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On the Number of Strong Dominating Sets in a Graph

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Abstract

This thesis investigates various problems related to the number of strong dominating sets in a graph. Given a graph G, a set of vertices D is said to be dominating if every vertex outside of D has a neighbor in D. Bród and Skupień proved that the number of dominating sets in a tree T on n vertices is at most $2^{n-1}+1$. A set S of vertices in a graph G is a strong dominating set if every vertex x outside of S has a neighbor $y \in S$ with $d(y) \geq d(x)$. We investigate the number of strong dominating sets in paths and binary trees. We also give bounds on the number of strong dominating sets in regular graphs and trees.

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On the number of strong dominating sets in a graph

by

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Chapter 1

Introduction

This chapter aims to introduce fundamental concepts of graph theory concerning strong dominating sets and to explore existing questions associated with these sets.

We begin by defining key graph theory terminology crucial for the results presented in this thesis. A *graph* G is an ordered pair $G = (V, E)$, where V is a non-empty, finite set and $E \subseteq \binom{V}{2}$, where $\binom{V}{2} = \{X \subseteq V : |X| = 2\}$. The *vertex set* of G, denoted $V = V(G)$, is the set of all vertices in G. The *edge set* is denoted $E = E(G)$ and the elements of E are called *edges* [10]. Suppose $u, v \in V$ are the endpoints of an edge. This edge is denoted uv and we say u and v are adjacent or they are *neighbors*, denoted $u \sim v$. If $e = uv$, then the edge e is incident to the vertices u and v. In the graph in Figure 1.1 the vertex set is $\{u, v, w, x, y\}$ and the edge set is $\{uv, uw, ux, vw, vx, wx, xy\}$. Furthermore, vertices u and v are neighbors whereas u and y are not neighbors.

We will only deal with simple undirected graphs, graphs with no loops or multiple edges. A simple graph is graph having no loops or multiple edges as shown in Figure 1.1. The *degree* of a vertex v in a graph G, denoted $d(v)$ is the number of edges incident to v. A *leaf* is a vertex of degree 1. The *open neighborhood* of v is the set of vertices

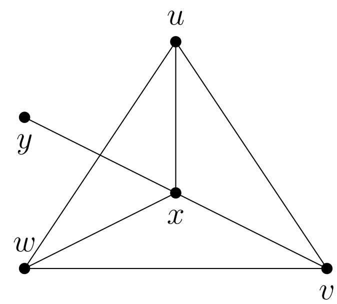

Figure 1.1: An example of a simple graph.

which are adjacent to v. Symbolically we can represent the open neighborhood of v as $N(v) = \{u \in V(G) : uv \in E(G)\}$, On the other hand, the *closed neighborhood* of a vertex v is $N[v] = N(v) \cup \{v\}$. For example, $N(x) = \{u, v, w, y\}$, whereas $N[x] = \{x, u, v, w, y\}$. The size of the neighborhood, denoted $|N(v)|$ is equal to $d(v)$. In the graph in Figure 1.1, we have $d(x) = 4$.

Our study is focused on dominating sets, specifically strong dominating sets in particular classes of graphs. The study of dominating sets in graphs began around 1960, however, the subject has roots dating back to 1862 when de Jaenisch studied the minimum number of queens necessary to dominate a chessboard [5]. In a broader context, dominating sets find applications in diverse fields such as determining bus routes, designing computer communication networks, and analyzing social network dynamics [5]. Moreover, dominating sets are important in routing computations, especially in the context of mobile networks.

Definition 1.0.1 (Dominating set)**.** A subset D of the vertex set $V(G)$, of a simple graph G, is a *dominating set* if every vertex $x \in V \setminus D$ has a neighbor in D. We denote the set of all dominating sets in a graph G by $\mathcal{D}(G)$ and $\partial_i(G) = |\mathcal{D}(G)|$ be the number of dominating sets in G.

In any simple graph G, there is at least one dominating set, namely the entire vertex set $V(G)$. For instance, consider the graph in Figure 1.1 once more. In this graph $V(G)$ is $\{u, v, w, x, y\}$ and to verify if $V(G)$ is a dominating set D, we need each vertex $z \in V(G) \setminus D$ to have a neighbor in D. But every vertex $z \in V(G)$ is an element of the dominating set D, so $V \setminus D$ is empty and by definition $V(G) = \{u, v, w, x, y\}$ is a dominating set. Another example of a dominating set is $\{x\}$. Since vertex x is adjacent to the all the vertices in the graph, every other vertex has a neighbor in the dominating set. To better understand dominating sets we will consider another graph called the complete graph. A complete graph is a simple graph where all vertices are pairwise adjacent; the complete graph with n vertices is denoted K_n. So, K_3 is the complete graph with 3 vertices as shown in Figure 1.2. We will list the dominating sets in K_3:

1. $\{x, y, z\}$; $V(G)$ of a simple graph G is always a dominating set.
2. $\{x, y\}$; z which is an element of $V \setminus D$ has two neighbors in D namely x and y or we say z is dominated by x and y.
3. $\{x, z\}$; $y \in V \setminus D$ is dominated by x and z.
4. $\{y, z\}$; $x \in V \setminus D$ is dominated by y and z.
5. $\{x\}$; $y, z \in V \setminus D$ are dominated by x
6. $\{y\}$; $x, z \in V \setminus D$ are dominated by y
7. $\{z\}$; $x, y \in V \setminus D$ are dominated by z

So, there are 7 dominating sets in K_3 or $\partial_i(K_3) = 7$. In short, the dominating sets of K_3 are all the subsets of $V(K_3)$ except for the empty set. So, $\partial_i(K_3) = 2^3 - 1 = 7$. Note, in general $\partial_i(K_n) = 2^{n-1}$.

Now that we listed the dominating sets in K_3, let's consider a graph with more dominating sets such as the graph in Figure 1.1. Notice x is adjacent to all the vertices in Figure 1.1 and is the only neighbor to y. Consequently every dominating set must include either x or y, otherwise y will not be dominated:

As we have seen $V(G)$ always forms a dominating set.

1. $\{u, v, w, x, y\}$

Any subset of $V(G)$ of 4 vertices is a dominating set and there are $\binom{5}{4} = 5$ such subsets.

2. $\{u, v, w, x\}$; y is dominated by x.
3. $\{u, v, w, y\}$; x is dominated by u, v, w, and y.
4. $\{u, v, x, y\}$; w is dominated by u, v, x, and y.
5. $\{u, w, x, y\}$; v is dominated by u, w, x, and y.
6. $\{v, w, x, y\}$; u is dominated by v, w, x, and y.

Any subset of $V(G)$ with 3 vertices will form a dominating set except for $\{u, v, w\}$. So there are $\binom{5}{3} - 1 = 10 - 1 = 9$ such subsets:

7. $\{u, v, x\}$; w, y are dominated by x.
8. $\{u, v, y\}$; x, w are dominated by u.
9. $\{u, x, y\}$; v, w are dominated by x.
10. $\{u, w, x\}$; v, y are dominated by x.
11. $\{v, w, x\}$; u, y are dominated by x.
12. $\{v, w, y\}$; u, x are dominated by w.
13. $\{w, x, y\}$;u, v are dominated by x
14. $\{v, x, y\}$; u, w are dominated by x
15. $\{u, w, y\}$; v, w are dominated by x

There are $\binom{5}{2} = 10$ subsets of size 2, however, $\{u, w\}, \{u, v\}$, and $\{v, w\}$ do not dominate y. The following subsets with 2 vertices do form dominating sets:

16. $\{x, u\}$; v, w, y are dominated by x
17. $\{x, v\}$; u, w, y are dominated by x
18. $\{x, w\}$; u, v, y are dominated by x
19. $\{x, y\}$; u, v, w are dominated by x
20. $\{y, u\}$; v, w, x are dominated by u
21. $\{y, w\}$; u, v, x are dominated by w
22. $\{y, v\}$; u, w, x are dominated by v

Finally, the only subset of $V(G)$ of size 1 that dominates G is $\{x\}$. There are 23 dominating sets in the graph in Figure 1.1.

A well-studied graph parameter in the context of domination is the domination number, the smallest size of a dominating set. The domination number of a graph denoted $\gamma(G)$, has been obtained for different classes of graphs. When we listed the dominating sets in the graph in Figure 1.1, the smallest set was $\{x\}$. In this case, the domination number is 1. Moreover, many upper and lower bounds on the domination number have been obtained over the years [2]. We are interested in graphs that have a substantial number of dominating sets or those with very few dominating sets. A natural question one might ask is can we find lower and upper bounds of $\partial_i(G)$ over various classes of graphs.

Since our focus has been on paths, trees, binary trees, and bipartite graphs we define them now. A path is a list $v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k$ of vertices and edges with no repeated vertices, where $e_i = v_{i-1}v_i$ for $i = 1, 2, \ldots, k$. A path on n vertices is denoted P_n. For example, P_4 is displayed in Figure 1.3. A graph G is connected if for each pair of vertices $u, v \in V(G)$ there is a path with endpoints u and v or a uv-path. A cycle is a closed path - that is a path where the endpoints are the same vertex. On the other hand a graph is acyclic if it contains no cycle. A forest is an acyclic graph and a tree is a connected acyclic graph. A graph G is bipartite if $V(G)$ is the union of two disjoint independent sets; the vertices can be partitioned into independent

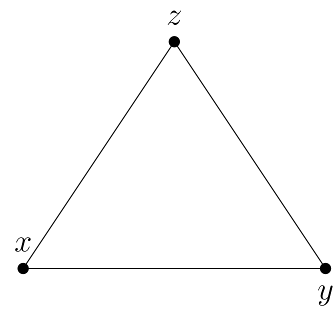

Figure 1.2: The complete graph on 3 vertices, K_3.

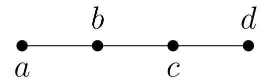

Figure 1.3: P_4

sets, called partite sets, such that there are no edges within the same partite set. Furthermore, a complete bipartite graph is a type of bipartite graph in which every vertex in one partition is connected to every vertex in the other partition, denoted $K_{r,s}$, where r and s represent the number of vertices in each partition. A complete balanced bipartite is a complete bipartite graph where the partitions are equal in size or $r = s$. When $r = s = 3$ we have $K_{3,3}$ as shown in Figure 1.5.

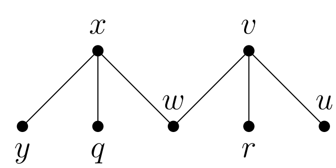

Figure 1.4: An example of a tree.

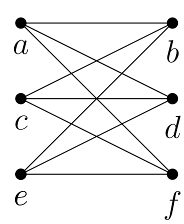

Figure 1.5: An example of a complete balanced bipartite graph, specifically $K_{3,3}$.

Proposition 1.0.1. *If G is a simple graph on n vertices then,*

$$1 \leq \partial_i(G) \leq 2^n - 1.$$

Proof. For any simple graph G, the vertex set of the graph $V(G)$ is a dominating set. Then any simple graph has at least one dominating set. Any subset of $V(G)$ could be a dominating set apart from the empty set; the empty set is not a dominating set D because each $v \in V \setminus D$ does not have a neighbor in D. For a simple graph G on n vertices there is $2^n - 1$ non-empty subsets of $V(G)$ or there is a most $2^n - 1$ dominating sets. □

In Proposition 1.1, equality is attained on the left by the empty graph and on the right by the complete graph. These bounds are in fact tight if there are no restrictions imposed on the graph other than the number of vertices. Bród and Skupień [2] determined upper and lower bounds for the number of dominating sets in a tree. Recall a leaf is a vertex of degree 1. In Figure 1.4 vertices q, r, u, and y are leaves.

Theorem 1.0.2. (Bród, Skupień 2006). *If T is a tree on n vertices, then*

$$c_n 5^{\lfloor n/3 \rfloor} \leq \partial_i(T) \leq 2^{n-1} + 1,$$

$$\text{where } c_n = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 3 \\ 9/5 & \text{if } n \equiv 1 \pmod 3 \\ 3 & \text{if } n \equiv 2 \pmod 3 \end{cases} \cdot$$

In Theorem 1.0.2 the maximum of the inequality, for a tree on n vertices, is attained by the star. A star is tree consisting of one vertex adjacent to all the others as shown in Figure 1.6. Note, when $n = 4$ and $n = 5$ the maximum is also attained

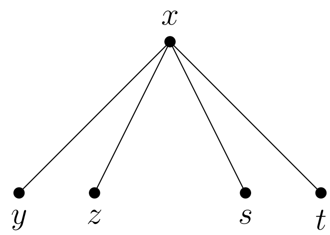

Figure 1.6: An example of a star.

by the path.

Inequalities that relate a graph and its complement are usually referred to as *Nordhaus-Gaddum* inequalities. The following theorem from Wagner [9] provides such an inequality for the number of dominating sets.

Theorem 1.0.3 (Wagner 2006)**.** *For any graph G on n vertices and its complement $\partial_i(\overline{G})$, the inequality*

$$\partial_i(G) + \partial_i(\overline{G}) \geq 2^n$$

holds, and this inequality is sharp.

Note equality is attained when G is the complete graph or the star. Wagner proposed that determining the maximum of $\partial_i(G)+\partial_i(\overline{G})$ as G ranges over all possible graphs on n vertices would be much more difficult. But Keough and Shane [6] proved the next theorem.

Theorem 1.0.4 (Keough , Shane 2019)**.** *If G is a graph on n vertices, then*

$$\partial_i(G) + \partial_i(\overline{G}) \leq 2^{n+1} - 2^{\lfloor \frac{n}{2} \rfloor} - 2^{\lceil \frac{n}{2} \rceil - 1}. \tag{1.1}$$

Keough and Shane [6] conjecture the extremal graph is the complete balanced bipartite graph, which leads to the following conjecture.

Conjecture 1. For a graph G on n vertices,

$$\partial_i(G) + \partial_i(\overline{G}) \leq 2\left(2^{\lfloor \frac{n}{2} \rfloor} - 1\right)\left(2^{\lceil \frac{n}{2} \rceil} - 1\right) + 2 \tag{1.2}$$

$$= 2^{n+1} - 2^{\lfloor \frac{n}{2} \rfloor + 1} - 2^{\lceil \frac{n}{2} \rceil + 1} \tag{1.3}$$

$$= \partial_i(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}) + \partial_i(\overline{K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}}) \tag{1.4}$$

We say $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is almost a complete balanced bipartite graph. If n is even, as described in the definition of a complete balanced bipartite graph, every vertex in one part is connected to every vertex in the other part, and both parts have the same number of vertices. If n is odd, then one part in $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ has one more vertex than the other part; we say it is as balanced as possible. Keough and Shane [6] verified the conjecture computationally for all graphs on at most 10 vertices. Wagner suggested that this conjecture is heuristically reasonable, as both the complete balanced bipartite graph and its complement can be dominated by only two vertices. We can compare the upper bound in Theorem 1.0.4 to the upper bound in the conjecture in Keough and Shane's conjecture by examining (1.1) and (3). Since, $2^{n+1} = 2^{n+1}$, $2^{\lfloor \frac{n}{2} \rfloor + 1} > 2^{\lfloor \frac{n}{2} \rfloor}$, and $2^{\lceil \frac{n}{2} \rceil + 1} > 2^{\lceil \frac{n}{2} \rceil - 1}$ we have,

$$2^{n+1} - 2^{\lfloor \frac{n}{2} \rfloor + 1} - 2^{\lceil \frac{n}{2} \rceil + 1} < 2^{n+1} - 2^{\lfloor \frac{n}{2} \rfloor} - 2^{\lceil \frac{n}{2} \rceil - 1}.$$

So Keough and Shane's conjecture provides a looser upper bound.

Definition 1.0.2. A set S of vertices of a simple graph $G = (V, E)$ is a *strong dominating set* if for every vertex $x \in V \setminus S$ there is a vertex $y \in S$ with $xy \in E(G)$ and $d(x) \leq d(y)$.

We denote the set of all strong dominating sets in G as $\mathcal{D}_{\text{st}}(G)$ and let $\partial_{\text{st}}(G) = |\mathcal{D}_{\text{st}}(G)|$ be the number of strong dominating sets. It was common to rely on our existing understanding of dominating sets, (see, e.g., [5]) and use similar techniques to determine $\partial_{\mathrm{st}}(G)$. We are concerned with determining the number of strong dominating sets and determining the bounds on the number of strong dominating sets in paths, binary trees, trees, and regular graphs. This thesis delves into an alternative perspective on dominating sets.

As with dominating sets, we begin with an example of finding the strong dominating sets of a graph. Consider the graph in Figure 1.1 once more. We will now determine its strong dominating sets. Notice $d(x)$ is greater than the degree of any of its neighbors and any vertex in the graph for that matter. Then a strong dominating set must include x. As with dominating sets, the entire vertex set or $\{u, v, w, x, y\}$ forms a strong dominating set. Any subset of four vertices will be a strong dominating set if x is an element of the set; such sets are $\{u, v, w, x\}$, $\{u, v, x, y\}$, $\{u, w, x, y\}$, and $\{v, w, x, y\}$. Similarly for subsets of size 3 we have $\{u, v, x\}$, $\{u, w, x\}$, $\{u, y, x\}$, $\{x, y, v\}$, $\{x, y, w\}$, and $\{w, v, x\}$. Any subset of two vertices, where one of the vertices is x forms a dominating set - that is $\{x, u\}$, $\{x, w\}$, $\{x, y\}$, and $\{x, v\}$ are dominating sets. The only singleton that forms a strong dominating set is $\{x\}$. We have 16 strong dominating sets. In this case, the number of strong dominating sets, 16, is less than the number of dominating sets, 23. Every strong dominating set is a dominating set, but not every dominating set is a strong dominating set as we saw in the previous example. In general for any simple graph G, we have $\partial_i(G) \geq \partial_{\mathrm{st}}(G)$. We attain equality if G is a regular graph, where all the vertex degrees are equal.

Chapter 2

Paths

Our focus lies in exploring the count of strong dominating sets within paths, initiating our investigation by counting the number of dominating sets. We would like to find a closed formula for $\partial_i(P_n)$ and begin by counting the number of dominating sets for P_n. For P_1 there is one dominating set, namely the set that contains the only vertex in P_1. A dominating set for P_2 is a set containing both vertices of $V(P_2)$ or any subset of $V(P_2)$ containing one vertex; so there are three dominating sets for P_2. In the case of P_3, any subset of $V(P_3)$ with two vertices will form a dominating set and there are three such subsets. The vertex that is not a leaf, which is then adjacent to the remaining two vertices, will form a dominating set. And all three vertices in $V(P_3)$ form a dominating set. Then we have five dominating sets for P_3. For P_4 any subset of three vertices will form a dominating set; so we have $\binom{4}{3} = 4$ dominating sets so far. The two leaves in $V(P_4)$ form a dominating set as well as the two non-leaves. For instance, for P_4 in Figure 1.3 subsets $\{a, c\}$ and $\{b, d\}$ form dominating sets. As we have seen in the previous cases, $V(P_4)$ forms a dominating set. There are nine dominating sets for P_4.

In Table 2.1 we have summarized our counts thus far and included the number of dominating sets for higher values of n. Notice the number of dominating sets appear

Table 2.1: The number of dominating sets in paths of lengths $n = 1$ to $n = 7$.

n	$\partial_i(P_n)$
1	1
2	3
3	5
4	9
5	17
6	31
7	57

to be the Tribonacci numbers, where the number of dominating sets of path n vertices is the sum of the three preceding counts. As an example we will list the number of dominating sets for P_5, see Figure 2.1. $V(P_5) = \{q, r, s, t, u\}$ forms a dominating set. Any subset of $V(P_5)$ with four vertices will form a dominating set and there are 5 such subsets: $\{q, r, s, t\}, \{q, r, s, u\}, \{q, r, t, u\}, \{q, s, t, u\}$, $\{r, s, t, u\}$. There are 10 subsets of size 3 of $V(P_5)$, but not all these subsets will form a dominating set. Take for instance $\{q, r, s\}$. Vertex u does not have a neighbor in the dominating set, so $\{q, r, s\}$ is not a dominating set. Similarly $\{s, t, u\}$ is not a dominating set. The following subsets with 3 vertices do form dominating sets: $\{q, r, t\}, \{q, r, u\}, \{q, s, t\}, \{q, s, u\}, \{q, t, u\}$, $\{r, s, t\}, \{r, s, u\}, \{r, t, u\}$. There are 10 subsets of size 2 of $V(P_5)$, but 7 of these subsets do not form dominating sets. The following subsets with 2 vertices do form dominating sets: $\{q, t\}, \{r, u\}$, and $\{r, t\}$. And no subset with 1 vertex forms a dominating set. There are 17 dominating sets of P_5. Because the number of dominating sets seem to be follow a tribonacci sequence we have,

$$\partial_i(P_5) = \partial_i(P_4) + \partial_i(P_3) + \partial_i(P_2) = 9 + 5 + 3 = 17$$

Notice the actual count of $\partial_i(P_5)$ equals the number of dominating sets obtained from the recursion. Before we prove Proposition 2.0.1, we define $\partial_i^*(P_n)$ as the number of dominating sets in P_n containing exactly one of its endpoints. Without loss of

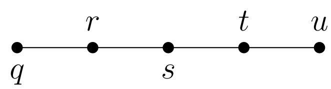

Figure 2.1: A path on 5 vertices, P_5

generality we choose the left endpoint for our illustration. Consider P_8 in Figure 2.2. Note that in Figure 2.2, a red vertex indicates the corresponding vertex will be an element of every dominating set. If we wanted to determine $\partial_i(P_8)$ we could count the number of dominating sets in the top path in Figure 2.2. Notice the endpoint a has to be in the dominating set or its neighbor b must be in the dominating set, otherwise a is not dominated. An alternative to counting the number of dominating sets in P_8 is to consider $\partial_i^*(P_8)$ and $\partial_i^*(P_7)$. The number of dominating sets containing a, denoted $\partial_i^*(P_8)$, which are the sets where a is dominated by itself. If a is not in the dominating set, it can only be dominated by its neighbor, so b must be in the dominating set. This latter case is represented by $\partial_i^*(P_7)$ the number of dominating sets on a path with one less vertex where its left endpoint is in every dominating set. In this scenario $n = 8$ but in general for a path on n vertices we have $\partial_i^*(P_{n-1})$. In either case $\partial_i^*(P_n)$ or $\partial_i^*(P_{n-1})$ contains the endpoint or its neighbor, respectively, in the dominating set so that the left endpoint is dominated.

Proposition 2.0.1. *For a path on n vertices,*

$$\partial_i(P_n) = \partial_i(P_{n-1}) + \partial_i(P_{n-2}) + \partial_i(P_{n-3})$$

Proof. Consider a path on n vertices, P_n. Let the left endpoint be v_1 and its neighbor be v_2. We will count the dominating sets in P_n according to whether the set contains v_1 or does not contain v_1. To ensure v_1 is dominated, v_1 is either in the dominating set or its neighbor v_2 must be in dominating set. With this objective in mind, let $\partial_i^*(P_n)$ be the number of dominating sets in P_n that contain v_1.We are left to count the number of dominating sets in P_n that do not contain v_1. If v_1 is not in the dominating set, then v_2 must be in the dominating set. Let $\partial_i^*(P_{n-1})$ be the number of dominating sets in P_n that do not contain v_1 but contain v_2 . So we can think of the number of dominating sets in P_n as the sum of the number of dominating sets in $\partial_i^*(P_n)$ and $\partial_i^*(P_{n-1})$ or

$$\partial_i(P_n) = \partial_i^*(P_n) + \partial_i^*(P_{n-1}). \tag{2.1}$$

We would like to find a recurrence for $\partial_i^*(P_n)$ so that we could rewrite (5) in terms of $\partial_i(P_k)$ for $k < n$. Recall for $\partial_i^*(P_n)$ we think of every dominating set containing one of its endpoint; in this case the endpoint is v_1, so we know v_1 dominates its neighbor v_2. Then for v_2 there is some choice in the sense that it can either be in or out of the dominating sets in P_n. If v_2 is in the dominating set, then $\partial_i^*(P_n)$ is equal to the number of dominating sets on a path of one less vertex, v_1, and its left endpoint is in every dominating set - that is $\partial_i^*(P_{n-1})$. Otherwise v_2 is not in the dominating set and $\partial_i^*(P_n)$ is equal to the number of dominating sets on a path with two less vertices, v_1 and v_2, or $\partial_i(P_{n-2})$. We have,

$$\partial_i^*(P_n) = \partial_i^*(P_{n-1}) + \partial_i(P_{n-2}). \tag{2.2}$$

By (2.1),

$$\partial_i(P_{n-2}) = \partial_i^*(P_{n-2}) + \partial_i^*(P_{n-3}). \tag{2.3}$$

By substituting (2.3) into (2.2), we obtain

$$\partial_i^*(P_n) = \partial_i^*(P_{n-1}) + \partial_i^*(P_{n-2}) + \partial_i^*(P_{n-3}). \tag{2.4}$$

Now we can rewrite $\partial_i(P_n)$ by applying (2.4) to $\partial_i^*(P_n)$ and $\partial_i^*(P_{n-1})$ in (2.1):

$$\partial_i(P_n) = \partial_i^*(P_{n-1}) + \partial_i^*(P_{n-2}) + \partial_i^*(P_{n-3}) + \partial_i^*(P_{n-2}) + \partial_i^*(P_{n-3}) + \partial_i^*(P_{n-4})$$

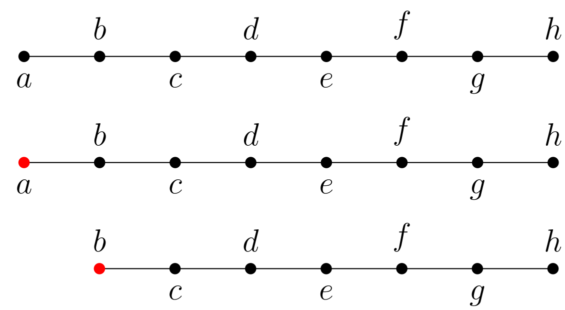

Figure 2.2: A visual to aid our discussion of $\partial_i^*(P_8)$

By rearranging and grouping terms we have,

$$\partial_i(P_n) = (\partial_i^*(P_{n-1}) + \partial_i^*(P_{n-2})) + (\partial_i^*(P_{n-2}) + \partial_i^*(P_{n-3})) + (\partial_i^*(P_{n-3}) + \partial_i^*(P_{n-4})) \,. \tag{2.5}$$

Notice by (2.1), $\partial_i(P_{n-1}) = \partial_i^*(P_{n-1}) + \partial_i^*(P_{n-2}), \partial_i(P_{n-2}) = \partial_i^*(P_{n-2}) + \partial_i^*(P_{n-3})$, and $\partial_i(P_{n-3}) = \partial_i^*(P_{n-3}) + \partial_i^*(P_{n-4})$. Then (2.5) becomes

$$\partial_i(P_n) = \partial_i(P_{n-1}) + \partial_i(P_{n-2}) + \partial_i(P_{n-3}).$$

Thus, the number of dominating sets in a path on n vertices is given by

$$\partial_i(P_n) = \partial_i(P_{n-1}) + \partial_i(P_{n-2}) + \partial_i(P_{n-3}).$$

□

We attempted to apply a similar idea to find a closed formula for the number of strong dominating sets in paths, however, it was not as clear. In the case of strong dominating sets we have to be more careful when it comes to the endpoints unlike the case with dominating sets. Strong dominating sets have an additional property related to the degree of the vertices in the graph; a vertex v in a graph G is strongly dominating an adjacent vertex, say x, only if $d(v) \geq d(x)$.

We begin by counting the number of strong dominating sets in paths P_n for small values of n. We start with a case that may initially seem redundant, but its relevance will become evident shortly. The path on no vertices, P_0, has one strong dominating set, the null set or $V(P_0)$. So $\partial_{\text{st}}(P_0) = 1$. For P_1, there is only strong dominating set $V(P_1)$ and $\partial_{\text{st}}(P_1) = 1$. For P_2, any non-empty subset of $V(P_2)$ will be a dominating set, so $\partial_{\text{st}}(P_2) = 2^2 - 1 = 3$. For $n \geq 3$ we are dealing with vertices of degree greater than 1. For a path on 3 vertices suppose we have v_1, e_1, v_2, e_2, v_3. Since $d(v_2) > d(v_1)$ and $d(v_2) > d(v_3)$, we must have v_2 in every strong dominating set because its neighbors v_1 and v_3 cannot dominate v_2. There are four subsets of $V(P_3)$ containing v_2, so $\partial_{\text{st}}(P_3) = 4$. We will refer to vertices that are not leaves as non-leaves. In this case of P_4, notice v_2 and v_3 are non-leaves. In the case of P_4 at least one the non-leaves, must be in every strong dominating set because the degree of a non-leaf is greater than the degree of a leaf. We know $V(P_4)$ forms a strong dominating set. Any subset of three vertices will include one of the non-leaves so any subset of three vertices will be a strong dominating set; there are 4 such sets. There are 6 subsets of $V(P_4)$ of size 2 but only 3 of them are strong dominating sets - namely $\{v_2, v_3\}, \{v_2, v_4\}$, and $\{v_1, v_3\}$. There is no subset containing one vertex that is a strong dominating set, so $\partial_{\text{st}}(P_4) = 8$. In Table 2.2 we have summarized our counts thus far and included the number of strong dominating sets for higher values of n.

Table 2.2: The number of strong dominating sets in paths of lengths $n = 1$ to $n = 7$.

n	$\partial_{\text{st}}(P_n)$
0	1
1	1
2	3
3	4
4	8
5	13
6	27
7	56

With the help of OEIS, we investigated further. Our values for $\partial_{\text{st}}(P_n)$ correspond to sequence A049893 in OEIS [7]. Earlier we included $\partial_{\text{st}}(P_0)$ so that the sequences in Table 2.2 and from OEIS would be easier to compare. Starting with $a(1) = a(2) = 1$ and $a(3) = 3$, the i^{th} term of the sequence is found by

$$a(i) = a(1) + a(2) + \cdots + a(i-1) - a(m), \tag{2.6}$$

for $i \geq 4$, where $m = 2^{p+1}+2-i$ and p is the unique integer such that $2^p < i-1 \leq 2^{p+1}$. Note in Table 2.2 we started our index with $n = 0$ but the sequence in OEIS starts with $i = 1$. This is a matter of offsetting the index, that is $i = n + 1$. For instance suppose we wanted to use the sequence to find the number of strong dominating sets for P_4. Then we want $i = 4 + 1 = 5$ and we compute $a(5)$:

$$a(5) = a(1) + a(2) + a(3) + a(4) - a(m),$$

where

$$m = 2^{p+1} + 2 - 5 \quad \text{and} \quad 2^p < 4 \leq 2^{p+1}. \tag{2.7}$$

The only value of p that satisfies the inequality in (2.7) is 1 because $2 = 2^1 < 4 \leq 2^2 = 4$. Then $m = 2^2 + 2 - 5 = 1$ and we have,

$$a(5) = a(1) + a(2) + a(3) + a(4) - a(1)$$

$$= 1 + 1 + 3 + 4 - 1$$

$$= 8$$

So $\partial_{\text{st}}(P_4) = 8$ and confirms our count from earlier. We explored the sequence more by considering the behavior of $a(i)$ as i increases. First we looked at the ratio of successive terms, $\frac{a(i+1)}{a(i)}$.

Table 2.3: Using $\frac{a(i+1)}{a(i)}$ to compare successive terms.

i	$a(i)$	$\frac{a(i+1)}{a(i)}$
1	1	—
2	1	1
3	3	3
4	4	1.333
5	8	2
6	13	1.6250
7	27	2.0769
8	56	2.07401
9	112	2
10	169	1.50893
11	367	2.1716
12	748	2.03815
13	1501	2.00668
14	3006	2.00266
15	6013	2.00033
16	12028	2.00033
17	24056	2
18	36085	1.50004
19	78185	2.16669

From Table 2.3, there is an observed growth factor ranging approximately from 1.5 to 2.2 between successive terms, $a(i)$ and $a(i+1)$. Notice for $i > 4$ the growth factor decreases until it reaches 1.5 and then jumps back up to 2. For instance at $i = 6$ the growth factor is 1.6250 but then for $i = 7$ the growth factors increases to 2.0769. There is a similar trend at $i = 7$, $i = 8$, $i = 9$, $i = 10$ where the growth factors are 2.0769, 2.07401, 2, and 1.50893 respectively. The next term $i = 11$ has a corresponding growth factor of 2.1716. We see this behavior as i increases to 18, the growth factor decreases to 1.50004 and then there is a spike once more. The trend is more evident in Figure 2.3. We attempted to create an exponential function to approximate the behavior, however, when we examined the percent error of the model it was too significant to establish a reliable model. The rationale behind this lies in the erratic behavior exhibited by the growth factors.

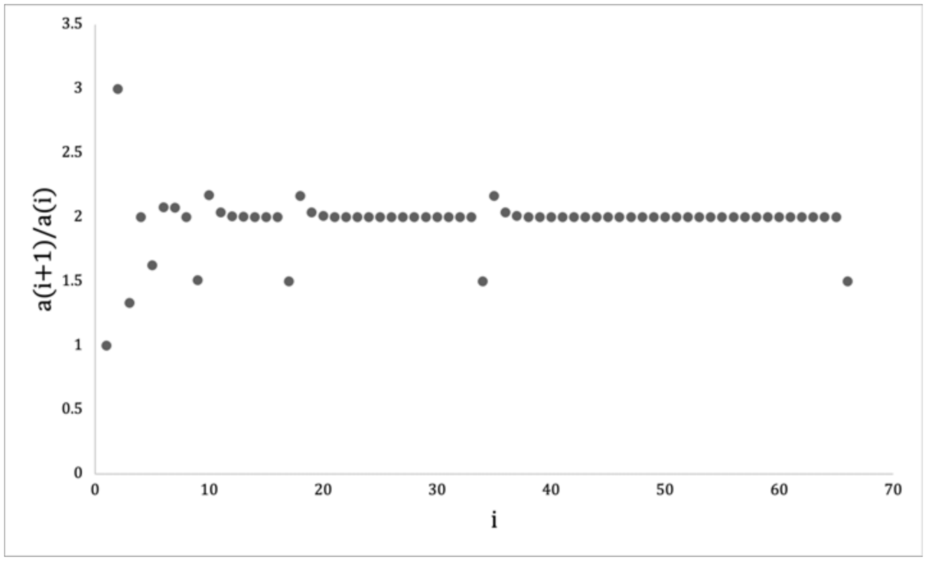

Figure 2.3: Scatterplot of $\frac{a(i+1)}{a(i)}$ versus i.

The left and right endpoints of a path have degree 1 while their neighbors have degree two (unless we are considering P_1 or P_2). For paths with more than 2 vertices, either endpoint could be in a strong dominating set, but neither would be strongly dominating its neighbor. In some sense knowing the left or right endpoint is in the strong dominating set does not help as it did before. Consider a path P_n as in Figure 2.4 with vertices $v_1, v_2, v_3, \ldots, v_{n-1}, v_n$. If the left point, v_1 is in the strong dominating set then $\partial_{\text{st}}(P_n)$ is equal to number of strong dominating sets in P_n containing the endpoint or $\partial^*_{\text{st}}(P_n)$. If v_1 is not in the strong dominating set then $\partial_{\text{st}}(P_n)$ is equal to the number of dominating sets where v_2 is contained in the strong dominating set, which we denote $\partial'_{\text{st}}(P_n)$. In general $\partial'_{\text{st}}(P_n)$ is the number of strong dominating sets such that the first vertex of $V(P_n)$ contained in the strong dominating set is the second vertex from the left endpoint. Then we have

$$\partial_{\text{st}}(P_n) = \partial^*_{\text{st}}(P_n) + \partial'_{\text{st}}(P_n). \tag{2.8}$$

We compare the recursive functions for the number of dominating sets versus

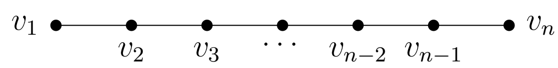

Figure 2.4: An illustration of P_n.

strong dominating sets. The recursion used to find the number of dominating sets in P_n is sum of three terms, namely the number of dominating sets for P_{n-1}, P_{n-2}, and P_{n-3}. Meanwhile, (2.6) shows the number of strong dominating sets in P_n is sum the number of dominating sets for P_1 up to P_n and then subtract a term depending on the value of m that satisfies the inequality. As the path grows, we sum more than three terms in (2.6). So for $n \geq 4$, it may appear that the number of strong dominating sets is greater than the number of dominating sets. However, we know that every strong dominating set is also a dominating set, but a dominating set is not necessarily a strong dominating set. This distinction arises because the degrees of the vertices significantly impact the establishment of a strong dominating set. We tried to translate the idea we used in the case of dominating sets to find a recursion for strong dominating sets. In our attempt, we included terms such as $\partial'_{\mathrm{st}}(P_{n-1})$ or $\partial'_{\mathrm{st}}(P_{n-2})$. As a result, we removed vertices and distorted the degrees of the remaining vertices, and complicated the process of finding a strong dominating set. In summary, the recursion may not be feasible because we are uncertain about the appropriate recursive approach in this case.

Chapter 3

Regular Graphs

We present the foundational concepts of entropy that will be used in this thesis. Though separate from the concept of entropy in physics, there exist similarities that warrant the shared name. In physics, entropy is a fundamental concept related to the measure of disorder or randomness in a system. Claude Shannon later integrated entropy into information theory [1]. In a structure that is randomized, entropy is a measure of uncertainty. In broad terms, greater uncertainty typically corresponds to increased entropy, while low entropy is characteristic of situations where everything is fully determined. Entropy has become a significant tool in combinatorics [4]; we will use entropy to help determine the maximum number of strong dominating sets in regular graphs. A graph G is k-regular is the common degree is k.

We introduce some probability vocabulary to formalize entropy. The set of all possible outcomes of an experiment is called the sample space and is denoted by Ω. A discrete random variable is a mapping $X : \Omega \to R \subseteq \mathbb{R}$. This means R is some countable subset of $\mathbb{R}$. The term "discrete" is included because of this restriction on R. The random variable X is just a function which attaches a number to each outcome in the sample space. In the following definition and throughout this thesis all logarithms are base two.

Definition 3.0.1. Suppose we have a finite sample space denoted $\Omega = \{x_1, x_2, \ldots, x_n\}$ and X is a discrete random variable where $\mathbb{P}(X = x_i) = p_i$. The *entropy* of X is given by

$$H(X) = \sum_{i=1}^{n} -p_i \log{(p_i)},$$

where we let $0 \cdot \log 0 = 0$.

To gain some insight into the nature of entropy consider the following example of tossing a coin. We will find the entropy $H(X)$ when the coin is biased so that the probability of a head is 0.95 and when the coin is fair so that the probability of a head is 0.5. In either case there are two outcomes, head or tail; let the probability of head and tail be p_1 and p_2, respectively. Then we have,

$$H(X) = -p_1 \log p_1 - p_2 \log p_2.$$

Because $p_2 = 1 - p_1$, we have

$$H(X) = -p_1 \log p_1 - (1 - p_1) \log{(1 - p_1)}.$$

In the case the coin is biased, where $p_1 = 0.95$ and $p_2 = 0.05$, the entropy $H_b(X)$ is

$$H_b(X) = -(0.95) \log 0.95 - (0.05) \log 0.05 \approx 0.2864.$$

When the coin is fair, $p_1 = p_2 = 0.5$ and the entropy $H_f(X)$ is

$$H_f(X) = -(0.5) \log 0.5 - (0.5) \log 0.5 = 1.$$

We can consider the previous two cases from the perspective of an individual who is trying to make some money by gambling with coin tosses. In the instance of a biased coin, the gambler is fairly certain of winning when it lands on heads, resulting in low entropy. When the coin is fair, the gambler in a state of maximum uncertainty and the entropy is greater.

We will now examine the two extreme cases of $H(X)$. Imagine X takes one of its values with certainty such that $p_1 = 1$ and $p_2 = \cdots = p_n = 0$. Then,

$$H(X) = \sum_{i=1}^{n} -p_i \log{(p_i)} = (-1)(\log 1) = 0.$$

On the other hand suppose X is uniformly distributed over $\{1, 2, 3, \ldots, n\}$ so that $p_1 = p_2 = \cdots = p_n = \frac{1}{n}$. Then we have

$$H(X) = \sum_{i=1}^{n} \frac{-1}{n} \log \frac{1}{n} = \frac{-n}{n} \log n = -\log \frac{1}{n} = \log n = \log |\text{range}(X)|.$$

The following theorem states the aforementioned cases are the extreme values of $H(X)$.

Theorem 3.0.1. *If X is a random variable then*

$$0 \leq H(X) \leq \log |\text{range}(X)|,$$

with equality if X is uniform on its range.

We will use entropy, $H(X)$, and *Shearer's Lemma* [3] to determine an upper bound on the number of strong dominating sets in a regular graph. Before introducing Shearer's Lemma, we'll outline some key vocabulary. A random vector $\mathbf{X} = (X_1, X_2, \ldots, X_n)$, where each coordinate of the vector is random variable. Let $A \subseteq \{1, 2, \ldots, n\}$ and $X_A = (X_i)_{i \in A}$. We denote the set of integers from 1 to n or $\{1, 2, 3, \ldots, n\}$ as $[n]$. Then $\mathscr{P}([n])$ denotes the power set of $[n]$, which is the power set of the integers from 1 to n.

Lemma 3.0.2 (Shearer's Lemma)**.** *Let* $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ *be a random vector. If* $\mathscr{A} \subseteq \mathscr{P}([n])$ *is a collection of subsets of* $[n]$ *such that for every* $i \in [n]$ *lies in at least* k *elements of* $\mathscr{A}$*, then*

$$H(X) \leq \frac{1}{k} \sum_{A \in \mathscr{A}} H(X_A).$$

We prove the following, which gives an upper bound on the number of strong dominating sets in a regular graph.

Theorem 3.0.3. *If* G *is* r*-regular on* n *vertices, then*

$$\partial_{\text{st}}(G) \leq \partial_{\text{st}}(K_{r+1})^{\frac{n}{r+1}} = (2^{r+1} - 1)^{\frac{n}{r+1}}.$$

Proof. If G is a regular graph, the number of dominating sets and the number of strong dominating sets are equal because any subset of $V(G)$ that is a dominating set is also a strong dominating set. We will show the number of dominating sets of G satisfies our claim and thereby show the number of strong dominating sets does as well. Let G be an r-regular graph and $V(G) = \{v_1, v_2, \ldots, v_n\}$. Define a random vector $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ on G by selecting a strong dominating set D uniformly at random from $\mathscr{D}_{st}(G)$ and letting

$$X_i = \begin{cases} 1 & \text{if } v_i \in D \\ 0 & \text{if } v_i \notin D \end{cases}$$

Then $H(X) = \log|\text{range}(X)| = \log \partial_{\text{st}}(G)$. Now let $\mathscr{A} = \{N[v] : v \in V(G)\}$, where $N[v] = N(v) \cup \{v\}$, the closed neighborhood of v. Then every vertex is in $r + 1$

elements of $\mathscr{A}$ since $\mathscr{A}$ is r-regular. By Shearer's Lemma,

$$H(X) = \log \partial_{\text{st}}(G) \leq \frac{1}{r+1} \sum_{v \in V(G)} H(X_{N[v]}).$$

Since X is uniformly distributed,

$$H(X_{N[v]}) \leq \log |\text{range}(X_{N[v]})|.$$

So,

$$\log \partial_{\text{st}}(G) \leq \frac{1}{r+1} \sum_{v \in V(G)} \log |\text{range}(X_{N[v]})|.$$

We want to find an upper bound for $\log \partial_{\text{st}}(G)$. In turn when we examine $|\text{range}(X_{N[v]})|$, we want to consider the greatest possible size of the range of $X_{N[v]}$. Since D is a strong dominating set, $X_{N[v]}$ cannot be all 0's, otherwise D would be empty. Then the random variable $X_{N[v]}$ is at most $2^{r+1} - 1$. Note, $2^{r+1} - 1$ is the number of dominating sets in the complete $(r+1)$-regular graph; in other words, $X_{N[v]}$ is at most $\partial_{\text{st}}(K_{r+1}) = 2^{r+1} - 1$. Then we have,

$$\begin{aligned}
\log \partial_{\text{st}}(G) &\leq \frac{1}{r+1} \sum_{v \in V(G)} \log(2^{r+1} - 1) \\
&= \frac{n}{r+1} \log(2^{r+1} - 1) \\
&= \frac{n}{r+1} \log \left(\partial_{\text{st}}(K_{r+1})\right) \\
&= \log \partial_{\text{st}}(K_{r+1})^{\frac{n}{r+1}}.
\end{aligned}$$

Thus, $\log \partial_{\text{st}}(G) \leq \log \partial_{\text{st}}(K_{r+1})^{\frac{n}{r+1}}$ or $\partial_{\text{st}}(G) \leq \partial_{\text{st}}(K_{r+1})^{\frac{n}{r+1}}$.

□

Chapter 4

Trees

We will now explain our process for establishing an upper bound on the number of strong dominating sets for trees. We attempted to exploit the structure of a tree to determine its maximum number of strong dominating sets, however, this proved to be difficult. Initially we tried using the degree of the vertices in trees to simplify the problem. For instance, we considered trees that contained vertices that we called *big*, which is a vertex v such that $d(v) > d(x), \forall x \in N(v)$.

Lemma 4.0.1. *If S is a strong dominating set in a simple graph G and v is big in G, then $v \in S$.*

Proof. Suppose v is a big vertex in a graph G. The degree of v is greater than the degree of any of its neighbors. No neighbors of v can strongly dominate v, so v must be in every strong dominating set. Alternatively, every big vertex belongs to every strong dominating set. □

Big vertices made it easier to count the number of strong dominating sets because big vertices are in every strong dominating set. So we could show our upper bound holds for trees with big vertices but we couldn't rely on big vertices in trees without any big vertices. To overcome this obstacle, we tried decomposing the tree into two components by splitting the tree at a vertex adjacent to a leaf; then we had hoped to use induction on the number of non-leaves in the tree. This issue arose when we split the tree, significantly altering its original structure and posing challenges in accurately counting strong dominating sets. Earlier we mentioned the maximum number of dominating sets for trees is $2^{n-1}+1$ in Theorem 1.0.2 [2]. We will use Theorem 1.0.2 to help determine an upper bound on the number of strong dominating sets. First we define important vocabulary used to prove the maximum number of strong dominating sets.

The *order* of a graph G, written $n(G)$ is the number of vertices in G. A *component* of a graph G is a connected subgraph that is not part of any larger connected subgraph. The components of any graph partition its vertices into disjoint sets, and are the induced subgraphs of those sets. A component is trivial if is has no edges; otherwise it is nontrivial. The maximum degree of a graph G is the maximum of the vertex degrees, denoted $\Delta(G)$. If G has a u,v-path, then the distance from u to v, written $d_G(u,v)$ is the least length of a u,v-path. The *diameter* (diam G) is $\max_{u,v\in V(G)} d(u,v)$.

Recall a tree is a connected acyclic graph. We will explore the structure of trees further and consider a deconstructed version. We begin by showing that every tree has at least two leaves.

Theorem 4.0.2 (Handshaking Lemma)**.** *Let T be a tree with n vertices. Then*

$$\sum_{v\in V(T)} d(v) = 2e(T) = 2(n-1).$$

Lemma 4.0.3. *A tree T on $n \geq 2$ vertices has at least two leaves.*

Proof. Suppose T is a tree on $n \geq 2$ vertices. Assume for contradiction T does not have two leaves and there only exists one leaf called x. Note $d(x) = 1$ while the degree of all other vertices in T are at least 2. Then we have,

$$\sum_{v \in V(T)} d(v) \geq 1 + 2(n-1) = 2n - 1 > 2(n-1). \tag{4.1}$$

From 4.1, we see $\sum_{v \in V(T)} d(v) > 2(n-1)$ but by the Handshaking Lemma $\sum_{v \in V(T)} d(v)$ should equal $2(n-1)$. We have a contradiction, so T has at least two leaves.

Theorem 4.0.4. *If T is a forest on $n \geq 6$ vertices, then $\partial_{\mathrm{st}}(T) \leq 2^{n-1}$.*

In Theorem 4.0.4 we had to specify the minimum number of vertices in T because there are some small cases of n that would not satisfy our claim. For example, when $n = 2$, T is a P_2 which we know has three strong dominating sets. However, $2^{2-1} = 2 \ngeq 3$. Consider another example where $n = 4$. In this case, T could be a star, path, or the union of two disjoint edges. If T is a star, then the vertex adjacent to the all the vertices in $V(T)$ must be in the strong dominating set while the leaves have a choice to be in or out of the set; this means there are $2^3 = 8$ strong dominating sets. We know the number of strong dominating sets in a path on 4 vertices is 8. So if T is a star or a path our claim is satisfied since $8 \leq 2^{4-1} = 2^3 = 8$. As we saw earlier an edge has three strong dominating sets but since we have two edges there are 9 strong dominating sets; in this case our claim is not satisfied because $9 \ngeq 2^{4-1} = 2^3 = 8$.

Proof. Suppose we have a forest T on $n \geq 6$ vertices. By Theorem 1.0.2, $\partial_i(T) \leq 2^{n-1} + 1$ and we want to show $\partial_{\mathrm{st}}(T) \leq 2^{n-1}$. So we need to find one dominating set that is not a strong dominating set.

If $\Delta(T) = 1$, then T is a disjoint union of edges. Each component in T has three dominating sets. If there are c components, then there are 3^c strong dominating sets. By our assumption n is at least 6, so the tree has at least 3 edges. In this case, there are $3^3 = 27$ strong dominating sets which is less than $2^{6-1} = 32$. Alternatively the number of components c could be expressed as $\lceil \frac{n}{2} \rceil$. In general, we want to show $3^{\lceil \frac{n}{2} \rceil} < 2^{n-1}$ when $n \geq 6$.

Suppose T is a disjoint union of stars such that $\Delta(T) \geq 2$. The set of all leaves in T gives a dominating set but is not a strong dominating set because the degree of any leaf is strictly less than the degree of its neighbor, provided there is a vertex of degree at least 2.

If T is neither a disjoint union of edges or stars, there exists a component with diameter of at least 3. Let P be a maximum path and v be adjacent to an endpoint of P. Let w be the non-leaf neighbor of v of P. Note, every neighbor of v is a leaf otherwise we have longer path and P would not be a maximum path. Then $D = V(T) - \{v, w\}$ is a dominating set but not a strong dominating set because v has no neighbor in D with larger degree.

□

Chapter 5

Further Directions

We examined the number of strong dominating sets in binary trees as well. Based on our findings there appears to be a bijection between the strong dominating sets in binary trees and multisets. Although we haven't identified a potential candidate yet we would like to delve deeper into this connection. To start, we introduce some key terms to establish a foundational understanding of binary trees. In computer science, trees are typically utilized as rooted trees since they enable efficient data storage for rapid access [10]. A rooted tree is a tree with one vertex r chosen as the root. For each vertex v, let $P(v)$ be the unique v, r-path. The *parent* of v is its neighbor on $P(v)$; its children are its other neighbors. The leaves are vertices with no children.

Definition 5.0.1. A *binary tree* is a rooted tree where each vertex has at most two children and each child of a vertex is designated at its left child or right child. We denote a binary tree with n vertices, B_n.

There is an example of a binary tree, in Figure 5.1, rooted at a with leaves d, e, f, and g. Broadly speaking we can think of binary trees as trees where each vertex has between 1 and 3 neighbors, inclusive. We begin by looking at a few examples of small binary trees and count the number of strong dominating sets in each tree. A binary

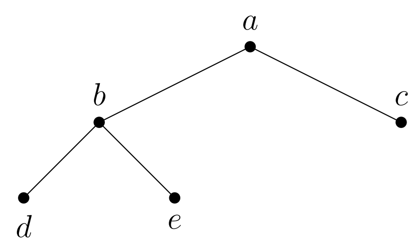

Figure 5.1: An example of B_5.

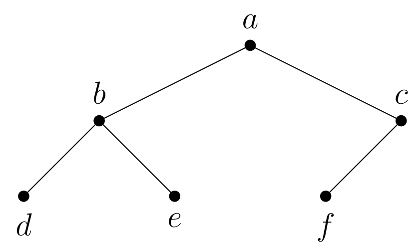

Figure 5.2: An example of B_6.

tree with one vertex is trivial, denoted B_1. Nonetheless, B_1 has one strong dominating set. We can build bigger trees by adding vertices, one at a time. Binary trees with two, three, and four vertices are the same as paths with two, three, and four vertices respectively. We have found that the number of strong dominating sets for P_2, P_3, P_4 are 3, 4, and 8 respectively. So the number of strong dominating sets for B_2, B_3, B_4 are 3, 4, and 8 respectively. The binary tree with five vertices, B_5, is the first case of new structure as shown in Figure 5.1. If vertices a and b are in the strong dominating set, then there are two choices for each leaf; each leaf can be in or out of the strong dominating set. This gives 2^3 or 8 strong dominating sets. Alternatively if vertices b and c are in the strong dominating set the remaining vertices can either be in or out.

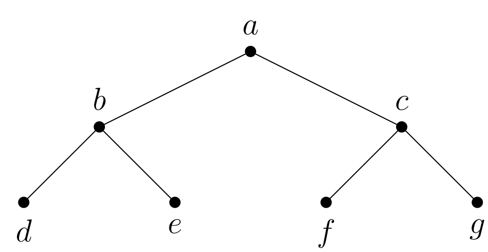

Figure 5.3: An example of B_7.

We do not want to overcount these cases twice: $\{a,b,c\}$, $\{a,b,c,d\}$, $\{a,b,c,e\}$, and $\{a,b,c,d,e\}$. So we will consider the ones where a is not in the strong dominating set and count the cases where leaves d and e are in or out of the strong dominating set. There are four such strong dominating sets. In total the number of strong dominating sets for B_5 is 12. We will consider the number of strong dominating sets in B_6 in a similar fashion. Notice either vertices b and c or vertices a, b, and f have to be in the strong dominating set while the remaining vertices can either be in or out. There are 16 strong dominating sets with b and c. There are 8 strong dominating sets including vertices a, b, and f but we have already included 4 of them in our count, namely $\{a,b,c,f\}$, $\{a,b,c,d,f\}$,$\{a,b,c,e,f\}$, and $\{a,b,c,d,e,f\}$. There is 20 strong dominating sets for B_6. Consider the binary tree on seven vertices in Figure 5.3. Notice, the degree of vertices b and c is greater than all of their neighbors. So b and c must be in every strong dominating set, but the remaining vertices can be either in or out. Then $\partial_{\text{st}}(B_7) = 2^5 = 32$. In summary, the number of strong dominating sets for B_1 to B_7 is 1, 3, 4, 8, 12, 20, 32. According to OEIS A349050 [8], the number of strong dominating sets can be found in a relevant sequence $a(n)$, which gives the number of multisets of size n that have no alternating permutations and cover an initial segment. Note, a sequence is alternating when it consistently alternates between being strictly increasing and strictly decreasing, irrespective of whether it begins with an increase or a decrease. An anti-run permutation is a permutation of a sequence in which no consecutive elements are in ascending order. For example, the sequence (3,2,2,2,1) has no alternating permutations, even though it does have the anti-run permutations (2,3,2,1,2) and (2,1,2,3,2).

In Table 5.1 when $n = 3$ the multiset is $\{1,1,1\}$ which corresponds to the one strong dominating set in B_1. When $n = 4$, the multisets are $\{1,1,1,1\}$, $\{1,2,2,2\}$, and $\{1,1,1,2\}$ which correspond to the three strong dominating sets in B_2. Jumping forward to when $n = 7$, these 12 multisets align with the 12 strong dominating

Table 5.1: The $a(2) = 1$ through $a(7) = 12$ multisets.

$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$
{1,1}	{1,1,1}	{1,1,1,1}	{1,1,1,1,1}	{1,1,1,1,1,1}	{1,1,1,1,1,1,1}
		{1,1,1,2}	{1,1,1,1,2}	{1,1,1,1,1,2}	{1,1,1,1,1,1,2}
		{1,2,2,2}	{1,2,2,2,2}	{1,1,1,1,2,2}	{1,1,1,1,1,2,2}
			{1,2,2,2,3}	{1,1,1,1,2,3}	{1,1,1,1,1,2,3}
				{1,1,2,2,2,2}	{1,1,2,2,2,2,2}
				{1,2,2,2,2,2}	{1,1,2,2,2,2,3}
				{1,2,2,2,2,3}	{1,2,2,2,2,2,2}
				{1,2,3,3,3,3}	{1,2,2,2,2,2,3}
					{1,2,2,2,2,3,3}
					{1,2,2,2,2,3,4}
					{1,2,3,3,3,3,3}
					{1,2,3,3,3,3,4}

sets in B_5. There seems to be relation between the strong dominating sets and the multisets but we could not find a bijection. This opens up a potential avenue for future exploration. Recently, Andrew Howroyd [8] found a closed form for the sequence:

$$a(n) = \begin{cases} (n+2)2^{\frac{n}{2}-3} & \text{for even } n > 0 \\ (n-1)2^{\frac{n-5}{2}} & \text{for odd } n \end{cases}$$

In the extremal case we suspect binary trees are the minimizer for trees.

We also would like to investigate the lower and upper bounds on the number of strong dominating sets in bipartite graphs. We suspect if G is a bipartite graph on n vertices, then

$$\partial_{\text{st}}\left(K_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n+1}{2} \rceil}\right) \leq \partial_{\text{st}}(G) \leq 2^{n-1}.$$

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