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Multicolor Bipartite Ramsey Numbers of Balanced Double Stars

Ella Oren-Dahan

Abstract

Given an integer $n \geq 1$, the balanced double star $S_{n,n}$ is a tree consisting of two vertex disjoint stars with n leaves each, connected at their central vertices by an edge. Given $r \geq 2$, we consider the problem of finding the smallest integer N such that every r-colored complete bipartite graph $K_{N,N}$ contains a monochromatic copy of the balanced double star $S_{n,n}$. This question is an instance of a problem within Ramsey theory. In this thesis, we cover the history of Ramsey theory and our problem in general, provide an alternative approach to prove the two colored case, prove new bounds as well as exact values when $r = 3$, and prove new bounds for $r > 3$.

MONTCLAIR STATE UNIVERSITY

Multicolor Bipartite Ramsey Numbers of Balanced Double Stars

by

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Contents

List of Figures

Chapter 1

Introduction

This chapter will introduce some graph theory terminology and concepts that relate to our problem. We also provide some background information about the problem.

1.1 Terminology

We begin with giving some basic graph theory definitions relevant to this thesis, as well as defining some key terms used. Any other necessary terminology will be defined later when needed.

A *graph* G is an ordered pair $G = (V, E)$ consisting of a *vertex set* $V = V(G)$ and an *edge set* $E = E(G)$. The elements of V are called *vertices* and the elements of E are called *edges*. Each edge is a 2-element subset of V. For every pair of distinct vertices $u, v \in V$, the edge between u and v is the subset $\{u, v\}$. If $\{u, v\} \in E$, then u and v are *adjacent*, and the edge $e = \{u, v\}$ is *incident* to both vertices. A graph H is a *subgraph* of the graph G if and only if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, denoted $H \subseteq G$, and we say "G contains H".

Families of graphs are collections of graphs with specific common properties and each family has a unique notation. For example, a graph G is *complete* if every pair of two distinct vertices in $V(G)$ are adjacent. The complete graph is denoted as K_n, with $n = |V(G)|$. A graph G is *bipartite* if $V(G)$ is the union of two disjoint independent sets called the *partite sets* of G. A *complete bipartite* graph is a bipartite graph such that vertices are adjacent if and only if they are in different partite sets. If G is a bipartite graph, then $X = X(G)$ and $Y = Y(G)$ will denote the partite sets of G, and G is denoted as $K_{x,y}$ where $|X| = x$ and $|Y| = y$. When $|X(G)| = |Y(G| = N$, the graph $G = K_{N,N}$ is a *balanced* complete bipartite graph.

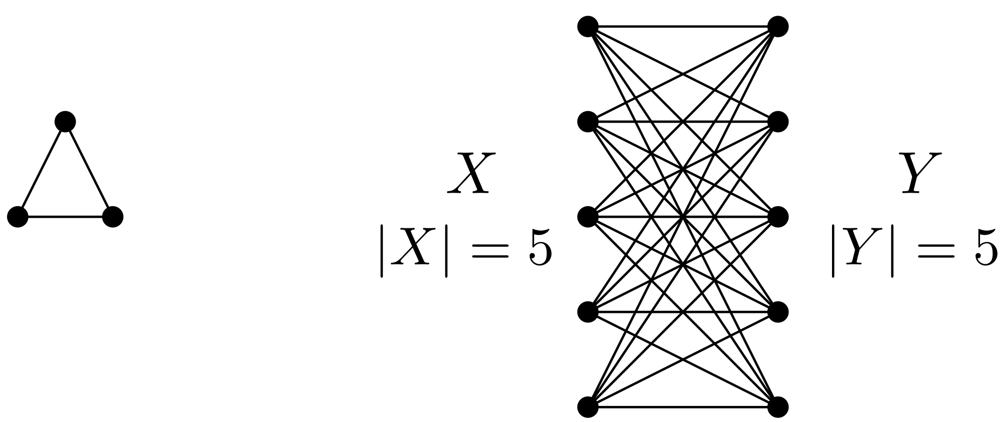

Figure 1.1: Example of a complete graph $G = K_3$ (left) and a complete bipartite graph, $G = K_{5,5}$ (right).

The number of edges incident to a particular vertex v is the *degree* of v, denoted $d(v)$. A *leaf* is a vertex with degree 1. For any graph G and vertex $v \in V(G)$, the subset of vertices that v is adjacent to is called the *neighborhood* of v, denoted $N(v)$. A *path* is a sequence of distinct vertices and edges $v_0, e_1, v_1, e_2, v_2, \ldots, e_{n-1}, v_{n-1}$ such that $e_i = \{v_{i-1}, v_i\}$ for $i = 1, 2, ..., n-1$. We call vertices v_0 and v_{n-1} *endpoints.* A *uv-path* is a path where u and v are endpoints. When the path is the entirety of the graph we denote the graph as P_n.

A *cycle* is a closed path, a path where the endpoints are the same. A graph is *acyclic* if it contains no cycle. A graph G is *connected* if for every pair of vertices u, v, there exists a uv-path in G. A graph is a *tree* if it is both connected and acyclic.

A *double star*, denoted $S_{m,n}$, is a tree with a singular edge incident to two vertices, called the *central vertices*, with degrees $m+1$ and $n+1$ respectively. The central vertices are respectively adjacent to m and n leaves. If $m = n$ the graph is called a *balanced double star*, and is denoted $S_{n,n}$.

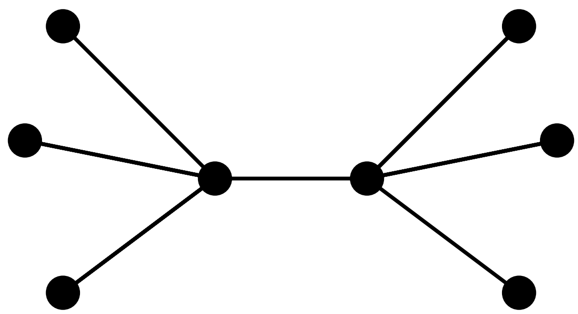

Figure 1.2: Example of a Double Star Graph, $G = S_{3,3}$.

An important element of the graphs we consider in this thesis is coloring. The following terms formally define how we describe the coloring of a graph. A graph G is called *single-colored* or *monochromatic* if all edges of G are colored with the same color. A *proper r edge-coloring* of a graph G is an assignment of r colors to edges such that no vertex is incident to two edges of the same color.

We can adjust the notation of both degree and neighborhood to specify only edges of a specific color. When specifying the degrees in a specific color c for vertex v, we denote $d_c(v)$. Additionally let $N_c(v) = \{w \in V(G) \mid w \text{ is adjacent to } v \text{ through a } c\text{-colored edge}\}$ be the c colored neighborhood of vertex v.

Remark 1.1.1. Every complete bipartite graph $K_{N,N}$ has a proper N-edge-coloring [10].

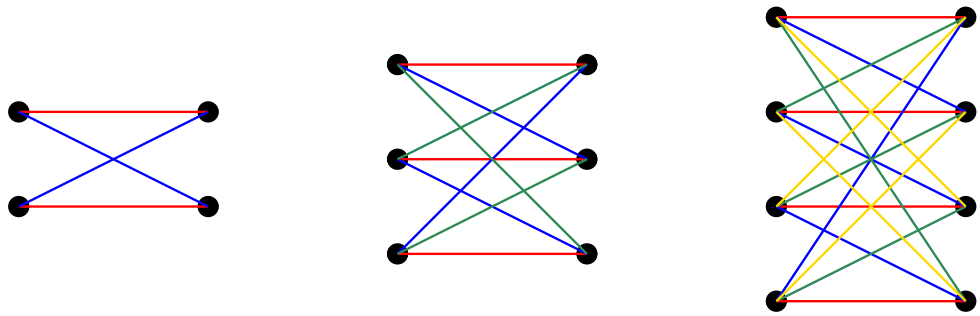

Figure 1.3: $K_{2,2}$ with a proper 2 edge coloring (left), $K_{3,3}$ with a proper 3 edge coloring (center), and $K_{4,4}$ with a proper 4 edge coloring.

1.2 Ramsey Theory

Ramsey theory is based on the Ramsey Theorem, originally published in Frank P. Ramsey's 1928 paper *On a Problem of Formal Logic* as a lemma [13]. The field refers to the study of partitions of large structures. Within graph theory, a typical example asks how large must a red and blue edge colored graph be to ensure the graph contains a specific monochromatic subgraph.

Roughly, Ramsey's Theorem can be stated as specific monochromatic subgraphs must be found in any sufficiently large arbitrarily colored complete graph. Formally,

Theorem 1.2.1 (Ramsey's Theorem)**.** *Given positive integers k and ℓ there exists a least positive integer $R(k, \ell)$ for which every red/blue coloring of the edges of the complete graph on $R(k, \ell)$ vertices contains a red clique on k vertices or a blue clique on ℓ vertices [13].*

The integers $R(k, \ell)$ are known as the Ramsey numbers [17].

A classic question within Ramsey theory is as follows:

Example 1.2.2. *"What is the smallest number of people needed in a room to guarantee that either at least three people know each other or at least three people do not know each other?"*

In this problem we represent people as vertices and the relationship of *knowing* another person as a blue edge between both vertices and *not knowing* another person as a red edge between both vertices. Thus, three people knowing one another would be represented by a monochromatic blue K_3, and three people not knowing one another would be a monochromatic red K_3. The question asks what is the smallest number of vertices v for K_v to always contain a monochromatic red or blue K_3, in other words, what is the Ramsey number $R(3,3)$?

Lemma 1.2.3 (The Pigeonhole Principle)**.** *Suppose n and m are positive integers such that $n > m$. If we distribute n objects into m sets, then the pigeonhole principle states at least one of the m sets must contain more than one item, specifically at least $\lceil \frac{n}{m} \rceil$ objects.*

Proof of Example 1.2.2. To prove $R(3,3) > 5$, it is sufficient to show a coloring of K_5 that does not contain a monochromatic K_3 in any color, this is done in Figure 1.4.

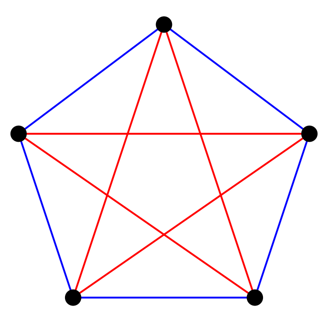

Figure 1.4: An edge coloring of K_5 which does not contain a monochromatic K_3.

To prove $R(3,3) = 6$, let a be an arbitrary vertex within $V(K_6)$. The vertex a is incident to five edges, using the pigeonhole principle we know at least three of these edges are the same color. Without loss of generality, suppose three edges of these edges are red. Let the three vertices incident to a in red be called b, c, and d respectively. If at least one of the edges $\{b, c\}, \{b, d\}$, or $\{c, d\}$ is red, then K_6 contains a monochromatic red K_3. Otherwise the edges $\{b, c\}, \{b, d\}$, and $\{c, d\}$ are all blue and K_6 contains a monochromatic blue K_3. Figure 1.5 illustrates both of these cases.

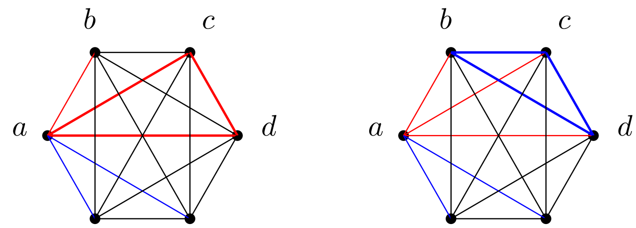

Figure 1.5: The edge coloring of K_6 on the left contains a red K_3 and the right contains a blue K_3.

Thus $v = 6$ is the smallest number of vertices for K_v to always contain a monochromatic red or blue K_3. □

No exact formula for any arbitrary Ramsey number is known, and to consider every possible edge coloring to identify Ramsey numbers rapidly becomes too large to compute.

In the case such that $k = \ell$, we call $R(k, k)$ the diagonal Ramsey number. The general bounds for $R(k, k)$ are given by $[1+o(1)]\frac{\sqrt{2}k}{e}2^{k/2} \leq R(k, k) \leq (4-\epsilon)^k$, where the lower bound is due to Spencer [15] and the upper bound was very recently proved in a major breakthrough by Campos, Morris, Griffiths and Sahasrabudhe [3].

More generally, given graphs G_1, G_2 we may define $R(G_1, G_2)$ as the smallest integer v such that every red/blue coloring of the edges of K_v contains a red copy of G_1 or a blue copy of G_2. These more general Ramsey numbers have been extensively studied with many results and references collected in Sections 4 and 5 of the dynamic survey *Small Ramsey Numbers* [16].

1.2.1 Extensions of Ramsey Theory

Ramsey theory can be extended in multiple fashions. Multicolor Ramsey theorem, for example, considers the case of any finite number of colors, $r \geq 2$, rather than the case of red/blue edge coloring. Formally stated, $v = R_r(G_1, G_2, \ldots, G_r)$ is the multicolored Ramsey number for an r-colored complete graph. That is, v is the least positive integer for which every r edge-coloring of K_v vertices contains a monochromatic copy of G_i in the ith color, for some $i \in \{1, 2, \cdots, r\}$. In the case that all the G_i are the same graph G, we write $R_r(G)$ for $R_r(G, G, \ldots, G)$.

We can also consider extensions utilizing different host graphs or adding conditions to the host graph. Examples of different host graphs include hypergraphs and complete bipartite graphs, which lead to hypergraph Ramsey numbers and bipartite Ramsey numbers respectively. Bipartite Ramsey theory was first introduced by Beineke and Schwenk [2].

Problems in Ramsey theory can consider multiple extensions simultaneously, further expanding the possible questions to consider. Multicolor bipartite Ramsey numbers are examined when the host graph is a balanced complete bipartite graph instead of a complete graph and we consider the case of r-colored edges.

Theorem 1.2.4 (Bipartite Ramsey Theorem)**.** *The multicolor bipartite Ramsey number* $v = B_r(G_1, G_2, \ldots, G_r)$*, is the least positive integer* v*, such that every coloring of the edges of* $K_{v,v}$ *with* r *colors, will result in a copy of the bipartite graph* G_i *in the* i*th color, for some* $i \in \{1, 2, \cdots, r\}$*.*

Again, in the case that all the G_i's are the same graph G, we write $B_r(G)$ for $B_r(G, G, \ldots, G)$.

The existence of bipartite Ramsey numbers follows from Erdős and Rado's results [6]. Furthermore, like general Ramsey numbers, many bipartite Ramsey numbers are known [11, 14, 18].

Chapter 2

The Problem

In this thesis we look at the multicolored bipartite Ramsey numbers of balanced double stars.

Based on the ideas from Section 1.2.1, we know we are looking for the smallest integer $N = B_r(S_{n,n})$ such that every r edge-coloring of $K_{N,N}$ will result in a copy of $S_{n,n}$ in the ith color, for some $i \in \{1, 2, \cdots, r\}$. It should be noted that $S_{n,n}$ is a bipartite graph, as are all trees, thus it is appropriate to use a bipartite Ramsey number.

To illustrate how a bipartite graph might contain a double star, Figure 2.1 shows a monochromatic red $S_{3,3}$ within a 3-colored $K_{5,5}$.

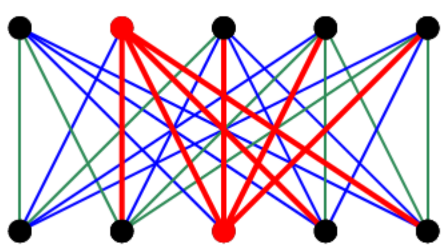

Figure 2.1: A 3-color edge coloring of $K_{5,5}$ that contains a red $S_{3,3}$.

In this thesis we call a vertex *colored* in i-color if and only if the i-color degree is greater or equal to $n+1$. Vertices can be colored in more than one color. In Figure 2.1 the two central vertices of $S_{3,3}$ are colored in red since $n=3$ in this case.

2.1 Previous Findings

While most multicolor bipartite Ramsey numbers of balanced double stars are unknown, the study of Ramsey numbers has applicable findings from similar problems. Furthermore, when variables are limited, some exact values have been proven.

When the colors are limited to a red/blue edge-coloring, the bipartite Ramsey numbers of all double stars are provided by Hattingh and Joubert [9]. They prove for any two positive integers n and m such that $n \geq m \geq 2$,

$$B_2(S_{n,n}, S_{m,m}) = n + m + 1.$$

It directly implies an equation for the bipartite Ramsey number of any balanced double star,

$$B_2(S_{n,n}) = 2n + 1. \tag{2.1}$$

When n is fixed to be 1, then $S_{1,1} = P_4$. Counting the Ramsey numbers of paths are problems that are often studied. The bipartite Ramsey number of $S_{1,1}$ for r colors was recently given by DeBiasio, Gyárfás, Krueger, Ruszinkó, and Sárközy [4].

Theorem 2.1.1 (DeBiasio, Gyárfás, Krueger, Ruszinkó, and Sárközy [4]). *For every positive* *integer* $r \geq 2$,

$$B_2(S_{1,1}) = 3,$$

$$B_3(S_{1,1}) = 4,$$

$$B_4(S_{1,1}) = 6, \ \ \textit{and}$$

$$B_r(S_{1,1}) = 2r - 3; \quad r \geq 5.$$

Regarding lower bounds on $B_r(S_{n,n})$ for any positive integer n, we first claim that $B_r(S_{n,n}) > rn$.

Claim 2.1.2. *For any positive integer* n*,* $B_r(S_{n,n}) > rn$*.*

Proof of claim. Take a proper r-edge-coloring of $K_{r,r}$ and blow up each vertex into a set of n vertices. Every vertex in the graph has at most $d_i(v) = n$ in every color $i \in \{1, 2, \cdots, r\}$. As no vertex can be incident to $n+1$ i-colored edges, the graph does not contain a monochromatic $S_{n,n}$. ∎

Figure 2.2 illustrates the process of blowing up an r-edge-coloring of $K_{r,r}$ with $r = 3$ and $n = 2$ as an example.

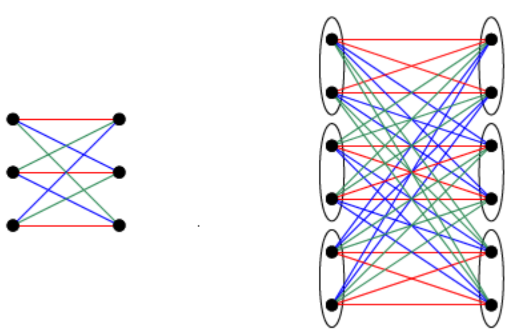

Figure 2.2: A 3-colored proper edge coloring of $K_{3,3}$ (left) blown up with $n = 2$ for the construction $B_3(S_{2,2}) > 6$ (right)

A result of DeBiasio, Gyárfás, Krueger, Ruszinkó, and Sárközy [4] implies an improvement on the lower bound on $B_r(S_{n,n})$ for $r \geq 4$.

Theorem 2.1.3. *For every balanced bipartite graph G on $2n+2$ vertices,*

$$B_r(G) \geq \begin{cases} rn+1, & 1 \leq r \leq 3 \\ 5n+1, & r = 4 \\ (2r-4)n+1, & r \geq 5 \end{cases}$$

Note how the double star graph $S_{n,n}$ is a balanced bipartite graph on $2n+2$ vertices, thus Theorem 2.1.3 is applicable.

Regarding the upper bound of $B_r(S_{n,n})$, a result from Hattingh and Joubert [9] proved $B_r(S_{n,n}) \leq \lceil rn + \sqrt{n2r(r-1) - 2r(n-1)} \rceil = n(r + \sqrt{r^2 - r}) - o(n); \quad r \geq 3,\, n \geq 3.$

Recently, Decamillis and Song [5] proved the following extremal result for double stars in balanced bipartite graphs.

Theorem 2.1.4 (Decamillis and Song [5])**.** *Let $n \geq m$ with $N \geq 3n+1$ and let G be a balanced bipartite graph on $2N$ vertices. If $e(G) > \max\{nN, 2m(N-m)\}$, then $S_{n,m} \subseteq G$. Furthermore, this result is best possible.*

From this they obtained the following corollary.

Corollary 2.1.5 (Decamillis and Song [5])**.** *Let $r \geq 2$ be an integer.*

1. *If $n \geq 2m$, then $B_r(S_{n,m}) \leq rn+1$.*

2. *If $m \leq n < 2m$, then $B_r(S_{n,m}) \leq (r + \sqrt{r(r-2)})m + 1 = (2r - 1 - \frac{1}{2r} - O(\frac{1}{r^2}))m + 1$.*

Corollary 2.1.5(2) improves the upper bound for $B_r(S_{n,n})$ when $n = m$.

Thus, the current general bounds known for $B_r(S_{n,n})$ given by the following:

$$\left.\begin{array}{ll} rn+1, & 1 \leq r \leq 3 \\ 5n+1, & r = 4 \\ (2r-4)n+1, & r \geq 5 \end{array}\right\} \leq B_r(S_{n,n}) \leq (r + \sqrt{r(r-2)})n + 1.$$

Chapter 3

The Two Colored Case

In this chapter we give an alternative approach to two edge colored bipartite Ramsey numbers of balanced double stars as a preview of the more complicated cases of multicolored Ramsey numbers and to illustrate the terminology for our approach. In this case edges are colored with colors $i \in \{1, 2\}$. Let $i = 1$ be red and $i = 2$ be blue.

Recall that we consider a vertex v to be colored with color $i \in \{1, 2\}$ if and only if v has $d_i(v) \geq n + 1$.

Observation 3.0.1. *A bipartite graph $G = K_{x,y}$ contains an i-colored $S_{n,n}$ if and only if there exists an i-colored vertex $v \in X$ and i-colored vertex $u \in Y$ incident to an i-colored edge.*

Theorem 3.0.2. *Let n be a positive integer, then $B_2(S_{n,n}) = 2n + 1$.*

Proof. We first prove $B_2(S_{n,n}) \geq 2n + 1$.

Let $G = K_{2n,2n}$, and note that we cannot guarantee by the pigeonhole principle that any vertex $v \in V(G)$ will be colored in red or blue. Following Observation 3.0.1 this already suggests that G may not contain a monochromatic $S_{n,n}$.

Partition sets $X(G)$ and $Y(G)$ into X_1 and X_2, and Y_1 and Y_2 respectively, such that each set contains exactly n vertices. Color the edges between X_i and Y_j red if $i = j$ and blue if $i \neq j$. The constructed graph contains no monochromatic $S_{n,n}$, thus $B_2(S_{n,n}) \geq 2n + 1$.

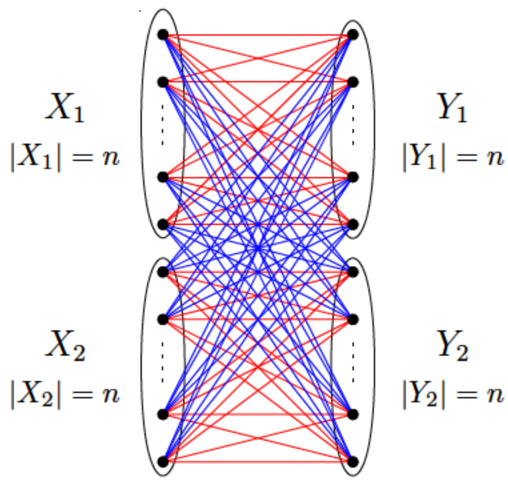

Figure 3.1: Construction of $K_{2n,2n}$ that does *not* contain a monochromatic $S_{n,n}$.

To prove $B_2(S_{n,n}) \leq 2n + 1$ we consider the graph $G = K_{2n+1,2n+1}$. By the pigeonhole principle in any red/blue edge coloring of G, every vertex $v \in V(G)$ will have at least degree $n + 1$ in color i. Thus every vertex v will be colored in color i for some $i \in [2]$. We partition set $X(G)$ into subsets X_1 and X_2, and set $Y(G)$ into subsets Y_1 and Y_2 such that X_i and Y_i contain i-colored vertices.

Additionally by the pigeonhole principle, we know that in X, one of X_1 or X_2 must contain the majority of the vertices. Thus either $|X_1| > |X_2|$ or $|X_1| < |X_2|$, likewise for set Y, either $|Y_1| > |Y_2|$ or $|Y_1| < |Y_2|$.

Case 1. Every vertex in X or Y is colored the same color.

Without loss of generality let $|X_1| = 2n+1$ and $X_2 = \emptyset$. In this case, X has at minimum $(n+1)(2n+1)$ incident red edges, and so by pigeonhole principle there must be at least 1 vertex in Y that is incident at least $n+1$ red edges. Thus there must exist a red vertex in Y and some red vertex in X both incident to a red edge. Following our observation 3.0.1, G contains a red $S_{n,n}$.

Case 2. There is at least one blue and one red vertex in both sets X and Y, thus X_1, X_2, Y_1, and Y_2 are all nonempty. Without loss of generality, $|X_1| \geq n+1$ and $|X_2| \leq n$. Given that Y_1 is not empty, there must exist some vertex $v \in Y_1$ that is incident to at least $n+1$ red edges. Since $|X_2| \leq n$, by the pigeonhole principle, vertex v must be adjacent to at least one vertex in X_1 through a red edge. Following Observation 3.0.1, G must contain a red $S_{n,n}$.

□

Chapter 4

The Three Colored Case

In the case that $r = 3$ we start with giving exact bipartite Ramsey numbers for small values of n. In this case edges are colored with color $i \in \{1, 2, 3\}$, and let $i = 1$ be red, $i = 2$ be blue, and $i = 3$ be green.

Example 4.0.1.

$$B_3(S_{1,1}) = 4. \tag{4.1}$$

Proof. To prove $B_3(S_{1,1}) \geq 4$ it is sufficient to construct an edge coloring of $K_{3,3}$ that does not contain a monochromatic $S_{1,1}$. A proper 3 edge coloring of $K_{3,3}$ is one such construction, displayed in Figure 4.1. Thus $B_3(S_{1,1}) \geq 4$.

To prove $B_3(S_{1,1}) \leq 4$, let $G = K_{4,4}$.

By the pigeonhole principle every vertex receives at least two i-colored edges for some $i \in \{1, 2, 3\}$. Additionally, at least two vertices in $X = X(G)$ will be colored the same color, creating a majority color class in X. Similarly, at least two vertices in $Y = Y(G)$ will be

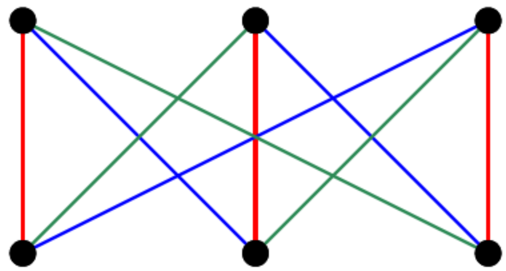

Figure 4.1: A construction of a 3 colored edge coloring on $K_{3,3}$ that does not contain a monochromatic $S_{1,1}$

colored in the same color, creating a majority color class in Y.

Thus, without loss of generality, let two vertices in X be colored red.

Case 1. If at least one vertex in Y is colored red, then G contains a red $S_{1,1}$.

Case 2. Suppose no vertex in Y is colored red. The greatest number of red edges that vertices in Y can be incident to is 4, and the least number of red edges vertices in X can be incident to is 4, thus the total number of red edges in G is 4. The remaining 12 edges in $E(G)$ must be colored blue or green. At least 6 of those edges are the same color by applying the pigeonhole principle. Without loss of generality let them be blue edges, this induces a blue $S_{1,1}$ in G. □

An alternative proof can be found in DeBiasio, Gyárfás, Krueger, Ruszinkó, and Sárközy [4] presented as the bipartite Ramsey number of P_n.

Example 4.0.2.

$$B_3(S_{2,2}) = 7. \tag{4.2}$$

It is important to note that in Example 4.0.2 because $G = K_{7,7}$ and $n = 2$ vertices can be colored with up to two distinct colors. For example if v has $d_1(v) = 3$ and $d_2(v) = 4$ then it would be colored both red and blue.

Remark 4.0.3. In general, a vertex can be colored in multiple colors. When a vertex v is incident to at least $n+1$ i-colored edges and at least $n+1$ j-colored edges such that $i \neq j$, we call the vertex *double colored*, or a *double*. When a vertex v is incident to at least $n+1$ i-colored edges, at least $n+1$ j-colored edges, and at least $n+1$ k-colored edges in three distinct colors i, j, k, we call the vertex *triple colored*, or a *triple*. Vertices colored in exactly one color are called *singles* or *single colored*.

We call the i-colored edges incident to i-colored vertices i-colored *important* edges.

Observation 4.0.4. *Without loss of generality, if the ratio of i-colored important edges in X to vertices not colored i in Y is greater than n, then an i-colored $S_{n,n}$ is induced. This ratio's formula is dependent on the existence of multi-colored vertices.*

Let the host graph be $G = K_{N,N}$ and consider the double star $S_{n,n}$. Partition $X(G)$ into subsets such that vertices in X_i are colored in i. Similarly partition $Y(G)$ into subsets such that vertices in Y_i are colored in i. Let D_x be the number of double colored vertices in X and D_y be the number of double colored vertices in Y.

1. *If there exists only single colored vertices in G then for some $i \in \{1, 2, 3\}$ when*

$$\left\lceil \frac{(N-2n)Y_i}{N-X_i} \right\rceil > n \tag{4.3}$$

or

$$\left\lceil \frac{(N-2n)X_i}{N-Y_i} \right\rceil > n \tag{4.4}$$

hold, G contains a monochromatic $S_{n,n}$.

2. If there exists double colored vertices in G then for some $i \in \{1, 2, 3\}$ when

$$\left\lceil \frac{(N-2n)(Y_i - D_y) + max(\lfloor \frac{N-n}{2} \rfloor, n+1)D_y}{N - X_i} \right\rceil > n \tag{4.5}$$

or

$$\left\lceil \frac{(N-2n)(X_i - X_d) + max(\lfloor \frac{N-n}{2} \rfloor, n+1)D_x}{N - Y_i} \right\rceil > n \tag{4.6}$$

hold, G contains a monochromatic $S_{n,n}$.

Following Observation 3.0.1 the edges incident to two i-colored vertices can induce a monochromatic $S_{n,n}$, we call these edges *forbidden* edges, specifically edges are forbidden in color i. The edges between X_1 and Y_1 are forbidden in color 1, similarly for the edges between X_2 and Y_2, and between X_3 and Y_3.

Observation 4.0.5. *It is apparent if an edge is both forbidden and important in i, then G contains an i-colored $S_{n,n}$. Thus if the number of forbidden edges is greater than the number of non-important edges, G contains a monochromatic $S_{n,n}$.*

Proof of Example 4.0.2. To prove $B_3(S_{2,2}) \geq 7$ we construct an edge coloring of $K_{6,6}$ that does not contain a monochromatic $S_{2,2}$. A 3 colored proper edge coloring of $K_{6,6}$ is one such construction, displayed in Figure 4.2. Thus $B_3(S_{2,2}) \geq 7$.

To prove $B_3(S_{2,2}) \leq 7$ we consider $G = K_{7,7}$. By the pigeonhole principle, every vertex in G must receive at least 3 same colored edges. Let sets X_1, X_2, and X_3 be subsets of X such that vertices in X_i are i-colored. Similarly, let sets Y_1, Y_2, and Y_3 be subsets of Y.

Case 1. Let each vertex be colored with at most one color. That is, $X_1, X_2, X_3, Y_1, Y_2,$

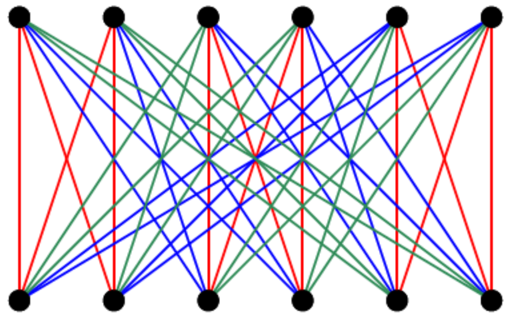

Figure 4.2: A construction of a 3 colored edge coloring on $K_{6,6}$ that does not contain a monochromatic $S_{2,2}$

and Y_3 are all disjoint.

The number of forbidden edges is exactly $\sum_{i=1}^{3} |X_i||Y_i|$. The number of important edges is at least 42 (3 from each vertex in G) and the number of non-important edges is at most 7 ($|E(G)| - 42 = 7^2 - 42$).

Thus following Observation 4.0.5 if

$$\sum_{i=1}^{3} |X_i||Y_i| > 7 \tag{4.7}$$

or Observation 4.0.4 (1) holds, then G contains a monochromatic double star.

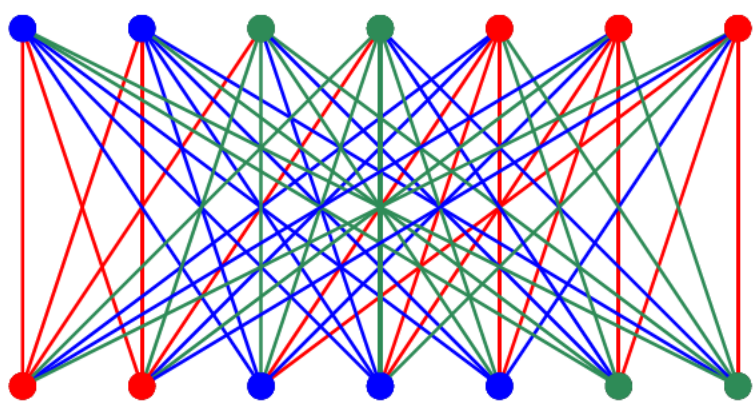

Figure 4.3: The graph shows an example of coloration in case 1, there are both green and blue $S_{2,2}$

Case 2. Let at least one vertex in $V(G)$ be a double colored vertex.

The number of doubles in X is $(\sum_{i=1}^{3}|X_i|)-7$, call this amount D_x. Similarly, the number of doubles in Y is $(\sum_{i=1}^{3}|Y_i|)-7$, call this amount D_y. Double colored vertices have at least $max(\lfloor\frac{N-n}{2}\rfloor, n+1)$ important edges in the same color.

The number of important edges in G is at least $3N$, with $3(N-D_x-D_y)$ important edges from single colored vertices and $3(D_x+D_y)$ important edges from the doubles. Thus the number of non important edges is at most $49-3N$. If there exists a vertex in X that is double colored in the same two colors as a double in Y, then the sum $\sum_{i=1}^{3}|X_i||Y_i|$ over counts the minimum number of forbidden edges. To avoid this we calculate $(\sum_{i=1}^{3}|X_i||Y_i|)-D_xD_y$ as our minimum number of forbidden edges.

Thus following Observation 4.0.5 if

$$\sum_{i=1}^{3}|X_i||Y_i|-D_xD_y>49-3N \tag{4.8}$$

or Observation 4.0.4 (2) hold, then G contains a monochromatic $S_{2,2}$.

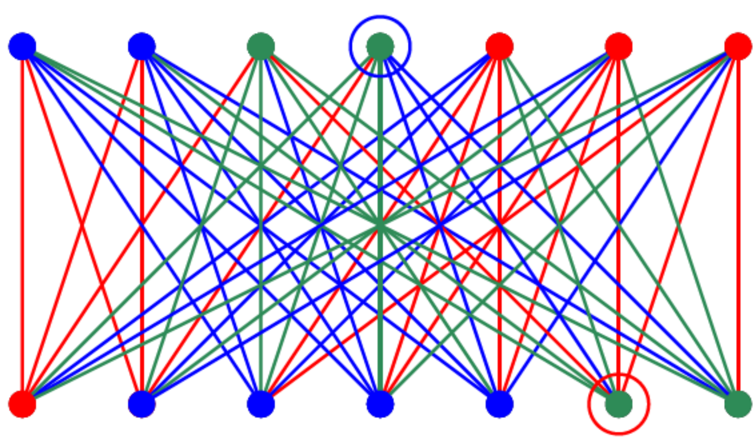

Figure 4.4: The graph shows an example of coloration in case 2, there are red, blue, and green $S_{2,2}$

Using a simple script we are able to compute all possible constructions.

```python
vtxs = [(x1, x2, x3) for x1 in range(5) for x2 in range(5) for x3 in range(5) if x1+x2+x3>= 7]
vectors = [(x1, x2, x3) for x1,x2,x3 in vtxs]
constructions = []
for X in vectors:
    for Y in vectors:
        Xvtx = X[0]+X[1]+X[2]
        Yvtx = Y[0]+Y[1]+Y[2]
        #case 1
        if Xvtx==7 and Yvtx==7:
            if 42<=49-(X[0]*Y[0] + X[1]*Y[1] +X[2]*Y[2]):
                if (math.ceil((3*Y[0])/(7-X[0]))<=2) and (math.ceil((3*Y[1])/(7-X[1]))<=2) and (math.ceil((3*Y[2])/(7-X[2]))<=2) and (math.ceil((3*X[0])/(7-Y[0]))<=2) and (math.ceil((3*X[1])/(7-Y[1]))<=2) and (math.ceil((3*X[2])/(7-Y[2]))<=2):
                    if(not [Y,X] in constructions):
                        constructions.append([X,Y])
        #case 2
        else:
            dY=Yvtx-7
            dX=Xvtx-7
            if 3*(14-dY-dX)+3*(dY+dX)<=49-(X[0]*Y[0]+X[1]*Y[1]+X[2]*Y[2])+dY*dX:
                if (math.ceil((3*(Y[0]-dY)+3*dY)/(7-X[0]))<=2) and (math.ceil((3*(Y[1]-dY)+3*dY)/(7-X[1]))<=2) and (math.ceil((3*(Y[2]-dY)+3*dY)/(7-X[2]))<=2) and (math.ceil((3*(X[0]-dX)+3*dX)/(7-Y[0]))<=2) and (math.ceil((3*(X[1]-dX)+3*dX)/(7-Y[1]))<=2) and (math.ceil((3*(X[2]-dX)+3*dX)/(7-Y[2]))<=2):
                    if(not [Y,X] in constructions):
                        constructions.append([X,Y])
```

Together both cases cover all possible three colored constructions, and the script results in no constructions that fails to hold Equation 4.8 or Observation 4.0.4 (2). Thus every coloring induces a monochromatic double star, therefore $B_3(S_{2,2}) = 7$. □

With specific cases examined and the two colored case as a base, we start with a lower bound for $B_3(S_{n,n})$.

Theorem 4.0.6. *For any positive integer n, $B_3(S_{n,n}) \geq 3n+1$.*

Proof. We start with the bipartite graph $G = K_{3n,3n}$. Partition $X(G)$ into three subsets of size n labeled X_1, X_2, and X_3. Similarly, partition the set $Y(G)$ into Y_1, Y_2, and Y_3. Color the edges between X_i and Y_i red if $i = j$, blue if $j = i+1$ or $j = i-2$, and green if $j = i+2$ or $j = i-1$. This construction is a proper 3 edge coloring of G and avoids a monochromatic $S_{n,n}$. Thus $B_3(S_{n,n}) \geq 3n+1$.

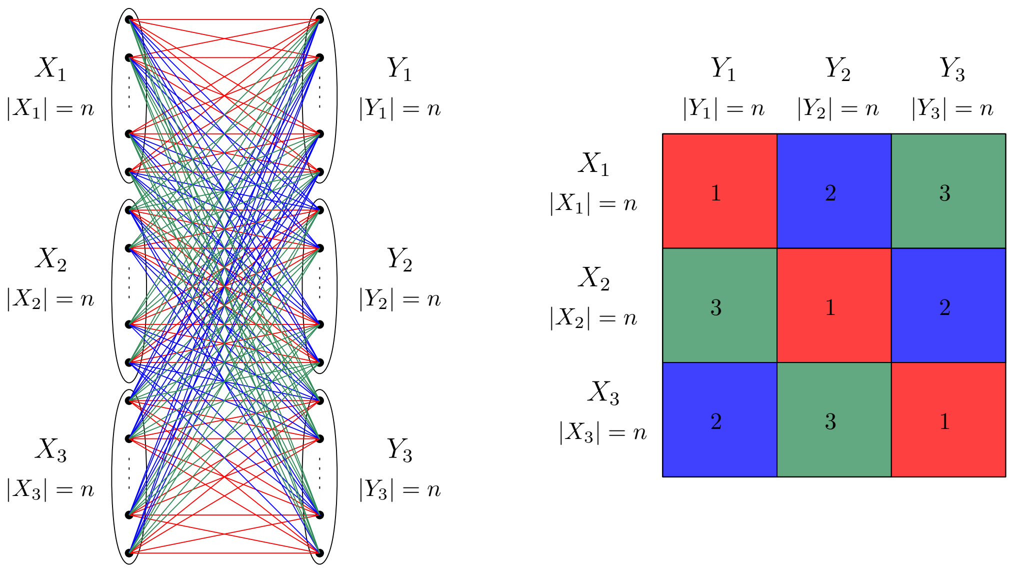

Figure 4.5: The graph (left) shows a proper 3 edge coloring of $K_{3n,3n}$, the matrix (right) displays the same construction.

$\square$

The bound matches the lower bound for $B_r(S_{n,n})$ proved by Alm, Hommowun, Schneider [1].

Remark 4.0.7. This lower bound and the pigeonhole principle imply if G is large enough to guarantee G contains a monochromatic $S_{n,n}$ then every vertex in G must be colored in at least one color.

To consider an upper bound for $B_r(S_{n,n})$, it is tempting to consider $3n+1$ and follow the proof for $r = 2$. Assuming such a bound would hold, then $B_3(S_{n,n}) \leq 3n+1$ and $B_3(S_{3,3}) \leq 10$. However, example 4.0.8 contradicts this claim.

Example 4.0.8. $B_3(S_{3,3}) > 10$.

Let $G = K_{10,10}$, partition $X(G)$ into X_1 and D_x such that vertices in X_1 are singles colored in red and vertices in D_x are colored in blue and green. Partition $Y(G)$ into Y_2 and Y_3 such that vertices in Y_2 are singles colored in blue and vertices in Y_3 are singles colored in green. Color edges between X_1 and Y_2 red and blue, similarly color the edges between X_1 and Y_3 red and green. Color the edges between D_x and Y_2 green and the edges between D_x and Y_3 blue.

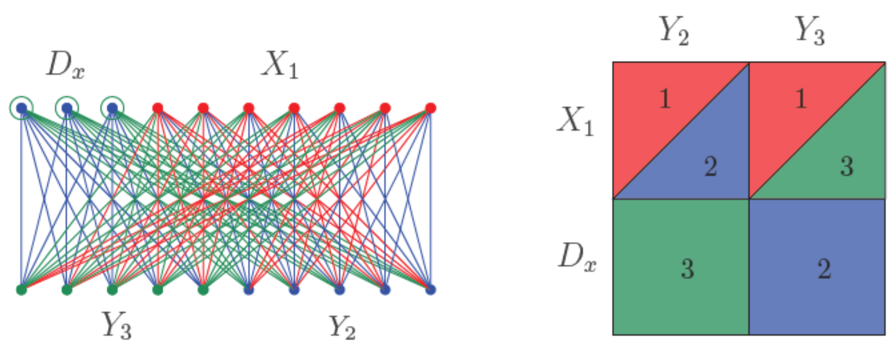

Figure 4.6: A coloration of $K_{10,10}$ which does not contain a monochromatic $S_{3,3}$.

However there must exist some real number, let us call it α, such that $B_3(S_{n,n}) \leq \alpha n$. From this claim we are able to prove the following upper bound.

Theorem 4.0.9. *For any positive integer n, $B_3(S_{n,n}) \leq (3+\sqrt{3})n \approx 4.7321n$.*

Proof. There must exist some real number, let us call it α, such that $B_3(S_{n,n}) \leq \alpha n$.

Let $G = K_{\alpha n, \alpha n}$, so that $|E(G)| = (\alpha n)^2$. The pigeonhole principle guarantees at least $\frac{|E(G)|}{3}$ edges will be colored by some color i. Without loss of generality, let that major color class be red.

Applying Decamillis and Song's [5] extremal result, we know if the number of i-colored edges in G are greater than $\max\{nN, 2n(N-n)\}$ then G contains an i-colored $S_{n,n}$.

Thus $\frac{\alpha^2 n^2}{3} < \max\{\alpha n^2, 2n^2(\alpha - 1)\}$.

Solving for α we have

$$\alpha^2 n^2 < 6n^2(\alpha - 1)$$

$$\alpha^2 < 6(\alpha - 1)$$

$$\alpha^2 - 6\alpha + 6 < 0$$

$$\alpha < 3 + \sqrt{3} \approx 4.7321$$

. Thus $\alpha < 3 + \sqrt{3} \approx 4.7321$ and the partite sets of G are of size $(3 + \sqrt{3})n \approx 4.7321n$. $\square$

Taking into consideration the important edges allows us to further improve the upper bound.

Theorem 4.0.10. *For any positive integer n, $B_3(S_{n,n}) \leq 4n$.*

Proof. Consider $G = K_{\alpha n, \alpha n}$ such that G contains monochromatic double star $S_{n,n}$.

The number of important edges are bounded above by $|E(G)| = (\alpha n)^2$ and bounded below by $|V(G)|(\alpha n - 2n)$. Thus,

$$2(\alpha n)(\alpha n - 2n) \leq (\alpha n)^2$$

$$2\alpha^2 n^2 - 4\alpha n^2 \leq \alpha^2 n^2$$

$$2\alpha^2 - 4\alpha \leq \alpha^2$$

$$\alpha^2 - 4\alpha \leq 0$$

$$\alpha(\alpha - 4) \leq 0$$

$$\alpha \leq 4.$$

Therefor $B_3(S_{n,n}) \leq 4n$.

□

Taking into consideration of doubles and triples (for larger graphs), we are able to improve the bounds on the number of important edges and thus also the upper bounds.

Theorem 4.0.11. *For any positive integer n, $B_3(S_{n,n}) \leq 3.6678 \cdot n$.*

The proof for Theorem 4.0.11 is a continuation of Theorem 5.0.2 and is given together.

Example 4.0.12.

$$B_3(S_{3,3}) = 11. \tag{4.9}$$

Proof of Example 4.0.12. To prove $B_3(S_{3,3}) \geq 11$ we must construct an edge coloring of $K_{10,10}$ that does not contain a monochromatic $S_{3,3}$. Example 4.0.8 presents such a case, thus $B_3(S_{3,3}) \geq 11$.

To prove $B_3(S_{3,3}) \leq 11$ we consider $G = K_{11,11}$. Partition X into X_1, X_2, and X_3 such that vertices in X_i are i-colored. Similarly, partition Y into Y_1, Y_2, and Y_3.

Case 1. Let each vertex be colored with at most one color. That is, X_1, X_2, X_3, Y_1, Y_2, and Y_3 are all disjoint.

The number of forbidden edges is exactly $\sum_{i=1}^{3} |X_i||Y_i|$. The number of important edges is at least 110 ($11 - 2n = 5$ from each vertex in G) and the number of non-important edges is at most 11 ($|E(G)| - 110 = 11^2 - 110$).

Thus following Observation 4.0.5 if

$$\sum_{i=1}^{3} |X_i||Y_i| > 11 \tag{4.10}$$

or Observation 4.0.4 (1) holds, then G contains a monochromatic double star.

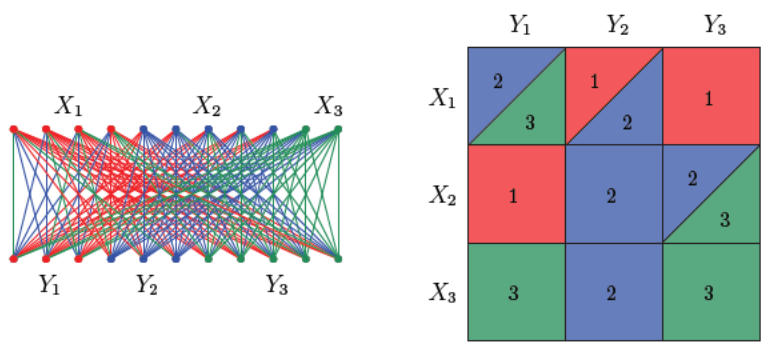

Figure 4.7: The graph shows an example of a coloration in case 1, G contains blue and green $S_{3,3}$.

Case 2. Let at least one vertex in $V(G)$ be a double colored vertex.

The number of doubles in X is $(\sum_{i=1}^{3} |X_i|) - 11$, call this amount D_x. Similarly, the number of doubles in Y is $(\sum_{i=1}^{3} |Y_i|) - 11$, call this amount D_y.

The number of important edges from singles is at least 5, and double colored vertices have at least 4 important edges in the same color. Therefore the number of important edges in G is at least $5(N - D_x - D_y) + 4(D_x + D_y)$. The number of non important edges is at most $121 - 5(N - D_x - D_y) + 4(D_x + D_y)$.

If there exists a vertex in X that is double colored in the same two colors as a double in Y, the sum $\sum_{i=1}^{3} |X_i||Y_i|$ over counts the number of forbidden edges in G. To avoid this $\sum_{i=1}^{3} |X_i||Y_i| - D_x D_y$ is our minimum number of forbidden edges.

Thus following Observation 4.0.5 if

$$\sum_{i=1}^{3} |X_i||Y_i| - D_x D_y > 121 - 5(N - D_x - D_y) + 4(D_x + D_y) \tag{4.11}$$

or Observation 4.0.4 (2) hold, then G contains a monochromatic $S_{3,3}$.

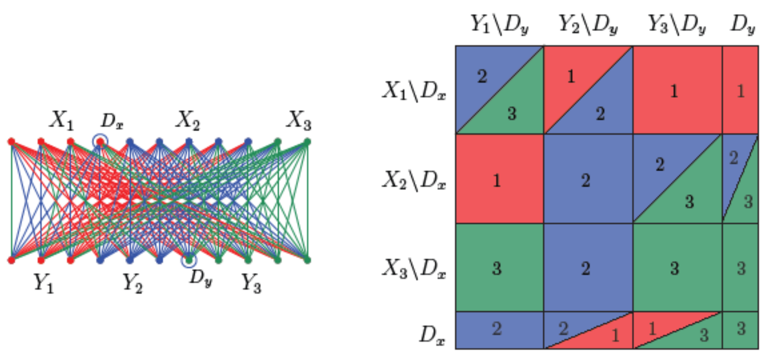

Figure 4.8: The graph shows an example of coloration in case 2, there are red, blue, and green $S_{3,3}$.

Using a simple script we are able to compute all possible constructions.

```
vtxs = [(x1, x2, x3) for x1 in range(9) for x2 in range(9) for x3 in range(9) if x1+x2+x3>= 11]
vectors = [(x1, x2, x3) for x1,x2,x3 in vtxs]
constructions = []
for X in vectors:
    for Y in vectors:
        Xvtx = X[0]+X[1]+X[2]
        Yvtx = Y[0]+Y[1]+Y[2]
        #case 1
        if Xvtx==11 and Yvtx==11:
            if 110<=121-(X[0]*Y[0] + X[1]*Y[1] +X[2]*Y[2]):
```

```
                if (math.ceil((5*Y[0])/(11-X[0]))<=3) and (math.ceil((5*Y[1])/(11-X[1]))<=3) and (math.ceil((5*Y[2])/(11-X[2]))<=3) and (math.ceil((5*X[0])/(11-Y[0]))<=3) and (math.ceil((5*X[1])/(11-Y[1]))<=3) and (math.ceil((5*X[2])/(11-Y[2]))<=3):
                    if(not [Y,X] in constructions):
                        constructions.append([X,Y])
        #case 2
        else:
            dY=Yvtx-11
            dX=Xvtx-11
            if 5*(22-dY-dX)+4*(dY+dX)<=121-(X[0]*Y[0]+X[1]*Y[1]+X[2]*Y[2])+dY*dX:
                if (math.ceil((5*(Y[0]-dY)+4*dY)/(11-X[0]))<=3) and (math.ceil((5*(Y[1]-dY)+4*dY)/(11-X[1]))<=3) and (math.ceil((5*(Y[2]-dY)+4*dY)/(11-X[2]))<=3) and (math.ceil((5*(X[0]-dX)+4*dX)/(11-Y[0]))<=3) and (math.ceil((5*(X[1]-dX)+4*dX)/(11-Y[1]))<=3) and (math.ceil((5*(X[2]-dX)+4*dX)/(11-Y[2]))<=3):
                    if(not [Y,X] in constructions):
                        constructions.append([X,Y])
```

Together both cases cover all possible three colored constructions, and the script results in no constructions that fails to hold Equation 4.11 or Observation 4.0.4 (2). Thus every coloring induces a monochromatic double star, therefore $B_3(S_{3,3}) = 11$. □

This example also breaks the pattern for the previous lower bound, thus we next prove a tighter lower bound.

Theorem 4.0.13. *For any positive integer n, $B_3(S_{n,n}) > \frac{10}{3}n$.*

Proof. Let $n = 3k$, thus $B_3(S_{n,n}) > 10k$.

For $k = 1$, we have $B_3(S_{3,3}) > 10$ proven in Example 4.0.8. Let this construction be called G_1.

For $k > 1$, we construct G_k by replacing each vertex of G_1 with k copies of the vertex, coloring each copy the in the same way. Figure 4.9 illustrates this process when $k = 2$.

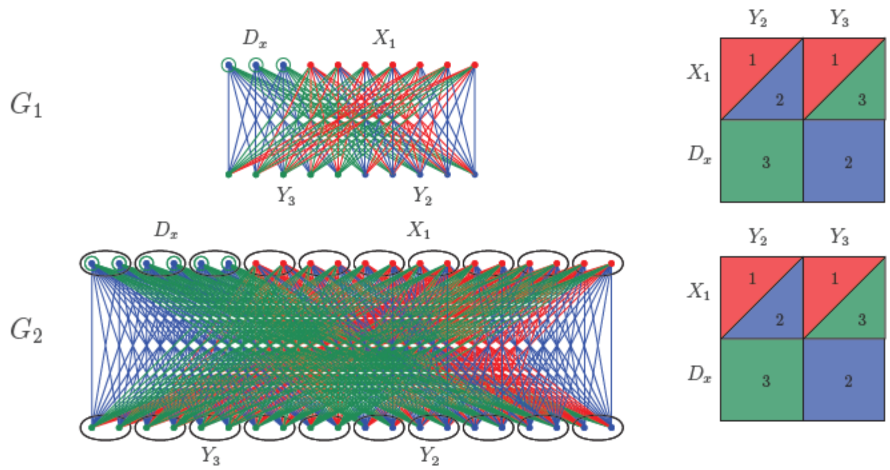

Figure 4.9: The graph both the matrix and graph for G_1 (top) and G_2 (bottom).

Note if vertex v is i-colored in G_1, then if vertex w is a copy of v in G_k, vertex w will be i-colored in G_k. Similarly if vertex v is not colored in i, v satisfies $d_i(v) \leq 3$ in G_1, then if vertex w is a copy of v in G_k, w satisfies $d_i(w) \leq 3k = n$ in G_k. In other words, a vertex is colored with i in G_k if and only if it is a copy of an i-colored vertex in G_1. $\square$

This construction is then blown up and optimized to prove the final lower bound on three colors.

Theorem 4.0.14. *For any positive integer n, $B_3(S_{n,n}) \geq (2+\sqrt{2})n \approx 3.4142n$.*

Proof. Consider $G = K_{(2+\sqrt{2})n,(2+\sqrt{2})n}$. Partition $X(G)$ into X_1 and X_{23} such that $|X_1| = (1+\sqrt{2})n$ and $|X_{23}| = n$. Partition $Y(G)$ into subsets of equal size Y_2 and Y_3 such that $|Y_2| = |Y_3| = (1+\frac{\sqrt{2}}{2})n$. Let $t = |X_1|$ and $s = |Y_2| = |Y_3|$. Let $X_1 = \{x_1, x_2, \ldots, x_t\}$ and $Y_2 = \{y_1, y_2, \ldots, y_s\}$, additionally let $Y_3 = \{y_1, y_2, \ldots, y_r\}$. For each $y_i \in Y_2$, let edges incident to vertices $x_{(i-1)n+1}, x_{(i-1)n+2} \ldots, x_{in}$ with indices all taken modulo t be colored red.

Let all the remaining edges between X_1 and Y_2 be colored blue. Similarly for each $y_i \in Y_3$, let edges incident to vertices $x_{(i-1)n+1}, x_{(i-1)n+2} \ldots, x_{in}$ with indices all taken modulo t be colored red. Let all the edges between X_1 and Y_3 be colored green. Color the edges between X_{23} and Y_2 green and the edges between X_{23} and Y_3 blue.

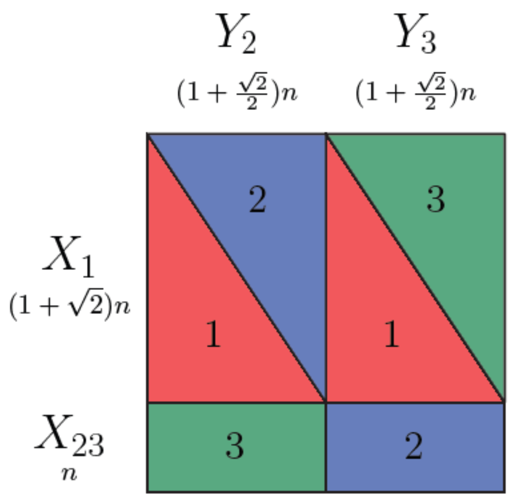

Figure 4.10: The matrix shows the coloring of $G = K_{(2+\sqrt{2})n,(2+\sqrt{2})n}$ that avoids a monochromatic $S_{n,n}$.

$\square$

Continuing to separate all possible constructions into cases for small values of n, examining them, and utilizing algorithms to verify every construction, we claim the following exact values.

Example 4.0.15.

$$B_3(S_{4,4}) = 14 \tag{4.12}$$

$$B_3(S_{5,5}) = 17 \tag{4.13}$$

$$B_3(S_{6,6}) = 21 \tag{4.14}$$

However it is not computationally reasonable to do so for all values of n.

Chapter 5

The Multicolored Case

In this chapter we consider the multicolored case with any $r \geq 2$.

Theorem 5.0.1. *For all $r \geq 2$,*

$$B_r(S_{n,n}) \geq \begin{cases} (\frac{3r}{2} - 1)n + 1, & \text{if } r \text{ is even} \\ \\ (r - 1 + \frac{\sqrt{r^2-1}}{2})n - \frac{r+1}{2}, & \text{if } r \text{ is odd.} \end{cases}$$

This lower bound beats the known lower bound from Example 2.1.3 when $r = 3$ and $r = 5$ (and matches the bound when $r = 4$ and $r = 6$). In the case of $r = 3$ Theorem 4.0.14 holds.

Theorem 5.0.2. *For all $r \geq 2$,*

$$B_r(S_{n,n}) \leq \left(\frac{3r - 5 + \sqrt{r^2 - 2r + 9}}{2}\right) n + 1 = \left(2r - 3 + \frac{2}{r} + O(\frac{1}{r^2})\right) n. \tag{5.1}$$

Note that when $r = 2$, we have $\left(\frac{3r-5+\sqrt{r^2-2r+9}}{2}\right) n + 1 = 2n + 1$, which recovers the known bound from Hattingh and Joubert [9].

Thus by combining Theorem 5.0.1, Theorem 4.0.14, Theorem 5.0.2, with Theorem 4.0.11 we have that for all $r \geq 3$,

$$\left.\begin{array}{ll} (2+\sqrt{2})n-2 & r=3 \\ 5n+1 & r=4 \\ (4+\sqrt{6})n-3 & r=5 \\ (2r-4)n+1 & r\geq 6 \end{array}\right\} \leq B_r(S_{n,n}) \leq \begin{cases} 3.6678n & r=3 \\ \left(\frac{3r-5+\sqrt{r^2-2r+9}}{2}\right)n+1 & r\geq 4 \end{cases}.$$

We begin generalizing the coloring used in Theorem 4.0.14.

Lemma 5.0.3. *Let $2 \leq n < s \leq t$ be integers and $G = K_{t,s}$. If $s - \lfloor \frac{sn}{t} \rfloor \leq n$, then there is a coloring of the edges of G with colors $\{1,2\}$ such that:*

1. *$d_1(v) \leq n$ for all $v \in X(G)$, and*

2. *$d_2(v) \leq n$ for all $v \in Y(G)$.*

Proof of Lemma. Let $X = \{x_1, \ldots, x_t\}$ and $Y = \{y_1, \ldots, y_s\}$. For each $i \in \{1, 2, \cdots, s\}$, let y_i have edges of color 2 to vertices $x_{(i-1)n+1}, x_{(i-1)n+2} \ldots, x_{in}$ where the indices are taken modulo t. Color the remaining edges of G with color 1. Condition (2) is then satisfied by construction.

To show Condition (1), note that for all $v \in X$, we have $d_2(v)$ is either $\lfloor \frac{sn}{t} \rfloor$ or $\lceil \frac{sn}{t} \rceil$. So for all $v \in X$, $d_1(v) = s - d_2(v) \leq s - \lfloor \frac{sn}{t} \rfloor$ which is at most n by assumption. $\square$

When $t = s = 2n$ note that G from Lemma 5.0.3 contains two disjoint copies of $K_{n,n}$ in each color.

Proof of Theorem 5.0.1. Let $G = K_{N,N}$ be colored with r colors.

Case 1. First suppose that r is even. Let R be the set of colors in G such that $|R| = r = 2k$ and let $N = (3k-1)n$. Partition R into two sets $A = \{1, \ldots, k\}$ and $B = \{k+1, \ldots 2k\}$. Also, let $A' = A \setminus \{k\}$ and $B' = B \setminus \{2k\}$.

We partition $X(G)$ into k sets $\{X_i : i \in A\}$, each of size $2n$ and $k-1$ sets $\{X_{j,2k} : j \in B'\}$, each of size n. We similarly partition $Y(G)$ into k single colored sets $\{Y_j : j \in B\}$, each of size $2n$ and $k-1$ double colored sets $\{Y_{i,k} : i \in A'\}$, each of size n. Vertices in a set X_i (or Y_i) are single colored in color i and have degree at most n in all other colors. Likewise, vertices in $X_{i,j}$ (or $Y_{i,j}$) are double colored in colors i and j and have degree at most n in all other colors. We call a set *single colored* if it has one subscript and *double colored* if it has two.

Between X_i and Y_j we color as described in Lemma 5.0.3 so that $d_j(v) \leq n$ for all $v \in X_i$ and $d_i(v) \leq n$ for all $v \in Y_j$. The hypothesis of the lemma is easy to check as both sets have order $2n$. These components cannot contain a copy of $S_{n,n}$.

Color all the edges between $X_{j,2k}$ and Y_i with color j unless $j = i$ in which case we use color $2k$. Color all edges between $Y_{j,k}$ and X_i with color j unless $j = i$ in which case we use color k. Finally, color all edges between $Y_{i,k}$ and $X_{j,2k}$ with color i. These components are all complete bipartite graphs with one side of size n, thus they also cannot contain a copy of $S_{n,n}$.

Case 2. Now consider when r is odd. Let R be the set of colors in G such that $|R| = r = 2k-1$ and let $N = \lfloor \alpha n \rfloor - k$ where $\alpha = r - 1 + \frac{\sqrt{r^2-1}}{2}$. We partition R into two sets $A = \{1, \ldots, k-1\}$ and $B = \{k, \ldots, 2k-1\}$. Let $A' = A \setminus \{k-1\}$ and $B' = B \setminus \{2k-1\}$.

Now we partition $X(G)$ into $k-1$ single colored sets $\{X_i : i \in A\}$, each of size $\lceil \frac{\alpha-(k-1)}{k-1} n \rceil$

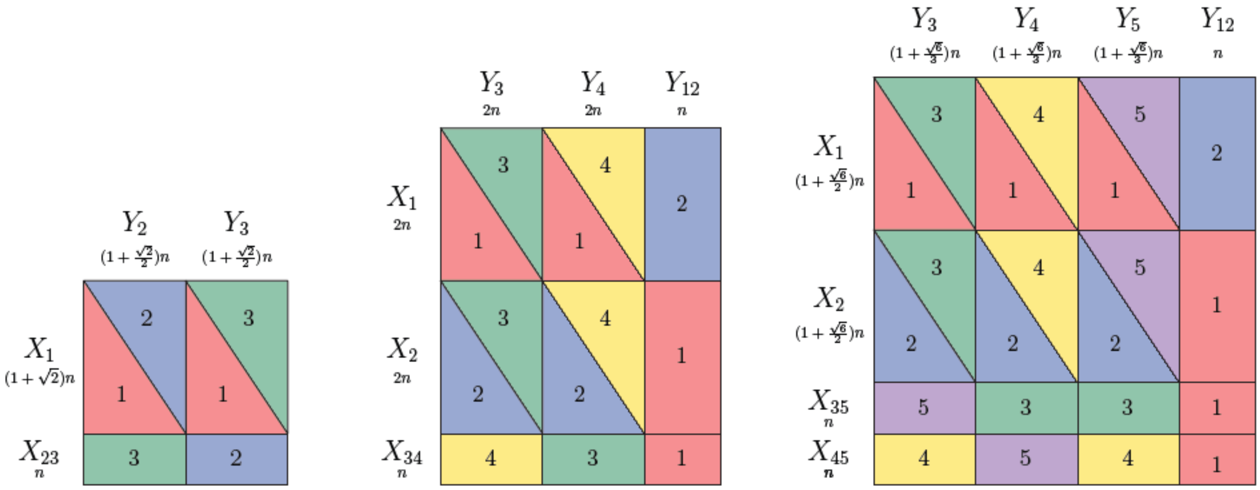

Figure 5.1: The edge coloring between sets used in the construction of the proof of Theorem 5.0.1 with three colors (left), four colors (center), and five colors (right). The size of each set is listed below the set name (ignoring floors and ceilings); for example, when $r = 3$, $|X_1| = (1 + \sqrt{2})n$.

and $k-1$ double colored sets $\{X_{j,2k-1} : j \in B'\}$, each of size at most n. We partition $Y(G)$ into k single colored sets $\{Y_j : j \in B\}$, each of size $\lfloor \frac{\alpha-(k-2)}{k} n \rfloor$ and $k-2$ double colored sets $\{Y_{i,k-1} : i \in A'\}$, each of size at most n.

Between X_i and Y_j we color as described in Lemma 5.0.3 so that $d_j(v) \le n$ for all $v \in X_i$ and $d_i(v) \le n$ for all $v \in Y_j$. To check that the hypothesis of Lemma 5.0.3 is satisfied in this case, first note that $|X_i| = \left\lceil \frac{\alpha-(k-1)}{k-1} n \right\rceil = \left\lceil \frac{k-1+\sqrt{k(k-1)}}{k-1} n \right\rceil = \left\lceil \left(1+\sqrt{\frac{k}{k-1}}\right) n \right\rceil$ and similarly, $|Y_j| = \left\lfloor \frac{\alpha-(k-2)}{k} n \right\rfloor = \left\lfloor \frac{k+\sqrt{k(k-1)}}{k} n \right\rfloor = \left\lfloor \left(1+\sqrt{\frac{k-1}{k}}\right) n \right\rfloor$.

Thus

$$|Y_j| - \left\lfloor \frac{|Y_j|n}{|X_i|} \right\rfloor \le \left\lfloor \left(1+\sqrt{\frac{k-1}{k}}\right) n \right\rfloor - \left\lfloor \frac{\left(1+\sqrt{\frac{k-1}{k}}\right)}{\left(1+\sqrt{\frac{k}{k-1}}\right)} n \right\rfloor = n,$$

where the last equality holds since $\frac{\left(1+\sqrt{\frac{k-1}{k}}\right)}{\left(1+\sqrt{\frac{k}{k-1}}\right)} = \sqrt{\frac{k-1}{k}}$.

Now color all the edges between $X_{j,2k-1}$ and Y_i with color j unless $j = i$ in which case we use color $2k-1$. Color all edges between $Y_{j,k-1}$ and X_i with color j unless $j = i$ in which case we use color $k-1$. Finally, color all edges between $Y_{i,k-1}$ and $X_{j,2k-1}$ with color i. As in the even case, this coloring contains no monochromatic $S_{n,n}$. $\square$

Now we prove the upper bound and main theorem in this thesis.

Proof of Theorem 5.0.2. Start with $G = K_{N,N}$ such that N is an integer with $N \geq rn+1$ and let $\alpha = \frac{N}{n}$. Suppose G is r-colored with no monochromatic $S_{n,n}$. We will later assume that N is larger, but first we prove claims with $N \geq rn+1$. Note that every vertex in G will be colored in at least one color.

For all $i \in [r]$, let z_i be the number of vertices which receive exactly i many colors. For all $\emptyset \neq S \subseteq [r]$, let X_S and Y_S be the set of vertices in $X(G)$ and $Y(G)$ respectively which are colored with exactly the colors in S and let $x_S = |X_S|$ and $y_S = |Y_S|$. For all $i \in [r]$, let $\mathcal{X}_i$ and $\mathcal{Y}_i$ be the set of vertices in X and Y respectively which receive color i (and possibly other colors). For $A \subseteq X$, $B \subseteq Y$, and $S \subseteq [r]$, let $e_S(A,B)$ be the number of edges between A and B which receive any color from S.

Due to our assumption, an important edge of color i is incident to exactly one vertex of color i, otherwise this edge would induce an i colored $S_{n,n}$. Let e^* be the number of important edges. Define σ such that σ^2 is the proportion of edges which are *not* important. We have

$$\sigma^2 N^2 = \sum_{\emptyset \neq S_1, S_2 \subseteq [r]} e_{[r]\setminus(S_1 \cup S_2)}(X_{S_1}, Y_{S_2}) \geq \sum_{i \in [r]} x_i y_i. \tag{5.2}$$

Note that by the definition of z_i, we have that for all $i \in [r]$ and all vertices v which receive exactly i colors, v is incident with at least $N-(r-i)n$ important edges. Thus we have the following bounds on e^*,

$$\sum_{i\in[r]} z_i(N-(r-i)n) \le e^* = (1-\sigma^2)N^2. \tag{5.3}$$

Our first claim gives an upper bound on the number of vertices which are colored with more than one color. Note that a higher proportion of non-important edges causes a smaller proportion of the vertices to have more than one color.

Claim 5.0.4.

$$\sum_{i=2}^{r} z_i \le (2r-2-\alpha(1+\sigma^2))N.$$

Proof of claim. Expanding, canceling, and simplifying (5.3) gives

$$\sum_{i=2}^{r} z_i \le z_2 + 2z_3 + \cdots + (r-1)z_r \le (2r-2-\alpha(1+\sigma^2))N. \qquad \blacksquare$$

The next claim gives an absolute upper bound on the order of an individual set X_i or Y_i.

Claim 5.0.5. *For all $i \in [r]$ we have $x_i \le \frac{N}{\alpha-(r-1)}$ and $y_i \le \frac{N}{\alpha-(r-1)}$.*

Proof of claim. For all $i \in [r]$ we have

$$x_i(N-(r-1)n) \le e_i(X_i, Y) = e_i(X_i, Y - \mathcal{Y}_i) \le n(N - |\mathcal{Y}_i|),$$

and thus

$$x_i \le \frac{N-|\mathcal{Y}_i|}{\alpha-(r-1)} \le \frac{N}{\alpha-(r-1)}.$$

Likewise for y_i,

$$y_i \leq \frac{N - |\mathcal{X}_i|}{\alpha - (r-1)} \leq \frac{N}{\alpha - (r-1)}.$$

■

The final claim gives an upper bound on the number of vertices which receive exactly one color.

Claim 5.0.6. *Let $C \in \mathbb{R}^+$. If there are exactly t indices $i \in [r]$ such that $\max\{x_i, y_i\} \geq \frac{\sigma N}{C}$, then*

$$z_1 = \sum_{i \in [r]} (x_i + y_i) \leq \left(\frac{t}{\alpha - (r-1)} + (r-t)\frac{\sigma}{C} + C\sigma \right) N.$$

Proof of claim. First note that if $\sigma = 0$, then $x_i > 0$ implies that $y_i = 0$ and vice versa. Hence Claim 5.0.5 implies that $z_1 \leq \frac{r}{\alpha-(r-1)} N$ and so the claim holds in this case. So we may assume that $\sigma > 0$ for the remainder. Without loss of generality, suppose that $\max\{x_i, y_i\} \geq \frac{\sigma N}{C}$ for all $i \in [t]$ and $\max\{x_i, y_i\} < \frac{\sigma N}{C}$ for all $i \in [r] \setminus [t]$.

Note that for all $i \in [t]$, we have $\max\{x_i, y_i\} \min\{x_i, y_i\} = x_i y_i$ and since $i \in [t]$, we have $\max\{x_i, y_i\} \geq \frac{\sigma N}{C}$ and thus

$$\min\{x_i, y_i\} \leq \frac{x_i y_i}{\frac{\sigma N}{C}}. \tag{5.4}$$

For all $i \in [r] \setminus [t]$, we have $\max\{x_i, y_i\} < \frac{\sigma N}{C}$ and thus $\frac{x_i}{\frac{\sigma N}{C}}, \frac{y_i}{\frac{\sigma N}{C}} < 1$. From this (and the fact that for all real numbers $0 \leq a, b \leq 1$, we have $a + b \leq 1 + ab$) we have

$$x_i + y_i \leq \frac{\sigma N}{C} + \frac{x_i y_i}{\frac{\sigma N}{C}}. \tag{5.5}$$

Using (5.4) and (5.5) together with Claim 5.0.5, we have

$$\begin{aligned} z_1 = \sum_{i\in[r]}(x_i+y_i) &= \sum_{i\in[t]}(\max\{x_i,y_i\}+\min\{x_i,y_i\}) + \sum_{i\in[r]\setminus[t]}(x_i+y_i) \\ &\le \frac{t}{\alpha-(r-1)}N + \sum_{i\in[t]}\frac{x_iy_i}{\frac{\sigma N}{C}} + \sum_{i\in[r]\setminus[t]}\frac{\sigma N}{C} + \frac{x_iy_i}{\frac{\sigma N}{C}} \\ &= \frac{t}{\alpha-(r-1)}N + (r-t)\frac{\sigma N}{C} + \sum_{i\in[r]}\frac{x_iy_i}{\frac{\sigma N}{C}} \\ &\overset{(5.2)}{\le} \frac{t}{\alpha-(r-1)}N + (r-t)\frac{\sigma N}{C} + C\sigma N, \end{aligned}$$

as desired. ■

Now we prove Theorem 5.0.2. Let N be an integer with $N > \left(\frac{3r-5+\sqrt{r^2-2r+9}}{2}\right)n$, set $\alpha = \frac{N}{n}$, and note that

$$\alpha > \frac{3r-5+\sqrt{r^2-2r+9}}{2}. \tag{5.6}$$

We now combine Claim 5.0.4 and Claim 5.0.6 to get a contradiction with (5.6).

Case 1. ($\sigma = 0$) Applying Claim 5.0.6 (with $C = 1$) we see that since $\sigma = 0$ we have that there are exactly r indices with $i \in [r]$ such that $\max\{x_i, y_i\} \ge 0 = \frac{\sigma N}{C}$ and thus Claim 5.0.6 together with Claim 5.0.4 gives

$$2N = z_1 + \sum_{i=2}^{r} z_i \le \frac{r}{\alpha-(r-1)}N + (2r-2-\alpha)N = \left(\frac{r}{\alpha-(r-1)} + 2r-2-\alpha\right)N$$

which contradicts (5.6).

Case 2. ($\sigma > 0$)

Set $C = (\alpha-(r-1))\sigma$. Let t be the number of indices where $\max\{x_i, y_i\} \ge \frac{\sigma N}{C} = \frac{N}{\alpha-(r-1)}$.

Now Claim 5.0.6 (with $C = (\alpha - (r-1))\sigma$) and Claim 5.0.4 implies

$$\begin{aligned}
2N = z_1 + \sum_{i=2}^{r} z_i &\leq \left(\frac{t}{\alpha - (r-1)} + (r-t)\frac{\sigma}{C} + C\sigma\right) N + (2r - 2 - \alpha(1+\sigma^2))N \\
&= \left(\frac{t}{\alpha - (r-1)} + \frac{r-t}{\alpha - (r-1)} + (\alpha - (r-1))\sigma^2 + 2r - 2 - \alpha(1+\sigma^2)\right) N \\
&= \left(\frac{r}{\alpha - (r-1)} + 2r - 2 - \alpha - \sigma^2(r-1)\right) N \\
&\leq \left(\frac{r}{\alpha - (r-1)} + 2r - 2 - \alpha\right) N
\end{aligned}$$

which, as before, contradicts (5.6). $\square$

Now we prove Theorem 4.0.11 using the claims proven.

Proof of Theorem 4.0.11. Let $G = K_{N,N}$ such that $N \geq 3.6678n$ and is an integer. Let $\alpha = \frac{N}{n}$, and note that $\alpha \geq 3.6678$. (The exact bound we will get from our calculations is actually the largest of the three real solutions to the cubic polynomial $4\alpha^3 - 20\alpha^2 + 19\alpha + 2 = 0$. However, the exact form of this solution is quite ugly, so we give the approximation 3.6678 instead).

Next note that for any positive integer k,

$$\sigma(k - \alpha\sigma) \leq \frac{k^2}{4\alpha} \tag{5.7}$$

with the maximum occurring when $\sigma = \frac{k}{2\alpha}$.

When $\sigma = 0$ we do the same as above, but note that since $r = 3$, there is one side, without loss of generality say $X(G)$, in which at most one of $\{X_1, X_2, X_3\}$ is non-empty. This fact together with Claim 5.0.5 and Claim 5.0.4 implies

$$N = |X| \leq \frac{1}{\alpha - 2}N + (4 - \alpha)N,$$

which is a contradiction when $\alpha > \frac{5+\sqrt{5}}{2} \approx 3.618$.

When $\sigma > 0$, Claim 5.0.6 (with $C = 1$) and Claim 5.0.4 imply

$$2N = z_1 + (z_2 + z_3) \leq (\frac{t}{\alpha - 2} + (3 - t)\sigma + \sigma)N + (4 - \alpha(1 + \sigma^2))N. \tag{5.8}$$

If $t = 0$, then (5.8) simplifies to

$$\begin{aligned} 2N = z_1 + (z_2 + z_3) \leq 4\sigma N + (4 - \alpha(1 + \sigma^2))N &= (4 - \alpha + \sigma(4 - \alpha\sigma))N \\ &\overset{(5.7)}{\leq} (4 - \alpha + \frac{4}{\alpha})N, \end{aligned}$$

which is a contradiction when $\alpha > 1 + \sqrt{5} \approx 3.2361$.

When $t \geq 1$, note that there is some set $W \in \{X_1, X_2, X_3, Y_1, Y_2, Y_3\}$ which has order at least σN. So by Claim 5.0.5, we have $\sigma N \leq |W| \leq \frac{1}{\alpha-2}N$, and thus

$$\sigma \leq \frac{1}{\alpha - 2}. \tag{5.9}$$

Now when $1 \leq t \leq 2$, (5.8) gives us

$$\begin{aligned}
2N = z_1 + (z_2 + z_3) &\leq (\frac{t}{\alpha - 2} + (3 - t)\sigma + \sigma)N + (4 - \alpha(1 + \sigma^2))N \\
&\overset{(5.9)}{\leq} \left(\frac{2}{\alpha - 2} + 4 - \alpha + \sigma(2 - \alpha\sigma)\right) N \\
&\overset{(5.7)}{\leq} \left(\frac{2}{\alpha - 2} + 4 - \alpha + \frac{1}{\alpha}\right) N
\end{aligned}$$

which is a contradiction when $\alpha > \frac{3+\sqrt{17}}{2} \approx 3.5616$.

Finally when $t = 3$, we may suppose without loss of generality that x_1, y_2, and y_3 are at least σN. Thus

$$\sigma^2 N^2 \geq \sum_{i=1}^{3} x_i y_i = \sum_{i=1}^{3} \min\{x_i, y_i\} \max\{x_i, y_i\} \geq \sigma N \sum_{i=1}^{3} \min\{x_i, y_i\},$$

which implies

$$x_2 + x_3 \leq y_1 + x_2 + x_3 \leq \sigma N. \tag{5.10}$$

Now by Claims 5.0.5 and (5.10) we have

$$\begin{aligned}
N = |X| = x_1 + (x_2 + x_3) &\leq \frac{N}{\alpha - 2} + \sigma N + (4 - \alpha(1 + \sigma^2))N \\
&= (\frac{1}{\alpha - 2} + 4 - \alpha + \sigma(1 - \alpha\sigma))N \\
&\overset{(5.7)}{\leq} (\frac{1}{\alpha - 2} + 4 - \alpha + \frac{1}{4\alpha})N
\end{aligned}$$

which is a contradiction when $\alpha \geq 3.6678$. $\square$

Chapter 6

Conclusion

6.1 Summary

Combining the known bounds and the bounds for $B_r(S_{n,n}$ proven in this thesis we have

$$\left.\begin{array}{ll} 2n+1 & r=2 \\ (2+\sqrt{2})n-2 & r=3 \\ 5n+1 & r=4 \\ (4+\sqrt{6})n-3 & r=5 \\ (2r-4)n+1 & r\geq 6 \end{array}\right\} \leq B_r(S_{n,n}) \leq \begin{cases} 2n+1 & r=2 \\ 3.6678n & r=3 \\ \left(\frac{3r-5+\sqrt{r^2-2r+9}}{2}\right)n+1 & r\geq 4 \end{cases}$$

with the new results proven in this thesis displayed in colored text.

6.2 Open Problems

The immediate continuation of the problem is to continue improving the bounds and solve for an exact equation to calculate $B_r(S_{n,n})$ for any $r \geq 2$. While examining the exact values determined and comparing to the current bounds, it appears that the lower bound is a tighter bound than the upper bound. Thus improving the upper bound will have the most impact in finishing the problem.

In addition to further improving $B_r(S_{n,n})$, it is interesting to consider the case of multicolored bipartite Ramsey numbers of unbalanced double stars, $B_r(S_{n,m})$. The unbalanced variation of this problem has been found to behave differently than the bounds for $B_r(S_{n,n})$.

Grossman, Harary and Klawe [8] proved that

$$R(S_{n,m}) = \begin{cases} \max\{2n+1, n+2m+2\}, & \text{if } n \text{ is odd and } m \leq 2 \\ \max\{2n+2, n+2m+2\}, & \text{if } n \text{ is even or } m \geq 3, \text{ and } n \leq \sqrt{2}m \text{ or } n \geq 3m \end{cases}$$

and conjectured that their result would hold when $\sqrt{2}m < n < 3m$. Recently Norin, Sun, and Zhao [12] proved this was not the case, when $n = 2m$ they showed that $S_{2m,m} \geq 4.2m$. In 2023, Flores Dubó and Stein [7] proved that $S_{2m,m} \leq 4.275m$.

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