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The Independence Polynomial of a Graph at −1

Phoebe Rose Zielonka

Abstract

The independence polynomial for a graph G is defined by $I(G;x) = \Sigma_{k=0}^{\alpha(G)} i_k(G)x^k$ where $i_k(G)$ is the number of independent sets in G of size k. Engström proved that the independent polynomial satisfies $|I(G;-1)| \leq 2^{\phi(G)}$ where $\phi(G)$ is the decycling number of G. The cyclomatic number of a graph G, denoted $\beta(G)$, is defined as $\beta(G) = e(G) - n(G) + q(G)$. If the length of a cycle in a graph G is divisible by 3, we call it $\tilde{3}$-*cycle*, otherwise a *non*-$\tilde{3}$-*cycle*. Cao and Ren proved that $|I(G;-1)| \leq 2^{\beta(G)} - \beta(G)$. Cutler and Kahl proved Levit and Mandrescu's conjecture that for every positive integer k and integer q, with $|q| \leq 2^k$, there exists a graph G with $\phi(G) = k$ and $I(G;-1) = q$. In this paper, we prove that $|I(G;-1)| \leq 2^{\beta(G)-1}$ for graphs with non-$\tilde{3}$-cycles. Furthermore, we prove a density result related to this upper bound by adapting Cutler and Kahl's results.

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The Independence Polynomial of a Graph at -1

by

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Where the crystal moon shines and roses bloom, so do I.

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Chapter 1

Introduction

1.1 Preliminary Definitions

We will begin with defining the graph theory terminologies essential to this thesis's results. First and foremost, we need to define a graph. A *graph* G is an ordered pair $G = (V, E)$ where $V(G)$ is the set of *vertices* and $E(G)$ is the set of *edges* in G. If we take a subset from $V(G)$ and $E(G)$, we can form a part of the graph, which is a subgraph. Formally, a *subgraph* of a graph G is a graph, say H, such that $V(H) \subset V(G)$ and $E(H) \subset E(G)$.

The vertices in $V(G)$ may be isolated or connected to others. Let $u, v \in V(G)$ be arbitrary vertices. If $u, v \in V(G)$ are connected in G, we denote the edge formed by $u, v \in V(G)$ as $uv \in E(G)$. In this case, u and v are said to be *adjacent*. That is to say if there is no edge uv connecting the vertices u and v, then u and v are non-adjacent. If we collect only the pairwise non-adjacent vertices in G, we get a subset called an independent set. This subset of $V(G)$ containing only independent vertices is formally defined as follows. An *independent set* of G, denoted $I(G)$, is a subset $I \subset V(G)$ such that for every $u, v \in I$, we have $uv \notin E(G)$. As described above, by definition, we know only non-adjacent vertices are contained in $I(G)$.

In this thesis, we study the independence polynomial of a graph. Therefore, it is essential to know what the independence polynomial of any graph G is. The independence number $\alpha(G)$ is the size of the largest independent set of a graph G. Throughout this thesis, we use the following definition of the independence polynomial. The independence polynomial of a graph is defined as

$$I(G;x) = \Sigma_{k=0}^{\alpha(G)} i_k(G)x^k,$$

where $i_k(G)$ is the number of independent sets of the size of k in G.

There are several families of graphs, and the specific independence polynomials for them vary accordingly. In this thesis, we focus our investigation on common graphs such as paths, cycles, and graphs that we can construct with them using the operation introduced in Chapter 3.

We will define the path and cycle class of graphs below. Let n be any arbitrary integer, then a *path* P_n with n vertices is a graph with the vertex set $V(P_n) = \{v_1, \dots, v_n\}$. In a *path* the first vertex v_1 and the last vertex v_n are not adjacent. Formally, a *path* is a list $v_0, e_1, v_1, e_2, ..., e_n, v_n$ of vertices and edges with no repeated vertices, where $e_i = v_i i - 1 v_i$ for $i = 1, 2, 3, ..., n$. In a *path*, v_0 and v_n are *endpoints*. One may consider a graph using the concept of the length. The *length* of a graph is the number of edges contained in it. A graph G is called *connected* if for each pair of vertices $u, v \in V(G)$ there exists a uv-path with u and v being endpoints in G. The *components* of a graph are its maximal connected subgraphs.

A connected graph is a graph G such that for any vertices u, v in G, there exists a path connecting them. Similarly, a cycle graph is like a path with the first vertex v_1 and the last vertex v_n are adjacent to each other (Figure 1.1).

Figure 1.1: Illustration of P_6 and C_5

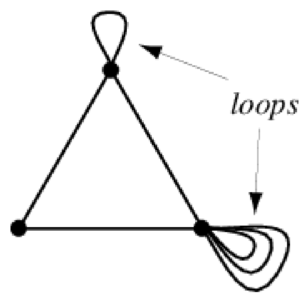

Figure 1.2: Loops in a graph

A vertex not only can be adjacent to other vertices but also to itself. When that setting happens, we call it a loop. A *loop* is an edge whose endpoints are equal. *Multiple edges* are edges having the same pair of endpoints (Figure 1.2). We define *simple graph* as a graph that has no loops or multiple edges. It is important to note that all graphs considered in this thesis are *simple graphs.*

In the following subsection, we will explore previous research on the independence polynomial of a graph which requires familiarity with concepts related to graphs with or without a cycle. Hence, it is worth to gain an understanding of these concepts. We will include the definitions and a relevant example. A graph G is considered *acyclic* if it contains no cycles. The decycling number of G, denoted $\phi(G)$, is the minimum size of a set $S \subset V(G)$ such that $G - S$ contains no cycles, i.e., is *acyclic.* For example, a *forest* is an acyclic graph. Figure 1.3 helps visualize acyclic graphs. A *tree* is considered a connected acyclic graph.

It is also essential to know the degree of a vertex since we will use it later. In a graph, each vertex may or may not be incident with edges. The *degree* of a vertex

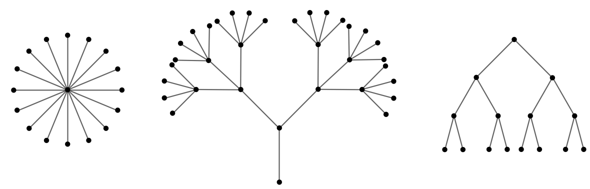

Figure 1.3: Trees - acyclic graphs

v in a graph G, denoted $d(v)$ or $d_G(v)$ is the number of edges incident to v. An important theorem in the next Subsection considers graphs that contain no vertex of degree one. To help with visualization, in Figure 1.3, all vertices adjacent to only one other vertex have degree one. In fact, in a tree, a vertex of degree one is called a *leaf*.

Just as importantly, we will get to know adjacent vertices with the goal of using them in computing the independence polynomial of a graph. Two adjacent vertices are called *neighbors*. The *neighborhood* or *open neighborhood* of v, denoted $N(v)$ is $N(v) = \{U \in V(G) : uv \in E(G)\}$. This neighborhood does not include v itself. When stated without any qualification, a neighborhood is assumed to be open. In this case, one may notice that $|N(v)| = d(v)$. The *closed neighborhood* of v is the neighborhood of v including v and is denoted as $N[v]$. The concept of $N[v]$ is crucial in the computation of independence polynomials of a graph G. We will refer to $N[v]$ in several tasks throughout this thesis.

1.2 Previous Results

In this Section, we will review the previous results on the independence polynomial of a graph at -1. The concept of independence polynomial was first defined by Gutman and Harary [5]. Since then, this topic has been gaining a significant level of attention in research for its usefulness in finding information on a graph, its connections with hard-sphere statistical mechanical theory, and other hard-particle models in physics. Therefore, further research on this topic is not only helpful in expanding mathematical knowledge but also in providing background for potential applications. In order to achieve that goal, a thorough review of what has been found is essential.

Consider an example for the independence number of a graph G below. For any graph G, at $x = -1$, then we have

$$I(G; -1) = i_0 - i_1 + ... + (-1)^{\alpha(G)} i_\alpha(G).$$

This equation determines the number of independent sets of even and odd sizes in the graph G above. Finding $\alpha(G)$ has been proved to be an *NP-complete* (nondeterministic polynomial-time complete) problem [4]. Polynomial time means its solution can be guessed and verified in polynomial time; nondeterministic means that no particular rule can be followed to make the guess. As a result of its complexity as a *NP-complete* problem, the computation of the independence polynomial is significantly difficult to determine.

There have been several projects aimed to shed light on independence polynomials. A classical question about the independence polynomial is its computation. Hopskin and Staton proved the formula for determining the independence polynomials for a path P_n while Engström's research resulted in significant progress. Specifically, Engström proved the upper bound on the independence polynomial of a graph G at -1. It is reasonable that we attempt to find bounds for the absolute value of the independence polynomial since its explicit computation is difficult to carry out. Hence, Engström's results opened up an important approach for tackling this topic.

Theorem 1.2.1 (Engström 2009)**.** *For any graph G we have*

$$|I(G; -1)| \leq 2^{\phi(G)}.$$

To achieve this result, Engström used techniques from topological combinatorics, which is originally the Discrete Morse Theory developed by Robin Forman [3] using ideas in algebraic topology. Levit and Mandrescu [7] gave an elementary proof of this same bound. Furthermore, they succeeded in finding a different upper bound for $I(G;-1)$ using the *cyclomatic number* of G.

Definition 1.2.1. The *cyclomatic number* of a graph G, denoted $\beta(G)$, is defined by

$$\beta(G) = E(G) - V(G) + q(G),$$

where $q(G)$ is the number of components in G.

They proved the following upper bound: relating the independence polynomials of G to $\beta(G)$ instead of decycling number $\phi(G)$.

Theorem 1.2.2 (Levit and Mandrescu 2013)**.** *If G is any graph, then*

$$|I(G;-1)| \leq 2^{\beta(G)}.$$

Levit and Mandrescu also explored the *maximum independent set* which is an independent set of maximum size. A graph G is said to be *well-covered* if all of its maximal sets are of the same cardinality defined by $\alpha(G)$. Their work led to a notable result.

Theorem 1.2.3 (Levit and Mandrescu 2013)**.** *If G is a unicyclic well-covered graph, and $G \neq C_3$, then we have $I(G;-1) \in \{-1, 0, 1\}$.*

Cutler and Kahl [2] proved the following Theorem which was proposed as a conjecture by Levit and Mandrescu [7]. The conjecture essentially asks how sharp the bound is.

Theorem 1.2.4 (Cutler and Kahl 2016)**.** *For every positive integer k and each integer q such that $|q| \leq 2^k$, there exists a graph G with $\phi(G) = k$ and $I(G;-1) = q$.*

For brevity, Cutler and Kahl defined the graph in Theorem 1.2.4 to be $(k,q)_\phi$-graph.

Definition 1.2.2. Let k be a positive integer and q be an integer such that $|q| \leq 2^k$. A graph with $\phi(G) = k$ and $I(G;-1) = q$ is called a $(k,q)_\phi$-graph.

Cao and Ren [1] generalized and improved Theorem 1.2.2 and figured out a sharper upper bound. Their method for determining a sharper bound on $|I(G;-1)|$ utilized the concept of non-$\tilde{3}$-cycles.

We will define the non-$\tilde{3}$-cycles used by Cao and Ren below.

Definition 1.2.3. If length of the graph is defined as the number of edges contained in the graph. We call a cycle in a graph G a $\tilde{3}$-*cycle* if its length is divisible by 3. If the length of a graph G is not divisible by 3, we call it a *non-*$\tilde{3}$*-cycle.*

The theorem below gives us a remarkable improvement in estimating the upper bound compared to previous studies. Estimation of $|I(G;-1)|$ is now considerably enhanced for graphs with non-$\tilde{3}$-cycles.

Theorem 1.2.5 (Cao and Ren 2020)**.** *If G contains a non-$\tilde{3}$-cycle, then*

$$|I(G;-1)| \leq 2^{\beta(G)} - \beta(G).$$

In addition, Cao and Ren [1] proposed an upper bound for graphs with non-$\tilde{3}$-cycles and no vertices of degree one.

Theorem 1.2.6 (Cao and Ren 2020)**.** *Let G be a graph with non-$\tilde{3}$-cycles. If G contains no vertices of degree one, then $|I(G;-1)| \leq 2^{\beta(G)-1}$.*

In another special case, if all cycles in G are vertex disjoint, the upper bound can be estimated using the following theorem also proved by Cao and Ren [1]. First, we will define *vertex disjoint*.

Definition 1.2.4. Two cycles in a graph G are *vertex disjoint* if they do not share any common vertices.

This leads to a notable result using $\tilde{3}$-cycles by Cao and Ren [1].

Theorem 1.2.7. *If all cycles of G are vertex disjoint, then $|I(G;-1)| \leq 2^k$, where k is the number of $\tilde{3}$-cycles of G.*

Since the beginning of research on the independence polynomials of a graph, several ventures have been made to pave the road to a clearer insight into this topic. The theorems above are highly constructive and serve well as the foundation for future investigations.

1.3 Our Research

Given the background above, one could ask how to improve the bound. As an attempt to generalize previous results, in this paper we propose a strengthening on the upper bound in Theorem 1.2.6. We also prove the density result related to this upper bound by adapting results of Theorem 1.2.4.

Theorem 1.3.1. *If the graph G contains a non-$\tilde{3}$-cycle, for all $\beta(G) \geq 1$, we have*

$$|I(G;-1)| \leq 2^{\beta(G)-1}.$$

where $\beta(G)$ is the cyclomatic number of G.

Further experiments suggested that this new bound is sharp. So, we also prove the density results for this new bound. We use $\beta(G)$ instead of $\phi(G)$ in our new density results. Accordingly, we define the adapted $(k,q)_\beta$-graph below.

Definition 1.3.1. Let k be a positive integer and q be an integer such that $|q| \leq 2^k$. A graph with $\beta(G) = k$ and $I(G; -1) = q$ is called a $(k, q)_\beta$-graph.

Theorem 1.3.2. *For each odd integer $q \in [0, 2^k]$, there is a connected (k, q)-graph G_v such that either*

$$I(G_v; -1) = \langle 2^{k-1}, 2^{k-1} - q \rangle$$

or

$$I(G_v; -1) = \langle -2^{k-1} + q, -2^{k-1} \rangle$$

The result in Theorem 1.3.2 helps us tremendously in obtaining the main density result below.

Theorem 1.3.3. *Given a positive integer k and an integer q with $|q| \leq 2^{k-1}$, there is a connected graph G with $\beta(G) = k$ and $I(G; -1) = q$.*

We will explore the proofs for these theorems in detail in Chapter 4.

Chapter 2

Graph Preliminaries

2.1 Independence Polynomials for P_n and C_n

2.1.1 A few results on determining the independence polynomials

We first consider paths and cycles. The proof provided by Hopkins and Staton [6] helps us determine the independence polynomial for a path P_n explicitly.

Theorem 2.1.1 (Hopkins and Staton). *For a graph P_n with n vertices where $n \geq 0$, the independence polynomial for any $x \in \mathbb{C}$ is given by,*

$$I(P_n; x) = \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} \binom{n-k+1}{k} x^k$$

When considering the cycle graph C_n, Hopkins and Staton also determined the independence polynomial for C_n.

Theorem 2.1.2 (Hopkins and Staton). *For a cycle C_n with n vertices where $n \geq 3$,* *the independence polynomial for* $x \in \mathbb{C}$ *is given by*

$$I(C_n; x) = 1 + \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{n}{k} \binom{n-k-1}{k-1} x^k.$$

2.1.2 Examples of computations for the independence polynomials

In order to demonstrate how the computation for independence polynomials is done, we will introduce a few basic examples. We compute the independence polynomials for path P_n with the number of vertices n ranging from 1 to 6.

$$I(P_1; x) = 1 + x$$

$$I(P_2; x) = 1 + 2x$$

$$I(P_3; x) = 1 + 3x + x^2$$

$$I(P_4; x) = 1 + 4x + 3x^2$$

$$I(P_5; x) = 1 + 5x + 6x^2 + x^3$$

$$I(P_5; x) = 1 + 6x + 10x^2 + 4x^3$$

Using these results, we can evaluate $I(P_n; x)$ at $x = -1$. When we substitute $x = -1$ into the polynomials, it easily follows that we get the following equations

$$I(P_1; -1) = 1 - 1 = 0$$
$$I(P_2; -1) = 1 - 2 = -1$$
$$I(P_3; -1) = 1 - 3 + 1 = -1$$
$$I(P_4; -1) = 1 - 4 + 3 = 0$$
$$I(P_5; -1) = 1 - 5 + 6 - 1 = 1$$
$$I(P_6; -1) = 1 - 6 + 10 - 4 = 1$$

We can observe from the list of the results above the pattern that the values of independence polynomials $I(P_n; -1) \in \{-1, 0, 1\}$. A more general pattern can be noted here if we partition the vertices of P_n into multiples of 3. To be specific, $I(P_{3n-2}; -1) = 0$ while

$$I(P_{3n-1}; -1) = I(P_{3n}; -1) = (-1)^n.$$

We will return to this interesting pattern later after introducing some handy tools to prove it. Similarly, we can compute the independence polynomials for cycle C_n except this time with the number of vertices n ranging from 3 to 8 or $n \geq 3$. Let's begin with C_3.

$$I(C_3; x) = 1 + 3x$$
$$I(C_4; x) = 1 + 4x + 2x^2$$
$$I(C_5; x) = 1 + 5x + 5x^2$$
$$I(C_6; x) = 1 + 6x + 9x^2 + 2x^3$$
$$I(C_7; x) = 1 + 7x + 14x^2 + 7x^3$$
$$I(C_8; x) = 1 + 8x + 20x^2 + 16x^3 + 2x^4.$$

Evaluating the values of the independence polynomials at -1 by substituting -1 yields the following results

$$I(C_3; -1) = 1 - 3 = -2$$
$$I(C_4; -1) = 1 - 4 + 2 = -1$$
$$I(C_5; -1) = 1 - 5 + 5 = 1$$
$$I(C_6; -1) = 1 - 6 + 9 - 2 = 2$$
$$I(C_7; -1) = 1 - 7 + 14 - 7 = 1$$
$$I(C_8; -1) = 1 - 8 + 20 - 16 + 2 = -1.$$

Again, we observe a unique pattern. We can generalize this pattern using the following identities

$$I(C_{3n}; -1) = 2(-1)^n,$$
$$I(C_{3n+1}; -1) = (-1)^n,$$
$$I(C_{3n+2}; -1) = (-1)^{n+1}.$$

Levit and Mandrescu [7] made notable progress on the topic of independence polynomials at -1 by proving these patterns. We will return to proving these patterns at the end of Section 2.2.

2.2 A Few Useful Tools

This section's goal is to provide techniques for constructions and operations on the considered graphs. We proceed with exploring some key definitions that will be useful later. First, consider the recursive formulas we found for paths and cycles [7]. We will start with partitioning the graph into independent sets that contain a selected vertex, say v, and the ones without it.

We proceed with exploring the subset of any graph G. Consider subset $S \subset V(G)$ of the vertex set and let $G - S$ be the subgraph with vertex set $v(G) - S$ and edges defined only for these vertices. So for a vertex $v \in V(G)$ of a graph G we can consider the subgraph with v removed, denoted $G - \{v\}$, or $G - v$ for convenience. This idea will be very helpful for proving a recursive identity for independence polynomials.

Also, in order to prove the recursive identity for the independence polynomials, will introduce the following general tools. First, we prove the formula for counting independent sets.

Proposition 2.2.1. *For any integer $k \geq 0$ and a graph G with $v \in V(G)$,*

$$i_k(G) = i_k(G - v) + i_{k-1}(G - N[v]).$$

Proof. Consider a graph G, let an integer $k \geq 0$, and let $v \in V(G)$. The idea behind this proof is to count the number of independent sets in G using partitioning. To be specific, we can partition the number of independent sets in G into the sets containing

v and those that do not contain v. The number of independent sets not containing v is given by $i_k(G-v)$. Notice that in order to count the number of independent sets containing v, we need to count the independent sets of size $k-1$ since v is already chosen. It follows that we need to exclude v and its neighbors. Hence, the number of independent sets containing v is given by $i_k(G-N[v])$. Putting the two results together yields

$$i_k(G) = i_k(G-v) + i_{k-1}(G-N[v]). \qquad \square$$

Using Proposition 2.2.1 we can prove the following result which will take us one step closer to the proof of recursive identity at the end of Section 2.

Proposition 2.2.2. *Let G be a graph with $v \in V(G)$, then*

$$I(G;x) = I(G-v;x) + xI(G-N[v];x).$$

Proof. Let G be a graph with $v \in V(G)$. Then, we have

$$I(G;x) = \sum_{k=0} i_k(G)x^k$$

Then, by Proposition 2.2.1, we have

$$\begin{aligned} I(G;x) &= \sum_k \left(i_k(G-v) + i_{k-1}(G-N[v])\right)x^k \\ &= \sum_k i_k(G-v)x^k + \sum_k i_{k-1}(G-N[v])x^k \\ &= \sum_k i_k(G-v)x^k + x\sum_k i_{k-1}(G-N[v])x^{k-1} \\ &= I(G-v;x) + xI(G-N[v];x). \qquad \square \end{aligned}$$

The formula in Proposition 2.2.2 is applicable at any value of x. Now, since our recursive identity only considers particular cases where $x = -1$, one may find the following corollary significantly advantageous.

Corollary 2.2.3. *Let G be a graph with $v \in V(G)$, then*

$$I(G; -1) = I(G - v; -1) - I(G - N[v]; -1).$$

Proof. Let G be a graph with $v \in V(G)$. Then, by Proposition 2.2.2, we have

$$I(G; x) = I(G - v; x) + xI(G - N[v]; x).$$

Substituting $x = -1$ into the equation in Proposition 2.2.2, yields

$$I(G; -1) = I(G - v; -1) - I(G - N[v]; -1). \qquad \square$$

Remark. With all the essential tools above, we can now begin exploring the proof of Levit and Mandrescu [8] for generalizing the patterns in our Examples 1 and 2 that demonstrated the independence polynomials of paths and cycles at -1. We observe that certain identities tend to hold if we partition the vertices into consider the length of modulo 3.

Theorem 2.2.4. *Let $n \geq 1$. For paths, we have*

$$I(P_{3n-2}; -1) = 0 \quad \textit{and} \quad I(P_{3n-1}; -1) = I(P_{3n}; -1) = (-1)^n.$$

For cycles, we have

$$I(C_{3n}; -1) = 2(-1)^n, \quad I(C_{3n+1}; -1) = (-1)^n, \quad \textit{and} \quad I(C_{3n+2}; -1) = (-1)^{n+1}.$$

Proof. We begin with proving the first identity using induction on n. Let $n = 1$ be given. From the first computation example in Section 2.1, we see that

$$I(P_1; x) = 1 + x$$
$$I(P_2; x) = 1 + 2x$$
$$I(P_3; x) = 1 + 3x + x^2.$$

Hence, it follows that when we substitute $x = -1$ into the independence polynomials above we get the following results

$$I(P_1; -1) = 1 + (-1) = 0$$
$$I(P_2; -1) = 1 + 2(-1) = 1 - 2 = (-1)^1$$
$$I(P_3; -1) = 1 + 3(-1) + (-1)^1 = 1 - 3 + 1 = -1 = (-1)^1$$

Thus, the result holds for $n = 1$. Suppose that the result holds for positive integers at most k. Then, evaluate a vertex v in each graph, we get,

$$I(P_{3(k+1)-2}; -1) = I(P_{3k+1}; -1).$$

By Corollary 2.2.3,

$$I(G; -1) = I(G - v; -1) - I(G - N[v]; -1).$$

Thus, taking v to be an endpoint at $P_{3k+1}-2$, we have

$$\begin{aligned} I(P_{3(k+1)-2};-1) &= I(P_{3k+1}-v;-1) - I(P_{3k+1}-N[v];-1) \\ &= I(P_{3k};-1) - I(P_{3k+1};-1) \\ &= (-1)^k - (-1)^k \\ &= 0. \end{aligned}$$

Using the above case, we also notice that,

$$\begin{aligned} I(P_{3(k+1)-1};-1) &= I(P_{3k+2};-1) \\ &= I(P_{3k+2}-v;-1) - I(P_{3k+2}-N[v];-1) \\ &= I(P_{3k+1};-1) - I(P_{3k});-1) \\ &= 0-(-1)^k \\ &= (-1)(-1)^k \\ &= (-1)^{k+1}. \end{aligned}$$

Similarly, using the two cases above we can also prove that

$$\begin{aligned} I(P_{3(k+1)};-1) &= I(P_{3k+3};-1) \\ I(P_{3(k+1)-1};-1) &= I(P_{3k+3}-v;-1) - I(P_{3k+3}-N[v];-1) \\ &= I(P_{3k+2};-1) - I(P_{3k+1});-1) \\ &= (-1)^{k+1} - 0 \\ &= (-1)^{k+1}. \end{aligned}$$

The result then follows by induction. For proof of part 2, we apply the result we achieved in part 1. First, let $n \geq 1$, then for every vertex v in each respective graph, we have the following.

$$\begin{aligned} I(C_{3n};-1) &= I(C_{3n}-v;-1) - I(C_{3n}-N[v];-1) \\ &= I(P_{3n-1};-1) - I(P_{3(n-1)};-1) \\ &= (-1)^n - (-1)^{n-1} \\ &= (-1)^n + (-1)(-1)^{n-1} \\ &= (-1)^n + (-1)^n \\ &= 2(-1)^n. \end{aligned}$$

Also,

$$\begin{aligned} I(C_{3n+1};-1) &= I(C_{3n+1}-v;-1) - I(C_{3n+1}-N[v];-1) \\ &= I(P_{3n};-1) - I(P_{3n-2};-1) \\ &= (-1)^n - (-1)^{n-1} \\ &= (-1)^n - 0 \\ &= (-1)^n. \end{aligned}$$

Finally,

$$\begin{aligned} I(C_{3n+2};-1) &= I(C_{3n+2}-v;-1) - I(C_{3n+2}-N[v];-1) \\ &= I(P_{3n+1};-1) - I(P_{3n-1};-1) \\ &= I(P_{3(n+1)-2};-1) - I(P_{3n-1};-1) \\ &= (-1)^n - (-1)^{n-1} \\ &= 0 - (-1)^n \\ &= (-1)(-1)^n \\ &= (-1)^{n+1}. \end{aligned}$$

□

In this Subsection, we have investigated the independence polynomial of a graph in general. One may question how the independence polynomial of the union of two disjoint graphs looks in relation to the independence polynomials of the individual graphs. The following result will help us take a closer look into the matter.

Proposition 2.2.5. *Let G_1 and G_2 be disjoint graphs and let $G = G_1 \cup G_2$, then*

$$I(G; x) = I(G_1; x)I(G_2; x).$$

Proof. Let two disjoint graphs G_1 and G_2 with $G_1 \cup G_2 = G$ be given. Consider the independence polynomial of the graph G we have

$$\begin{aligned}
I(G; x) &= \sum_k i_k(G)x^k \\
&= \sum_k i_k(G_1 \cup G_2)x^k \\
&= \sum_k \left(\sum_{l=0}^{k} i_l(G_1)i_{k-l}(G_2) \right) x^k \\
&= \left(\sum_k i_k(G_1)x^k \right) \left(\sum_k i_k(G_2)x^k \right) \\
&= I(G_1; x)I(G_2; x).
\end{aligned}$$

□

Now, we have gathered several useful tools that we can use in the exploration of Cutler and Kahl's density results in the next section.

Chapter 3

Density Results by Cutler and Kahl

We aim to prove the following density result.

Theorem 3.0.1. *For each odd integer $q \in [0, 2^k]$, there is a connected $(k, q)_\phi$-graph G_v such that either*

$$I(Gv; -1) = \langle 2^{k-1}, 2^{k-1} - q \rangle$$

or

$$I(Gv; -1) = \langle -2^{k-1} + q, -2^{k-1} \rangle.$$

In order to achieve that goal, we take a closer look at the techniques contributed by Cutler and Kahl [2] in attempts to prove the following result that Levit and Mandrescu first conjectured.

Theorem 3.0.2 (Cutler and Kahl 2016)**.** *For every positive integer k and each integer q such that $|d| \leq 2^k$, there is a graph G with $\phi(G) = k$ and $I(G; -1) = q$.*

We will see that the techniques used by Cutler and Kahl are noticeably useful in proving the density results for our new upper bound.

3.1 $(k, q)_\phi$-graphs, Brackets, Extensions, and Pasting

In order to prove the density results, Cutler and Kahl developed a few operation tools and techniques. These techniques include a special type of graph called a $(k, q)_\phi$*-graph* along with a few construction tools such as bracket, extension, and pasting. We begin by introducing the key definitions and then providing examples and illustrations for each concept.

Definition 3.1.1. Let G be a graph and $v \in V(G)$. The *rooted graph* of G at v, denoted G_v is defined to be simply the graph G with the vertex v labeled.

Suppose G is a non-empty graph. Consider an arbitrary vertex $v \in G$. By Corollary 2.2.3, we have

$$I(G; -1) = I(G - v, -1) - I(G - N[v]; -1)$$

This understanding leads us to the following definition.

Definition 3.1.2. (Bracket). Let G be a graph and $v \in V(G)$ be a vertex of graph G. Let

$$I(G - v; -1) = a$$

and

$$I(G - N[v]; -1) = b$$

We have,

$$I(G; -1) = a - b$$

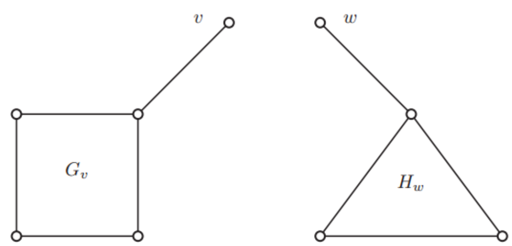

Figure 3.1: Rooted graphs G_v and H_w before pasting

Now, we can define the bracket of G as $I(G;-1) = a - b = \langle a, b\rangle$. We will define the graph obtained by pasting two rooted graphs together below.

Definition 3.1.3. If G_v and H_w are two rooted graphs, we define the graph $G_v \wedge H_w$ to be the graph obtained by pasting G_v and H_w together by identifying v with w. We write $u = v$ for short.

We will consider the example below for a better comprehension of how the pasting operation works.

Example 1. Consider the following example of the pasting operation. Let G_v and H_w be two distinct graphs rooted at v and w respectively. The figures below demonstrate the pasting operation that joins G_v and H_w.

Figures 2 and 3 give us a clear visualization of the pasting operation. The operation opens up our path for constructing graphs with the brackets we are looking for and thus, guides us to the pasting lemma below.

Lemma 3.1.1. *(Pasting Lemma) Let G_v and H_w be rooted graphs which are disjoint. Let the brackets of G_v and H_w have brackets $I(G_w;-1) = \langle a, b\rangle$ and $I(H_w,-1) = \langle c, d\rangle$ then*

$$I(G_v \wedge H_w;-1) = ac - bd = \langle ac, bd\rangle.$$

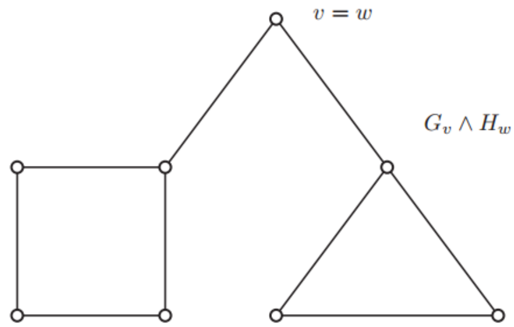

Figure 3.2: Graph $G_v \wedge H_w$

Proof. By definition of the bracket we have

$$I(G - v; -1) = a$$

$$I(G_v - N[v]; -1) = b$$

and

$$I(H_w - w; -1) = c$$

$$I(H_w - N[w], -1) = d.$$

Then, because G_v and H_w are disjoint and pasted together at v and w, we have

$$I(G_v \wedge H_w; -1) = I(G_v \wedge H_w - v; -1) - I(G_v \wedge H_w - N[v]; -1)$$

Then, by Proposition 2.2.5, we obtain

$$I(G_v \wedge H_w; -1) = I(G_v - v; -1)I(H_v - v; -1) - I(G_v - N[v]; -1)I(H_v - N[v]; -1).$$

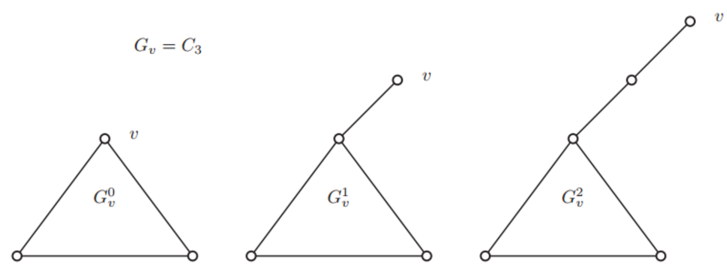

Figure 3.3: The l-extension of C_3

Since we know $v = w$, we can rewrite the equation above as

$$\begin{aligned} I(G_v \wedge H_w; -1) &= I(G_v - v; -1)I(H_W - w; -1) \\ &\qquad - I(G_v - N[v]; -1)I(H_w - N[w]; -1) \\ &= ac - bd \\ &= \langle ac, bd \rangle. \end{aligned}$$ □

Another helpful graph construction tool in our toolbox is the extension operation which is defined below.

Definition 3.1.4. Let G_v be a rooted graph and integer $l \geq 0$. An l-extension, denoted G_v^l is defined to be the graph obtained by pasting a path of length l to G_v at v and an endpoint of the path. In this manner, G_v^0 is an alternative way to denote G_v.

For a better understanding of how this tool works, we will illustrate this useful operation in the example below.

Example 2. (l-extension of C_3). We hereby illustrate a graph C_3. Consider $G_v = C_3$, then its extensions are $G_v^0 = C_3^0, G_v^1 = C_3^1, G_v^2 = C_3^2$ for $l = 0, 1, 2$ respectively (Figure 3.3).

Having the above construction tools in hand, we now see the relationship between

$\phi(G)$ and $\beta(G)$ with more clarity. The following example is one of the observations that now become more obvious using the pasting operation.

Example 3. An interesting observation is that $\beta(G) = \phi(G)$ for infinitely many graphs G with non-$\tilde{3}$-cycles. We can use the pasting operation to construct an example that allows us to join disjoint cycles n times, with $n \geq 1$. Consider a non-$\tilde{3}$-cycle C_5 extended by 1, we can let

$$G = C_5^1 \wedge C_5^1 \wedge ... \wedge C_5^1.$$

Then, for each integer $n \geq 1$, we observe that there exist n disjoint cycles, so the cyclomatic number for G is $\beta(G) = n$. In addition, to make G acyclic, or to make sure there are no more cycles in G, we need to delete at least one vertex from each cycle. Therefore, the decycling number $\phi(G)$ is also n. Since there are infinitely many integers n, we have as many graphs G with $\beta(G) = \phi(G)$. We used C_5^1 only, so each cycle is a C_5 where 5 is not divisible by 3, so each cycle in G is a non-$\tilde{3}$-cycle.

We have seen how the tool of pasting sheds light on our exploration of graphs with cycles. Now, one may ask what happens to the brackets when we apply the l-extension on a graph. Interestingly, there are changes in the brackets of the graph in that case. The intriguing pattern for them is shown in the lemma below.

Lemma 3.1.2 (Extension Lemma)**.** *Let G_v be a rooted graph with bracket $I(G_v; -1) = \langle a, b\rangle = a - b$, then*

$$I(G_v^1; -1) = -b = \langle a - b, a\rangle$$

$$I(G_v^2; -1) = -a = \langle -b, a - b\rangle$$

$$I(G_v^3; -1) = b - a = \langle -a, -b\rangle = -\langle a, b\rangle = -I(G_v; -1)$$

Proof. Suppose $I(G_v; -1) = a - b = \langle a, b\rangle$, then, by corollary 2.4

$$\begin{aligned} I(G_v^1; -1) &= I(G_v^1 - v; -1) - I(G_v^1 - N[v]; -1) \\ &= I(G_v^0; -1) - I(G_v^0 - v; -1) \\ &= (a - b) - a \\ &= -b \\ &= \langle a - b, a\rangle. \end{aligned}$$

Similarly,

$$\begin{aligned} I(G_v^2; -1) &= I(G_v^2 - v; -1) - I(G_v^2 - N[v]; -1) \\ &= I(G_v^1; -1) - I(G_v^0 - v; -1) \\ &= -b - (a - b) \\ &= \langle -b, a - b\rangle, \end{aligned}$$

and

$$\begin{aligned} I(G_v^3; -1) &= I(G_v^3 - v; -1) - I(G_v^3 - N[v]; -1) \\ &= I(G_v^2; -1) - I(G_v^1 - v; -1) \\ &= -a - (-b) \\ &= \langle -a, -b\rangle. \end{aligned}$$

We notice that

$$\begin{aligned}-a-(-b) &= -(a-b)\\ &= -\langle a,b\rangle\\ &= -I(G_v;-1).\end{aligned}$$

Hence, we have $I(G_v^3; -1 = b - a = \langle -a, -b\rangle = -\langle a, b\rangle = -I(G_v; -1)$. □

Remark. Alternatively, we can also observe that for a rooted graph G_v, we have $(G_v^l)^1 = G_v^{l+1}$ for any integer $l \geq 0$. For example, if we extend G_v by 2, we get the same graph G_v^2 as extending G_v^1 by 1. Thus $G_v^2 = (C_v^1)^1$. So, if we have $I(G_v^1; -1) = \langle a-b, a\rangle$, we can easily obtain that $I(G_v^1; -1) = \langle a-b, a\rangle$. We can reapply the first statement to the later extension of the graph G_v.

Example 4. Brackets of C_6. Now, we will work with an example to get a better hold of how the bracket and extension work on a cycle graph. Let a vertex $v \in V(C_6)$ then,

$$\begin{aligned}I(C_6;-1) &= I(C_6-v;-1) - I(C_6-N[v];-1)\\ &= I(P_5;-1) - I(P_3;-1)\\ &= (-1)^2 - (-1)\\ &= 1-(-1)\\ &= \langle 1,-1\rangle.\end{aligned}$$

Then using the extension lemma gives us the following brackets for each extension of

C_6

$$
\begin{aligned}
I(C_6^1) &= \langle 1-(-1), 1\rangle \\
&= \langle 2, 1\rangle, \\
I(C_6^2) &= \langle 2-1, 2\rangle \\
&= \langle 1, 2\rangle, \\
I(C_6^3) &= \langle 1-2, 1\rangle \\
&= \langle -1, 1\rangle, \\
I(C_6^4) &= \langle -1-1, 1\rangle \\
&= \langle -2, 1\rangle, \\
I(C_6^5) &= \langle -2-(-1), -2\rangle \\
&= \langle -1, -2\rangle, \\
I(C_6^6) &= \langle -1-(-2), -1\rangle \\
&= \langle 1, -1\rangle \\
&= I(C_6; -1).
\end{aligned}
$$

We observe that $I(C_6^6; -1) = \langle 1, -1\rangle = I(C_6; -1)$. This observation demonstrates that the patterns of the brackets for extensions of a graph repeat after six times of extension. We can see that there is a cyclic tendency in the nature of the set of brackets for these extended graphs. Using the same method, we obtain several brackets for extensions of a few common graphs summarized in the following table.

l	$I(C_6^l;-1)$	$I(C_5^l;-1)$	$I(P_5^l;-1)$
0	$\langle 1,-1\rangle$	$\langle 0,-1\rangle$	$\langle 0,-1\rangle$
1	$\langle 2,1\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$
2	$\langle 1,2\rangle$	$\langle 1,1\rangle$	$\langle 1,1\rangle$
3	$\langle -1,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$
4	$\langle -2,-1\rangle$	$\langle -1,0\rangle$	$\langle -1,0\rangle$
5	$\langle -1,-2\rangle$	$\langle -1,-1\rangle$	$\langle -1,-1\rangle$
6	$\langle 1,-1\rangle$	$\langle 0,-1\rangle$	$\langle 0,-1\rangle$

Example 5. We consider the extension operation on the path graph P_5. Let $v \in V(P_5)$ be an endpoint of the path. Then,

$$\begin{aligned} I(P_5;1) &= I(P_5 - v;-1) - I(P_5 - N[v];-1) \\ &= I(P_4;-1) - I(P_3;-1) \\ &= 0 - (-1) \\ &= \langle 0,-1\rangle. \end{aligned}$$

Applying the Extension Lemma yields,

$$\begin{aligned} I(P_5^0;-1) &= \langle 0,-1\rangle, \\ I(P_5^1;-1) &= \langle 1,0\rangle, \\ I(P_5^2;-1) &= \langle 1,1\rangle, \\ I(P_5^3;-1) &= \langle 0,1\rangle, \\ I(P_5^4;-1) &= \langle -1,0\rangle, \\ I(P_5^5;-1) &= \langle -1,-1\rangle, \\ I(P_5^6;-1) &= \langle 0,-1\rangle. \end{aligned}$$

Remark. We again notice that the bracket of an extension of length 6 P_5^6 is the same as the bracket of the original graph P_5^0. In fact, we can prove this pattern in a generalized manner.

Proposition 3.1.3. *Let G_v be a rooted graph with $I(G_v;-1) = a - b = \langle a, b\rangle$, then*

$$I(G_v;-1) = I(G_v^6;-1).$$

Proof. If G_v is a rooted graph at v with $I(G_v;-1) = a - b = \langle a, b\rangle$, then by the extension lemma, we have

$$\begin{aligned}
I(G_v^3;-1) &= b - a = \langle -a, -b\rangle = -\langle a, b\rangle = -I(G_v;-1),\\
I(G_v^4;-1) &= \langle -a - (-b), -a\rangle = \langle b - a, -a\rangle,\\
I(G_v^5;-1) &= \langle b - a - (-a), b - a\rangle = \langle b, b - a\rangle,\\
I(G_v^6;-1) &= \langle b - (b - a), b\rangle = \langle a, b\rangle.
\end{aligned}$$

Recall that $I(G_v;-1) = a - b = \langle a, b\rangle$, we conclude that $I(G_v^6;-1) = \langle a, b\rangle = I(G_v;-1)$. □

This proposition illustrates that the bracket of extensions to a graph shows a cycling phenomenon. In other words, the brackets for the extensions of a graph are cyclic and repeat every time we complete an extension of 6. Therefore, we can use this result to find the $I(G_v^l;-1)$ of any length l of extension using mod 6. The bracket for any length of extension $l \geq 6$ can be found in the table of brackets for $l < 6$.

3.2 Results on $(k, q)_\phi$-graphs

Cutler and Kahl were successful at proving the density result that was first conjectured by Levit and Mandrescu. The techniques they used and their results are helpful for our density proof. Now that we explored the techniques, we can look at the outline of the density proof. Since we will alter their result and walk through a detailed version in Chapter 4 when we show our results, in this subsection we will only briefly discuss the logic behind their proof to avoid repetition. For brevity, Cutler and Kahl introduced the $(k, q)_\phi$-graph definition and a relevant lemma.

Definition 3.2.1. Let k be a positive integer and q be an integer such that $|q| \leq 2^k$. A graph with $\phi(G) = k$ and $I(G; -1) = q$ is called a $(k, q)_\phi$-graph.

Lemma 3.2.1. *Let G and H be disjoint connected $(k_1, q_1)_\phi$ and $(k_2, q_2)_\phi$-graphs, respectively. Set $k_1 + k_2 = k$ and $q_1 q_2 = q$, then there exists a connected $(k, q)_\phi$-graph F such that $\phi(F) = k_1 + k_2 = k$ and $I(F; -1) = q_1 q_2 = q = I(G \cup H; -1)$.*

Proof. Let $G = G_v$ and $H = H_w$ and let

$$I(G_v; -1) = \langle a, b \rangle = q_1$$

$$I(G_v; -1) = \langle c, d \rangle = q_2.$$

By the Extension Lemma, we have that

$$I(G_v^1; -1) = \langle a - b, a \rangle = \langle q_1, a \rangle$$

$$I(G_v^2; -1) = \langle -b, q_1 \rangle.$$

Similarly, we also have

$$I(H_w^1; -1) = \langle c - d, c \rangle = \langle q_2, c \rangle$$

$$I(H_w^2; -1) = \langle -d, q_2. \rangle.$$

Now, applying the Pasting Lemma yields

$$I(G_v^2 \wedge H_w^2; -1) = \langle bd, q_1 q_2 \rangle.$$

We apply the Extension Lemma again to get

$$I((G_v^2 \wedge H_w^2)^1; -1) = \langle bd - q_1 q_2, bd \rangle = -q_1 q_2.$$

Recall that we can do more than one application of Extension Lemma for the length of 3,

$$\begin{aligned} I(((G_v^2 \wedge H_w^2)^1)^3; -1) &= I((G_v^2 \wedge H_w^2)^4; -1) \\ &= -I((G_v^2 \wedge H_w^2)^1; -1) \\ &= -(-q_1 q_2) \\ &= q. \end{aligned}$$

Set $(G_v^2 \wedge H_w^2)^4 = F$ for brevity. Then, we have $I(F, -1) = q_1 q_2 = q$. Since G and H are disjoint, we have $I(H \cup G; -1) = I(G; -1) I(H; -1) = q_1 q_2 = q$. The operations of extension and pasting do not add any cycles to the graph F so the decycling number of F remains unchanged. Thus, we have $\phi(F) = k_1 + k_2 = k$. Hence, F is a $(k, q)_\phi$-graph. □

Corollary 3.2.2. *If G is a $(k, q)_\phi$-graph, then there exists*

1. *a connected* $(k+1, 2q)_\phi$*-graph*

2. *a connected* $(k, -q)_\phi$*-graph.*

Lemma 3.2.3. *Let* $k \geq 1$ *be an integer. For every odd integer* $g \in [0, 2^k]$ *there is a* $(k, q)_\phi$*-graph* G_v *such that either* $I(G_v; -1) = \langle 2^k, 2^k - q\rangle$ *or* $I(G_v; -1) = \langle -2^k + q; -2^k\rangle$*.*

Proof. The idea is to use induction on k. If $k = 1$, then $\phi(C_6^1) = 1$ and $I(C_6^1; -1) = \langle 2, 1\rangle = 2 - 1 = 1$. Therefore, we have a graph G_v with $q = 1$ in the form of

$$I(G_v; -1) = \langle 2^1, 2^1 - q\rangle.$$

Thus, the result holds for $k = 1$. Suppose the result holds for $k - 1$ and let $q \in [0, 2^k]$ be an odd integer. We now consider two cases: $q \in [2^{k-1}, 2^k]$ and $q \in [0, 2^k]$.

Case 1: In the first case, if $q \in [2^{k-1}, 2^k]$ is an odd integer, then there exists an odd integer $r \in [0, 2^{k-1}]$ with $q = 2^k - r$. By assumption, there exists a $(k-1, 2^{k-1} - r)_\phi$-graph G_v such that either

$$\begin{aligned} I(G_v; -1) &= \langle 2^{k-1}, 2^{k-1} - (2^{k-1} - r)\rangle \\ &= \langle 2^{k-1}, r\rangle, \end{aligned}$$

or

$$\begin{aligned} I(G_v, -1) &= \langle -2^{k-1} + (2^{k-1} - r, -2^{k-1}\rangle \\ &= \langle -r, -2^{k-1}\rangle. \end{aligned}$$

If $I(G_v; -1) = \langle 2^{k-1}, r\rangle$, since we have $I(C_6^1; -1) = \langle 2, 1\rangle$ (see Example 4 for computation) and $\phi(C_6^1) = 1$, we can apply the Pasting Lemma to join G_v and

C_6^1 which yields

$$\begin{aligned} I(G_v \wedge C_6^1; -1) &= \langle (2)2^{k-1}, (1)r \rangle \\ &= \langle 2^k, r \rangle \\ &= \langle 2^k, 2^k - q \rangle. \end{aligned}$$

If $I(G_v, -1) = \langle -r, -r^{k-1} \rangle$, then we can, again, use the Pasting Lemma to join C_6^2 to G_v, and achieve

$$\begin{aligned} I(G_v \wedge C_6^2; -1) &= \langle (1)(-r), (2)(-2^{k-1}) \rangle \\ &= \langle -r, -2^k \rangle \\ &= \langle -2^k + q, -2^k \rangle. \end{aligned}$$

Additionally, we have $\phi(G_v \wedge C_6^2) = l - 1 + 1 = k$. Hence, $G_v \wedge C_6^2$ is a k, q-graph with $I(G_v \wedge C_6^2) = \langle -2^k + 1, -2^k \rangle$. Thus, it follows by induction for odd integers $q \in [2^k - 1, 2^k]$.

Case 2: If $q \in [0, 2^{k-1}]$ is an odd integer, then $q = 2^k - r$ for an odd integer $r \in [2^{k-1}, 2^k]$. Now, by Case 1, there is a $(k, q)_\phi$-graph G_v such that either

$$I(G_v; -1) = \langle 2^k, 2^k - r \rangle$$

or

$$I(G_v; -1) = \langle -2^k + r, -2^k \rangle.$$

If $I(G_v; -1) = \langle 2^k, 2^k - r \rangle = \langle 2^k, q \rangle$, then by Extension Lemma, we have

$$\begin{aligned} I(G_v^3; -1) &= \langle -2^k, -q \rangle, \\ I(G_v^4; -1) &= \langle -2^k + q, -2^k \rangle \end{aligned}$$

Therefore, we have G_v^4 as a $(k,q)_\phi$-graph with $I(G_v^2;-1) = \langle -2^k+q, -2^k\rangle$ which yields q using the bracket. Furthermore, if $I(G_v;-1) = \langle -2^k+r, -2^k\rangle = \langle -q, -2^k\rangle$, then by the Extension Lemma, we have

$$I(G_v^1;-1) = \langle 2^k - q, -q\rangle$$

$$I(G_v^2;-1) = \langle 2^k, 2^k - q\rangle.$$

Thus, we have G_v^2 as a (k, q_ϕ)-graph with $I(G_v^2 : -1) = \langle 2^k, 2^k - q\rangle$. The results then will follow by induction for odd integer $q \in [0, 2^{k-1}]$. ☐

3.3 Density Results

In this subsection, we will prove the density results below using previous results from Chapter 3. This result is foundational to our adaptation.

Theorem 3.3.1 (Cutler and Kahl 2016). *Given a positive integer k and an integer q with $|q| \leq 2^k$, there is a connected graph G with $\phi(G) = k$ and $I(G;-1) = q$.*

Proof. By 3.2.1, we know disconnected (k,q)-graphs are suffice. Thus, it is not required to produce connected (k,q)-graphs for all $|q| \leq 2^k$.

Since $I(G \cup K_1;-1) = 0$ for all G, we can consider the case $q = 0$ done for all integers k. The goal now is to prove this claim using induction on k. When $k = 1$ then $I(C_6;-1) = 2 = \langle 1, -1\rangle$ and, as noted in Table 1, when taking extensions of C_6 we rotate through all members of $\{2, 1, -1, -2\}$. Thus, the theorem holds for $k = 1$.

Suppose $(k,q)_\phi$-graphs are constructible for all q with $|q| \leq 2^{k-1}$. By Corollary 3.2.2 (a), we have $(k,q)_\phi$-graphs that are constructible for all even integers q. By Corollary 3.3.2.2(b), we only need to construct a $(k,q)_\phi$-graphs with positive $q \leq 2^k$.

Thus, if we can construct $(k, q)_\phi$-graphs for each odd integer $q \in [0, 2^k]$. Now, in the proof of Lemma 3.2.3 we already show that it is constructible. Thus, by Lemma 3.2.3, we complete the induction which completes the proof. $\square$

Chapter 4

Our Results

We have gathered the tools and concepts needed to prove the results we introduced earlier in the first section. We will present our proofs of the upper bound, the density proof related to it, and the two lemmas that help us arrive at the density proof for which we adapt Cutler and Kahl's method in the previous sections. We begin with a step-by-step proof of the new bound in Section 4.1. The idea is to use previous results by Levit and Mandrescu on the graphs with non-$\tilde{3}$-cycles used by Cao and Ren. Then we prove the relevant density results in Section 4.2.

4.1 Our upper bound

Using previous results by Cao and Ren, and Levit and Mandrescu, we are able to remove the condition that G must contain no vertices of degree one. This result helps us strengthen the upper bound which will be proved below using induction on the cyclomatic number $\beta(G)$. Given that the graph G contains non-$\tilde{3}$-cycle, we will consider two main cases of whether or not there are intersecting cycles in G.

Theorem 4.1.1. *If a graph G contains a non-$\tilde{3}$-cycle, for all $\beta(G) \geq 1$, we have*

$|I(G;-1)| \leq 2^{\beta(G)-1}$.

Proof. We want to show that in graphs with non-$\tilde{3}$-cycles, for all $\beta(G) \geq 1$,

$$|I(G;-1)| \leq 2^{\beta(G)-1}.$$

We establish the proof by induction on $\beta(G)$. If $\beta(G) = 1$, we can evaluate both sides as follows. For the left hand side, by Theorem 1.2.5 we have $|I(G;-1)| \leq 2^{\beta(G)} - \beta(G)$. Thus,

$$|I(G;-1)| \leq 2^1 - 1 = 1.$$

Computing the right hand side yields $2^{\beta(G)-1} = 2^{1-1} = 2^0 = 1$. Therefore, we have that $|I(G;-1)| \leq 2^{\beta(G)-1}$ holds for $\beta(G) = 1$.

Now, let G be a graph with $\beta(G) \geq 2$. Let $\beta(G) = k+1$ where $\beta(G) \geq 2$, then we have $k = \beta(G) - 1$. We need to consider two cases: if G has intersecting cycles and if all cycles in G are pairwise disjoint.

Case 1: If there are intersecting cycles, then there exists a vertex v that is on at least two cycles. It follows that when we remove this vertex v, we will delete at least two cycles in G. Thus, we can choose this vertex v such that $\beta(G - v) \leq k - 1$. Using Proposition 2.2.2 we know $I(G;-1) = I(G - v;-1) - I(G - N[v];-1)$. But then, by triangle inequality we have

$I(G;-1) = |I(G-v;-1) - I(G-N[v];-1| \leq |I(G-v;-1)| + |I(G-N[v];-1|$.

Thus, we can write

$$|I(G;-1)| \leq |I(G-v;-1)| + |I(G-N[v];-1)| = 2^{k-1} + 2^{k-1} = 2(2^{k-1}) = 2^k.$$

Since $k = \beta(G) - 1$, by substituting $k = \beta(G) - 1$, we get

$$|I(G; -1)| \leq 2^{\beta(G)-1}.$$

Case 2a If there are $\tilde{3}$-cycles in G and all cycles are vertex disjoint. Then, there exists an arbitrary $\tilde{3}$-cycle, say C, in G. We can pick a vertex u on cycle C in G. Then, by induction hypothesis $\beta(G - u) \leq k$ and $G - u$ contains a non-$\tilde{3}$-cycle. We consider when $\beta(G - N[u]) \leq k - 1$ and when $\beta(G - N[u]) = k$.

Subcase 2a(i) Consider when removing the neighborhood including u, we have $\beta(G - [u]) \leq k - 1$.

Then,

$$|I(G; -1) \leq |I(G - u; -1)| + |I(G - N[u], -1)|.$$

It is enough to prove that $I(G - u; -1) \leq 2^{k-1}$ and $|I(G - N[u], -1)| \leq 2^{k-1}$. By induction hypothesis, we have $I(G-u; -1) \leq 2^{k-1}$. Also, by Theorem 1.2.2, $|(G - N[u], -1)| \leq 2^{k-1}$. Thus, we have

$$|I(G; -1)| \leq 2^{k-1} + 2^{k-1} = (2)2^{k-1} = 2^k$$

Recall that $\beta(G) = k + 1$ by assumption. We get

$$|I(G; -1)| \leq 2^{\beta(G)-1}.$$

Subcase 2a(ii) If $\beta(G) - N[u]) = k$ then $G - N[u]$ contains a non-$\tilde{3}$-cycle. In that case, we have

$$|I(G; -1) \leq |I(G - u; -1)| + |I(G - N[u], -1)|.$$

Then by induction hypothesis we have,

$$|I(G;-1) \leq 2^{k-1} + 2^{k-1} = 2(2^{k-1}) = 2^k = 2^{\beta(G)-1}.$$

Case 2b If all cycles in G are non-$\tilde{3}$-cycles, then we can apply the induction hypothesis. Specifically, we have $\beta(G-u) \leq k$ and $\beta(G-N[u]) \leq k$ Let u be a vertex on the $\tilde{3}$-cycle, then $\beta(G-u) \leq k$ and $G-u$ contains non-$\tilde{3}$-cycles.

Then by induction hypothesis, we have $I(G-u;-1) \leq 2^{k-1}$. Also, if $G-N[u]$ contains a cycle then it must be a non-$\tilde{3}$-cycle, thus, induction hypothesis applies, yields $I(G-N[u];-1) \leq 2^{k-1}$. If there is no cycle, then using Theorem 1.2.2 we have

$$|I(G-N[u];-1) \leq 2^{\beta(G)} = 2^0 = 1.$$

Otherwise, we know $1 < 2^{k-1}$. Thus, using Theorem 1.2.2, we achieve

$$|I(G;-1)| \leq 2^{k-1} + 2^{k-1} = 2(2^{k-1}) = 2^k = 2^{\beta(G)-1}. \qquad \square$$

We have proved the upper bound for all graph G that contains non-$\tilde{3}$-cycle. In the next subsection, we will prove the density results related to this strengthened bound.

4.2 Our density result

In this subsection, we aim to adapt Cutler and Kahl's density result [2] for the new bound above. We will include a detailed proof after introducing and proving two lemmas below. These two lemmas will be useful in proving the density theorem later.

Lemma 4.2.1. *Let G and H be two disjoint graphs. The cyclomatic number of the*

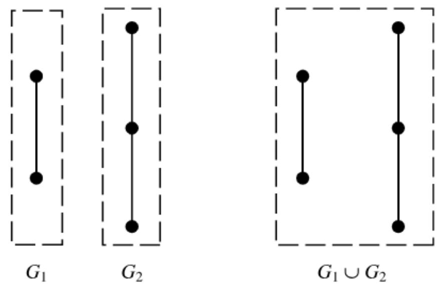

Figure 4.1: P_2, P_3 and their disjoint union

disjoint union of G *and* H *is equal to the sum of their respective cyclomatic numbers.*

$$\beta(G \cup H) = \beta(G) + \beta(H)$$

Proof. The cyclomatic number β of a graph G is given as,

$$\beta(G) = e(G) - n(G) + q(G),$$

where $e(G)$ is the number of edges, $n(G)$ is the number of vertices, and $q(G)$ is the number of components of G. Let G be a graph and H be another graph disjoint from G. Let I be the disjoint union of graphs G and H, so that $I = G \cup H$. (See Figure 5 for an example.)

Then, we have

$$e(I) = e(G) + e(H),$$
$$n(I) = n(G) + n(H),$$
$$q(I) = q(G) + q(H).$$

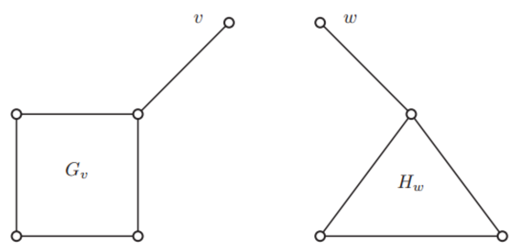

Figure 4.2: Graph G_v and H_w

Then the cyclomatic number $\beta(I)$ is given by

$$\begin{aligned}\beta(I) &= e(I) - n(I) + q(I)\\ &= e(G) + e(H) - (n(G) + n(H)) + q(G) + q(H)\\ &= e(G) - n(G) + p(G) + e(H) - n(H) + q(H)\\ &= \beta(G) + \beta(H).\end{aligned}$$

□

The next lemma shows the cyclomatic number of a graph constructed by pasting two rooted graphs together.

Lemma 4.2.2. *Let two rooted graphs G_v and H_w be rooted at v and w respectively. If we paste G_v and H_w together at v and w, then*

$$\beta(G_v \wedge H_w) = \beta(G_v) + \beta(H_w).$$

Proof. The two graphs G_v and H_w are pasted at v and w. Consider $e(G_v \wedge H_w)$. The number of edges remains unchanged, so

$$e(G_v \wedge H_w) = e(G_v) + e(H_w).$$

Now, consider the number of vertices, since after pasting $v = w$, we lose one vertex. Thus, we have $n(G_v \wedge H_w) = n(G_v) + n(H_w) - 1$. Consider the number of components

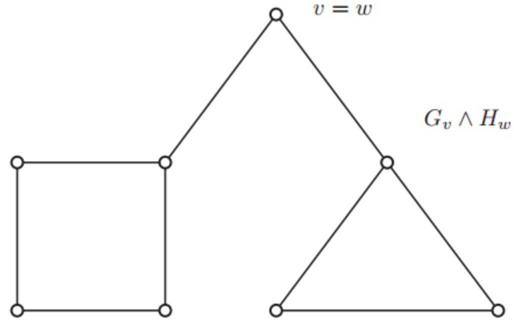

Figure 4.3: G_v and H_w pasted at $v = w$

of $G_v \wedge H_w$. When the two disjoint graphs are pasted together, two components are now joined at $v = w$, so, the initial disjoint two components containing v and w become one component. Hence, we have

$$q(G_v \wedge H_w) = q(G_v) + q(H_w) - 1.$$

Now, by definition, the cyclomatic number of $G_v \wedge H_w$ is given as:

$$\beta(G_v \wedge H_w) = e(G_v \wedge H_w) - n(G_v \wedge H_w) + q(G_v \wedge H_w).$$

Hence, the cyclomatic number of $(G_v \wedge H_w)$ can also be expressed as:

$$\begin{aligned}
\beta(G_v \wedge H_w) &= e(G_v) + e(H_w) - [n(G_v) + n(H_w) - 1] + [q(G) + q(H_w) - 1] \\
&= e(G_v) + e(H_w) - [n(G_v) + n(H_w)] + 1 + [q(G) + q(H_w)] - 1 \\
&= e(G_v) + e(H_w) - n(G_v) - n(H_w) + q(G) + q(H_w) \\
&= e(G_v) - n(G_v) + q(G_v) + e(H_w) - n(H_w) + q(H_w) \\
&= \beta(G_v) + \beta(H_w)
\end{aligned}$$

Recall that by Lemma 4.2.1, we have

$$\beta(G_v \cup H_w) = \beta(G_v) + \beta(H_w).$$

So, we also have

$$\beta(G_v \wedge H_w) = \beta(G_v) + \beta(H_w) = \beta(G_v \cup H_w). \qquad \square$$

Definition 4.2.1. Let k be a positive integer and q be an integer such that $|q| \leq 2^k$. A graph with $\beta(G) = k$ and $I(G; -1) = q$ is called a $(k, q)_\beta$-graph.

With Lemma 4.2.1 and Lemma 4.2.2 confirmed, we now begin proving the density result for the following theorem adapted from Cutler and Kahl. The idea is to find $(k, q)_\beta$-graph for each pair of integer k, q. We define $(k, q)_\beta$-graph above. Our goal is to prove that for every pair of integers k and q we can find the desired $(k, q)_\beta$-graph, including the cases for even and odd integers. To achieve this goal, we first prove that it is possible to construct such a graph for odd integers in the theorem below.

Theorem 4.2.3. *For each odd integer $q \in [0, 2^k]$, there is a connected $(k, q)_\beta$-graph G_v such that*

1. $I(G_v; -1) = \langle 2^{k-1}, 2^{k-1} - q \rangle$
2. $I(G_v; -1) = \langle -2^{k-1} + q, -2^{k-1} \rangle$.

Proof. We will follow the proof of Theorem 3.2.3 by Cutler and Kahl [2] explained in Chapter 3. The proof uses induction on k. For $k = 1$, we see that that the bracket C_5^0 has the necessary form,

$$I(C_5^0; -1) = \langle 0, -1 \rangle = 0 - (-1) = 1$$

and

$$\beta(C_5^0) = \langle 1, 0\rangle = \langle 2^0, 2^0 - 1\rangle.$$

Suppose the hypothesis of the statement is true for $k-1$; we aim to find a $(k,q)_\beta$-graph for each odd $q \in [0, 2^{k-1}]$ with bracket $\langle 2^{k-1}, 2^{k-1} - q\rangle$ or $\langle -2^{k-1} + q, -2^{k-1}\rangle$. We consider two cases as follows.

Case 1: If $q \in [2^{k-2}, 2^{k-1}]$, then $q = 2^{k-1} - r$ for some $r \in [0, 2^{k-2}]$ then we can prove the lemma by induction. Specifically, by induction, there exists a $(k-1, 2^{k-2} - r)_\beta$-graph H_w such that either (subcase A)

$$I(H_w; -1) = \langle 2^{k-2}, 2^{k-2} - (2^{k-2} - r)\rangle$$

or (subcase B)

$$I(H_w; -1) = \langle -2^{k-2} + (2^{k-2} - r, -2^{k-2}\rangle.$$

For subcase A, we have

$$I(H_w; -1) = \langle 2^{k-2}, 2^{k-2} - (2^{k-2} - r)\rangle = \langle 2^{k-2}, r\rangle.$$

But then, we know $I(C_6^1; -1) = \langle 2, 1\rangle$ (see Example 4 for computation) and $\beta(C_6^1) = 1$. Thus, using the Pasting Lemma we get

$$\begin{aligned} I(H_w \wedge C_6^1; -1) &= \langle 2^{k-2}(2), r(1)\rangle \\ &= \langle 2^{k-1}, r\rangle \\ &= \langle 2^{k-1}, 2^{k-1} - q\rangle. \end{aligned}$$

Note that by Lemma 4.2.2, we proved that for any pair of distinct rooted graphs

G_v and H_w,

$$\beta(H_w \wedge C_6^1) = \beta(H_w) + \beta(C_6^1) = k - 1 + 1 = k.$$

Hence, $H_w \wedge C_6^1$ is a $(k, q)_\beta$-graph with $I(H_w \wedge C_6^1) = \langle 2^{k-1}, 2^{k-1} - q \rangle$ which yields q using the bracket. Thus, it follows by induction for odd integers $q \in [2^{k-2}, 2^{k-1}]$.

For subcase B, we have

$$I(H_w; -1) = \langle -r, -2^{k-2} \rangle.$$

But then, we can use the pasting operation and Pasting Lemma on H_w and C_6^2. Since $I(C_6^2) = \langle 1, 2 \rangle$, we have

$$\begin{aligned} I(H_w \wedge C_6^2; -1) &= \langle -r(1), -2^{k-2}(2) \rangle \\ &= \langle -r, -2^{k-1} \rangle \\ &= \langle -2^{k-1} + q, -2^{k-1} \rangle. \end{aligned}$$

Note that by Lemma 4.2.2, we also know

$$\begin{aligned} \beta(H_w \wedge C_6^2) &= \beta(H_w) + \beta(C_6^2) \\ &= k - 1 + 1 = k. \end{aligned}$$

Case 2: If $q \in [0, 2^{k-2}]$, then there exists an odd $r \in [2^{k-2}, 2^{k-1}]$ such that

$$q = 2^{k-1} - r.$$

By Case 1, we know that there exists at least a $(k, q)_\beta$-graph H_w with a bracket determined as either (subcase A)

$$I(H_w; -1) = \langle 2^{k-1}, 2^{k-1} - r \rangle$$

or (subcase B)

$$I(H_w; -1) = \langle -2^{k-1} + r, -2^{k-1} \rangle.$$

For subcase A we have independence polynomial for H_w at -1 determined with bracket

$$I(H_w; -1) = \langle 2^{k-1}, q \rangle \ = \langle 2^{k-1}, 2^{k-1} - r \rangle$$

Recall that we can extend any rooted graph with extension operation as many times as we need. Then, using the Extension Lemma, we can determine brackets for each case as follows,

$$\begin{aligned}
I(H_w^0; -1) &= \langle 2^{k-1}, q \rangle \\
I(H_w^1; -1) &= \langle 2^{k-1} - q, 2^{k-1} \rangle \\
I(H_w^2; -1) &= \langle -q, 2^{k-1} - q \rangle \\
I(H_w^3; -1) &= \langle -2^{k-1}, -q \rangle \\
I(H_w^4; -1) &= \langle -2^{k-1} + q, -2^{k-1} \rangle.
\end{aligned}$$

Note that when we use extension operation on H_w, each extended vertex adds one edge to the graph. In addition, we add no new components in this operation. Thus, the cyclomatic number

$$\begin{aligned}
\beta(H_w^2) &= e(H_w^2) - n(H_w^2) + q(H_w^2) \\
&= \beta(H_w) + 2 - 2 \\
&= k - 2 + 2 = k.
\end{aligned}$$

Therefore, we have $\beta(H_w) = \beta(H_w^2) = \beta(H_w^4) = k$. Thus H_w^4 is a $(k,q)_\beta$-graph with $I(H_w^4;-1) = \langle -2^{k-1}+q, -2^{k-1}\rangle$ which yields q using the bracket.

For subcase B, we have that the independence polynomial for H_w can be determined using the bracket $I(H_w;-1) = \langle -q, -2^{k-1}\rangle$. Then, using the Extension Lemma, we get the following results,

$$I(H_w^0;-1) = \langle -q, -2^{k-1}\rangle$$
$$I(H_w^1;-1) = \langle 2^{k-1}-q, q\rangle$$
$$I(H_w^2;-1) = \langle 2^{k-1}, 2^{k-1}-q\rangle.$$

Thus, we have H_w^2 as a $(k,q)_\beta$-graph with $I(H_w^2;-1) = \langle 2^{k-1}, 2^{k-1}-q\rangle$. The results then follow by induction for odd integer $q \in [0, 2^k-1]$. □

We now confirmed that the $(k,q)_\beta$-graphs are constructible for all odd integers. With 4.2.3 in hands, we can prove the density result below.

Theorem 4.2.4. *Given a positive integer k and an integer q with $|q| \leq 2^{k-1}$, there is a connected graph G with $\beta(G) = k$ and $I(G;-1) = q$.*

Proof. We follow the proof to Theorem 3.3.1. Let $G = \wedge_{i=1}^k C_5^2$ which means joining C_5^2 together k times using the pasting operation. We have

$$G = \wedge_{i=1}^k C_5^2 = C_5^2 \wedge C_5^2 \wedge \ldots \wedge C_5^2.$$

Then, we have $\beta(G) = k$ and $I(G;-1) = \langle 1,1\rangle = 0$. Thus, consider $q = 0$ true for all k.

Our proof proceeds inductively on k. When $k = 1$ then $I(C_6;-1)$. When $k = 1$ then $I(C_6^1;-1) = \langle 2,1\rangle = 2-1 = 2^0 = 1$ as noted in Table 3.1. Now, assume that $(k-1,q)_\beta$-graphs are constructible for all $q \leq 2^{k-1}$. We have that $(k,q)_\beta$-graphs for even integers and negative integers q with $|q| \leq 2^{k-1}$ are constructible. Thus, we only need to construct $(k, q)_\beta$-graphs for odd integers in $[0, 2^{k-1}]$. However, in the proof of Theorem 4.2.3 above, we have proved, in detail, that such graphs are constructible. Thus, by Theorem 4.2.3, we complete the induction for the proof. $\square$

Chapter 5

Future Research

In this section, we discuss some questions for further research in both theoretical and applied manners. As mentioned above, a significant result by Cao and Ren [1] has led to new paths for investigation in the topic.

Theorem 5.0.1 (Cao and Ren 2020)**.** *Let G be a graph with non $\tilde{3}$-cyles. If G contains no vertices of degree one, then $|I(G;-1)| \leq 2^{\beta(G)-1}$.*

Our results improved the bound by removing the condition of having no vertices of degree one. Additionally, we proved a density result for the new bound. Cao and Ren proved several other interesting results in their research. These results significantly contribute to our knowledge of the independence polynomial of a graph at -1. For example, Cao and Ren also proved the following results.

Theorem 5.0.2 (Cao and Ren 2020)**.** *If all cycles of G are pairwise disjoint, then $|I(G;-1)| \leq 2^k$, where k is the number of $\tilde{3}$-cycles of G.*

This theorem by Cao and Ren suggests an interesting question for further investigation. The bound in Theorem 5.0.2 is for graphs whose all cycles are pairwise disjoint. It will be helpful to check if we can improve this theorem to include graphs whose cycles are not pairwise disjoint. Furthermore, it is worth investigating whether or not there is a density result for this new theorem. Exploration in these directions will potentially help generalize the results and expand our current knowledge of the topic.

In this thesis, we considered the upper bound of $|I(G;-1)|$ in graphs that contain non-$\tilde{3}$-cycles. A graph may contain several cycles with different lengths. It is possible that a graph has both non-$\tilde{3}$-cycles and $\tilde{3}$-cycles. We are interested in exploring how the upper bound looks like for a graph containing both non-$\tilde{3}$-cycles and $\tilde{3}$-cycles. Let $\phi_{\tilde{3}}(G)$ be the minimum number of vertices needed to remove from G so that G contains no $\tilde{3}$-cycles. Is it true that $|I(G,-1)| \leq 2^{\phi_{\tilde{3}}(G)}$?

On the mission of further investigation on the independence polynomial of a graph at -1, we tested a preliminary question for this problem. We checked if it is true that if $\phi_{\tilde{3}}(G) = 0$ then $|I(G;-1) \leq 1$. Some experiments on the preliminary questions suggest that the statement holds for some graphs with $\phi_{\tilde{3}} = 0$.

Consider the following examples to further explore this topic. The graph C_4 contains no $\tilde{3}$-cycles, hence, we do not need to remove any vertex to destroy its $\tilde{3}$-cycles. Thus, we have $\phi_{\tilde{3}}(C_4) = 0$. Evaluate the independence polynomial of C_4 at -1 yields

$$\begin{aligned} I(C_4;-1) &= 1 + 4x + 2x^2 \\ &= 1 + 4(-1) + 2(-1)^2 \\ &= 1 - 4 + 2 = -1. \end{aligned}$$

Similarly, we have the independence polynomials of a few other non-$\tilde{3}$-cycles as follows.

Computing the independence polynomial of C_5, we get

$$\begin{aligned} I(C_5; -1) &= 1 + 5x + 5x^2 \\ &= 1 + 5(-1) + 5(-1)^2 \\ &= 1 - 5 + 5 = 1. \end{aligned}$$

Carrying out the same task for C_7 yields

$$\begin{aligned} I(C_7; -1) &= 1 + 7x + 14x^2 + 7x^3 \\ &= 1 + 7(-1) + 14(-1)^2 + 7(-1)^3 \\ &= 1 - 7 + 14 - 7 = 1. \end{aligned}$$

Similarly, for C_8 we have

$$\begin{aligned} I(C_8; -1) &= 1 + 8x + 20x^2 + 16x^3 + 2x^4 \\ &= 1 + 8(-1) + 20(-1)^2 + 16(-1)^3 + 2(-1)^4 \\ &= 1 - 8 + 20 - 16 + 2 = -1. \end{aligned}$$

There seems to be a pattern that $I(G, -1) \leq 1$ and rotating between -1 and 1 for cycle graphs with $\phi_{\bar{3}}(G) = 0$. The intriguing pattern above suggests that further investigation into this direction is necessary. Note that besides cycle graphs like the examples above, it will be helpful to experiment with more classes of graph with $\phi_{\bar{3}}(G) = 0$ such as book graphs (see Figure 5.1). If the base case can be proved, the proof for graphs with $\phi_{\bar{3}}(G) \geq 1$ can be achieved by induction.

The topic of independence polynomials of graphs at -1 has been researched not only for theoretical reasons but also for its usefulness in hard-sphere particle physics. Therefore, collaborations between graph theorists and statistical physicists may also

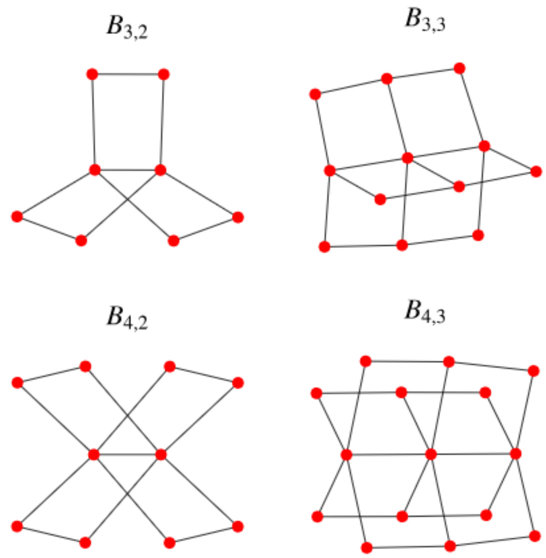

Figure 5.1: Book graphs

help with developing effective applied research and more applications of the topic.

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