical content knowledge: A study of Chinese pre-service teachers on the topic of three-term ratio. *The Mathematics Educator*, *15*(2), 50-76.

Doerr, H. M., & Thompson, T. (2004). Understanding teacher educators and their pre-service teachers through multi-media case studies of practice. *Journal of Mathematics Teacher Education*, *7*(3), 175-201.

Fennema, E., Carpenter, T. P., Franke, M. L., Levi, L., Jacobs, V. R., & Empson, S. B. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education*, *27*(4), 403-434.

Fennema, E., & Franke, M. L. (1992). Teachers' knowledge and its impact. In D.A, *Handbook of research on mathematics teaching and learning. A Project of the National Council of Teachers of Mathematics* (pp 147-164). Macmillan Publishers Co, Inc.

Franke, M. L., Carpenter, T. P., Levi, L., & Fennema, E. (2001). Capturing teachers' generative change: A follow-up study of professional development in mathematics. *American Educational Research Journal*, *38*(3), 653-689.

Franke, M. L., & Kazemi, E. (2001). Learning to teach mathematics: Focus on student thinking. *Theory into Practice*, *40*(2), 102-109.

Friedrichsen, P., Driel, J. H. V., & Abell, S. K. (2011). Taking a closer look at science teaching orientations. *Science Education*, *95*(2), 358-376.

Garet, M. S., Heppen, J. B., Walters, K., Parkinson, J., Smith, T. M., Song, M., . . . Borman, G. D. (2016). Focusing on Mathematical Knowledge: The Impact of Content-Intensive Teacher Professional Development. NCEE 2016-4010. *National Center for Education Evaluation and Regional Assistance*.

Geertz, C. (1973). Thick description: Toward an interpretive theory of culture. *Turning points in qualitative research: Tying knots in a handkerchief*, *3*, 143-168.

Gess-Newsome, J. (1999). Pedagogical content knowledge: An introduction and orientation. In *Examining pedagogical content knowledge* (pp. 3-17). Springer.

Glaser, B. G., & Strauss, A. L. (1967). The discovery of Grounded. *Theory: Strategies of Qualitative Research,, Wiedenfeld &, Nicholson, London, United Kingdom*.

Goodwin, C. (1994). Professional Vision. *American Anthropologist*, 606-633.

Gravemeijer, K., & Doorman, M. (1999). Context problems in realistic mathematics education: A calculus course as an example. *Educational studies in mathematics*, *39*(1), 111-129.

Greeno, J. G. (1988). The situated activities of learning and knowing mathematics. Proceedings of the 10th annual meetings of PME-NA,

Grossman, P. L., Wilson, S. M., & Shulman, L. S. (1989). Teachers of substance: Subject matter knowledge for teaching. *Profesorado, Revista de currículum y formación del profesorado*, *9*(2), 1-25.

Guba, E. G. (1981). Criteria for assessing the trustworthiness of naturalistic inquiries. *Ectj*, *29*(2), 75-91.

Guba, E. G., & Lincoln, Y. S. (1989). Fourth generation evaluation.

Hashweh*, M. Z. (2005). Teacher pedagogical constructions: a reconfiguration of pedagogical content knowledge. *Teachers and Teaching*, *11*(3), 273-292.

Herbel-Eisenmann, B. A., & Breyfogle, M. L. (2005). Questioning our patterns of questioning. *Mathematics teaching in the middle school*, *10*(9), 484-489.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. *Handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics*, 65-97.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for research in mathematics education*, 372-400.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American educational research journal*, *42*(2), 371-406.

Howell, H., & Mikeska, J. N. (2021). Approximations of practice as a framework for understanding authenticity in simulations of teaching. *Journal of Research on Technology in Education*, *53*(1), 8-20.

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for research in mathematics education*, 169-202.

Jones, S. R., & Watson, K. L. (2018). Recommendations for a "target understanding" of the derivative concept for first-semester calculus teaching and learning. *International Journal of Research in Undergraduate Mathematics Education*, *4*(2), 199-227.

Kazemi, E., Franke, M., & Lampert, M. (2009). Developing pedagogies in teacher education to support novice teachers' ability to enact ambitious instruction. Crossing divides: Proceedings of the 32nd annual conference of the Mathematics Education Research Group of Australasia,

Kilic, H. (2011). The nature of pre-service teachers' pedagogical content knowledge. Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education–CERME7,

Kolb, S. M. (2012). Grounded theory and the constant comparative method: Valid research strategies for educators. *Journal of emerging trends in educational research and policy studies*, *3*(1), 83-86.

Krauss, S., Brunner, M., Kunter, M., Baumert, J., Blum, W., Neubrand, M., & Jordan, A. (2008). Pedagogical content knowledge and content knowledge of secondary mathematics teachers. *Journal of Educational Psychology*, *100*(3), 716.

Krupa, E. E., Huey, M., Lesseig, K., Casey, S., & Monson, D. (2017). Investigating secondary preservice teacher noticing of students' mathematical thinking. In *Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks* (pp. 49-72). Springer.

Lave, J. (1991). Situating learning in communities of practice.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge university press.

Lew, K., Fukawa-Connelly, T. P., Mejía-Ramos, J. P., & Weber, K. (2016). Lectures in advanced mathematics: Why students might not understand what the mathematics professor is trying to convey. *Journal for Research in Mathematics Education*, *47*(2), 162-198.

Lobato, J., Hohensee, C., & Rhodehamel, B. (2013). Students' mathematical noticing. *Journal for Research in Mathematics Education*, *44*(5), 809-850.

Lobato, J., & Siebert, D. (2002). Quantitative reasoning in a reconceived view of transfer. *The Journal of Mathematical Behavior*, *21*(1), 87-116.

Lobato, J., & Thanheiser, E. (2002). Developing understanding of ratio as measure as a foundation for slope. *Making sense of fractions, ratios, and proportions*, 162-175.

Loewenberg Ball, D., & Bass, H. (2009). With an Eye on the Mathematical Horizon. *Beiträge zum Mathematikunterricht 2009*.

Lortie, D. C. (1977). Schoolteacher: A Sociological Study.

Ma, X. (1999). A meta-analysis of the relationship between anxiety toward mathematics and achievement in mathematics. *Journal for research in mathematics education*, 520-540.

Maanen, J. V. (1988). *Tales of the fieldon writing ethnography* (0226849627).

Magiera, M. T., Van den Kieboom, L. A., & Moyer, J. C. (2013). An exploratory study of pre-service middle school teachers' knowledge of algebraic thinking. *Educational Studies in Mathematics*, *84*(1), 93-113.

Merriam, S. B., & Tisdell, E. J. (2015). *Qualitative research: A guide to design and implementation*. John Wiley & Sons.

Moon, K., Brewer, T. D., Januchowski-Hartley, S. R., Adams, V. M., & Blackman, D. A. (2016). A guideline to improve qualitative social science publishing in ecology and conservation journals. *Ecology and Society*, *21*(3).

Moore, K. C., Carlson, M. P., & Oehrtman, M. (2009). The role of quantitative reasoning in solving applied precalculus problems. Twelfth Annual Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education (SIGMAA on RUME) Conference, Raleigh, NC: North Carolina State University,

Moore, K. C., & Thompson, P. W. (2015). Shape thinking and students' graphing activity. Proceedings of the 18th meeting of the MAA special interest group on research in undergraduate mathematics education.

Moore-Russo, D., & Viglietti, J. M. (2011). Teachers' reactions to animations as representations of geometry instruction. *ZDM*, *43*(1), 161-173.

Nagle, C., & Moore-Russo, D. (2013). The concept of slope: Comparing teachers' concept images and instructional content. *Investigations in Mathematics Learning*, *6*(2), 1-18.

Nagle, C., Moore-Russo, D., Viglietti, J., & Martin, K. (2013). Calculus students' and instructors' conceptualization of slope: A comparison across academic levels. *International Journal of Science and Mathematics Education*, *11*(6), 1491–1515.

NCTM, P. (2000). standards for school mathematics, Reston, VA. *EE. UU.*

Park, J. (2013). Is the derivative a function? If so, how do students talk about it? *International Journal of Mathematical Education in Science and Technology*, *44*(5), 624-640.

Park, S., & Oliver, J. S. (2008). Revisiting the conceptualisation of pedagogical content knowledge (PCK): PCK as a conceptual tool to understand teachers as professionals. *Research in Science Education*, *38*(3), 261–284.

Patton, M. Q. (2015). The sociological roots of utilization-focused evaluation. *The American Sociologist*, *46*(4), 457-462.

Piaget, J. (1955). The language and thought of the child. Cleveland. *Ohio: World Publishing*.

Piaget, J. (1970). *Genetic epistemology*. Columbia University Press.

Piccolo, D. (2008). Views of content and pedagogical knowledges for teaching mathematics. *School Science and Mathematics*, *108*(2), 46–49.

Pino-Fan, L. R., Godino, J. D., & Font, V. (2018). Assessing key epistemic features of didactic-mathematical knowledge of prospective teachers: the case of the derivative. *Journal of Mathematics Teacher Education*, *21*(1), 63–94.

Putman, R., & Borko, H. (2000). What do new views of knowledge and thinking have to say. *Qualitative Health Research*, *9*(1), 112-121.

Rodgers, C. (2002). Voices inside schools. *Harvard Educational Review*, *72*(2), 230–254.

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education*, *8*(3), 255–281.

Scheiner, T. (2015). Lessons we have (not) learned from past and current conceptualizations of mathematics teachers' knowledge. Ninth Congress of the European Society for Research in Mathematics Education (CERME9, 4-8 February 2015),

Scheiner, T., Montes, M. A., Godino, J. D., Carrillo, J., & Pino-Fan, L. R. (2019). What makes mathematics teacher knowledge specialized? Offering alternative views. *International Journal of Science and Mathematics Education*, *17*(1), 153-172.

Schoenfeld, A. H., & Kilpatrick, J. (2008). Toward a theory of proficiency in teaching mathematics. In *International Handbook of Mathematics Teacher Education: Volume 2* (pp. 321-354). Brill Sense.

Schön, D. A. (1992). Designing as reflective conversation with the materials of a design situation. *Knowledge-based systems*, *5*(1), 3-14.

Sherin, M., & van Es, E. (2002). Using video to support teachers' ability to interpret classroom interactions. society for information technology & teacher education international conference.

Sherin, M. G. (2002). When teaching becomes learning. *Cognition and Instruction*, *20*(2), 119–150.

Sherin, M. G. (2004). New perspectives on the role of video in teacher education. In *Using video in teacher education* (pp. 1-27). Elsevier Science.

Sherin, M. G. (2007). The development of teachers' professional vision in video clubs. *Video research in the learning sciences*, 383-395.

Sherin, M. G., & Han, S. Y. (2004). Teacher learning in the context of a video club. *Teaching and Teacher education*, *20*(2), 163-183.

Shulman, L. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, *57*(1), 1–23.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4–14.

Star, J. R., & Strickland, S. K. (2008). Learning to observe: Using video to improve preservice mathematics teachers' ability to notice. *Journal of Mathematics Teacher Education*, *11*(2), 107-125.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. *Handbook of research design in mathematics and science education*, 267-306.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research*. Sage publications.

Strauss, A. L. (1987). *Qualitative analysis for social scientists*. Cambridge university press.

Stump, S. L. (2001). Developing preservice teachers' pedagogical content knowledge of slope. *The Journal of Mathematical Behavior*, *20*(2), 207-227.

Stylianides, G. J., & Stylianides, A. J. (2010). Mathematics for teaching: A form of applied mathematics. *Teaching and Teacher Education*, *26*(2), 161-172.

Sun, J., & Van Es, E. A. (2015). An exploratory study of the influence that analyzing teaching has on preservice teachers' classroom practice. *Journal of Teacher Education*, *66*(3), 201-214.

Sánchez-Matamoros, G., Fernández, C., & Llinares, S. (2015). Developing pre-service teachers' noticing of students' understanding of the derivative concept. *International journal of science and mathematics education*, *13*(6), 1305-1329.

Tall, D., & Vinner, S. (1981). Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity. *Educational Studies in Mathematics*, *12*(2), 151-169. https://doi.org/10.2307/3482362

Teuscher, D., Moore, K. C., & Carlson, M. P. (2016). Decentering: A construct to analyze and explain teacher actions as they relate to student thinking. *Journal of Mathematics Teacher Education*, *19*(5), 433-456.

Thomas, J., Jong, C., Fisher, M. H., & Schack, E. O. (2017). Noticing and knowledge: Exploring theoretical connections between professional noticing and mathematical knowledge for teaching. *The Mathematics Educator*, *26*(2).

Thompson, A. G. (1984). The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practice. *Educational Studies in Mathematics*, *15*(2), 105–127.

Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. *The development of multiplicative reasoning in the learning of mathematics*, 179-234.

Tsamir, P. (2005). Enhancing prospective teachers' knowledge of learners' intuitive conceptions: The case of same A–same B. *Journal of Mathematics Teacher Education*, *8*(6), 469-497.

Tunney, J. W., & van Es, E. A. (2016). Using video for teacher-educator professional development. *The New Educator*, *12*(1), 105-127.

van Es, E. A. (2011). A framework for learning to notice student thinking. In *Mathematics teacher noticing* (pp. 164-181). Routledge.

Van Es, E. A., & Sherin, M. G. (2002). Learning to notice: Scaffolding new teachers' interpretations of classroom interactions. *Journal of Technology and Teacher Education*, *10*(4), 571-596.

Vygotsky, L. (1978). Internalization of higher psychological functions. *Mind in society: The development of higher psychological processes*, *37*(2), 52-57.

Walters, C. D. (2017). *The development of mathematical knowledge for teaching for quantitative reasoning using video-based instruction*. University of California, San Diego.

Wenger, E. (2009). Communities of practice. *Communities*, *22*(5), 57-80.

Wilson, P. H., Lee, H. S., & Hollebrands, K. F. (2011). Understanding prospective mathematics teachers' processes for making sense of students' work with technology. *Journal for Research in Mathematics Education*, *42*(1), 39-64.

Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. *CBMS Issues in Mathematics Education*, *8*, 103-127.

Zandieh, M. J., & Knapp, J. (2006). Exploring the role of metonymy in mathematical understanding and reasoning: The concept of derivative as an example. *The Journal of Mathematical Behavior*, *25*(1), 1-17.

Zembylas, M. (2007). Emotional ecology: The intersection of emotional knowledge and pedagogical content knowledge in teaching. *Teaching and Teacher Education*, *23*(4), 355-367.

# Appendices

## Appendix A: Interview Contact Summary Form

Location: ___________________ Interview Date: ___________________ Today's Date:

___________________

1. What were the main issues or themes that struck you in this interview?
2. Summarize the information you got (or failed to get) on each of the target questions you had for this contact.

- Noticing student thinking. How do PMTs notice student thinking?
- Learning. What skills of noticing student thinking PMTs have learned?
- Image of a learning trajectory. How is PMTs learning trajectory unfolding?
- Interventions. Which interventions seemed to support PMTs to notice student thinking?

3. Anything else that struck you as salient, interesting, illuminating, or important in this interview?
4. What new (or remaining) questions have come up as a result of this interview? What changes to the protocol or problem-solving videos might you make in future interviews?

## Appendix B: Salient Episodes in the Problem-solving Videos

Session 1: The constant rate of change

| Timestamp | Episode | Interactions |
|---|---|---|

| | | |
|---|---|---|
| 0:54 | J: Started off getting faster, then at some point, around here, is getting slower | |
| 1:30 | A: Is slower but increasing because we are not talking out water | |
| 1.58 | J: If its faster here, and then slower, is it like increasing and then decreasing there? | |
| 3.30 | A: If we add 1milimeter of water, then the height increases by 1cm. If we add another 1mm, the height increases by another 1cm. And then it makes it constant | |
| 4.00 | J: at t=1, it might have gone up one more step. At t=2, then with a faster pour it should have more water, and height should be like 3.5. | |
| Session 2: The varying rate of change | | |
| 0.42 | Julian: I'm thinking if I wouldn't have to pour, in relation to like the water entering the flask, like the pour was constant the entire time. | |
| 0.8 | Julian: its constant | |
| 1.04 | Julian: it doesn't actually change like how much water is being poured in from the picture. | |
| 1.09 | J & A: it would expand somehow | |
| 1.15 | J: I'm thinking of like, constant rate. | |
| 1.42 | A: I agree that the pour is constant but that doesn't really take into consideration like the shape of the flask there. Like the area of that like, little slice of the flask is different than the base, this is a lot bigger. I mean, this up here this are a lot smaller. | |

| | | |
|---|---|---|
| 2.04 | A: I agree that the pour is constant but that doesn't really take into consideration like the shape of the flask there. Like the area of that like, little slice of the flask is different than the base, this is a lot bigger. I mean, this up here this are a lot smaller. | |
| 2.09 | what I was saying was like down here, you need more water to make the height go up higher than up here which would make it increasing. But you think that... | |
| 2.35 | there is less volume up here…you need less water | |
| 3.02 | A: I mean I get what you are saying that like there's less volume up here so the rate of the volume is decreasing as we add more water, is that how you looking at it? | |
| 3.12 | A: I guess that makes sense. I'm not sure though, cause it doesn't really talk about the height that much | |
| Session 3: Approximating the Instantaneous Rate of Change Transcript | | |
| 00.53 | A: Yeah, that's like, like distance over time, but if we don't have any time, we can't really be moving. | |
| 01.00 | J: I just I think what I'm struggling with is like in a baseball game, you know, the pitcher is throwing the ball from the lounge to the capture. So, it's like from point A to B and that's how the ball like swing. | |
| 01.17 | J: like there is movement and like, there is a change in time | |

| | | |
|---|---|---|
| 1.24 | A: But we are only seeing like that one little, like, like instantaneous moment | |
| 1.42 | J: But I think maybe like to show, two seconds to show the change of time and the change of position. I guess the picture should, in my opinion, have some blur, like some motion and it like have that motion, right? | |
| 1.56 | A: I mean it's kinda hard to see right now cause probably still in, but we definitely need to know like how long it took the camera to take the photo. | |
| 2.01 | A: Cause that will give us like the time aspect that we need | |
| 2.15 | J: I'm taking we also need to like again, the point of A to B, so I'm assuming the first blurry scene or last part blurry scene, so everything is blurry. Just go end to end. | |
| 2.57 | A: we don't want to take like, like, so this point here is like the size of the ball, so we wouldn't want to take like that, like very end of the blur. I think we need to take out the length or the diameter of the ball. Is this a better word? So that we can get it from like, like, um, so we can get like the distance but not like... | |
| 3.10 | A: our speed is going to be like change in distance over change in time. | |
| 3.21 | A: to find the change in distance we take the length of the blurry minus the diameter of the baseball. | |

| | | |
|---|---|---|
| 3.30 | J: At the moment the picture was taken, the ball was travelling 65 miles per hour. | |
| 3.46 | A: I don't think it's like at that moment, because we did say it was, um, like we did incorporate like the change in time, so it's not at that moment, it's still like that interval even if it is like such so small. | |
| 3.51 | A: I guess it's the velocity over the interval | |
| 4.10 | A: I think our best approximation for the moment the photo was taken because we have such a small interval, because like we said before we can't really have like a change of time over like in instantaneous moment. | |
| 4.17 | J: so that is approximately 65 miles per hour because of the interval | |
| Session 4: Limit Definition of Derivative | | |
| 1.15 | J: let's take a T that is one smaller and then a T that's one bigger, and do like the change of Y and change of X. S | |
| 1.35 | J: if we use this kind of formula, we can do T equals 3 and T equals 5… And then get that slope | |
| 2.10 | A: I think that will be good approximation… but like, like could we make it better by making it like a smaller… like intervals maybe like you do like three to four. | |

| | | |
|---|---|---|
| 2.21 | A: I'm not good at drawing science but I guess that looks like way to steep now | |
| 2.30 | A: I guess it'd be the same like four and five, but it'd be less steep | |
| | A: so like, how can I get a more accurate, like approximation | |
| 2.58 | J: without having it to be… so steep or… | |
| 3.10 | J: So, we've said so far that T equals three and T equals five, but we also said T equals three and T equals four. | |
| 3.16 | J: And we want get, like we want to be as close to four as possible | |
| 3.29 | J: could we possibly do something like, like, I guess like T equals 3.9 | |
| 3.50 | J: it's like, we will do like T one equals 3.9 and then T two equals 4.1, because then that would get us something like there. And then we would get... | |
| 3.51 | A: be a lot closer | |
| 4.10 | J: Yeah, which we got us this line here, which, which seems a little bit more accurate than I guess the larger intervals. So, it's like, the smaller the intervals the more accurate it could be... | |
| 4.16 | A: Yeah, like how small do we need to make it, to make it | |
| 4.19 | J: Like do we have to go smaller than this? We might. | |
| 4.22 | A: I mean, like we could always make it go smaller | |
| Using Limit to Compute Derivative | | |

| | | |
|---|---|---|
| 00:41 | Maria (M): we want to find the rate of change of the side length with respect to the area. So it's a derivative. | |
| 00:46 | Kelly (K): So, we are looking for a derivative | |
| 00.50 | M: and we know that we are looking at when the area is five | |
| 00.55 | and we know the area is just the side squared. I know a lot of things... | |
| 01.03 | K: Sort of, area equals side square, we know that the side is equal to root A | |
| 01.10 | M: Hmm that's true. Then the limit of the change in area as opposed to zero. | |
| 01.17 | K: And then its S of five plus delta A | |
| 01.22 | M & K: minus S of five | |
| 01.24 | K: all over delta A | |
| 01.59 | M: I feel like that we know. So, let's see if we can replace any of these. So, we know the area, when or if we have S of five plus delta. That's just asking what is the side, when the area is five plus delta A | |
| 02.09 | So, we are looking at the limit of delta A. Um, we know S of five is just gonna be root five, right? | |
| 02.55 | M: I feel like we are just kinda stuck with the delta A | |
| Slope of Secant and Tangent lines | | |
| 00.29 | A: well I remember derivative is, uhm the slope of the tangent line. | |

| | | |
|---|---|---|
| 00.32 | J: can you remind me what a tangent line is again? | |
| 00.36 | A: Yeah, so it is a line that touches the graph at just one point. | |
| 00.46 | A: for like uhm this point A, it goes through that one point and then not touch any of the other points. | |
| 01.02 | J: But what, like why can't it go through like there or like there? Like it's not touching the graph. | |
| 01.18 | A: Yeah, it's true. Uh, maybe it's like the other one. Uh, so like for B to go through the point, the tangent line will be like that I think | |
| 01.34 | J: I think like, my only like question is like, if it hits here, that also hits along the entire thing is it still a tangible line? | |
| 03.10 | J: So even though it's like, like this is actually going through the graph, but it's going through the graph at t. So, is that still a tangent, if it goes through the graph? | |
| 03.23 | J: Cause like, this one doesn't go through the graph, like A and C don't go through the graph | |
| | 03.26: But D is actually like intersecting other points | |
| Graphing the Derivative | | |
| 00:18 | K: That's the speed of the football for all moments in time | |
| 00:45 | M: we have distance over time, uhm isn't that just drawing us the speed already? Well, you'd have distance divided by time, uhm, so I | |

| | | |
|---|---|---|
| | guess that wouldn't, wouldn't technically be the speed because you have to calculate what the distance is over time. | |
| 00:47 | M: the speed would be different from what's on that line. | |
| 00:51 | K: Wait, isn't speed the derivative? | |
| 00:57 | M: Okay. So, we are just finding the derivative of this thing? | |
| 01:00 | M: so, I guess we are just trying to graph a derivative | |
| 01:13 | K: I think that it's just like slow and then speeds up and then he releases the ball | |
| 01:18 | M: that's just kind of showing that it's a slow rate at the beginning | |
| 01:22 | M:And then it goes up, and it's always positive. So that works. | |
| 01:27 | K: so I guess that works but I don't really know how precise it is… it's just not really precise enough | |
| 01.42 | M: Okay, I think that if we were to pick a couple of points in time | |
| 01:45 | K: and say calculate the speed that would give us a more precise graph | |
| 01:57 | M: So, if we, if we take like at one second... The distance is about half. So, half over one second would be half speed | |
| 01:59 | K: because speed equals distance over time | |
| 02:07 | M: That's what I'm thinking. So, we are looking for distance over time if that’s one second. It's a half, it'll be like here | |
| 02.24 | K: you know what, that's the change in distance over the change in time and we are only doing a point. | |

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| 02.28 | M: So instead of D over T you want to do delta T over delta T? | |
| 2.34 | so for Delta T we just want like a small-time interval? | |
| 03.36 | M: So now let's do two. So, we get zero… it's kind of like a flat… it's like a plateau there | |
| 03.39 | Right, cause we are finding the slope… when we are taking the average rate of change | |
| 04.25 | M: So right now, if I were to just connect the points, I'm not sure what it'd look like connecting from zero, like what it just connects the origin to that, I guess | |
| 04:31 | K: I guess so, because from like zero to point five is kind of flat like we were talking about with the slope | |
| 04.36 | M: So is this kind of what our speed look like or the derivative of that function | |
| 04.42 | M: I mean it makes sense for the points we've plotted anyway but I don't know about the ones in between | |
| Interpreting derivative | | |
| 00:34 | M: At five, so we are looking at five hours after she left home | |
| 00.42 | M: Does that just mean that she used 2.5 gallons since she left home? | |

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| 00:49 | K: No, I think that would just be f of five cause... f of t is the amount of fuel her car has consumed | |
| 00:54 | M: That makes sense. So, f of t or f of five, would be she has used 2.5 gallons, essentially? | |
| 01:14 | M: But this is describing the rate at five hours, so is it saying that at exactly five hours, she's using 2.5 gallons so like the instant rate of change would be 2.5 gallons? | |
| 01:43 | K: You just like reading. Instead of reading the derivative of, instead of reading the derivative at five is two point five, I think you are just reading it as the instantaneous rate of change... | |
| 02:04 | K: Uhm, then maybe at every hour she has had two point five gallons consumed. So, the amount of fuel consumed increased by two point five after each hour | |
| 02.11 | M: So, you are saying like at hour one, two, three, four, five, how much.... how many gallons did she consume? | |
| 02.18 | K: Two point five, and then two point five. Then for three, two point five, and for four, two point five and five, two point five | |
| 02.27 | M: So, in the span of five hours she used, ten, eleven, twelve point five gallons? | |
| 02:31 | M: That sounds like a lot of gas | |

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| 02.48 | K: Oh, you know what, that would mean she is travelling at a constant speed. We didn't know that she's on constant speed. | |
| 02:52 | M: Okay. So, she could be using more gallons | |
| 02:55 | M: or less like per hour | |
| 02.58 | K: Like we are just concerned with our five | |
| 03:03 | M: so, we just care about t equal five. We can't say what happened before that. | |
| 03.35 | M: And that also makes sense because I feel like for our last guess, so when we were concerned about the gallons consumed overall, our answer would have been in just gallons and not like gallons per hour which I think is the rate we want in the end | |
| 04:00 | K: Oh however, that would say that she would have to travel at a constant rate at hour five. So, if we had like a timeline down here when we have hour one, hour two, three, four and five, that would mean between hour five and six she would have to use the constant rate. Well, like if she drove for five minutes after hour five... | |
| 04.04 | K: and then didn't drive again until five minutes before our six | |
| 04.18 | M: maybe if we didn't say over the next hour but we said... like an average, are you looking for like an average | |
| 04.32 | K: Maybe not an average, but instead of saying Uhm, over the course of the next hour maybe we say like over the course of the | |

| | | |
|---|---|---|
| | next five seconds she was going to use two point five gallons per hour | |
| 05.06 | M: you are using two point five gallons... per hour. So, I suppose, is that...can we say that? | |

**Appendix C: A List of Problem-solving Videos.**

1. Constant Rate of change Student Problem Solving: Pouring Water into a Cylinder
2. Varying rate of change Student Problem Solving: Pouring Water into an Erlenmeyer Flask
3. Approximating instantaneous rate of change Student Problem Solving: The Stationary Baseball
4. Limit definition of the derivative: Student Problem Solving: Rate of Absorbing Ibuprofen
5. Using limits to compute derivatives: Student Problem Solving: Using Limits to Compute Derivatives
6. Slopes of secant and tangent lines: Student Problem Solving: The Imprecision of Tangents
7. Graphing the derivative: Student Problem Solving: Graphing the Speed of a Baseball
8. Interpreting derivative: Student Problem Solving: Interpreting Derivative